ESTIMATING DISCRETE CURVATURES IN TERMS OF BETA NUMBERS

SŁAWOMIR KOLASIŃSKI

Abstract. For an arbitrary Radon measure μ we estimate the integrated discrete curvature of μ in terms of its centred variant of Jones' beta numbers. We farther relate integrals of centred and non-centred beta numbers. As a corollary, employing the recent result of Tolsa [Calc. Var. PDE, 2015], we obtain a partial converse of the theorem of Meurer [arXiv:1510.04523].

1. Introduction

Let μ be a Radon measure over $\mathbb{R}^n$. Whenever $x_0, x_1, \dots, x_{m+1} \in \mathbb{R}^n$, let $h_{\min}(x_0, \dots, x_{m+1})$ be the minimal length of a height of the simplex spanned by $x_0, \dots, x_{m+1}$, that is,

$$h_{\min}(x_0, \dots, x_{m+1}) = \min\bigl\{\operatorname{dist}\bigl(x_j, \operatorname{aff}(\{x_0, \dots, x_{m+1}\} \sim \{x_j\})\bigr) : j = 0, 1, \dots, m+1\bigr\},$$

where $\operatorname{aff}(A)$ denotes the smallest affine plane containing the set $A \subseteq \mathbb{R}^n$. For $p \in [1, \infty)$, and $\alpha \in [0,1]$, and $x \in \mathbb{R}^n$, and $r \in (0, \infty)$ define

$$\mathcal{K}^{\alpha}_{\mu,p}(x,r) = \int_{\mathbf{B}(x,r)} \cdots \int_{\mathbf{B}(x,r)} \frac{h_{\min}(x_0, \dots, x_{m+1})^p}{\operatorname{diam}(\{x_0, \dots, x_{m+1}\})^{m(m+1)+(1+\alpha)p}} \,\mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_{m+1}).$$

The functionals obtained by integrating $\mathcal{K}^{\alpha}_{\mu,p}(x, \infty)$ with respect to $x \in \mathbb{R}^n$ were studied in [Kol15b, KSv15, KS13, SvdM11] in the context of solving variational problems with topological constraints and the search for a canonical embedding of a manifold into $\mathbb{R}^n$. Relation to the Sobolev and Sobolev-Slobodeckij spaces were found in [BK12, KSvdM13]. Rectifiability properties of measures for which $\mathcal{K}^{\alpha}_{\mu,p}(x,r)$ is finite μ almost everywhere were obtained in [Meu15, Kol15a]. Criteria involving $\mathcal{K}^{0}_{\mu,p}(x,r)$ for uniform rectifiability and the geometric (p,p) property (see [DS93, Definition 1.2 on p. 313]) were given in [LW11, LW09]. Similar expressions were also used in [LW12] to approximate the least square error of a measure.

We denote by $\mathbf{A}(n,m)$ the set of m dimensional affine planes in $\mathbb{R}^n$. For $L \subseteq \mathbb{R}^n$, and $x \in \mathbb{R}^n$, and $r \in (0,\infty)$, and $p \in [1, \infty)$ define

$$\beta_{\mu,p}(x,r,L) = \frac{1}{r}\left(\frac{1}{r^m}\int_{\mathbf{B}(x,r)} \operatorname{dist}(y,L)^p \,\mathrm{d}\mu(y)\right)^{1/p},$$

$$\beta_{\mu,p}(x,r) = \inf\bigl\{\beta_{\mu,p}(x,r,L) : L \in \mathbf{A}(n,m)\bigr\},$$

$$\mathring{\beta}_{\mu,p}(x,r) = \inf\bigl\{\beta_{\mu,p}(x,r,L) : L \in \mathbf{A}(n,m),\ x \in L\bigr\}.$$

The numbers $\beta_{\mu,\infty}$ were first introduced in [Jon90] and the $\beta_{\mu,p}$ numbers in [DS91, DS93]. They play an important role in harmonic analysis and, most notably, in questions involving boundedness of singular integral operators on $L^2(\mu)$; see, e.g., [DS93, Paj02, Tol14]. They

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also appear naturally in regularity theory for stationary and minimising harmonic maps; see [NV15, (1.31)]. The same notion under the name "height-excess" is extensively used in regularity theory for varifolds; see, e.g., [All72, 8.16(9)] and [Sch04, Sch09, Men09, Men10, Men11, Men12, KM15].

Following [Fed69, 3.2.14] we say that $E \subseteq \mathbb{R}^n$ is *countably* (μ, m) *rectifiable* if there exists a countable family $\mathcal{A}$ of m-dimensional submanifolds of $\mathbb{R}^n$ of class $\mathscr{C}^1$ such that $\mu(E \sim \bigcup \mathcal{A}) = 0$. For $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ we define the densities

$$\boldsymbol{\Theta}^m(\mu, x, r) = \frac{\mu(\mathbf{B}(x, r))}{\boldsymbol{\alpha}(m) r^m}, \quad \boldsymbol{\Theta}^{m*}(\mu, x) = \limsup_{r \downarrow 0} \boldsymbol{\Theta}^m(\mu, x, r),$$

where $\boldsymbol{\alpha}(m)$ denotes the Lebesgue measure of a unit ball in $\mathbb{R}^m$.

