Spin-multipole effects in binary black holes and the test-body limit

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(Dated: June 29, 2016)

We discuss the Hamiltonian for the conservative dynamics of generic-orbit arbitrary-mass-ratio spinning binary black holes, at the leading post-Newtonian orders at each order in an expansion in spins, to all orders in the spins. The leading-order couplings can all be obtained from a map to the motion of a test black hole (a test body with the spin-induced multipoles of a Kerr black hole) in the Kerr spacetime, as is confirmed with direct post-Newtonian calculations for arbitrary mass ratios. Furthermore, all of the couplings can be "deduced" from those of a pole-dipole test body in Kerr.

Binary black holes (BBH's) [or something very much like them] have provided us with the first gravitational waves detected at Earth [1, 2]. So far, the detected signals lie within the error bars of our expecations—that black holes exist, that binaries of them emit gravitational waves which become ever stronger as they spiral into one another and eventually merge into one bigger black hole, and that all of this is governed by Einstein's theory of general relativity (GR) [3].

Being able to make such statements requires that we know exactly what it is that GR predicts. Decades of work in numerical relativity (solving Einstein's equations directly on a supercomputer), combined with analytic approximation schemes for treating the two-body problem, have led to the current understanding of BBH's which has allowed the analysis of the detected signals. Yet, there are still many respects in which the relativistic two-body problem is not yet solved (see e.g. Fig. 1).

For analytic attacks, two complimentary approximation schemes are available: The post-Newtonian (PN) approximation expands about the Newtonian (weak-field, slow-motion) limit but is valid for arbitrary mass ratios [4, 5], while the extreme-mass-ratio approximation, encompassing the "self-force paradigm" [6–12], expands about the test-body limit but is valid in the strong-field, relativistic regime.

The claims made in this paper's abstract point out an interplay between the PN limit and the test-body limit, which relies on very special properties (seemingly) specific to BBH's in GR. This interplay ultimately allows one to determine the leading-PN-order BBH dynamics, to all orders in the BHs' spins, using only information from the test-body limit, in two different ways. This points toward structure in the BBH dynamics which has yet to be recognized.

The leading-PN-order, all-orders-in-spin BBH Hamiltonian is given explicitly by (23) below. This result (with its conceivable generalizations beyond leading order) is of particular interest for the case of large BH spins, and is thus relevant to LIGO's ability to test GR with strong, nonlinear spin/precession effects in BBH's [3].

We can begin to explain and substantiate the above claims by reviewing the results of PN calculations which describe the conservative dynamics of binaries of compact objects with spin-induced multipole moments.

In the PN approximation, we can describe a binary of compact objects, bodies $A = 1, 2$, in terms of

- their worldlines $\boldsymbol{x} = \boldsymbol{z}_A(t)$ in a post-Newtonian spacetime with coordinates $x^\mu = (t, x^i) = (t, \boldsymbol{x})$, defining the relative position $\boldsymbol{R} = \boldsymbol{z}_2 - \boldsymbol{z}_1$ and distance $R = |\boldsymbol{R}|$,
- their masses m_A, defining the total mass $M = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/M$, and the symmetric mass ratio $\nu = \mu/M$, taking $m_1 \geq m_2$, with the "test-body limit" defined by $m_2 \to 0$,
- their intrinsic angular momentum (or spin) vectors $\boldsymbol{S}_A = S^i_A$, defining the rescaled spin vectors $\boldsymbol{a}_A = \boldsymbol{S}_A/m_A c$ with dimensions of length,
- and their higher-order multipole moments, which begin with the mass quadrupole tensors Q^{ij}_A,

with appropriate definitions for these quantities, sufficient for our purposes here, given e.g. in [13, 14] or [15].

In general, the bodies' quadrupoles and higher-order moments can be dynamical, depending on further internal degrees of freedom [16–19]. But the leading-order effects in the post-Newtonian regime arise from (i) intrinsic spin-induced quadrupolar deformations scaling as the square of the spin, and (ii) quadrupolar tidal deformations which are adiabatically induced by the external field, which contribute to the quadrupole tensors as follows [20],

$$Q^{ij}_A = -\kappa_A m_A a_A^{<i} a_A^{j>} - \lambda_A \mathcal{E}^{ij}(\boldsymbol{z}_A). \tag{1}$$

Here, $\mathcal{E}_{ij}$ is the electric tidal tensor, with $\mathcal{E}_{ij} = -\partial_i \partial_j U$ in the Newtonian limit, with U being the Newtonian potential with convention $U = Gm/R$ for a monopole. Angle brackets denote symmetric-tracefree (STF) projection.

The constants κ_A and λ_A are linear response coefficients, measuring the leading-order quadrupolar deformation of body A, due to its rotation/spin and due to the external tidal field, respectively. The values appropriate for a black hole,

$$\kappa_{\rm BH} = 1, \qquad \lambda_{\rm BH} = 0, \tag{2}$$

have been established through several arguments and derivations, e.g. [20–26].

