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Krylov subspace methods for model reduction of quadratic-bilinear systems

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Abstract: The authors propose a two sided moment matching method for model reduction of quadratic-bilinear descriptor systems. The goal is to approximate some of the generalised transfer functions that appear in the input–output representation of the non-linear system. Existing techniques achieve this by utilising moment matching for the first two generalised transfer functions. In this study, they derive an equivalent representation that simplifies the structure of the generalised transfer functions. This allows them to extend the idea of two sided moment matching to higher subsystems which was difficult in the previous approaches. Numerical results are given for some benchmark examples of quadratic-bilinear systems.

1 Introduction

Consider a multi-input multi-output (MIMO) quadratic-bilinear descriptor system of the form

$$\Sigma: \begin{cases} E\dot{x}(t) &= Ax(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + H[x(t) \otimes x(t)] + Bu(t), \\ y(t) &= Cx(t). \end{cases} \tag{1}$$

Here $E, A, N_i \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{n \times n^2}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$ are the state-space matrices and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. This class of non-linear systems has applications in many areas including the simulation of fluid flow, electrical circuits and some biological systems, cf. [1]. Also, a large class of non-linear systems can be written as quadratic-bilinear differential algebraic equations (QBDAEs) by utilising exact transformations [2], extending its application areas even further.

In this paper, we discuss the approximation of QBDAEs by constructing reduced order models. That is, we construct

$$\hat{\Sigma}: \begin{cases} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \sum_{i=1}^{m} \hat{N}_i \hat{x}(t) u_i(t) + \hat{H}[\hat{x}(t) \otimes \hat{x}(t)] + \hat{B}u(t), \\ \hat{y}(t) &= \hat{C}\hat{x}(t), \end{cases} \tag{2}$$

with $\hat{x}(t) \in \mathbb{R}^r$ and $r \ll n$ such that $\hat{y}(t)$ is close to $y(t)$ in an appropriate norm. Recently, projection based (generalised-) moment matching techniques [2–4] have been used in the literature to construct such reduced order systems, in contrast to trajectory-based methods such as proper orthogonal decomposition (POD) [5], the reduced basis method, POD with discrete empirical interpolation method [6], and, the trajectory piecewise linear method [7]. Since all these trajectory based methods share the disadvantage of input dependency, generalised moment matching techniques are particularly useful for systems with varied input function, which is common in control and optimisation problems. For details on non-linear model reduction techniques, we refer a recent survey paper [8].

Projection involves identifying suitable basis matrices V and W (the columns of each matrix span a particular subspace), approximating $x(t)$ by $V\hat{x}(t)$ and ensuring Petrov–Galerkin conditions. This leads to the following reduced state-space matrices:

$$\begin{aligned} &\hat{E} = W^{\mathrm{T}}EV, \hat{A} = W^{\mathrm{T}}AV, \quad \hat{H} = W^{\mathrm{T}}H(V \otimes V), \\ &\hat{N}_i = W^{\mathrm{T}}N_iV, \quad i = 1, \dots, m, \quad \hat{B} = W^{\mathrm{T}}B, \quad \hat{C} = CV. \end{aligned} \tag{3}$$

Clearly the reduced system matrices depend on the choice of the basis matrices V and W for a given Σ. In moment matching techniques, these projection matrices are constructed such that the reduced system matches some of the generalised moments associated with the underlying generalised transfer functions of the QBDAE system at fixed interpolation points. In [2], the procedure of moment matching is restricted to orthogonal projection (i.e. $W = V$) with the moments matched for each of the first two generalised transfer functions. This idea has been extended recently to oblique projection [4] which improves the approximation quality of the reduced system, but because of the complex structure of the moments corresponding to third and higher dimensional generalised transfer functions, the higher dimensional moments are again ignored.

In this paper, we utilise the known connection between different forms of the generalised transfer functions in order to identify an equivalent representation of the generalised transfer functions which simplifies the structure of the moments. This allows us to use two sided projection based moment matching techniques also for higher dimensional transfer functions.

In Section 2, we briefly overview some background theory and existing moment matching techniques for model reduction of QBDAEs. To extend the idea of moment matching to higher subsystems, we derive a simplified form of multivariate transfer functions in Section 3. Based on the simplified form, we propose our moment matching framework using orthogonal as well as oblique projections. These results are given in Section 4. Numerical results are presented in Section 5 and, finally, the conclusions and future work are given in Section 6.

2 Background

The input–output representation for single input quadratic-bilinear systems can be expressed by the Volterra series expansion of the output $y(t)$ with quantities analogous to the standard convolution operator. That is (see (4)) where it is assumed that the input signal is one-sided, $u(t) = 0$ for $t < 0$. In addition, each of the generalised impulse responses, $h_k(t_1, \ldots, t_k)$, also called the k-dimensional kernel of the subsystem, is assumed to be one-sided. In terms of the multivariable Laplace transform, the k-dimensional subsystem can be represented as

$$Y_k(s_1, \ldots, s_k) = H_k(s_1, \ldots, s_k)U(s_1)\cdots U(s_k), \tag{5}$$

where $H_k(s_1, \ldots, s_k)$ is the multivariable transfer function of the k-dimensional subsystem. The above equation follows by applying the convolution property of the multivariable Laplace transform to (4), see [9] for details. If the multivariable Laplace transforms of all subsystems, that are $H_k(s_1, \ldots, s_k)$'s, and the input, $U(s)$, are known, then the inverse Laplace transforms can be computed to identify $y_k(t_1, \ldots, t_k)$. The output $y(t)$ becomes

$$y(t) = \sum_{k=1}^{\infty} y_k(t_1, \ldots, t_k)\,|_{t_1 = \cdots = t_k = t} = \sum_{k=1}^{\infty} y_k(t, \ldots, t) \tag{6}$$

The generalised transfer functions in the output expression (5) are in what is called the triangular form [9] of the functions. We denote the k-dimensional triangular form by $H^{[k]}_{\text{tri}}(s_1, \ldots, s_k)$. There are some other useful forms such as the symmetric and the regular forms of the multivariable transfer functions as discussed in [9]. The triangular form is related to the symmetric form by the following expression:

$$H^{[k]}_{\text{sym}}(s_1, \ldots, s_k) = \frac{1}{n!}\sum_{\pi(\cdot)} H^{[k]}_{\text{tri}}(s_{\pi(1)}, \ldots, s_{\pi(k)}), \tag{7}$$

where the summation includes all $k!$ permutations of $s_1, \ldots, s_k$. Also, the triangular form can be connected to the regular form of the transfer function by using

$$H^{[k]}_{\text{tri}}(s_1, \ldots, s_k) = H^{[k]}_{\text{reg}}(s_1, s_1 + s_2, \ldots, s_1 + s_2 + \cdots + s_k)\,. \tag{8}$$