In [Tol15, AT15] the authors show that if $\mu(\mathbb{R}^n) < \infty$, and $0 < \boldsymbol{\Theta}^{m*}(\mu, x) < \infty$ for μ almost all x, and $\gamma \in [0, \infty)$, then $\mathbb{R}^n$ is countably (μ, m) rectifiable if and only if

$$\int_0^1 \boldsymbol{\Theta}^m(\mu, x, r)^\gamma \beta_{\mu,2}(x, r)^2 \frac{\mathrm{d}r}{r} < \infty \quad \text{for } \mu \text{ almost all } x. \tag{1}$$

In [AT15, Theorem 1.4] they also show that in case $m = 1$ and $n = 2$ we have

$$\int c^2 \, \mathrm{d}\mu^3 + \mu(\mathbb{R}^2) \approx \int \int_0^1 \boldsymbol{\Theta}^m(\mu, x, r) \beta_{\mu,2}(x, r)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\mu(x) + \mu(\mathbb{R}^2),$$

where $c(x, y, z) = 4\mathcal{H}^2(\operatorname{conv}\{x, y, z\})(|x - y||y - z||z - x|)^{-1}$ is the *Menger curvature* of the triple $(x, y, z) \in (\mathbb{R}^2)^3$ and $A \approx B$ means that there exists a constant $\Delta > 0$ such that $A \le \Delta B$ and $B \le \Delta A$. The Menger curvature of μ, i.e., $\int c^2 \, \mathrm{d}\mu^3$ has played a crucial role in the proof of the Vitushkin's conjecture on removable sets for bounded analytic functions; see [Dav98, Lég99]. The expression $\int \mathcal{K}^0_{\mu,2}(x, \infty) \, \mathrm{d}\mu(x)$ can be seen as a generalisation of $\int c^2 \, \mathrm{d}\mu^3$ to the case $m > 1$, although it does not coincide with the Menger curvature if $m = 1$. Different expressions, which do coincide with Menger curvature for $m = 1$, were suggested in [LW09, LW11]. In [Meu15] it is also shown that if $E \subseteq \mathbb{R}^n$ is Borel, and $\mu = \mathcal{H}^m \llcorner E$, and $\int \mathcal{K}^0_{\mu,2}(x, \infty) \, \mathrm{d}\mu(x) < \infty$, then E is countably $(\mathcal{H}^m, m)$ rectifiable.

In this note we prove the following two lemmas

1.1. **Lemma.** *Let* $\alpha \in [0, 1]$*, and* $p \in [1, \infty)$*, and* $R \in (0, \infty]$*, and* $x \in \mathbb{R}^n$*. Then*

$$\mathcal{K}^\alpha_{\mu,p}(x, R) \le \Gamma \int_0^{2R} \boldsymbol{\Theta}^m(\mu, x, r)^m \frac{\mathring{\beta}_{\mu,p}(x, r)^p}{r^{\alpha p}} \frac{\mathrm{d}r}{r},$$

where $\Gamma = \Gamma(m, p, \alpha) \in [1, \infty)$*.*

and

1.2. **Lemma.** *Let* $p, q \in [1, \infty]$ *satisfy* $q \le p$*, and* $\gamma \in [0, \infty)$*, and* $\alpha \in [0, 1]$*, and* $\rho \in (0, \infty]$*. Then*

$$\int_0^\rho \boldsymbol{\Theta}^m(\mu, x, r)^\gamma \frac{\beta_{\mu,p}(x, r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} < \infty \quad \textit{for } \mu \textit{ almost all } x \tag{2}$$

if and only if

$$\int_0^\rho \boldsymbol{\Theta}^m(\mu, x, r)^\gamma \frac{\mathring{\beta}_{\mu,p}(x, r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} < \infty \quad \textit{for } \mu \textit{ almost all } x. \tag{3}$$

Moreover, there exists $\Gamma = \Gamma(n, m, p, q, \alpha, \gamma) \in (0, \infty)$ such that for any cube $Q \subseteq \mathbb{R}^n$

$$\int_Q \int_0^\rho \mathbf{\Theta}^m(\mu, x, r)^\gamma \frac{\mathring{\beta}_{\mu,p}(x, r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \le \Gamma \int_{3Q} \int_0^{12\rho\sqrt{n}} \mathbf{\Theta}^m(\mu, x, r)^\gamma \frac{\beta_{\mu,p}(x, r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \,,$$

whenever the last integral is finite.

Combining 1.1 and 1.2 and [Tol15] we obtain a partial converse of [Meu15].

