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Complete conservative dynamics for inspiralling compact binaries with spins at fourth post-Newtonian order

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Abstract. In this work we complete the spin dependent conservative dynamics of inspiralling compact binaries at the fourth post-Newtonian order, and in particular the recent derivation of the next-to-next-to-leading order spin-squared interaction potential. We derive the physical equations of motion of the position and the spin from a direct variation of the action. Further, we derive the quadratic in spin Hamiltonians, as well as their expressions in the center of mass frame. We construct the conserved integrals of motion, which form the Poincaré algebra. This construction provided a consistency check for the validity of our result, which is crucial in particular in the current absence of another independent derivation of the next-to-next-to-leading order spin-squared interaction. Finally, we provide here the complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to the fourth post-Newtonian order. These high post-Newtonian orders, in particular taking into account the spins of the binary constituents, will enable to gain more accurate information on the constituents from even more sensitive gravitational wave detections to come.

Contents

1 Introduction

The recent direct detections of gravitational waves (GWs) from binary black hole mergers GW150914 [1] and GW151226 [2] by the twin Advanced LIGO detectors in the US [3] has opened a new era of observational astronomy. With the upcoming operation of the advanced Virgo detector in Europe [4], and the second observational run of LIGO with an improved signal sensitivity, more events are expected to be detected, including also neutron stars in the binary components. Using the matched filtering technique to detect the GW signal, and gain from it as much information as possible, requires accurate theoretical waveforms, where the continuous signal is modeled via the Effective-One-Body (EOB) approach [5]. The initial part of the waveform corresponding to the inspiral phase of the binaries evolution is analytically described by the post-Newtonian (PN) approximation of General Relativity [6, 7].

In order to enable an improved analysis of the GW events, and consequently have an improved parameter estimation [8, 9], i.e. to gain more accurate information about the inner structure of the constituents, high order PN corrections are required, taking into account in particular the spin of the objects. Considering the recent completion of the 4PN order point-mass correction [10–12], a recent series of works [13–18] based on an effective field theory (EFT) approach for the binary inspiral [19, 20] has completed the same PN accuracy for binaries with generic compact spinning objects. This line of work includes all spin interactions linear and quadratic in spin up to the next-to-leading order (NLO) via an EFT for spinning objects, which was formulated in [16], and further new results were obtained to next-to-NLO (NNLO), as well as the leading order (LO) cubic and quartic in spin interactions, all via the EFT for spin [16].

In this paper we complete this series of works to the 4PN order, and in particular the recent derivation of the NNLO spin-squared interaction potential [18]. First, we derive the physical equations of motion (EOMs) of the position and the spin, which are both obtained directly via the action approach [14, 16]. Then we derive the Hamiltonian following the procedure outlined in previous works [14, 17], and specify it also for the center of mass frame. We provide here also an equivalent potential and Hamiltonian to those of [13, 14] for the NNLO spin1-spin2 sector for a total result consistent with the formulation and gauge choices of [16] (which are also different than in [21, 22]). We proceed to construct the conserved integrals of motion, which form the Poincaré algebra. As there is currently no other independent derivation of the NNLO spin-squared sector, this construction actually provides a crucial consistency check of our result. Finally, we provide complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to the 4PN order.

The paper is organized as follows. In section 2 we obtain the equations of motion of the position and the spin. In section 3, we provide the NNLO quadratic in spin Hamiltonians consistent with the gauge choices in [16], and also specify them for the center of mass frame. In section 4 we find the conserved integrals of motion, which constitute a strong check of our potential and Hamiltonian result from [18]. In section 5 we complete the gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to 4PN order. We conclude in section 6. In addition in appendix A we provide the NNLO spin1-spin2 potential with the gauge choices of [16], and show its equivalence to the Hamiltonians provided in [14, 23].