We will restrict attention to spin-induced multipoles, as is appropriate for black holes at the PN orders discussed here. Spinning bodies (like black holes) generally have

- even-order mass multipoles $\mathcal{I}^L$,
  quadrupole $Q^{ij} \equiv \mathcal{I}^{ij}$, hexadecapole $\mathcal{I}^{ijkl}$, $\ldots$, 2^ℓ-pole $\mathcal{I}^L = \mathcal{I}^{i_1 \cdots i_\ell}$ with ℓ even, and
- odd-order current multipoles $\mathcal{J}^L$,
  dipole $S^i \equiv \mathcal{J}^i$, octupole $\mathcal{J}^{ijk}$, $\ldots$, 2^ℓ-pole $\mathcal{J}^L = \mathcal{J}^{i_1 \cdots i_\ell}$ with ℓ odd,

which are induced by their rotation, and which are generally proportional to STF outer products of ℓ copies of the spin vector. $L = i_1 \ldots i_\ell$ is a spatial multi-index. The proportionality constants (like κ for $\ell = 2$) vary with the composition and structure of the bodies. The multipoles of a black hole (with certain normalizations), including the mass monopole $m \equiv \mathcal{I}$, are given by [21]

$$\left(\mathcal{I}^L + \frac{i}{c}\mathcal{J}^L\right)_{\rm BH} = i^\ell \, m \, a^{<L>}, \tag{3}$$

where $a^{<L>} = a^{<i_1} \ldots a^{i_\ell>} = S^{<L>}/(mc)^\ell$, with $\ell = 0, 1, \ldots, \infty$. With $\ell = 2$, this reproduces (1) with $\kappa = 1$ (and $\lambda = 0$).

With only such spin-induced multipoles, the only degrees of freedom of the binary are the relative position $\boldsymbol{R}(t)$ (in the center-of-mass frame) and the spins $\boldsymbol{S}_1(t)$ and $\boldsymbol{S}_2(t)$. The PN conservative dynamics can be encoded in a Hamiltonian $H(\boldsymbol{R}, \boldsymbol{P}, \boldsymbol{S}_1, \boldsymbol{S}_2)$, where $\boldsymbol{P}$ is the linear momentum canonically conjugate to $\boldsymbol{R}$, and the equations of motion are determined from

$$\dot{R}^i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial R^i}, \quad \dot{S}^i_A = \epsilon^{ij}{}_k \frac{\partial H}{\partial S^j_A} S^k_A, \tag{4}$$

with $A = 1, 2$.

The Hamiltonian, expanded in PN orders and in powers of the spins, takes the following form, $H =$

$$\begin{aligned} & H^{(0\rm PN)}_{\text{LO-S}^0} \\ & + H^{(1\rm PN)}_{\text{NLO-S}^0} + H^{(1.5\rm PN)}_{\text{LO-S}^1} + H^{(2\rm PN)}_{\text{LO-S}^2} \\ & + H^{(2\rm PN)}_{\text{NNLO-S}^0} + H^{(2.5\rm PN)}_{\text{NLO-S}^1} + H^{(3\rm PN)}_{\text{NLO-S}^2} + H^{(3.5\rm PN)}_{\text{LO-S}^3} + H^{(4\rm PN)}_{\text{LO-S}^4} \\ & + \ldots \end{aligned} \tag{5}$$

Here, $H_{\text{LO-S}^0} \equiv H_{\rm N}$ is the Newtonian (0PN) point-mass (no-spin) Hamiltonian, and the other terms are at higher orders in the PN parameter $\epsilon \sim Gm/c^2R \sim v^2/c^2$ and higher orders in the spin. The PN order counting here, with an nPN contribution scaling as $\epsilon^n H_{\rm N}$, assumes rapidly rotating bodies, with spin magnitudes $S \sim Gm^2/c$.

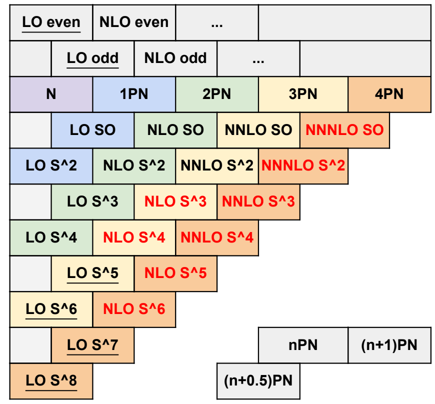

FIG. 1. Contributions to the two-body Hamiltonian in the PN-spin expansion, for arbitrary-mass-ratio binaries with spin-induced multipole moments (such as BBH's). Terms in red text are unknown. Terms in black text have been calculated, and confirmed by independent groups, all except for (i) the recent NNLO S^2 calculations of [27], (ii) the recent 4PN calculations of [28–31], and (iii) the underlined LO-S^n terms with $n \geq 5$, which are presented here for BBH's.