1.3. **Corollary.** *Assume $\mathbb{R}^n$ is countably (μ, m) rectifiable and $\mu(\mathbb{R}^n) < \infty$ and there exists $M \in (0, \infty)$ such that $\mu(\mathbf{B}(x, r)) < M\boldsymbol{\alpha}(m)r^m$ for all $x \in \operatorname{spt}\mu$ and all $r > 0$. Then $\mathcal{K}^0_{\mu,2}(x, \infty)$ is finite for μ almost all x.*

Our result can be directly compared with [LW11] where the authors provide similar comparison between discrete curvatures and beta numbers. However, in [LW11] it is assumed a priori that μ is Ahlfors-David regular, i.e., that there exists a constant $C \in [1, \infty)$ such that

$$C^{-1} r^m \le \mu(\mathbf{B}(x, r)) \le C r^m \quad \text{for } x \in \operatorname{spt}\mu \text{ and } 0 < r \le \operatorname{diam}(\operatorname{spt}\mu) \,.$$

In 1.1 and 1.2 we do not assume any bounds on $\mu(\mathbf{B}(x, r))$ but we obtain the density term $\mathbf{\Theta}^m(\mu, x, r)$ in the estimates. The second difference is that in [LW11] the comparison is proven for integrals of the type $\int_{\mathbf{B}(y,r)} \mathcal{K}^0_{\mu,2}(x, r) \,\mathrm{d}\mu(x)$ rather than for $\mathcal{K}^0_{\mu,2}(x, r)$ itself. Our results can be applied even if $\mathcal{K}^0_{\mu,2}(x, r)$ is not integrable on any ball. In particular, to derive 1.3 we use [Tol15] which states that countable (μ, m) rectifiability of $\mathbb{R}^n$ implies merely (1). Thirdly, we provide the comparison for arbitrary values of $\alpha \in [0, 1]$ and $p \in [1, \infty)$ which allows to translate some of the results of [Kol15b, BK12, KSv15, KSvdM13, KS13] to the language of beta numbers; see 3.2. On the other hand we use a simpler integrand than that used in [LW11] which is less singular because it does not include the product of side lengths of the simplex $\operatorname{conv}\{x_0, \dots, x_{m+1}\}$ in the denominator. This actually simplifies dramatically the analysis and allows us to omit the inventive “geometric multipoles” construction of [LW11].

2. Preliminaries

The integers are denoted by $\mathbb{Z}$ and $\mathbb{N}$ is the set of non-negative integers. The numbers $m, n \in \mathbb{N}$ such that $0 < m < n$ will be fixed throughtout the paper. The symbol μ shall always denote a Radon measure over $\mathbb{R}^n$.

We say that $Q \subseteq \mathbb{R}^n$ is a cube if there exist $a \in \mathbb{R}^n$ and $\lambda \in (0, \infty)$ such that $Q = \{a + \lambda x : x \in [0, 1)^n\}$; in such case λ is said to be the *side length of* Q and is denoted by $l(Q)$. Whenever Q is a cube and $k \in (0, \infty)$ we denote by kQ the cube with the same centre as Q and side length equal $kl(Q)$. We fix a lattice $\mathcal{D}$ of dyadic cubes in $\mathbb{R}^n$ by setting

$$\mathcal{D} = \big\{ \{2^{-k}(a + x) : x \in [0, 1)^n\} : \text{for some } k \in \mathbb{Z} \text{ and } a \in \mathbb{Z}^n \big\} \,.$$

Observe that whenever $Q, R \in \mathcal{D}$ and $l(Q) = l(R)$, then $Q \cap R = \varnothing$.

For $L \subseteq \mathbb{R}^n$ and a cube $Q \subseteq \mathbb{R}^n$ we define

$$\mathbf{\Theta}^m(\mu, Q) = \frac{\mu(Q)}{l(Q)^m} \,, \quad \beta_{\mu,p}(Q, L) = \frac{1}{l(Q)} \bigg(\frac{1}{l(Q)^m} \int_Q \operatorname{dist}(y, L)^p \,\mathrm{d}\mu(y) \bigg)^{1/p} \,,$$
$$\beta_{\mu,p}(Q) = \inf\big\{ \beta_{\mu,p}(Q, L) : L \in \mathbf{A}(n, m) \big\} \,.$$

3. Controlling the Menger-like curvature by the beta numbers

Proof of 1.1. For $y \in \mathbb{R}^n$ set

$$U(x,y) = \big\{ (z_1, \ldots, z_m) \in (\mathbb{R}^n)^m : |z_j - x| \le |y - x| \text{ for } j = 1,2,\ldots,m \big\} \,,$$
$$\mathcal{E}(x,y) = \int_{U(x,y)} \frac{h_{\min}(x,y,z_1,\ldots,z_m)^p}{\operatorname{diam}(\{x,y,z_1,\ldots,z_m\})^{m(m+1)+(1+\alpha)p}} \, \mathrm{d}\mu^m(z_1,\ldots,z_m) \,.$$

Since the integrand of $\mathcal{K}^{\alpha}_{\mu,p}(x,r)$ is invariant under permutations of $(x_1,\ldots,x_{m+1})$ we have

$$\mathcal{K}^{\alpha}_{\mu,p}(x,r) = (m+1) \int_{\mathbf{B}(x,r)} \mathcal{E}(x,y) \, \mathrm{d}\mu(y) \,. \tag{4}$$