2 EOMs of position and spin

As noted in [14, 16] the derivation of the EOMs of the positions is straightforward using a variation on the action, which contains higher order time derivatives of positions up to $\dot{a}$. Hence we use:

$$\frac{\delta \mathcal{S}(x_i, v_i, a_i, \dot{a}_i, S_{ij}, \dot{S}_{ij}, \ddot{S}_{ij})}{\delta x_i} = -m_i a_i - \left(\frac{\partial V}{\partial x_i} - \frac{d}{dt}\frac{\partial V}{\partial v_i} + \frac{d^2}{dt^2}\frac{\partial V}{\partial a_i} - \frac{d^3}{dt^3}\frac{\partial V}{\partial \dot{a}_i} \right) = 0. \tag{2.1}$$

The EOMs of the spin are also directly obtained in the EFT of spin [16] in a simple form via a variation of the action. Apart from time derivatives of the squared spin length, which can be dropped at this stage, the potential contains higher order time derivatives of the spin up to $\ddot{S}_i$. Hence we use:

$$\dot{S}^{ij} = -4 S^{k[i}\delta^{j]l} \frac{\delta \int dt\, V}{\delta S^{kl}} = -4 S^{k[i}\delta^{j]l} \left[\frac{\partial V}{\partial S^{kl}} - \frac{d}{dt}\frac{\partial V}{\partial \dot{S}^{kl}} + \frac{d^2}{dt^2}\frac{\partial V}{\partial \ddot{S}^{kl}} \right]. \tag{2.2}$$

The complete EOMs up to 4PN order can be provided in a Mathematica notebook upon request.

3 Hamiltonian

Although it is beneficial to have the potential, given in terms of the widely used harmonic coordinates, an order reduction at the level of the EOMs is required in order to numerically integrate the dynamics. In this section we perform an order reduction at the level of the action, which enables to transform to a Hamiltonian. The Hamiltonian has the advantage that it leads directly to the first-order Hamilton's equations. Furthermore, the Hamiltonian is most useful for implementation to the Effective-One-Body model for GWs from compact binary coalescence.

The spin-dependent Hamiltonian and its center of mass expression up to the 3.5PN order can be found in [14, 17]. In order to complete the 4PN order Hamiltonian, we need to add the NNLO spin1-spin2, NNLO spin-squared, and LO quartic in spin Hamiltonians. The latter can be found in [15]. For the NNLO spin-squared part we start with the potential recently derived in [18] using the formulation developed in [16]. For the NNLO spin1-spin2 part we use the potential given in appendix A, which we derived with the formulation and gauge choices in [16] in order to have a consistent expression for the spin-dependent Hamiltonian to 4PN order. The NNLO spin1-spin2 potentials and Hamiltonians in [13, 14, 23] are canonically equivalent.

For the NNLO Hamiltonians we follow the computational steps outlined in our previous work [16, 17]. The first step is a reduction of higher order time derivatives at the level of the action through suitable variable transformations [24], with its extension to the spinning case found in [14]. We are eliminating the higher order time derivatives successively at each PN order This reduction is generally not equivalent to a substitution of lower order EOM, which is crucial for the nonlinear in spin sectors. That is, the transformation of the position required at LO spin-orbit [16, 25–27] given by

$$\vec{y}_1 \to \vec{y}_1 + \frac{1}{2m_1}\vec{S}_1 \times \vec{v}_1, \tag{3.1}$$

adds nonlinear contributions.

The next step is a standard Legendre transformation, with the relation between velocity and canonical momentum given by