According to [9], the structure of the generalised symmetric transfer functions can be identified by the growing exponential approach. The structure of these symmetric transfer functions for the first three subsystems of the quadratic-bilinear system (1) can be written as (see (9)) . Clearly, if a model reduction approach can ensure that

$$H^{[k]}_{\text{sym}}(s_1, \ldots, s_k) \simeq \hat{H}^{[k]}_{\text{sym}}(s_1, \ldots, s_k), \quad \text{for } k = 1, \ldots, K, \tag{10}$$

with $\hat{H}^{[k]}_{\text{sym}}(s_1, \ldots, s_k)$ being the multivariate transfer functions of the reduced system $\hat{\Sigma}$, we can expect that the output $y(t)$ is well approximated by $\hat{y}(t)$. This idea was initially utilised in [2] to construct a reduced quadratic-bilinear system with orthogonal projection of the first two subsystems using the symmetric transfer functions. Recently, this approach was extended in [4] to the oblique projection framework in order to improve the quality of the reduced model. The complex structure of the third and higher symmetric transfer functions has again restricted these projection techniques to the moment matching of the first two subsystems only. In the following we briefly review the oblique projection framework. Before proceeding further, we discuss some properties and notations:

$$F(s) := (sE - A), \tag{11}$$

$$w^{\mathrm{T}}H(u \otimes v) = u^{\mathrm{T}}H^{(2)}(v \otimes w) \tag{12}$$

where $w, u, v \in \mathbb{R}^n$ are arbitrary and it is assumed that $H(u \otimes v) = H(v \otimes u)$ holds. The matrix $H^{(2)}$ is the 2-matricisation of the three-dimensional tensor $\mathcal{H} \in \mathbb{R}^{n \times n \times n}$ having H as its 1-matricisation, see [10]. To recycle vectors for approximation subspaces, it is assumed in [4] that $s_1 = s_2 = s$. With these settings, the second symmetric transfer function becomes

$$H^{[2]}_{\text{sym}}(s, s) = CF(2s)^{-1}\Big(H\big(F(s)^{-1}B \otimes F(s)^{-1}B\big) + NF(s)^{-1}B\Big).$$

The following summarises the result introduced in [4].

Lemma 1: Let $\sigma_i \in \mathbb{C}$ be the interpolation points and $\sigma_i \notin \{\Lambda(A, E), \Lambda(A_r, E_r)\}$, where $\Lambda(A, E)$ represents the generalised eigenvalues of the matrix pencil $\lambda E - A$. Assume that $\hat{E} = W^{\mathrm{T}}EV$ is non-singular and $\hat{A}$, $\hat{H}$, $\hat{N}$, $\hat{B}$, $\hat{C}$ are as in (3) with full rank matrices $V, W \in \mathbb{R}^{n \times r}$ such that (see equation below). Then the reduced QBDAE satisfies the following (Hermite) interpolation conditions: (see equation below).

Proof: See [4] for a proof. □

In the remaining part of this paper, our goal is to identify the regular form of the multivariate transfer functions that can hopefully simplify the moment matching concept and allows us to use this new framework for third and higher subsystems.

3 Regular form of multivariate transfer functions

In this section, we utilise the connections between different forms of the multivariate transfer functions discussed in the previous section to identify the regular form of the corresponding functions.

$$y(t) = \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} h_k(t_1, \ldots, t_k)u(t - t_1)\cdots u(t - t_k)dt_k \cdots dt_1, \tag{4}$$

$$\begin{aligned}
H^{[1]}_{sym}(s_1) &= C(s_1 I - A)^{-1}B =: CG^{[1]}_{sym}(s_1)\\
H^{[2]}_{sym}(s_1, s_2) &= \frac{1}{2!}C((s_1 + s_2)I - A)^{-1}\Big(N[G^{[1]}_{sym}(s_1) + G^{[1]}_{sym}(s_2)]\\
&\qquad + H[G^{[1]}_{sym}(s_1) \otimes G^{[1]}_{sym}(s_2) + G^{[1]}_{sym}(s_2) \otimes G^{[1]}_{sym}(s_1)]\Big)\\
&=: CG^{[2]}_{sym}(s_1, s_2)\\
H^{[3]}_{sym}(s_1, s_2, s_3) &= \frac{1}{3!}C((s_1 + s_2 + s_3)I - A)^{-1}\Big(N\big[G^{[2]}_{sym}(s_1, s_2) + G^{[2]}_{sym}(s_2, s_3)\\
&\qquad + G^{[2]}_{sym}(s_1, s_3)\big] + H\big[G^{[1]}_{sym}(s_1) \otimes G^{[2]}_{sym}(s_2, s_3) + G^{[2]}_{sym}(s_2, s_3) \otimes G^{[1]}_{sym}(s_1)\\
&\qquad + G^{[1]}_{sym}(s_2) \otimes G^{[2]}_{sym}(s_1, s_3) + G^{[2]}_{sym}(s_1, s_3) \otimes G^{[1]}_{sym}(s_2)\\
&\qquad + G^{[1]}_{sym}(s_3) \otimes G^{[2]}_{sym}(s_1, s_2) + G^{[2]}_{sym}(s_1, s_2) \otimes G^{[1]}_{sym}(s_3)\big]\Big)
\end{aligned} \tag{9}$$

The following theorem gives the regular form, which is one of our main results.

Theorem 1: Given a quadratic-bilinear descriptor system, the transfer functions in regular form are given as $H_{\text{reg}}^{[1]}(s_1) = CF(s_1)^{-1}B$ for $k = 1$ and for $k \geq 2$

$$H_{\text{reg}}^{[k]}(s_1, \ldots, s_k) = CF(s_k)^{-1}\Big(NG_{\text{reg}}^{[k-1]}(s_1, \ldots, s_{k-1}) + H\Big[\sum_{p=1}^{k-1} G_{\text{reg}}^{[p]}(\bar{s}_1, \ldots, \bar{s}_p) \otimes G_{\text{reg}}^{[k-p]}(s_1, \ldots, s_{k-p})\Big]\Big), \quad (13)$$

where $G_{\text{reg}}^{[k]}$ is defined such that $H_{\text{reg}}^{[k]} = CG_{\text{reg}}^{[k]}$ and

$$\{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_p\} := \{s_{k-p+1} - s_{k-p}, s_{k-p+2} - s_{k-p}, \ldots, s_k - s_{k-p}\}.$$

Proof: We begin with the $k = 2$ case where the regular form becomes

$$\begin{aligned}H_{\text{reg}}^{[2]}(s_1, s_2) &= CF(s_2)^{-1}\Big(HG_{\text{reg}}^{[1]}(s_2 - s_1) \otimes G_{\text{reg}}^{[1]}(s_1) + NG_{\text{reg}}^{[1]}(s_1)\Big) \\ &= CF(s_2)^{-1}\Big(HF(s_2 - s_1)^{-1}B \otimes F(s_1)^{-1}B + NF(s_1)^{-1}B\Big)\end{aligned}$$