For $x_0, x_1 \in \mathbb{R}^n$, and $L \in \mathbf{A}(n,m)$, and $j \in \{0,1,\ldots,m+1\}$ define

$$U^L_j(x_0,x_1) = \left\{ (x_2,\ldots,x_{m+1}) \in U(x_0,x_1) : \begin{array}{l} \operatorname{dist}(x_i,L) \le \operatorname{dist}(x_j,L) \text{ and } \operatorname{dist}(x_j,L) > 0 \\ \text{for each } i \in \{0,1,\ldots,m+1\} \end{array} \right\} .$$

Observe that for any $L \in \mathbf{A}(n,m)$ and $x_0,\ldots,x_{m+1} \in \mathbb{R}^n$, using [Kol15a, 8.4],

$$h_{\min}(x_0,\ldots,x_{m+1}) \le 2(m+2) \max\big\{ \operatorname{dist}(x_i,L) : i = 0,1,\ldots,m+1 \big\}$$
$$\text{and} \quad \operatorname{diam}(\{x_0,\ldots,x_{m+1}\}) \ge \max\big\{ |x_i - x_0| : i = 0,1,\ldots,m+1 \big\} \,;$$

hence, if $x_1 \in \mathbb{R}^n$ and $j \in \{0,1,\ldots,m+1\}$, then

$$\mathcal{E}(x_0,x_1) \le 2(m+2) \sum_{j=0}^{m+1} \int_{U^L_j(x_0,x_1)} \frac{\operatorname{dist}(x_j,L)^p}{|x_0 - x_1|^{m(m+1)+(1+\alpha)p}} \, \mathrm{d}\mu^m(x_2,\ldots,x_{m+1}) \,. \tag{5}$$

If $j \in \{2,3,\ldots,m+1\}$, and $L \in \mathbf{A}(n,m)$, and $y \in \mathbb{R}^n$ we obtain

$$\begin{aligned} &\int_{U^L_j(x,y)} \frac{\operatorname{dist}(x_j,L)^p}{|x-y|^{m(m+1)+(1+\alpha)p}} \, \mathrm{d}\mu^m(x_2,\ldots,x_{m+1}) \\ &\quad = \boldsymbol{\alpha}(m)^{m-1} \boldsymbol{\Theta}^m(\mu,x,|x-y|)^{m-1} \int_{\mathbf{B}(x,|x-y|)} \frac{\operatorname{dist}(z,L)^p}{|x-y|^{2m+(1+\alpha)p}} \, \mathrm{d}\mu(z) \\ &\quad = \boldsymbol{\alpha}(m)^{m-1} \boldsymbol{\Theta}^m(\mu,x,|x-y|)^{m-1} \frac{\beta_{\mu,p}(x,|x-y|,L)^p}{|x-y|^{m+\alpha p}} \,. \end{aligned} \tag{6}$$

For $j = 1$ we get

$$\int_{U^L_1(x,y)} \frac{\operatorname{dist}(y,L)^p}{|x-y|^{m(m+1)+(1+\alpha)p}} \, \mathrm{d}\mu^m = \boldsymbol{\alpha}(m)^m \boldsymbol{\Theta}^m(\mu,x,|x-y|)^m \frac{\operatorname{dist}(y,L)^p}{|x-y|^{m+(1+\alpha)p}} \,. \tag{7}$$

Assume now that $x \in L$, then for $j = 0$ we have

$$U^L_0(x_0,x_1) = \varnothing \,. \tag{8}$$

Combining (5), (6), (7), (8) we get

$$\begin{aligned} \mathcal{E}(x,y) &\le 2(m+2) \boldsymbol{\alpha}(m)^{m-1} \boldsymbol{\Theta}^m(\mu,x,|x-y|)^{m-1} \frac{\beta_{\mu,p}(x,|x-y|,L)^p}{|x-y|^{m+\alpha p}} \\ &\quad + 2(m+2) \boldsymbol{\alpha}(m)^m \boldsymbol{\Theta}^m(\mu,x,|x-y|)^m \frac{\operatorname{dist}(y,L)^p}{|x-y|^{m+(1+\alpha)p}} \,, \end{aligned} \tag{9}$$

for any $L \in \mathbf{A}(n,m)$ such that $x \in L$.