We now discuss in turn the contributions in (5). To ease the notation, we henceforth set $G = c = 1$ and define new momenta rescaled by the reduced mass,

$$\bar{H} = \frac{H}{\mu}, \qquad \bar{\boldsymbol{P}} = \frac{\boldsymbol{P}}{\mu}, \qquad \bar{\boldsymbol{L}} = \frac{\boldsymbol{L}}{\mu} = \boldsymbol{R} \times \bar{\boldsymbol{P}}, \tag{6}$$

where $\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P}$ is the orbital angular momentum. We follow e.g. [32, 33] for the results summarized in the following three subsections.

Nonspinning. The Newtonian Hamiltonian [i.e. the leading-order (LO) no-spin (LO-S^0) Hamiltonian] reads

$$\bar{H}_{\rm N} = \frac{\bar{\boldsymbol{P}}^2}{2} - \frac{M}{R}, \tag{7}$$

and the 1PN point-mass Hamiltonian [i.e. the next-to-leading-order (NLO) no-spin (NLO-S^0) Hamiltonian], in harmonic/ADM gauge, reads

$$\begin{aligned} \bar{H}_{\rm 1PN} = {} & (-1 + 3\nu)\frac{\bar{\boldsymbol{P}}^4}{8} + (-3 - 2\nu)\frac{M\bar{\boldsymbol{P}}^2}{2R} \\ & + (0 + \nu)\frac{M\bar{\boldsymbol{L}}^2}{2R^3} + (1 + 0\nu)\frac{M^2}{2R^2}. \end{aligned} \tag{8}$$

The Newtonian Hamiltonian could be said to be equal to its test-body limit, in the sense that it has no dependence on the mass ratio. The same is not true of $H_{\rm 1PN}$, and one can recover only the first terms in parentheses from the test-body limit, with the terms $\propto \nu$ being "self-force corrections" [6–10].

More precisely, even $H_{\rm N}$ is not literally equal to its test-body limit, if this is defined as the limit $m_2 \to 0$, because then we obtain only the m_1 term in $M = m_1 + m_2$ [among other subtleties]. Rather, the arbitrary-mass-ratio Newtonian dynamics in the center of mass frame is *equivalent* to the dynamics of a test body with mass μ in the field of a stationary mass M, under the map $M = m_1 + m_2$ and $\mu = m_1 m_2/M$. This map and this fortuitous coincidence allow us to obtain from the test-body limit what could be considered a self-force correction (the m_2 in M) [and less handwavingly, the full exact Newtonian dynamics].

Leading-order spin-orbit couplings. Next we have the leading-order "spin-orbit" (linear-in-spin) Hamiltonian, at 1.5PN,

$$\bar{H}_{\text{LO-S}^1} = \left(2m_1 + \frac{3}{2}m_2\right)\frac{\bar{\boldsymbol{L}}\cdot\boldsymbol{a}_1}{R^3} + \left(\frac{3}{2}m_1 + 2m_2\right)\frac{\bar{\boldsymbol{L}}\cdot\boldsymbol{a}_2}{R^3}. \tag{9}$$

The m_2 terms are self-force corrections which drop out in the test-body limit. But under the map

$$\begin{aligned} \boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 &= m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 = M\boldsymbol{a}, \\ \frac{\boldsymbol{S}_{\rm test}}{\nu} = \boldsymbol{S}^* = \frac{m_1}{m_2}\boldsymbol{S}_2 + \frac{m_2}{m_1}\boldsymbol{S}_1 &= m_1\boldsymbol{a}_2 + m_2\boldsymbol{a}_1 = M\boldsymbol{\sigma}, \end{aligned} \tag{10}$$

this can be rewritten exactly as the LO linear-in-spins part of the Hamiltonian of a test body with mass μ and spin $\boldsymbol{S}_{\rm test} = \mu\boldsymbol{\sigma} = \nu\boldsymbol{S}^*$ in the field of a stationary body with mass M and spin $\boldsymbol{S} = M\boldsymbol{a}$,

$$\begin{aligned} \bar{H}_{\text{LO-S}^1}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) &= \bar{H}^{\rm test}_{\text{LO-S}^1}(M, \boldsymbol{a}, \mu, \boldsymbol{\sigma}) \qquad (11) \\ &= \bar{\boldsymbol{L}}\cdot\left(2\boldsymbol{a} + \frac{3}{2}\boldsymbol{\sigma}\right)\frac{M}{R^3} \\ &= -\bar{\boldsymbol{P}}\times\left(2\boldsymbol{a} + \frac{3}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,\frac{M}{R}, \end{aligned}$$

where $\boldsymbol{\partial} = \partial_i = \partial/\partial R^i$.