Using (8) we have

$$\begin{aligned}H_{\text{tri}}^{[2]}(s_1, s_2) &= H_{\text{reg}}^{[2]}(s_1, s_1 + s_2) \\ &= CF(s_1 + s_2)^{-1}\Big(HG_{\text{reg}}^{[1]}(s_2) \otimes G_{\text{reg}}^{[1]}(s_1) + NG_{\text{reg}}^{[1]}(s_1)\Big).\end{aligned}$$

Since $G_{\text{reg}}^{[1]}(s_1) = G_{\text{tri}}^{[1]}(s_1)$, we have $G_{\text{reg}}^{[1]}(s_2) = G_{\text{tri}}^{[1]}(s_2)$. Therefore, the triangular form becomes

$$H_{\text{tri}}^{[2]}(s_1, s_2) = CF(s_1 + s_2)^{-1}\Big(HG_{\text{tri}}^{[1]}(s_2) \otimes G_{\text{tri}}^{[1]}(s_1) + NG_{\text{tri}}^{[1]}(s_1)\Big). \quad (14)$$

Now using the connection between triangular and symmetric forms given in (7), we get

$$H_{\text{sym}}^{[2]}(s_1, s_2) = \frac{1}{2}CF(s_1 + s_2)^{-1}\Big(HF(s_2)^{-1}B \otimes F(s_1)^{-1}B + NF(s_1)^{-1}B + HF(s_1)^{-1}B \otimes F(s_2)^{-1}B + NF(s_2)^{-1}B\Big),$$

which is exactly equal to the known symmetric form. Thus (13) holds for $k = 2$.
Now we check the $k = 3$ case

$$H_{\text{reg}}^{[3]}(s_1, s_2, s_3) = CF(s_3)^{-1}\Big(H\Big[G_{\text{reg}}^{[1]}(s_3 - s_2) \otimes G_{\text{reg}}^{[2]}(s_1, s_2) + G_{\text{reg}}^{[2]}(s_2 - s_1, s_3 - s_1) \otimes G_{\text{reg}}^{[1]}(s_1)\Big] + NG_{\text{reg}}^{[2]}(s_1, s_2)\Big).$$

This means that

$$\begin{aligned}H_{\text{tri}}^{[3]}(s_1, s_2, s_3) &= H_{\text{reg}}^{[3]}(s_1, s_1 + s_2, s_1 + s_2 + s_3) \\ &= CF(s_1 + s_2 + s_3)^{-1}\Big(H\Big[G_{\text{reg}}^{[1]}(s_3) \otimes G_{\text{reg}}^{[2]}(s_1, s_1 + s_2) + G_{\text{reg}}^{[2]}(s_2, s_2 + s_3) \otimes G_{\text{reg}}^{[1]}(s_1)\Big] + NG_{\text{reg}}^{[2]}(s_1, s_1 + s_2)\Big).\end{aligned}$$

As $G_{\text{tri}}^{[2]}(s_1, s_2) = G_{\text{reg}}^{[2]}(s_1, s_1 + s_2)$, we can write $G_{\text{tri}}^{[2]}(s_2, s_3) = G_{\text{reg}}^{[2]}(s_2, s_2 + s_3)$. The triangular form is then

$$H_{\text{tri}}^{[3]}(s_1, s_2, s_3) = CF(s_1 + s_2 + s_3)^{-1}\Big(H\Big[G_{\text{tri}}^{[1]}(s_3) \otimes G_{\text{tri}}^{[2]}(s_1, s_2) + G_{\text{tri}}^{[2]}(s_2, s_3) \otimes G_{\text{tri}}^{[1]}(s_1)\Big] + NG_{\text{tri}}^{[2]}(s_1, s_2)\Big).$$

Using (7), we observe that the equivalent symmetric form is the same as in (9). Similarly for higher values of k, one can show that the regular form in (13) holds. □

Remark 1: The regular and triangular forms include $k - 1$ sums of Kronecker products which is much smaller as compared to the corresponding symmetric form. Also in the symmetric form, it is difficult, if not impossible, to represent a general kth-dimensional multivariate transfer function.

Remark 2: The symmetric form is exactly equal to the triangular form if we assume that $s_1 = \cdots = s_k = s$ and the two forms are equal to the regular form if the regular variables are $s_1 = s, s_2 = 2s, \ldots, s_k = ks$.

Example 1: Consider the second subsystem in symmetric form with $s_2 = s_1 = \sigma$. That is

$$H_{\text{sym}}^{[2]}(\sigma, \sigma) = CF(2\sigma)^{-1}\Big(NF(\sigma)^{-1}B + HF(\sigma)^{-1}B \otimes F(\sigma)^{-1}B\Big).$$

From (7) we have $H_{\text{sym}}^{[2]}(\sigma, \sigma) = H_{\text{tri}}^{[2]}(\sigma, \sigma)$, which clearly holds by setting $s_2 = s_1 = \sigma$ in (14). Also from (8), we know that $H_{\text{tri}}^{[2]}(\sigma, \sigma) = H_{\text{reg}}^{[2]}(\sigma, 2\sigma)$. Thus by fixing k to 2 and $s_1 = \sigma, s_2 = 2\sigma$ in Theorem 1, the regular form $H_{\text{reg}}^{[2]}(\sigma, 2\sigma)$ is equal to $H_{\text{sym}}^{[2]}(\sigma, \sigma)$ and $H_{\text{tri}}^{[2]}(\sigma, \sigma)$.

In the following, we use the regular form of the multivariate transfer functions to construct a reduced interpolating quadratic-bilinear system.

4 Multimoment-matching with the regular form

In this section, we propose orthogonal as well as oblique projection techniques for model reduction of quadratic-bilinear systems using the regular form. We begin with the case of multimoment-matching for the first two subsystems only. The general case, where the multimoments associated with third and higher subsystems are also matched, is discussed later.

4.1 Case $K = 2$

The first two transfer functions in the regular form with the assumption that $s_1 = \sigma, s_2 = 2\sigma$ can be written as

$$\text{span}(V) = \underset{i=1,\ldots,k}{\text{span}}\{F(\sigma_i)^{-1}B, F(2\sigma_i)^{-1}[H(F(\sigma_i)^{-1}B \otimes F(\sigma_i)^{-1}B) + NF(\sigma_i)^{-1}B]\}$$

$$\text{span}(W) = \underset{i=1,\ldots,k}{\text{span}}\Big\{F(2\sigma_i)^{-\text{T}}C^{\text{T}}, F(\sigma_i)^{-\text{T}}\Big[H^{(2)}(F(\sigma_i)^{-1}B \otimes F(2\sigma_i)^{-\text{T}}C^{\text{T}}) + \frac{1}{2}N^{\text{T}}F(2\sigma_i)^{-\text{T}}C^{\text{T}}\Big]\Big\}.$$

$$\begin{aligned}&H_{\text{sym}}^{[1]}(\sigma_i) = \hat{H}_{\text{sym}}^{[1]}(\sigma_i), \quad H_{\text{sym}}^{[1]}(2\sigma_i) = \hat{H}_{\text{sym}}^{[1]}(2\sigma_i), \\ &H_{\text{sym}}^{[2]}(\sigma_i, \sigma_i) = \hat{H}_{\text{sym}}^{[2]}(\sigma_i, \sigma_i), \frac{\partial}{\partial s_j}H_{\text{sym}}^{[2]}(\sigma_i, \sigma_i) = \frac{\partial}{\partial s_j}\hat{H}_{\text{sym}}^{[2]}(\sigma_i, \sigma_i), \quad j = 1, 2.\end{aligned}$$