Whenever $l \in \mathbb{N}$, and $r \in (0,\infty)$, and $y \in \mathbf{B}(x,R)$ define

$$D_l = \mathbf{B}(x, 2^{-l}R) \sim \mathbf{U}(x, 2^{-l-1}R)\,,$$
$$L(s) \in \mathbf{A}(n,m) \text{ such that } \beta_{\mu,p}(x,r,L(r)) = \mathring{\beta}_{\mu,p}(x,r)\,,$$
$$T(y) = L(2^{-k}R) \quad \text{where } k \in \mathbb{N} \text{ is such that } 2^{-k-1}R < |y-x| \le 2^{-k}R\,.$$

Observe that for any $s,t \in \mathbb{R}$ with $0 < s/2 \le t \le s$ and any $L \in \mathbf{A}(n,m)$ we have

$$\beta_{\mu,p}(x,t,L)^p \le \left(\frac{s}{t}\right)^{m+p} \beta_{\mu,p}(x,s,L)^p \le 2^{m+p}\beta_{\mu,p}(x,s,L)^p$$
$$\text{and} \quad \mathbf{\Theta}^m(\mu,x,t) \le \left(\frac{s}{t}\right)^{m} \mathbf{\Theta}^m(\mu,x,s) \le 2^m \mathbf{\Theta}^m(\mu,x,s)\,.$$

Hence,

$$\begin{aligned}
(10)\quad & \int_{\mathbf{B}(x,R)} \mathbf{\Theta}^m(\mu,x,|x-y|)^{m-1} \frac{\beta_{\mu,p}(x,|y-x|,T(y))^p}{|y-x|^{m+\alpha p}}\,\mathrm{d}\mu(y) \\
&= \sum_{k=0}^{\infty} \int_{D_k} \mathbf{\Theta}^m(\mu,x,|x-y|)^{m-1} \frac{\beta_{\mu,p}(x,|y-x|,L(2^{-k}R))^p}{|y-x|^{m+\alpha p}}\,\mathrm{d}\mu(y) \\
&\le 2^{2m+(1+\alpha)p+m(m-1)}\boldsymbol{\alpha}(m) \sum_{k=0}^{\infty} (2^{-k}R)^{-\alpha p} \mathbf{\Theta}^m(\mu,x,2^{-k}R)^m \mathring{\beta}_{\mu,p}(x,2^{-k}R)^p \\
&\le 4^{m+(1+\alpha)p+m\cdot m}\boldsymbol{\alpha}(m) \sum_{k=0}^{\infty} \fint_{2^{-k}R}^{2^{-k+1}R} \mathbf{\Theta}^m(\mu,x,r)^m \frac{\mathring{\beta}_{\mu,p}(x,r)^p}{r^{\alpha p}}\,\mathrm{d}r \\
&\le 4^{m+(1+\alpha)p+m\cdot m+1}\boldsymbol{\alpha}(m) \int_0^{2R} \mathbf{\Theta}^m(\mu,x,r)^m \frac{\mathring{\beta}_{\mu,p}(x,r)^p}{r^{\alpha p}} \frac{\mathrm{d}r}{r}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(11)\quad & \int_{\mathbf{B}(x,R)} \mathbf{\Theta}^m(\mu,x,|x-y|)^{m} \frac{\operatorname{dist}(y,T(y))^p}{|x-y|^{m+(1+\alpha)p}}\,\mathrm{d}\mu(y) \\
&\le 2^{m+(1+\alpha)p+m\cdot m} \sum_{k=0}^{\infty} \mathbf{\Theta}^m(\mu,x,2^{-k}R)^m \int_{D_k} \frac{\operatorname{dist}(y,L(2^{-k}R))^p}{(2^{-k}R)^{m+(1+\alpha)p}}\,\mathrm{d}\mu(y) \\
&\le 2^{m+(1+\alpha)p+m\cdot m} \sum_{k=0}^{\infty} (2^{-k}R)^{-\alpha p} \mathbf{\Theta}^m(\mu,x,2^{-k}R)^m \mathring{\beta}_{\mu,p}(x,2^{-k}R)^p \\
&\le 4^{m+(1+\alpha)p+m\cdot m+1} \int_0^{2R} \mathbf{\Theta}^m(\mu,x,r)^m \frac{\mathring{\beta}_{\mu,p}(x,r)^p}{r^{\alpha p}} \frac{\mathrm{d}r}{r}\,.
\end{aligned}$$

Combining (4), (9), (10), and (11) we finally obtain

$$\mathcal{K}^{\alpha}_{\mu,p}(x,R) \le \Gamma \int_0^{2R} \mathbf{\Theta}^m(\mu,x,r)^m \frac{\mathring{\beta}_{\mu,p}(x,r)^p}{r^{\alpha p}} \frac{\mathrm{d}r}{r}\,,$$

where $\Gamma = 2(m+1)(m+2)\boldsymbol{\alpha}(m)^m 4^{m+(1+\alpha)p+m\cdot m+1}$. □

Proof of 1.2. Clearly $\beta_{\mu,p}(x,r) \le \mathring{\beta}_{\mu,p}(x,r)$ for each $x \in \mathbb{R}^n$ and $r \in (0,\infty)$ so, for the first part of the lemma, we only need to prove that (2) implies (3).