Note that the dynamics defined by $H_{\rm N}$, $H_{\rm 1PN}$, and $H_{\text{LO-S}^1}$ (and more generally, through linear order in spin) is universal, independent of the nature of the bodies.

Leading-order spin-squared couplings. The next contribution in (5) is the LO-S^2 Hamiltonian at 2PN,

$$\bar{H}_{\text{LO-S}^2} = \frac{1}{2}\Big(\kappa_1 a_1^i a_1^j + 2a_1^i a_2^j + \kappa_2 a_2^i a_2^j\Big)\partial_i\partial_j\frac{M}{R}, \tag{12}$$

which begins to depend on the bodies' internal structure through the response coefficients $\kappa_{1,2}$. Note that the κ terms encode the coupling of the spin-induced quadrupole of one body to the monopole of the other, while the a_1-a_2 term encodes the (universal) coupling between the bodies' spins.

Remarkably, for the very special case of a binary black hole, $\kappa_1 = \kappa_2 = 1$, this factorizes into [33]

$$\begin{aligned} &\bar{H}^{\rm BBH}_{\text{LO-S}^2}(m_1, \boldsymbol{a}_1, m_2, \boldsymbol{a}_2) = \frac{1}{2}\big(\boldsymbol{a}_1 + \boldsymbol{a}_2\big)^i\big(\boldsymbol{a}_1 + \boldsymbol{a}_2\big)^j\partial_i\partial_j\frac{M}{R} \\ &= \bar{H}^{\rm BBH,test}_{\text{LO-S}^2}(M, \boldsymbol{a}, \mu, \boldsymbol{\sigma}) = \frac{1}{2}\big((\boldsymbol{a}+\boldsymbol{\sigma})\cdot\boldsymbol{\partial}\big)^2\frac{M}{R} \qquad (13) \\ &= \bar{H}^{\rm BBH,test}_{\text{LO-S}^2}(M, \boldsymbol{a}_0, \mu, 0) = \frac{1}{2}(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^2\frac{M}{R} \qquad (14) \end{aligned}$$

noting that spin vectors commute with spatial derivatives $\boldsymbol{\partial}$, and where

$$\boldsymbol{a}_0 = \boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a} + \boldsymbol{\sigma} = \frac{\boldsymbol{S} + \boldsymbol{S}^*}{M} = \frac{\boldsymbol{S}_0}{M} \tag{15}$$

is the combination of the spins whose importance was noted in [33, 34].

The LO-S^2 BBH Hamiltonian is equivalent to that of a test-body in two different ways. On the one hand, as in (13), it is the LO quadratic-in-spins part of the Hamiltonian of a "test black hole" with mass μ and spin $\mu\boldsymbol{\sigma}$ (and quadrupole $Q^{ij} = -\mu\sigma^{<i}\sigma^{j>}$) in the field of a stationary Kerr black hole with mass M and spin $M\boldsymbol{a}$. On the other hand, as in (14), it is the LO-S^2 part of the Hamiltonian of a structureless point mass (following a geodesic) in the field of a Kerr black hole with mass M and spin $M\boldsymbol{a}_0$ [33, 34].

Leading-order couplings for binary black holes through fourth order in spin. The LO-S^3 (3.5PN) and LO-S^4 (4PN) contributions in (5) have been computed and confirmed by a variety of methods in [35–39]. To the authors' knowledge, there are no previous results for the PN dynamics of arbitrary-mass-ratio binaries at fifth or sixth order in the spins (5.5PN or 6PN at LO) or beyond, though much is known (at least in principle) from the test body limit.

The LO-S^3 contributions arise from (i) a body's spin-induced current octupole coupling to its companion's mass monopole, (ii) the mass quadrupole coupling to the companion's spin, and (iii) more subtle kinematical effects. These kinematical effects, like those encountered for the spin-orbit couplings (11) which are linked to Thomas precession [40], are related to the transport of the local frame in which the spin is defined and its interplay with the spin supplementary condition [41].

The LO-S^4 contributions arise from hexadecapole-monopole, octupole-dipole, and quadrupole-quadrupole couplings. As with the LO-S^2 couplings, there is no dependence on $\bar{\boldsymbol{P}}$, only on $\boldsymbol{R}$, and there are no subtle kinematical effects.

Like the LO-S^2 part, the LO-S^3 and LO-S^4 parts undergo remarkable simplifications in the special case when the spin-induced multipole moments match those of a Kerr black hole.