Algorithm 1

1: **Inputs:** $E,\ A,\ b,\ c,\ N,\ H,\ \sigma_i \in \mathbb{C},\ i = 1, \ldots, n_r$
2: **Outputs:** $\hat{E},\ \hat{A},\ \hat{B},\ \hat{C},\ \hat{N},\ \hat{H}$
3: For $i = 1, \ldots, n_r$, construct

$$\begin{aligned} V_1^i &= (\sigma_i E - A)^{-1} b, \\ V_2^i &= (2\sigma_i E - A)^{-1}\big(NV_1^i + H(V_1^i \otimes V_1^i)\big), \\ W_1^i &= (\sigma_i E - A)^{-T} c^T \\ W_2^i &= (2\sigma_i E - A)^{-T} N^T W_1^i + (\sigma_i E - A)^{-T} H^{(2)}(V_1^i \otimes W_1^i)\big). \end{aligned}$$

4: $V = \mathrm{orth}([V_1^i\ V_2^i]),\ W = \mathrm{orth}([W_1^i\ W_2^i])$
5: $\hat{E} = W^T E V,\ \hat{A} = W^T A V,\ \hat{b} = W^T b,\ \hat{c} = cV,\ \hat{N} = W^T N V,\ \hat{H} = W^T H(V \otimes V).$

Fig. 1 *Algorithm 1 Oblique projection with regular transfer functions, K = 2*

$$H_{\mathrm{reg}}^1(\sigma) = C(\sigma E - A)^{-1}B,$$
$$H_{\mathrm{reg}}^2(\sigma, 2\sigma) = C(2\sigma E - A)^{-1}\Big(NG_{\mathrm{reg}}^1(\sigma) + H\big[G_{\mathrm{reg}}^1(\sigma) \otimes G_{\mathrm{reg}}^1(\sigma)\big]\Big).$$

The above transfer functions are similar to symmetric transfer functions, if in the symmetric case $s_1 = s_2 = \sigma$, which is assumed in Lemma 1. Thus, the framework for interpolation of the transfer functions in the regular form is similar to Lemma 1. However, the interpolation of the partial derivatives with respect to s_1 or s_2 varies. The following theorem shows the interpolation conditions in the regular case for the first two subsystems.

Theorem 2: Let $\sigma_i \in \mathbb{C}$ be the interpolation points and $j \times \sigma_i \notin \{\Lambda(A, E), \Lambda(A_r, E_r)\},\ j = 1, \ldots, k$. Assume that $\hat{E} = W^{\mathrm{T}}EV$ is non-singular and $\hat{A}$, $\hat{H}$, $\hat{N}$, $\hat{B}$, $\hat{C}$ are as defined in (3) with $V, W \in \mathbb{R}^{n \times r}$ such that (see (15)) Then the reduced QBDAE ensures that the following holds:

$$H_{\mathrm{reg}}^{[1]}(\sigma_i) = \hat{H}_{\mathrm{reg}}^{[1]}(\sigma_i), \quad H_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) = \hat{H}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) \tag{16}$$

and, in addition, interpolates all combinations of the multimoments that can be written as $w^{\mathrm{T}}Ev$ and $w^{\mathrm{T}}(Nv + H(v \otimes v))$, where $v \in \mathrm{span}(V)$ and $w \in \mathrm{span}(W)$.

Proof: Equation (16) holds due to the structure of V and its proof is as in Lemma 1. To prove the second part, let $\hat{V} \in \mathbb{R}^{n \times n_r}$ and $\hat{W} \in \mathbb{R}^{n \times n_r}$ be defined such that (see (17)) Also let $\hat{v} \in \mathrm{span}(\hat{V})$ and $\hat{w} \in \mathrm{span}(\hat{W})$. Assume that v, $\hat{v}$ (and w, $\hat{w}$) are the same linear combination of the columns of the matrices on the right-hand side of (15) and (17), respectively. Then analogous to the discussion of one-sided projection, it is easy to show that

$$v = V\hat{v}, w = W\hat{w}. \tag{18}$$

This means that

$$\begin{aligned} w^{\mathrm{T}}Ev &= \hat{w}^{\mathrm{T}}W^{\mathrm{T}}EV\hat{v}, \\ &= \hat{w}^{\mathrm{T}}\hat{E}\hat{v}. \end{aligned}$$

Similarly, $w^{\mathrm{T}}(Nv + H(v \otimes v)) = \hat{w}^{\mathrm{T}}(\hat{N}\hat{v} + \hat{H}(\hat{v} \otimes \hat{v}))$. □

The complete approach of one-sided projection for interpolatory model reduction of quadratic-bilinear systems, using regular generalised transfer functions, is shown in algorithm 1 (Fig. 1).

4.2 General case

The transfer function of the kth subsystem in the regular form with the assumption that $s_1 = s, s_2 = 2s, \ldots, s_k = ks$ can be written as

$$H_{\mathrm{reg}}^k(s, \ldots, ks) = C(ksE - A)^{-1}\Bigg(NG_{\mathrm{reg}}^{k-1}(s, \ldots, (k-1)s) + H\Big[\sum_{p=1}^{k-1} G_{\mathrm{reg}}^p(s, \ldots, ps) \otimes G_{\mathrm{reg}}^{k-p}(s, \ldots, (k-p)s)\Big]\Bigg),$$

The transfer function shows that one can recycle vectors in the construction of the projection matrix. This is shown in the following lemma which extends the orthogonal projection technique [2] for mutimoment matching to third and higher subsystems.