Assume (2) and choose a compact set $F \subseteq \mathbb{R}^n$ such that

$$\int_F \int_0^{12\rho\sqrt{n}} \mathbf{\Theta}^m(\mu,x,r)^\gamma \frac{\beta_{\mu,p}(x,r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) < \infty \,. \tag{12}$$

For $\varepsilon > 0$ define

$$G_\varepsilon = \big\{ x \in F : \mu(Q \cap F) \ge \varepsilon \mu(Q) \text{ for all } Q \in \mathcal{D} \text{ with } l(Q) \le \varepsilon \text{ and } x \in Q \big\} \,.$$

Applying the Lebesgue points theorem (cf. [Fed69, 2.8.19, 2.9.8]) to the characteristic function of F we see that

$$\mu\big(F \sim \textstyle\bigcup_{\varepsilon>0} G_\varepsilon\big) = 0 \,. \tag{13}$$

For any cube $Q \subseteq \mathbb{R}^n$ let $L(Q) \in \mathbf{A}(n,m)$ be such that $\beta_{\mu,p}(Q) = \beta_{\mu,p}(Q, L(Q))$. Observe that if $Q \subseteq \mathbb{R}^n$ is a cube and $x \in \mathbb{R}^n$, $r \in (0,\infty)$ are such that $\mathbf{B}(x,r) \subseteq Q$, and $v \in \mathbb{R}^n$ is such that $\operatorname{dist}(x, L(Q)) = |v|$ and $x - v \in L$, then

$$\begin{aligned} \mathring{\beta}_{\mu,p}(x,r)^q &\le \beta_{\mu,p}(x,r,v+L(Q))^q \\ &\le \left(\frac{l(Q)}{r}\right)^{q+mq/p} l(Q)^{-q} \left(l(Q)^{-m} \int_Q \operatorname{dist}(y,L(Q))^p \,\mathrm{d}\mu(y) + \frac{\mu(Q)}{l(Q)^m} \operatorname{dist}(x,L(Q))^p \right)^{q/p} \\ &\le \left(\frac{l(Q)}{r}\right)^{q+mq/p} \left(\beta_{\mu,p}(Q)^q + \left(\frac{\mu(Q)}{l(Q)^m}\right)^{q/p} \frac{\operatorname{dist}(x,L(Q))^q}{l(Q)^q} \right) . \end{aligned} \tag{14}$$

Fix $\varepsilon > 0$. If $\rho < \infty$, find an integer k_0 such that $2^{-k_0} \ge \rho > 2^{-k_0-1}$, if $\rho = \infty$, set $k_0 = -\infty$. We define $\mathcal{D}_k = \big\{Q \in \mathcal{D} : l(Q) = 2^{-k}\big\}$ and $\mathcal{D}(\rho) = \big\{Q \in \mathcal{D} : l(Q) < 2\rho\big\}$. Using (14) we can write

$$\begin{aligned} &\int_{G_\varepsilon} \int_0^\rho \mathbf{\Theta}^m(\mu,x,r)^\gamma \frac{\mathring{\beta}_{\mu,p}(x,r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \\ &\quad\le \sum_{k=k_0}^\infty \sum_{Q \in \mathcal{D}_k} \int_{G_\varepsilon \cap Q} \int_{l(Q)/2}^{l(Q)} \mathbf{\Theta}^m(\mu,x,r)^\gamma \frac{\mathring{\beta}_{\mu,p}(x,r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \\ &\quad\le 6^{q+mq/p+\gamma m} 2^{2+\alpha q} \sum_{Q \in \mathcal{D}(\rho),\, \mu(Q\cap G_\varepsilon)>0} \frac{\mathbf{\Theta}^m(\mu,3Q)^\gamma}{l(3Q)^{\alpha q}} \Big[\mu(G_\varepsilon \cap 3Q) \beta_{\mu,p}(3Q)^q \\ &\qquad + \left(\frac{\mu(3Q)}{l(3Q)^m}\right)^{q/p} \int_{G_\varepsilon \cap Q} \frac{\operatorname{dist}(x,L(3Q))^q}{l(3Q)^q} \,\mathrm{d}\mu(x) \Big] . \end{aligned} \tag{15}$$

For any $Q \in \mathcal{D}(\rho)$ with $\mu(Q \cap G_\varepsilon) > 0$ we use Hölder's inequality to derive

$$\begin{aligned} &\left(\frac{\mu(3Q)}{l(3Q)^m}\right)^{q/p} \int_{G_\varepsilon \cap Q} \frac{\operatorname{dist}(x,L(3Q))^q}{l(3Q)^q} \,\mathrm{d}\mu(x) \\ &\quad\le \left(\frac{\mu(3Q)}{l(3Q)^m}\right)^{q/p} \left(\int_{G_\varepsilon \cap Q} \frac{\operatorname{dist}(x,L(3Q))^p}{l(3Q)^p} \,\mathrm{d}\mu(x) \right)^{q/p} \mu(G_\varepsilon \cap Q)^{1-q/p} \\ &\quad\le \mu(F \cap 3Q) \left(\frac{\mu(3Q)}{\mu(F \cap 3Q)} \right)^{q/p} \beta_{\mu,p}(3Q)^q \,. \end{aligned} \tag{16}$$