$$\begin{aligned}
v_1^i = \cdots &- \frac{3G}{2m_2 r^3}\Big[2S_2^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_2 - S_2^i \vec{\hat{p}}_2\cdot\vec{S}_2 + n^i \vec{n}\cdot\vec{S}_2 \vec{\hat{p}}_2\cdot\vec{S}_2 + \hat{p}_2^i S_2^2 - n^i \vec{n}\cdot\vec{\hat{p}}_2 S_2^2 \\
&- 2\hat{p}_2^i (\vec{n}\cdot\vec{S}_2)^2\Big] + \frac{C_{2ES^2}G}{4m_2 r^3}\Big[6S_2^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_2 - 2S_2^i \vec{\hat{p}}_2\cdot\vec{S}_2 + 6n^i \vec{n}\cdot\vec{S}_2 \vec{\hat{p}}_2\cdot\vec{S}_2 - 6\hat{p}_1^i S_2^2 \\
&+ 9\hat{p}_2^i S_2^2 - 3n^i \vec{n}\cdot\vec{\hat{p}}_2 S_2^2 + 18\hat{p}_1^i (\vec{n}\cdot\vec{S}_2)^2 - 21\hat{p}_2^i (\vec{n}\cdot\vec{S}_2)^2 - 15n^i \vec{n}\cdot\vec{\hat{p}}_2 (\vec{n}\cdot\vec{S}_2)^2\Big] \\
&+ \frac{G}{4m_1 r^3}\Big[-24S_2^i \vec{n}\cdot\vec{\hat{p}}_1 \vec{n}\cdot\vec{S}_1 + 21S_2^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_1 + 10S_2^i \vec{\hat{p}}_1\cdot\vec{S}_1 - 12S_2^i \vec{\hat{p}}_2\cdot\vec{S}_1 \\
&- 6S_1^i \vec{n}\cdot\vec{\hat{p}}_1 \vec{n}\cdot\vec{S}_2 + 18S_1^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_2 + 12\hat{p}_1^i \vec{n}\cdot\vec{S}_1 \vec{n}\cdot\vec{S}_2 - 21\hat{p}_2^i \vec{n}\cdot\vec{S}_1 \vec{n}\cdot\vec{S}_2 \\
&- 30n^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_1 \vec{n}\cdot\vec{S}_2 - 6n^i \vec{\hat{p}}_1\cdot\vec{S}_1 \vec{n}\cdot\vec{S}_2 + 21n^i \vec{\hat{p}}_2\cdot\vec{S}_1 \vec{n}\cdot\vec{S}_2 + 10S_1^i \vec{\hat{p}}_1\cdot\vec{S}_2 \\
&- 24n^i \vec{n}\cdot\vec{S}_1 \vec{\hat{p}}_1\cdot\vec{S}_2 - 10S_1^i \vec{\hat{p}}_2\cdot\vec{S}_2 + 18n^i \vec{n}\cdot\vec{S}_1 \vec{\hat{p}}_2\cdot\vec{S}_2 - 20\hat{p}_1^i \vec{S}_1\cdot\vec{S}_2 + 24\hat{p}_2^i \vec{S}_1\cdot\vec{S}_2 \\
&+ 48n^i \vec{n}\cdot\vec{\hat{p}}_1 \vec{S}_1\cdot\vec{S}_2 - 45n^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{S}_1\cdot\vec{S}_2\Big] + \frac{Gm_2}{4m_1^2 r^3}\Big[15S_1^i \vec{n}\cdot\vec{\hat{p}}_1 \vec{n}\cdot\vec{S}_1 - 6S_1^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_1 \\
&- 10S_1^i \vec{\hat{p}}_1\cdot\vec{S}_1 + 15n^i \vec{n}\cdot\vec{S}_1 \vec{\hat{p}}_1\cdot\vec{S}_1 + 6S_1^i \vec{\hat{p}}_2\cdot\vec{S}_1 - 12n^i \vec{n}\cdot\vec{S}_1 \vec{\hat{p}}_2\cdot\vec{S}_1 + 10\hat{p}_1^i S_1^2 - 6\hat{p}_2^i S_1^2 \\
&- 9n^i \vec{n}\cdot\vec{\hat{p}}_1 S_1^2 + 6n^i \vec{n}\cdot\vec{\hat{p}}_2 S_1^2 - 21\hat{p}_1^i (\vec{n}\cdot\vec{S}_1)^2 + 12\hat{p}_2^i (\vec{n}\cdot\vec{S}_1)^2\Big] \\
&+ \frac{C_{1ES^2}Gm_2}{4m_1^2 r^3}\Big[-6S_1^i \vec{n}\cdot\vec{\hat{p}}_1 \vec{n}\cdot\vec{S}_1 + 6S_1^i \vec{n}\cdot\vec{\hat{p}}_2 \vec{n}\cdot\vec{S}_1 + 4S_1^i \vec{\hat{p}}_1\cdot\vec{S}_1 - 6n^i \vec{n}\cdot\vec{S}_1 \vec{\hat{p}}_1\cdot\vec{S}_1 \\
&- 2S_1^i \vec{\hat{p}}_2\cdot\vec{S}_1 + 6n^i \vec{n}\cdot\vec{S}_1 \vec{\hat{p}}_2\cdot\vec{S}_1 - 10\hat{p}_1^i S_1^2 + 9\hat{p}_2^i S_1^2 + 12n^i \vec{n}\cdot\vec{\hat{p}}_1 S_1^2 - 3n^i \vec{n}\cdot\vec{\hat{p}}_2 S_1^2 \\
&+ 18\hat{p}_1^i (\vec{n}\cdot\vec{S}_1)^2 - 21\hat{p}_2^i (\vec{n}\cdot\vec{S}_1)^2 - 15n^i \vec{n}\cdot\vec{\hat{p}}_2 (\vec{n}\cdot\vec{S}_1)^2\Big],
\end{aligned} \tag{3.2}$$