Lemma 2: Let $\sigma_i \in \mathbb{C}$ be the interpolation points and $j \times \sigma_i \notin \{\Lambda(A, E), \Lambda(A_r, E_r)\},\ j = 1, \ldots, k$. Assume that $\tilde{E} = V^{\mathrm{T}}EV$ is non-singular and $\tilde{A} = V^{\mathrm{T}}AV$, $\tilde{H} = V^{\mathrm{T}}H(V \otimes V)$, $\tilde{N} = V^{\mathrm{T}}NV$, $\tilde{B} = V^{\mathrm{T}}B$, $\tilde{C} = CV$ with $V \in \mathbb{R}^{n \times r}$ having full rank such that

$$\mathrm{span}(V) = \underset{i=1,\ldots,n_r}{\mathrm{span}}\Bigg(\Bigg[\underbrace{F(\sigma_i)^{-1}B}_{V_1^i}, \ldots, \underbrace{F(K\sigma_i)^{-1}\Bigg(NV_{K-1}^i + H\Bigg(\sum_{p=1}^{K-1} V_p^i \otimes V_{K-p}^i\Bigg)\Bigg)}_{V_K^i}\Bigg]\Bigg) \tag{19}$$

Then the reduced QBDAE ensures moment matching for the first K subsystems. That is, the following holds:

$$H_{\mathrm{reg}}^{[k]}(\sigma_i, 2\sigma_i, \ldots, k\sigma_i) = \tilde{H}_{\mathrm{reg}}^{[k]}(\sigma_i, 2\sigma_i, \ldots, k\sigma_i), \tag{20}$$

$$\begin{aligned} \mathrm{span}(V) &= \underset{i=1,\ldots,n_r}{\mathrm{span}}\big([\underbrace{F(\sigma_i)^{-1}b}_{V_1^i}, F(2\sigma_i)^{-1}(NV_1^i + H(V_1^i \otimes V_1^i))]\big), \\ \mathrm{span}(W) &= \underset{i=1,\ldots,n_r}{\mathrm{span}}\big([\underbrace{F(\sigma_i)^{-T}c^T}_{W_1^i}, \underbrace{F(2\sigma_i)^{-T}N^TW_1^i + F(\sigma_i)^{-T}H^{(2)}(V_1^i \otimes W_1^i)}_{W_2^i}]\big). \end{aligned} \tag{15}$$

$$\begin{aligned} \hat{V} &= \Big(\Big[\underbrace{\hat{F}(\sigma_i)^{-1}\hat{b}}_{\hat{V}_1^i}, \hat{F}(2\sigma_i)^{-1}\big(\hat{N}\hat{V}_1^i + \hat{H}(\hat{V}_1^i \otimes \hat{V}_1^i)\big)\Big]\Big)_{i=1,\ldots,n_r} \\ \hat{W} &= \Big(\Big[\underbrace{\hat{F}(\sigma_i)^{-\mathrm{T}}\hat{c}^{\mathrm{T}}}_{\hat{W}_1^i}, \underbrace{\hat{F}(2\sigma_i)^{-\mathrm{T}}\hat{N}^T\hat{W}_1^i + \hat{F}(\sigma_i)^{-\mathrm{T}}\hat{H}^{(2)}(\hat{V}_1^i \otimes \hat{W}_1^i)}_{\hat{W}_2^i}\Big]\Big)_{i=1,\ldots,n_r} \end{aligned} \tag{17}$$

Algorithm 2

Inputs: $E,\ A,\ b,\ c,\ N,\ H,\ \sigma_i \in \mathbb{C},\ i = 1, \ldots, n_r$

Outputs: $\hat{E},\ \hat{A},\ \hat{B},\ \hat{C},\ \hat{N},\ \hat{H}$

For $i = 1, \ldots, n_r$, construct

$$
\begin{aligned}
V_1^i &= (\sigma_i E - A)^{-1} b, \\
V_2^i &= (2\sigma_i E - A)^{-1}\big(NV_1^i + H(V_1^i \otimes V_1^i)\big), \\
&\ \vdots \\
V_K^i &= (K\sigma_i E - A)^{-T}\big(NV_{K-1}^i + H(\sum_{p=1}^{K-1} V_p^i \otimes V_{K-p}^i)\big).
\end{aligned}
$$

$V = \mathrm{orth}([V_1^i \cdots V_K^i])$

$\hat{E} = V^T E V$, $\hat{A} = V^T A V$, $\hat{b} = V^T b$, $\hat{c} = cV$, $\hat{N} = V^T N V$, $\hat{H} = V^T H(V \otimes V)$.

Fig. 2 *Algorithm 2 Single sided projection with regular transfer functions*

for $i = 1, \ldots, n_r$ and $k = 1, \ldots, K$.

Proof: We define $\tilde{G}_{\mathrm{reg}}^{[k]}(s, 2s, \ldots, ks)$ to be such that

$$\tilde{H}_{\mathrm{reg}}^{[k]}(s, 2s, \ldots, ks) = C\tilde{G}_{\mathrm{reg}}^{[k]}(s, 2s, \ldots, ks).$$

If $k = 1, 2$, then (20) holds, since

$$V\tilde{G}_{\mathrm{reg}}^{[1]}(\sigma_i) = G_{\mathrm{reg}}^{[1]}(\sigma_i), \tag{21}$$

$$V\tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) = G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i), \tag{22}$$

from Lemma 1 since $G_{\mathrm{sym}}^{[1]}(\sigma_i) = G_{\mathrm{reg}}^{[1]}(\sigma_i)$ and $G_{\mathrm{sym}}^{[2]}(\sigma_i, \sigma_i) = G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i)$. For the third and higher dimensional transfer functions, we proceed in the same way. To demonstrate this, we provide the proof for the third subsystem. Define $\tilde{P} = V(V^{\mathrm{T}}EV)^{-1}V^{\mathrm{T}}E$, so that $\tilde{P}^2 = \tilde{P}$ and $\mathrm{range}(\tilde{P}) = \mathrm{range}(V)$. Thus, if $x \in \mathrm{range}(V)$, then $\tilde{P}x = x$. Note that (see equation below). Using (21) and (22) and using the fact that $F(3\sigma_i)F(3\sigma_i)^{-1}$ is equal to the identity matrix, we have (see equation below). Since $\tilde{P}$ is a projector onto $\mathrm{range}(V)$, it follows that:

$$V\tilde{G}_{\mathrm{reg}}^{[3]}(\sigma_i, 2\sigma_i, 3\sigma_i) = G_{\mathrm{reg}}^{[3]}(\sigma_i, 2\sigma_i, 3\sigma_i). \tag{23}$$

Premultiplying the above equation by C proves the statement for the third subsystem. Similarly we can prove the result for higher subsystems. □

Based on Lemma 2, we propose Algorithm 2 that shows the required steps for constructing a reduced interpolation-based quadratic-bilinear model (Fig. 2).

Thus we can construct a reduced quadratic-bilinear system that is interpolating in the sense of (20). The following theorem discusses an oblique projection framework in terms of the regular form for model reduction of quadratic-bilinear systems.