Now we gather all of the results for the leading-PN-order Hamiltonians at each order in spin, available from [35–39] through fourth order in spin, specializing to the

BBH case. This is as in (5), but where we will neglect the NLO terms $H_{1\rm PN}$, $H_{\rm NLO\text{-}S^1}$, $H_{\rm NLO\text{-}S^2}$ and (at NNLO) $H_{2\rm PN}$, as well as all other NLO terms. Working from the Hamiltonians of [37], after a canonical transformation affecting only the S^3 terms, and after some simplification, using (10), the leading-order Hamiltonian can be written as $\bar{H}^{\rm BBH}_{\rm LO} = \bar{H}^{\rm BBH}_{\rm LO,even} + \bar{H}^{\rm BBH}_{\rm LO,odd}$, with the even-in-spins part

$$\bar{H}^{\rm BBH}_{\rm LO,even} = \frac{\bar{\boldsymbol{P}}^2}{2} - \frac{M}{R} + \frac{1}{2!}(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^2\,\frac{M}{R} - \frac{1}{4!}(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^4\,\frac{M}{R} + \mathcal{O}(S^6), \tag{16}$$

and the odd-in-spins part

$$\bar{H}^{\rm BBH}_{\rm LO,odd} = -\frac{1}{1!}\bar{\boldsymbol{P}}\times\left(2\boldsymbol{a}+\frac{3}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,\frac{M}{R} + \frac{1}{3!}\bar{\boldsymbol{P}}\times\left(2\boldsymbol{a}+\frac{1}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^2\,\frac{M}{R} + \mathcal{O}(S^5). \tag{17}$$

Note that hidden within these "factorized" forms is a considerable network of multipole-multipole couplings and kinematical effects, as well as "self-force corrections."

All of these LO PN results (for arbitrary mass ratios), even and odd, are obtained from the test-body limit according to

$$\bar{H}^{\rm BBH}_{\rm LO}(m_1,\boldsymbol{a}_1,m_2,\boldsymbol{a}_2) = \bar{H}^{\rm BBH,test}_{\rm LO}(M,\boldsymbol{a},\mu,\boldsymbol{\sigma}), \tag{18}$$

with (10), where $\bar{H}^{\rm BBH,test}$ is the Hamiltonian of a "*test black hole*" with mass μ and spin $\mu\boldsymbol{\sigma}$—having all of the spin-induced multipoles of a black hole, keeping $\boldsymbol{\sigma}$ finite as $\mu\to 0$, noting that all of the LO couplings end up with one factor of μ which scales away as in (6)—in a Kerr spacetime with mass M and spin $M\boldsymbol{a}$. The even part has the further feature

$$\bar{H}^{\rm BBH}_{\rm LO,even}(m_1,\boldsymbol{a}_1,m_2,\boldsymbol{a}_2) = \bar{H}^{\rm BBH,test}_{\rm LO,even}(M,\boldsymbol{a}_0,\mu,0), \tag{19}$$

so that it is obtained from geodesic motion in a Kerr spacetime with mass M and spin $M\boldsymbol{a}_0$; this is not true of the odd part. These hold for the above results through fourth order in spin, as can be confirmed from [42], and we will see that they hold to all orders. Note that this means that an effective-one-body Hamiltonian which uses $M\boldsymbol{a}_0$ as the spin for an effective (ν-deformed) Kerr metric entering the geodesic Hamiltonian (a recent example being that in [43]) correctly encodes all of the LO even-in-spin couplings.

To all orders in spin, at the leading post-Newtonian orders, for binary black holes. There is a clear pattern developing in the even part (16). In light of (3), one is well-motivated to argue that this pattern continues,

$$\bar{H}^{\rm BBH}_{\rm LO,even} = \frac{\bar{\boldsymbol{P}}^2}{2} - \sum_{\ell}^{\rm even}\frac{i^\ell}{\ell!}(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^\ell\,\frac{M}{R} = \frac{\bar{\boldsymbol{P}}^2}{2} - \cos(\boldsymbol{a}_0\cdot\boldsymbol{\partial})\,\frac{M}{R}. \tag{20}$$

For the odd part, one could argue from the limited data in (17) that there is an analogous pattern developing, with only two new coefficients at each order in spin (the coefficients of $\boldsymbol{a}$ and $\boldsymbol{\sigma}$ in the cross product). If we were to assume that this pattern holds to all orders, then these coefficients would all be fixed by matching to the dynamics of a pole-dipole test body in the Kerr spacetime, obeying the Mathisson-Papapetrou-Dixon (MPD) equations [44–46] to linear order in the spin of the test body. The resultant coefficients are available in principle from [42, 47] and are derived in detail in [48]. This yields

$$\begin{aligned}\bar{H}^{\rm BBH}_{\rm LO,odd} &= \sum_{\ell}^{\rm odd}\frac{i^{\ell-1}}{\ell!}\bar{\boldsymbol{P}}\times\left(-2\boldsymbol{a}+\frac{\ell-4}{2}\boldsymbol{\sigma}\right)\cdot\boldsymbol{\partial}\,(\boldsymbol{a}_0\cdot\boldsymbol{\partial})^{\ell-1}\,\frac{M}{R}\\ &= \bigg[-2\,\bar{\boldsymbol{P}}\times\boldsymbol{a}_0\cdot\boldsymbol{\partial}\,\frac{\sin(\boldsymbol{a}_0\cdot\boldsymbol{\partial})}{\boldsymbol{a}_0\cdot\boldsymbol{\partial}} + \frac{1}{2}\,\bar{\boldsymbol{P}}\times\boldsymbol{\sigma}\cdot\boldsymbol{\partial}\,\cos(\boldsymbol{a}_0\cdot\boldsymbol{\partial})\bigg]\frac{M}{R}.\end{aligned} \tag{21}$$