where we use the abbreviation $\vec{\hat{p}}_a \equiv \vec{p}_a/m_a$, and the dots denote the terms up to 3.5PN order given in eq. (4.10) of [17].

The NNLO spin1-spin2 Hamiltonian now reads

$$
\begin{aligned}
H^{\mathrm{NNLO}}_{\mathrm{S_1S_2}} = &-\frac{G}{16r^3}\Big[22\hat{p}_1^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 - 8\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 - 18\hat{p}_2^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 \\
&- 10\hat{p}_1^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 + 14\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 - 10\hat{p}_2^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 - 20\hat{p}_1^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 \\
&+ 46\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 - 20\hat{p}_2^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 - 18\hat{p}_1^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 - 8\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 \\
&+ 22\hat{p}_2^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 + 42\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 + 8\hat{p}_1^2\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 + 42\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 \\
&- 34\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{p}_2)^2 - 22\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2\hat{p}_1^4 - 22\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2\hat{p}_2^4 - 93\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n} + 42\hat{p}_1^2\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n} \\
&- 93\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n} - 18\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} + 36\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} \\
&+ 96\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} - 102\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} - 60\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} \\
&+ 72\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} - 75\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} + 66\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} \\
&+ 78\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} - 48\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} + 33\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} \\
&- 48\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n} - 48\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2 + 33\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2 \\
&- 48\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2 + 78\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2 + 66\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2 \\
&- 75\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2 + 144\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 - 360\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 \\
&+ 72\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2 - 60\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2 - 102\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2 \\
&+ 96\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2 + 36\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2 - 18\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2 \\
&- 72\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 + 144\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 - 153\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 \\
&+ 306\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 - 153\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 + 120\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 \\
&- 36\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 - 30\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 + 48\hat{p}_1^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 \\
&- 72\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 + 36\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 + 78\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2 \\
&- 30\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 + 120\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 - 36\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 \\
&+ 36\hat{p}_1^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 - 72\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 + 48\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 + 18\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}\hat{p}_1^4 \\
&+ 18\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}\hat{p}_2^4 + 120\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n} - 390\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n} \\
&+ 120\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n} + 120\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2 - 90\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2 \\
&- 180\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2 + 330\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2 \\
&- 360\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 + 300\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2 + 240\vec{p}_2\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^3 \\
&- 90\hat{p}_1^2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2 + 120\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2 + 300\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2 \\
&- 360\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_1\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2 + 330\vec{p}_1\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 \\
&- 180\vec{p}_1\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^2 - 480\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2 + 240\vec{p}_1\cdot\vec{n}\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2(\vec{p}_2\cdot\vec{n})^3 \\
&- 210\vec{\hat{S}}_1\cdot\vec{n}\vec{\hat{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\Big] - \frac{G^2m_1}{8r^4}\Big[115\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 - 110\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_1\cdot\vec{\hat{S}}_2 \\
&- 166\vec{p}_1\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 + 124\vec{p}_2\cdot\vec{\hat{S}}_1\vec{p}_2\cdot\vec{\hat{S}}_2 - 163\hat{p}_1^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 + 348\vec{p}_1\cdot\vec{p}_2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2 - 126\hat{p}_2^2\vec{\hat{S}}_1\cdot\vec{\hat{S}}_2
\end{aligned}
$$