Theorem 3: Let $\sigma_i \in \mathbb{C}$ be the interpolation points and $j \times \sigma_i \notin \{\Lambda(A, E), \Lambda(A_r, E_r)\}$, $j = 1, \ldots, k$. Also let $s_{kj} := s_k - s_j$ for $k = 1, \ldots, K$ and $j = 1, \ldots, K-1$. Assume that $\hat{E} = W^{\mathrm{T}}EV$ is non-singular and $\hat{A} = W^{\mathrm{T}}AV$, $\hat{H} = W^{\mathrm{T}}H(V \otimes V)$, $\hat{N} = W^{\mathrm{T}}NV$, $\hat{B} = W^{\mathrm{T}}B$, $\hat{C} = CV$ with $V \in \mathbb{R}^{n \times r}$ as defined in Lemma 2 and $W \in \mathbb{R}^{n \times r}$ is such that (see (24)) Then the reduced QBDAE obtained via oblique projection ensures that for $i = 1, \ldots, n_r$

$$H_{\mathrm{reg}}^{[k]}(\sigma_i, \ldots, k\sigma_i) = \hat{H}_{\mathrm{reg}}^{[k]}(\sigma_i, \ldots, k\sigma_i), k = 1, \ldots, K, \tag{25}$$

and, in addition, interpolates all combinations of the multimoments that can be written as $w^{\mathrm{T}}Ev$ and $w^{\mathrm{T}}(Nv + H(\sum_{p=1}^{k-1} v \otimes v))$, where $v \in \mathrm{span}(V)$ and $w \in \mathrm{span}(W)$.

Proof: Equation (25) holds due to the structure of V and its proof is given in Lemma 2. To prove the second part, the same argument as used in Theorem 2 is valid here. That is $\hat{w}^{\mathrm{T}}\hat{E}\hat{v} = w^{\mathrm{T}}Ev$ and $\hat{w}^{\mathrm{T}}(\hat{N}\hat{v} + \hat{H}(\sum_{p=1}^{k-1} \hat{v} \otimes \hat{v})) = w^{\mathrm{T}}(Nv + H(\sum_{p=1}^{k-1} v \otimes v))$, where v, w, $\hat{v}$ and $\hat{w}$ are as defined before. □

Remark 3: It is easy to see that for $K = 2$, Theorem 3 reduces to the result shown in Theorem 2.

Using Theorem 3, an algorithm for the oblique projection framework can be easily identified by extending Algorithm 1 to the

$$
\begin{aligned}
V\tilde{G}_{\mathrm{reg}}^{[3]}(\sigma_i, 2\sigma_i, 3\sigma_i) &= V\tilde{F}(3\sigma_i)^{-1}\Big(\tilde{N}\tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + \tilde{H}(\tilde{G}_{\mathrm{reg}}^{[1]}(\sigma_i) \otimes \tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + \tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) \otimes \tilde{G}_{\mathrm{reg}}^{[1]}(\sigma_i))\Big) \\
&= V\tilde{F}(3\sigma_i)^{-1}V^{\mathrm{T}}\Big(NV\tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + H(V\tilde{G}_{\mathrm{reg}}^{[1]}(\sigma_i) \otimes V\tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + V\tilde{G}_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) \otimes V\tilde{G}_{\mathrm{reg}}^{[1]}(\sigma_i))\Big).
\end{aligned}
$$

$$
\begin{aligned}
V\tilde{G}_{\mathrm{reg}}^{[3]}(\sigma_i, 2\sigma_i, 3\sigma_i) &= V\tilde{F}(3\sigma_i)^{-1}V^{\mathrm{T}}F(3\sigma_i)F(3\sigma_i)^{-1}\Big(NG_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + H(G_{\mathrm{reg}}^{[1]}(\sigma_i) \otimes G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) \otimes G_{\mathrm{reg}}^{[1]}(\sigma_i))\Big) \\
&= V\tilde{F}(3\sigma_i)^{-1}V^{\mathrm{T}}F(3\sigma_i)\tilde{P}F(3\sigma_i)^{-1}\Big(NG_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + H(G_{\mathrm{reg}}^{[1]}(\sigma_i) \otimes G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) \otimes G_{\mathrm{reg}}^{[1]}(\sigma_i))\Big) \\
&= \tilde{P}F(3\sigma_i)^{-1}\Big(NG_{\mathrm{reg}}^{2}(\sigma_i, 2\sigma_i) + H(G_{\mathrm{reg}}^{[1]}(\sigma_i) \otimes G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) + G_{\mathrm{reg}}^{[2]}(\sigma_i, 2\sigma_i) \otimes G_{\mathrm{reg}}^{[1]}(\sigma_i))\Big).
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{span}(W) = \underset{i=1,\ldots,n_r}{\mathrm{span}}\Bigg(\Big[&\underbrace{F(\sigma_i)^{-\mathrm{T}}C^{\mathrm{T}}}_{W_1^i}, \underbrace{F(2\sigma_i)^{-\mathrm{T}}N^T W_1^i + F(\sigma_i)^{-\mathrm{T}}H^{(2)}V_1^i \otimes W_1^i}_{W_2^i}, \ldots, \\
&\underbrace{F(K\sigma_i)^{-\mathrm{T}}N^T W_{K-1}^i + (\sum_{p=1}^{K-1} F(p\sigma_i)^{-\mathrm{T}})H^{(2)}(\sum_{p=1}^{K-1} V_{K-p}^i \otimes W_{K-1}^i)}_{W_K^i}\Big]\Bigg)
\end{aligned}
\tag{24}
$$

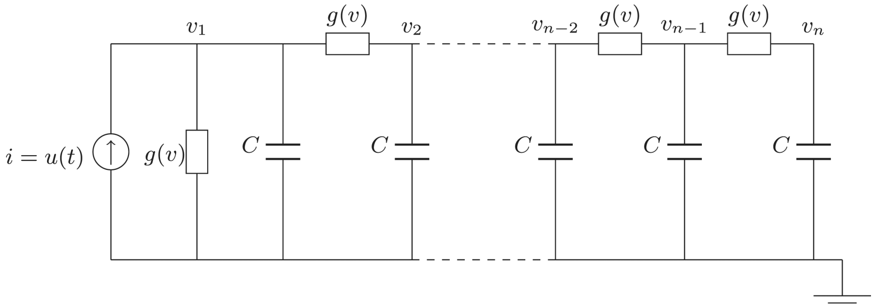

Fig. 3 *RC circuit diagram*

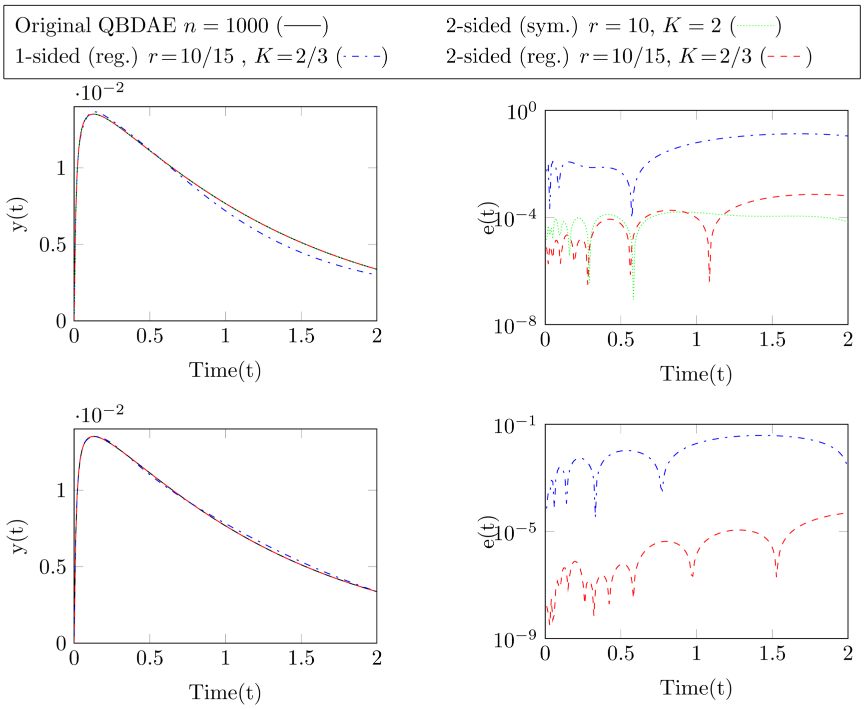

Fig. 4 *Multimoment matching for non-linear RC-circuit for* $u(t) = e^{-t}$
(a,c) Transient responses, ***(b,d)*** Relative errors