The results (20) and (21) can also be derived without relying on such seemingly unjustified extrapolation.

We show in [48] how these Hamiltonians $H^{\rm BBH}_{\rm LO}(m_1,\boldsymbol{a}_1,m_2,\boldsymbol{a}_2)$ are obtained as the leading-PN-order part of the Hamiltonian of a test black hole with mass $\mu = m_1m_2/M$ and spin $\mu\boldsymbol{\sigma} = \nu(m_1\boldsymbol{a}_2+m_2\boldsymbol{a}_1)$ in a Kerr spacetime with mass $M = m_1+m_2$ and spin $M\boldsymbol{a} = m_1\boldsymbol{a}_1+m_2\boldsymbol{a}_2$, as in (18).

We obtain the same results, (20) and (21) under the map (10), from a direct PN calculation for arbitrary mass ratios in [49].

The calculations of [48, 49] are both based on generalizations of the well-developed action description of spinning bodies in general relativity [41, 50–54], which results in a form of the MPD dynamics [44–46]. The action encodes both the bodies' motion in an effective external field and the effective stress-energy which sources the field equations—at the least, in the leading-PN-order context. In [48, 49], we draw in particular from the analyses of spin-multipole effects in [39, 42, 55–57], and [41] which derived all LO spin-induced multipole couplings. We obtain all needed coupling constants in the action by matching the effective black holes' spin-multipole structure to that of a Kerr black hole, and by ensuring the kinematical consistency of the MPD dynamics.

We can also present the results (20) and (21) in a simple explicit closed form by introducing new coordinates on the (flat) 3-space; starting from Cartesian coordinates (X,Y,Z), as illustrated in Fig. 2, we have

$$\begin{aligned}&\text{cylindrical } (\rho,\Phi,Z), && X=\rho\cos\Phi, && Y=\rho\sin\Phi,\\ &\text{spherical } (R,\Theta,\Phi), && \rho=R\sin\Theta, && Z=R\cos\Theta,\\ &\text{spheroidal } (r,\theta,\Phi), && \rho=\sqrt{r^2+a_0^2}\,\sin\theta, && Z=r\cos\theta.\end{aligned} \tag{22}$$

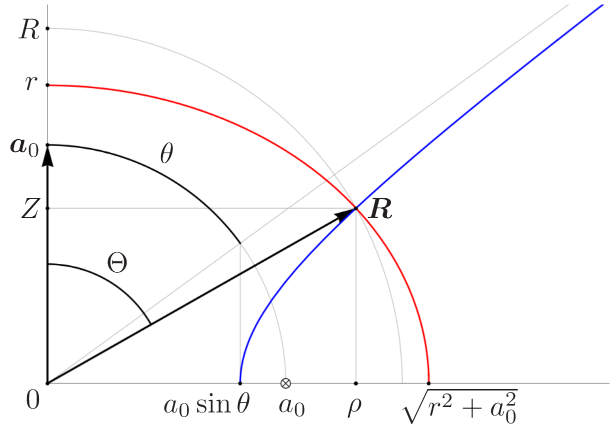

FIG. 2. Relationships between the coordinates (22) on flat 3-space, with (r, θ, Φ) being oblate spheroidal coordinates with "ring-radius" $a_0 = |\boldsymbol{a}_0|$. Shown here is a quadrant of a vertical plane, with the Z-axis vertical and the ρ-axis horizontal. The surfaces of constant r are oblate ellipsoids with foci on "the ring" $\rho^2 = X^2 + Y^2 = a_0^2$ in the $Z = 0$ plane, with a cross-section shown in red. The ring pierces this plane orthogonally at the $\otimes$ symbol, with its center at the origin. The locus $r = 0$ is the disk $Z = 0$, $\rho < a_0$ bounded by the ring. The surfaces of constant θ are half- one-sheeted hyperboloids with foci on the same ring, with a cross-section shown in blue; as $r \to \infty$, they asymptote to cones opening an angle θ from the $+Z$-axis. The locus $\theta = \pi/2$ is the plane $Z = 0$ minus the disk $\rho < a_0$. The ring represents the "ring singularity" of an effective Kerr BH with rescaled spin $\boldsymbol{a}_0$.