Set $\Delta_1 = 6^{q+mq/p+\gamma m} 2^{2+\alpha q}$. Plugging (16) into (15) and using the definition of G_ε we obtain (17)

$$\int_{G_\varepsilon} \int_0^\rho \mathbf{\Theta}^m(\mu,x,r)^\gamma \frac{\mathring{\beta}_{\mu,p}(x,r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \le \frac{\Delta_1}{\varepsilon^{q/p}} \sum_{Q\in\mathcal{D}(\rho)} \frac{\mathbf{\Theta}^m(\mu,3Q)^\gamma}{l(3Q)^{\alpha q}} \beta_{\mu,p}(3Q)^q \mu(F\cap 3Q)\,.$$

Now we only need to show that the last term in (17) is controlled by (12). If $Q \in \mathcal{D}_k$ for some $k\in\mathbb{N}$, then there exist exactly 3^n cubes in $\mathcal{D}_k$ which intersect $3Q$; hence,

$$\begin{aligned}
(18)\quad & \sum_{Q\in\mathcal{D}(\rho)} \frac{\mathbf{\Theta}^m(\mu,3Q)^\gamma}{l(3Q)^{\alpha q}} \beta_{\mu,p}(3Q)^q \mu(F\cap 3Q) \\
&= \sum_{k=k_0}^\infty \sum_{Q\in\mathcal{D}_k} \int_{F\cap 3Q} \fint_{l(3Q)\sqrt{n}}^{2l(3Q)\sqrt{n}} \frac{\mathbf{\Theta}^m(\mu,3Q)^\gamma}{l(3Q)^{\alpha q}} \beta_{\mu,p}(3Q)^q \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \\
&\le \frac{3^n(2\sqrt{n})^{\gamma m+\alpha q+mq/p+q}}{\log(2)} \int_F \int_0^{12\rho\sqrt{n}} \frac{\mathbf{\Theta}^m(\mu,x,r)^\gamma}{r^{\alpha q}} \beta_{\mu,p}(x,r)^q \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) < \infty\,.
\end{aligned}$$

Now we see that the right-hand side of (17) is finite for each $\varepsilon > 0$. Recalling (13) the first part of the lemma is proven.

To prove the second part let $R \subseteq \mathbb{R}^n$ be a cube. Rescaling and translating the dyadic lattice $\mathcal{D}$ we can assume $R \in \mathcal{D}$. We first estimate $\int_R \int_0^\rho \mathbf{\Theta}^m(\mu,x,r)^\gamma \mathring{\beta}_{\mu,p}(x,r)^q r^{-\alpha q} \frac{\mathrm{d}r}{r}\,\mathrm{d}\mu(x)$ the same way as in (15) putting R in place of G_ε. Then instead of (16) for $Q \in \mathcal{D}$ we write

$$\begin{aligned}
&\left(\frac{\mu(3Q)}{l(3Q)^m}\right)^{q/p} \int_{R\cap Q} \frac{\operatorname{dist}(x,L(3Q))^q}{l(3Q)^q} \,\mathrm{d}\mu(x) \\
&\quad\le \left(\frac{\mu(3Q)}{l(3Q)^m}\right)^{q/p} \left(\int_{3Q} \frac{\operatorname{dist}(x,L(3Q))^p}{l(3Q)^p} \,\mathrm{d}\mu(x)\right)^{q/p} \mu(3Q)^{1-q/p} \le \mu(3Q)\beta_{\mu,p}(3Q)^q\,.
\end{aligned}$$

Then in place of (17) we obtain

$$\int_R \int_0^\rho \mathbf{\Theta}^m(\mu,x,r)^\gamma \frac{\mathring{\beta}_{\mu,p}(x,r)^q}{r^{\alpha q}} \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) \le \Delta_1 \sum_{Q\in\mathcal{D}(\rho),\, \mu(Q\cap R)>0} \frac{\mathbf{\Theta}^m(\mu,3Q)^\gamma}{l(3Q)^{\alpha q}} \beta_{\mu,p}(3Q)^q \mu(3Q)\,.$$

Set $\Delta_2 = 3^n(2\sqrt{n})^{\gamma m+\alpha q+mq/p+q}/\log(2)$. Estimating as in (18)

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}(\rho),\, \mu(Q\cap R)>0} \frac{\mathbf{\Theta}^m(\mu,3Q)^\gamma}{l(3Q)^{\alpha q}} \beta_{\mu,p}(3Q)^q \mu(3Q) \\
&\qquad\le \Delta_2 \int_{3R} \int_0^{12\rho\sqrt{n}} \frac{\mathbf{\Theta}^m(\mu,x,r)^\gamma}{r^{\alpha q}} \beta_{\mu,p}(x,r)^q \frac{\mathrm{d}r}{r} \,\mathrm{d}\mu(x) < \infty\,.
\end{aligned}$$

Hence, we can set $\Gamma = \Delta_1\Delta_2$. □

Now we can derive a partial converse of the theorem of [Meu15], i.e., we prove 1.3.