general K value, as done in the case of single sided projection given in Algorithm 2

5 Numerical results

We demonstrate our results with two benchmark examples. These include the non-linear RC circuit example [1] and the 1D Burgers' equation [11]. In each case, we use the iterative rational Krylov algorithm [12] on the corresponding linear part of the example to identify Süditaliens interpolation points.

5.1 Non-linear RC circuit

We consider a non-linear RC circuit as shown in Fig. 3, which is a benchmark example for non-linear model reduction [2, 4]

The non-linearity in the system is due to the diode *I–V* characteristics, given by $g(v) = e^{40v} - 1$, where v is the node voltage. The current i is treated as the input and the voltage $v_1(t)$ at node 1 as the output of the system. Using Kirchhoff's current law at each of the n nodes and assuming a normalised capacitance, $C = 1$, we have

$$\dot{v}(t) = f(v(t)) + bu(t), \quad y(t) = cu(t),$$

where $f(v(t))$ is the non-linear function and $b = c^{\mathrm{T}}$ is the first column of the $N \times N$ identity matrix. This non-linear model can be transformed [2] to an equivalent quadratic-bilinear descriptor system with size $n = 2N$. We choose $N = 500$, which results in a quadratic-bilinear system of order 1000.

We reduce the order of the QBDAE system to $r = 10$ by using orthogonal as well as oblique projections such that interpolation for the first two subsystems with the regular form of the transfer functions is ensured. The results are compared with the existing interpolation results based on the symmetric form, for the exponential decay function e^{-t} as system input. The output response and the relative error are shown in Fig. 4*a* and *b*, respectively.

As discussed in Remark 2, the symmetric form is equal to the regular form under some interpolation conditions. Since we are using these interpolation conditions in Section 4, the projection matrix V will not change in the symmetric and the regular forms. Thus the one sided projection results are the same in both the symmetric and regular forms. The partial derivatives of the regular and symmetric forms with respect to s_1 and s_2 are, however, different and therefore the oblique projection matrix W is different in the symmetric and regular forms. As shown in Figs. 4*a* and *b*, the relative error associated with the proposed two-sided projection technique in the regular case is comparable to the existing two-sided projection technique in the symmetric case.

Interpolation of third and higher subsystems requires some simplifications in order to use the two-sided symmetric projection approach. Since the primary source of two-sided symmetric projection approach [4] is restricted to the first two subsystems, for interpolation of higher subsystems, we are not comparing our results with the two-sided symmetric case. The reduced model through the one-sided symmetric projection, however, would be the same as obtained from the use of regular form. Results for our proposed one-sided and two-sided regular projection approaches

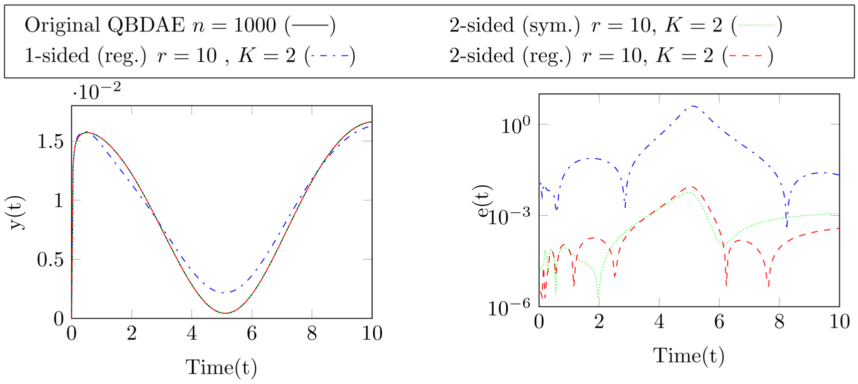

Fig. 5 *Multimoment matching for non-linear RC-circuit for* $u(t) = \cos(2\pi(t/10) + 1)/2$
(a) Transient response for $K = 2$, ***(b)*** Relative error $e(t) = (y(t) - \hat{y}(t))/y(t)$

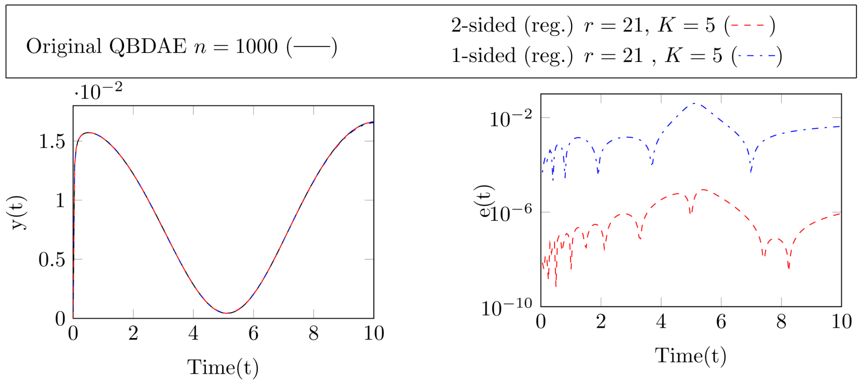

Fig. 6 *Multimoment matching for non-linear RC-circuit for* $u(t) = \cos(2\pi(t/10) + 1)/2$
(a) Transient response and, ***(b)*** Relative error

are shown in Figs. 4*c* and *d*, where interpolation is achieved for the first three subsystems, $K = 3$. The size of the reduced system becomes $r = 15$.

The results show that the additional use of the third transfer function improves the error of both the one-sided projection approach and the two-sided approach. Although the improvement is not significant in the one-sided projection, we gain basically one order of accuracy for the two-sided approach. We believe that this might be due to the choice of interpolation points. Since in the two-sided approach, we are matching derivatives as well for the same choice of interpolation points, the effect of interpolating the third transfer function is more obvious in the two-sided case.