$$
\begin{aligned}
&+123\hat{p}_1^2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}-304\vec{p}_1\cdot\vec{p}_2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}+86\hat{p}_2^2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}+17\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}\\
&+256\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}+126\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}-276\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}\\
&-314\vec{p}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_2+267\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_2+226\vec{p}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_2-84\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_2\\
&-565\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2+342\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{p}_1\cdot\vec{n})^2+276\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{p}_2\cdot\vec{n})^2-198\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\\
&-108\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\Big]-\frac{G^2m_2}{8r^4}\Big[124\vec{p}_1\cdot\vec{\tilde{S}}_1\vec{p}_1\cdot\vec{\tilde{S}}_2-110\vec{p}_2\cdot\vec{\tilde{S}}_1\vec{p}_1\cdot\vec{\tilde{S}}_2-166\vec{p}_1\cdot\vec{\tilde{S}}_1\vec{p}_2\cdot\vec{\tilde{S}}_2\\
&+115\vec{p}_2\cdot\vec{\tilde{S}}_1\vec{p}_2\cdot\vec{\tilde{S}}_2-126\hat{p}_1^2\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2+348\vec{p}_1\cdot\vec{p}_2\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-163\hat{p}_2^2\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2+86\hat{p}_1^2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\\
&-304\vec{p}_1\cdot\vec{p}_2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}+123\hat{p}_2^2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}-84\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}+226\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}\\
&+267\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}-314\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{n}-276\vec{p}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_2\\
&+126\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_1\cdot\vec{\tilde{S}}_2+256\vec{p}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_2+17\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{p}_2\cdot\vec{\tilde{S}}_2-565\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\\
&+276\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{p}_1\cdot\vec{n})^2+342\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{p}_2\cdot\vec{n})^2-198\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\\
&-108\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\Big]-\frac{G^3m_1m_2}{4r^5}\Big[179\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-339\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\Big]-\frac{3G^3m_1^2}{2r^5}\Big[13\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\\
&-21\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\Big]-\frac{3G^3m_2^2}{2r^5}\Big[13\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-21\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\Big],
\end{aligned}
\tag{3.3}
$$

which as we noted is in agreement with previous results [13, 14, 23] as we detail in the appendix A.

The NNLO spin-squared Hamiltonian is given by

$$
\begin{aligned}
H_{\rm SS}^{\rm NNLO}=&-\frac{Gm_2}{16m_1r^3}\Big[7\hat{p}_1^2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1+24\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-12\hat{p}_2^2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\\
&+15\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2S_1^2-18\hat{p}_1^2\hat{p}_2^2S_1^2+12\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2S_1^2-24S_1^2(\vec{p}_1\cdot\vec{p}_2)^2-11\hat{p}_1^2(\vec{p}_1\cdot\vec{S}_1)^2\\
&-22\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{S}_1)^2+18\hat{p}_2^2(\vec{p}_1\cdot\vec{S}_1)^2+11S_1^2\hat{p}_1^4+21\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\\
&-9\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1+36\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1-24\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\\
&-30\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1+12\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1+54\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\\
&-96\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1+24\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1-24\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\\
&-33\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}S_1^2+48\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2S_1^2-12\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2S_1^2-66\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n})^2\\
&-15\hat{p}_1^2S_1^2(\vec{p}_1\cdot\vec{n})^2+15\vec{p}_1\cdot\vec{p}_2S_1^2(\vec{p}_1\cdot\vec{n})^2-9\hat{p}_2^2S_1^2(\vec{p}_1\cdot\vec{n})^2+60\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(\vec{p}_2\cdot\vec{n})^2\\
&+60\hat{p}_1^2S_1^2(\vec{p}_2\cdot\vec{n})^2-60\vec{p}_1\cdot\vec{p}_2S_1^2(\vec{p}_2\cdot\vec{n})^2-39\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2+48(\vec{p}_1\cdot\vec{p}_2)^2(\vec{S}_1\cdot\vec{n})^2\\
&+39\hat{p}_1^2\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2-24\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2+12(\vec{p}_1\cdot\vec{n})^2(\vec{p}_1\cdot\vec{S}_1)^2+42\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2\\
&-60(\vec{p}_2\cdot\vec{n})^2(\vec{p}_1\cdot\vec{S}_1)^2+48(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{S}_1)^2-18(\vec{S}_1\cdot\vec{n})^2\hat{p}_1^4-30\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n})^2\\
&+15\vec{p}_2\cdot\vec{n}S_1^2(\vec{p}_1\cdot\vec{n})^3+180\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(\vec{p}_2\cdot\vec{n})^2-60\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(\vec{p}_2\cdot\vec{n})^3\\
&-120\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(\vec{p}_2\cdot\vec{n})^2-60S_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+60\vec{p}_1\cdot\vec{n}S_1^2(\vec{p}_2\cdot\vec{n})^3\\
&+15\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2-120\hat{p}_1^2(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2+120\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{GC_{1(\rm ES^2)}m_2}{16m_1r^3}\Big[4\hat{p}_1^2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-24\vec{p}_1\cdot\vec{p}_2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1+8\hat{p}_2^2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-24\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2S_1^2\\
&+11\hat{p}_1^2\hat{p}_2^2S_1^2-4\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2S_1^2+26S_1^2(\vec{p}_1\cdot\vec{p}_2)^2-4\hat{p}_1^2(\vec{p}_1\cdot\vec{S}_1)^2+24\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{S}_1)^2
\end{aligned}
$$