Proof of 1.3. Since $\mathbb{R}^n$ is countably (μ, m) rectifiable the result of Tolsa [Tol15] implies that

$$\int_0^\infty \beta_{\mu,2}(x,r)^2 \frac{\mathrm{d}r}{r} < \infty \quad \text{for } \mu \text{ almost all } x\,.$$

Hence, the first part of 1.2 yields

$$\int_0^\infty \mathring{\beta}_{\mu,2}(x,r)^2 \frac{\mathrm{d}r}{r} < \infty \quad \text{for } \mu \text{ almost all } x\,.$$

Then from 1.1 it follows that

$$\mathcal{K}^0_{\mu,2}(x,\infty) \le \Gamma_{1.1} \int_0^\infty \mathbf{\Theta}^m(\mu,x,r)^m \mathring{\beta}_{\mu,2}(x,r)^2 \frac{\mathrm{d}r}{r} \le M^m \Gamma_{1.1} \int_0^\infty \mathring{\beta}_{\mu,2}(x,r)^2 \frac{\mathrm{d}r}{r} < \infty$$

for μ almost all x. □

We also obtain an analogue of [LW11, Theorem 1.1 and §10].

3.1. **Corollary.** *For a cube $Q \subseteq \mathbb{R}^n$, and $\alpha \in [0,1]$, and $p \in [1,\infty)$, and $R \in (0,\infty]$*

$$\int_Q \mathcal{K}^\alpha_{\mu,p}(x,R)\,\mathrm{d}\mu(x) \le \Gamma_{1.1}\Gamma_{1.2} \int_{3Q} \int_0^{24R\sqrt{n}} \mathbf{\Theta}^m(\mu,x,r)^m \frac{\beta_{\mu,p}(x,r)^p}{r^{\alpha p}} \frac{\mathrm{d}r}{r}\,\mathrm{d}\mu(x)\,.$$

3.2. *Remark.* Define

$$\kappa(x_0,\dots,x_{m+1}) = \frac{\mathcal{H}^{m+1}(\operatorname{conv}\{x_0,\dots,x_{m+1}\})}{\operatorname{diam}(x_0,\dots,x_{m+1})^{m+1}} \quad \text{for } x_0,\dots,x_{m+1} \in \mathbb{R}^n\,,$$

$$\mathcal{M}_p(\mu) = \int \cdots \int \frac{\kappa(x_0,\dots,x_{m+1})^p}{\operatorname{diam}(\{x_0,\dots,x_{m+1}\})^p}\,\mathrm{d}\mu(x_0)\cdots\mathrm{d}\mu(x_{m+1})\,,$$

If $\mu = \mathcal{H}^m \llcorner \Sigma$ for some Borel set $\Sigma \subseteq \mathbb{R}^n$ with $\mathcal{H}^m(\Sigma) < \infty$, then set $\mathcal{M}_p(\Sigma) = \mathcal{M}_p(\mathcal{H}^m \llcorner \Sigma)$. The functional $\mathcal{M}_p$ has been studied in [Kol15b, BK12, KSv15, KSvdM13, KS13].

Assume $M \in (0,\infty)$, and $p \in [m(m+1),\infty)$, and $\alpha = 1 - m(m+1)/p$, and $\mu = \mathcal{H}^m \llcorner \Sigma$ for some Borel set $\Sigma \subseteq \mathbb{R}^n$ such that $\mu(\mathbf{B}(x,r)) \le M\boldsymbol{\alpha}(m)r^m$ for each $x \in \mathbb{R}^n$ and $r \in (0,\infty)$. Observe that [Kol15a, 8.10] gives

$$\kappa(x_0,\dots,x_{m+1}) \le \Gamma \frac{h_{\min}(x_0,\dots,x_{m+1})}{\operatorname{diam}(\{x_0,\dots,x_{m+1}\})}\,, \quad \text{where } \Gamma = \Gamma(m) \in [1,\infty)\,.$$

Hence,

$$\begin{aligned}\mathcal{M}_p(\Sigma) &\le \Gamma \int \cdots \int \frac{h_{\min}(x_0,\dots,x_{m+1})^p}{\operatorname{diam}(\{x_0,\dots,x_{m+1}\})^{2p}}\,\mathrm{d}\mu(x_0)\cdots\mathrm{d}\mu(x_{m+1}) \\ &= \Gamma \int \mathcal{K}^\alpha_{\mu,p}(x,\infty)\,\mathrm{d}\mu(x) \le \Gamma\Gamma_{1.1}\Gamma_{1.2}M^m \int\int_0^\infty r^{m(m+1)-p}\beta_{\mu,p}(x,r)^p \frac{\mathrm{d}r}{r}\,\mathrm{d}\mu(x)\,.\end{aligned}$$

Therefore, all the conclusions drawn from finiteness of $\mathcal{M}_p(\Sigma)$ in [Kol15b, BK12, KSv15, KSvdM13, KS13] apply also whenever $\int\int_0^\infty r^{m(m+1)-p}\beta_{\mu,p}(x,r)^p \frac{\mathrm{d}r}{r}\,\mathrm{d}\mu(x)$ is finite.

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Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Am Mühlenberg 1, D-14476 Golm, Germany

E-mail address: skola@mimuw.edu.pl