The LO Hamiltonian, the sum of (20) and (21), can then be written as [48]

$$H^{\rm BBH}_{\rm LO} = \frac{\bar{\boldsymbol{P}}^2}{2} - \frac{Mr}{r^2 + a_0^2\cos^2\theta}\left(1 - \frac{2\boldsymbol{R}\times\bar{\boldsymbol{P}}\cdot\boldsymbol{a}_0}{r^2+a_0^2}\right) - \frac{M}{4}\bar{\boldsymbol{P}}\times\boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{R}+i\boldsymbol{a}_0}{(r+ia_0\cos\theta)^3} + c.c.\right), \quad (23)$$

where *c.c.* denotes the complex conjugate. The manipulations linking (20)–(21) to (23) are similar to those in [58].

The dynamics defined by (23) is likely to have to have further unique properties, an exploration of which we leave to future work. One is led to wonder, for example, if it admits an analog of the Carter constant [59].

Conclusion. We have presented and argued for the leading-PN-order, all-orders-in-spin Hamiltonian for a binary black hole, which is derived in detail [49]. This Hamiltonian can also be obtained from the test-body limit in Kerr in two different ways, as is demonstrated in [48]. These results are clearly relevant to efforts to develop effective-one-body Hamiltonians for BBH's (see e.g. [33, 34, 43, 60, 61]).

Acknowledgements. We thank Alejandro Bohé, Alessandra Buonanno, Éanna Flanagan, Abraham Harte, Tanja Hinderer, and Sylvain Marsat for their comments on this draft and for our many fruitful conversations.

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[1] L. S. Collaboration and V. Collaboration, Phys. Rev. Lett. **116**, 061102 (2016).
[2] B. P. Abbott *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. **116**, 241103 (2016).
[3] B. P. Abbott *et al.* (Virgo, LIGO Scientific), Phys. Rev. Lett. **116**, 221101 (2016), arXiv:1602.03841 [gr-qc].
[4] L. Blanchet, Living Rev.Rel. **9**, 4 (2006).
[5] E. Poisson and C. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic* (Cambridge University Press, 2014).
[6] S. L. Detweiler and E. Poisson, Phys. Rev. **D69**, 084019 (2004), arXiv:gr-qc/0312010 [gr-qc].
[7] E. Poisson, A. Pound, and I. Vega, Living Reviews in Relativity **14**, 7 (2011), arXiv:1102.0529 [gr-qc].
[8] L. Barack, in *General Relativity, Cosmology and Astrophysics*, Fundamental Theories of Physics, Vol. 177 (Springer International Publishing, 2014) pp. 147–168.
[9] A. I. Harte, *Proceedings, 524th WE-Heraeus-Seminar: Equations of Motion in Relativistic Gravity (EOM 2013)*, Fund. Theor. Phys. **179**, 327 (2015), arXiv:1405.5077 [gr-qc].
[10] A. Pound, ArXiv e-prints (2015), arXiv:1506.06245 [gr-qc].
[11] D. Bini and T. Damour, Phys. Rev. **D93**, 104040 (2016), arXiv:1603.09175 [gr-qc].
[12] D. Bini, T. Damour, and A. Geralico, (2016), arXiv:1602.08282 [gr-qc].
[13] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **43**, 3273 (1991).
[14] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992).
[15] E. Racine and E. E. Flanagan, Phys. Rev. **D71**, 044010 (2005), [Erratum: Phys. Rev.D88,no.8,089903(2013)], arXiv:gr-qc/0404101 [gr-qc].
[16] E. E. Flanagan and T. Hinderer, Phys. Rev. **D77**, 021502 (2008), arXiv:0709.1915 [astro-ph].
[17] S. Chakrabarti, T. Delsate, and J. Steinhoff, Phys. Rev. **D88**, 084038 (2013), arXiv:1306.5820 [gr-qc].
[18] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. **D73**, 104030 (2006), arXiv:hep-th/0511133 [hep-th].
[19] T. Hinderer *et al.*, Phys. Rev. Lett. **116**, 181101 (2016), arXiv:1602.00599 [gr-qc].
[20] E. Poisson, Phys. Rev. **D57**, 5287 (1998), arXiv:gr-qc/9709032 [gr-qc].
[21] R. O. Hansen, J. Math. Phys. **15**, 46 (1974).
[22] R. A. Porto and I. Z. Rothstein, Phys. Rev. **D78**, 044013 (2008), [Erratum: Phys. Rev.D81,029905(2010)], arXiv:0804.0260 [gr-qc].
[23] S. Taylor and E. Poisson, Phys. Rev. **D78**, 084016 (2008), arXiv:0806.3052 [gr-qc].
[24] T. Damour and A. Nagar, Phys. Rev. **D80**, 084035 (2009), arXiv:0906.0096 [gr-qc].
[25] B. Kol and M. Smolkin, JHEP **02**, 010 (2012), arXiv:1110.3764 [hep-th].