Now we change the system input to $u(t) = \cos(2\pi(t/10) + 1)/2$ and use the same reduced models as before for the $K = 2$ case to obtain the results are shown in Figs. 5*a* and *b*. Unlike trajectory based methods for model order reduction, the results show that the approximation quality of the projection based reduced models is not effected by the variation in the control input. Next we consider interpolation of the first five subsystems. The size of the reduced system is now $r = 21$ (which should be ≤ 25). The results are shown in Figs. 6*a* and *b*. Although we are using the same set of interpolation points as those used in Fig. 4, here we get a significant improvement of the one-sided and two-sided approximation errors. This is because we are now interpolating the first five subsystems in the regular form.

5.2 Burgers' equation

As a second example, we consider a 1D Burgers' equation on $\Omega = (0, 1) \times (0, T)$, resulting in a set of equations, cf. [11],

$$\begin{aligned} v_t + v \cdot v_x &= \nu \cdot v_{xx}, \quad \text{in } (0,1) \times (0,T), \\ \alpha v(0, \cdot) + \beta v(0, \cdot) &= u(t), \quad \text{in } (0,T), \\ v_x(1,0) &= 0, \quad \text{in } ((0,T), \\ v(x,0) &= v_0(x), \quad \text{in } (0,1), \end{aligned} \tag{26}$$

where ν is the viscosity and $v(0, x)$ is the initial condition of the system. A semi-discretisation of the above PDE generates a quadratic-bilinear system. We choose $\nu = 0.05$ and $n = 1000$ points for spatial discretisation of the system and reduce the order of the quadratic-bilinear system to $r = 10$ using symmetric as well as regular form for $K = 2$. The results are shown in Figs. 7 and 8 for $u(t) = e^{-t}$ and $u(t) = \cos(2\pi(t/10) + 1)/2$, respectively. If in addition, we ensure the interpolation of the third subsystem, our reduced model is of size $r = 15$. These are also shown in Figs. 5 and 6 for two different inputs.

The results clearly show that the additional interpolation of the third subsystem improves the approximation error significantly in both one-sided and two-sided projection approaches. The quality of the reduced model varies with the choice of the interpolation points. We used IRKA on the linear part of the system to select and fix a choice of interpolation points.

6 Conclusion

We extended the orthogonal and oblique projection framework for model reduction of quadratic-bilinear systems to multimoment matching of higher subsystems. For this, we derive the regular form of the multivariate transfer functions associated with the quadratic-bilinear system. The structure of the regular multivariate transfer functions is simpler as compared to the symmetric form of the transfer functions. Multimoment matching of the regular form is therefore easy to ensure for higher subsystems. The choice of interpolation points is an important issue. We selected the interpolation points so that the basis vectors can be reused for other basis vectors. An important future work would be to improve the choice of interpolation points.

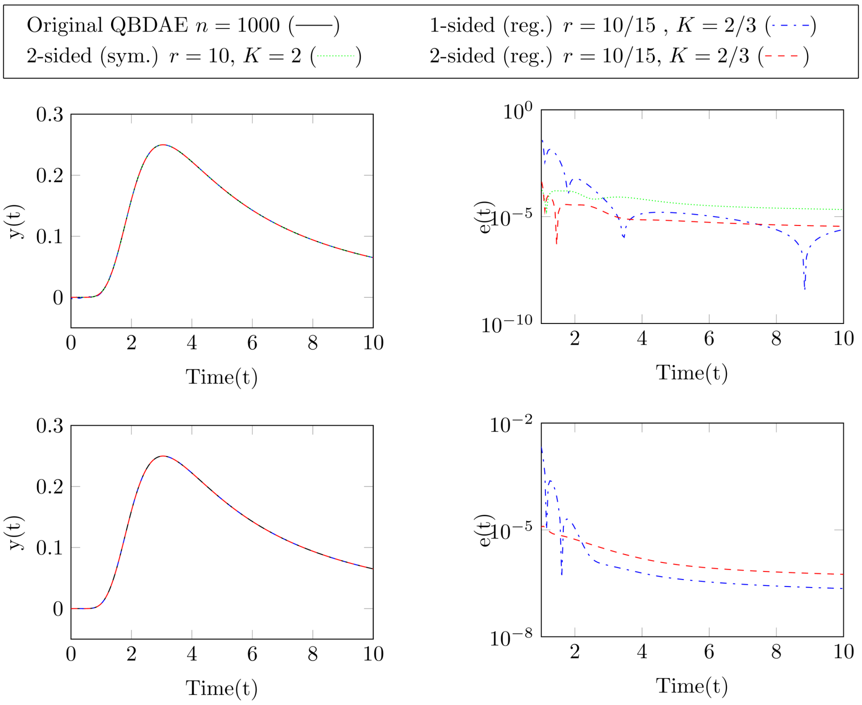

Fig. 7 *Multimoment matching for Burgers' equation with* $u(t) = e^{-t}$
(a,c) Transient responses, ***(b,d)*** Relative errors

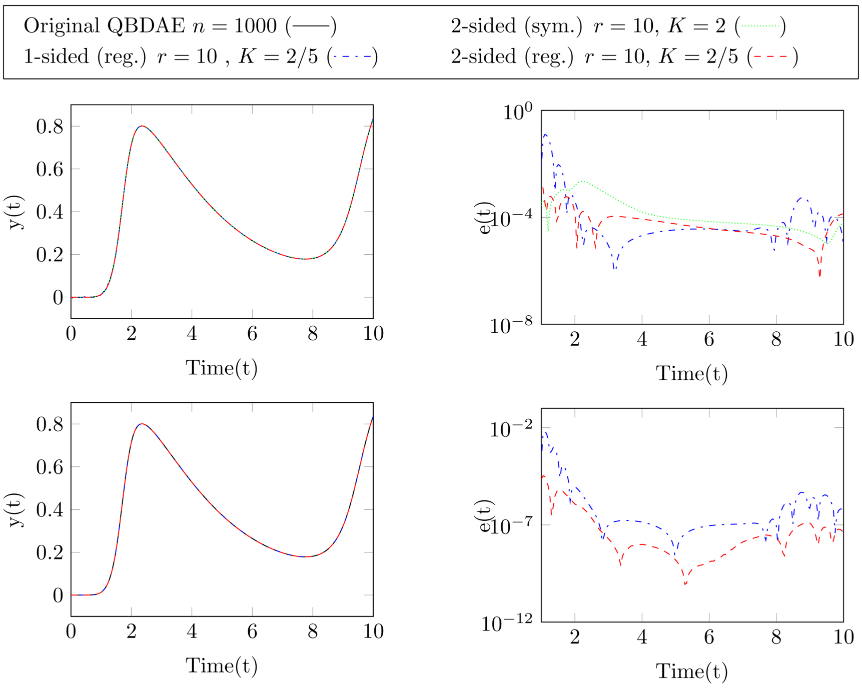

Fig. 8 *Multimoment matching for Burgers' equation with* $u(t) = \cos(2\pi(t/10) + 1)/2$
(a,c) Transient responses, ***(b,d)*** Relative errors

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