[26] P. Pani, L. Gualtieri, A. Maselli, and V. Ferrari, Phys. Rev. **D92**, 024010 (2015), arXiv:1503.07365 [gr-qc].
[27] M. Levi and J. Steinhoff, JCAP **1601**, 008 (2016), arXiv:1506.05794 [gr-qc].
[28] T. Damour, P. Jaranowski, and G. Schfer, Phys. Rev. **D89**, 064058 (2014), arXiv:1401.4548 [gr-qc].
[29] T. Damour, P. Jaranowski, and G. Schfer, Phys. Rev. **D91**, 084024 (2015), arXiv:1502.07245 [gr-qc].
[30] L. Bernard, L. Blanchet, A. Boh, G. Faye, and S. Marsat, Phys. Rev. **D93**, 084037 (2016), arXiv:1512.02876 [gr-qc].
[31] T. Damour, P. Jaranowski, and G. Schfer, Phys. Rev. **D93**, 084014 (2016), arXiv:1601.01283 [gr-qc].
[32] L. E. Kidder, Phys. Rev. **D52**, 821 (1995), arXiv:gr-qc/9506022 [gr-qc].
[33] T. Damour, Phys. Rev. D **64**, 124013 (2001), gr-qc/0103018.
[34] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **78**, 024009 (2008), arXiv:0803.0915 [gr-qc].
[35] S. Hergt and G. Schäfer, Phys. Rev. **D77**, 104001 (2008), arXiv:0712.1515 [gr-qc].
[36] S. Hergt and G. Schäfer, Phys. Rev. **D78**, 124004 (2008), arXiv:0809.2208 [gr-qc].
[37] M. Levi and J. Steinhoff, JHEP **06**, 059 (2015), arXiv:1410.2601 [gr-qc].
[38] V. Vaidya, Phys. Rev. **D91**, 024017 (2015), arXiv:1410.5348 [hep-th].
[39] S. Marsat, Class. Quant. Grav. **32**, 085008 (2015), arXiv:1411.4118 [gr-qc].
[40] T. Damour and J. H. Taylor, Phys. Rev. **D45**, 1840 (1992).
[41] M. Levi and J. Steinhoff, JHEP **09**, 219 (2015), arXiv:1501.04956 [gr-qc].
[42] J. Vines, D. Kunst, J. Steinhoff, and T. Hinderer, Phys. Rev. **D93**, 103008 (2016), arXiv:1601.07529 [gr-qc].
[43] S. Balmelli and T. Damour, Phys. Rev. D **92**, 124022 (2015), arXiv:1509.08135 [gr-qc].
[44] M. Mathisson, Acta Physica Polonica **6**, 163 (1937).
[45] A. Papapetrou, Proc. R. Soc. London A **209**, 248 (1951).
[46] W. G. Dixon, in *Proceedings of the International School of Physics Enrico Fermi LXVII*, edited by J. Ehlers (North Holland, Amsterdam, 1979) pp. 156–219.
[47] E. Barausse, É. Racine, and A. Buonanno, Phys. Rev. D **80**, 104025 (2009), arXiv:0907.4745 [gr-qc].
[48] J. Vines *et al.*, in preparation (2016).
[49] J. Vines *et al.*, in preparation (2016).
[50] A. Hanson and T. Regge, Annals of Physics **87**, 498 (1974).
[51] I. Bailey and W. Israel, Communications in Mathematical Physics **42**, 65 (1975).
[52] I. Bailey and W. Israel, Ann. Phys. (N.Y.) **130**, 188 (1980).
[53] R. A. Porto, Phys. Rev. **D73**, 104031 (2006), arXiv:gr-qc/0511061 [gr-qc].
[54] J. Steinhoff, G. Schäfer, and S. Hergt, Phys. Rev. **D77**, 104018 (2008), arXiv:0805.3136 [gr-qc].
[55] J. Steinhoff, Ann. Phys. (Berlin) **523**, 296 (2011), arXiv:1106.4203 [gr-qc].
[56] J. Steinhoff, ArXiv e-prints (2015), arXiv:1501.04951 [gr-qc].
[57] A. Bohé, G. Faye, S. Marsat, and E. K. Porter, Class. Quant. Grav. **32**, 195010 (2015), arXiv:1501.01529 [gr-qc].
[58] W. Israel, Phys. Rev. D **2**, 641 (1970).
[59] C. M. Will, Phys. Rev. Lett. **102**, 061101 (2009), arXiv:0812.0110 [gr-qc].
[60] A. Buonanno and T. Damour, Phys. Rev. D **59**, 084006 (1999), gr-qc/9811091.
[61] E. Barausse and A. Buonanno, Phys. Rev. D **81**, 084024 (2010), gr-qc/0912.3517.