$$
\begin{aligned}
&- 10\hat{p}_2^2(\vec{p}_1\cdot\vec{S}_1)^2 + 2\hat{p}_1^2(\vec{p}_2\cdot\vec{S}_1)^2 - S_1^2\hat{p}_1^4 - 5S_1^2\hat{p}_2^4 + 18\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
&- 12\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 72\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 + 72\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
&+ 24\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 24\vec{p}_2\cdot\vec{n}\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 24\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 \\
&- 12\hat{p}_1^2\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 + 72\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 - 24\vec{p}_1\cdot\vec{n}\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 \\
&- 24\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 - 24\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}S_1^2 - 12\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2S_1^2 + 12\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2S_1^2 \\
&+ 12\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n})^2 - 12\hat{p}_1^2S_1^2(\vec{p}_1\cdot\vec{n})^2 + 60\vec{p}_1\cdot\vec{p}_2S_1^2(\vec{p}_1\cdot\vec{n})^2 - 27\hat{p}_2^2S_1^2(\vec{p}_1\cdot\vec{n})^2 \\
&+ 9\hat{p}_1^2S_1^2(\vec{p}_2\cdot\vec{n})^2 - 12\hat{p}_1^2\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 - 6(\vec{p}_1\cdot\vec{p}_2)^2(\vec{S}_1\cdot\vec{n})^2 - 9\hat{p}_1^2\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 \\
&- 12\vec{p}_1\cdot\vec{p}_2\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 - 6(\vec{p}_1\cdot\vec{n})^2(\vec{p}_1\cdot\vec{S}_1)^2 + 24\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2 - 6(\vec{p}_2\cdot\vec{n})^2(\vec{p}_1\cdot\vec{S}_1)^2 \\
&- 6(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{S}_1)^2 + 15(\vec{S}_1\cdot\vec{n})^2\hat{p}_1^4 + 15(\vec{S}_1\cdot\vec{n})^2\hat{p}_2^4 - 60\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n})^2 \\
&+ 60\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n})^2 + 60\vec{p}_2\cdot\vec{n}S_1^2(\vec{p}_1\cdot\vec{n})^3 + 60\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(\vec{p}_2\cdot\vec{n})^2 \\
&- 45S_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2 + 60\vec{p}_1\cdot\vec{n}\hat{p}_1^2\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 + 15\hat{p}_1^2(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2 \\
&- 180\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 + 15\hat{p}_2^2(\vec{p}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2 + 60\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 \\
&- 105(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2\Big] + \frac{G^2C_{1(\mathrm{ES}^2)}m_2^2}{8m_1r^4}\Big[72\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 + 64\hat{p}_1^2S_1^2 - 216\vec{p}_1\cdot\vec{p}_2S_1^2 \\
&+ 143\hat{p}_2^2S_1^2 + 4(\vec{p}_1\cdot\vec{S}_1)^2 - 73(\vec{p}_2\cdot\vec{S}_1)^2 - 40\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 + 48\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
&- 402\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 + 356\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 + 368\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}S_1^2 - 64S_1^2(\vec{p}_1\cdot\vec{n})^2 \\
&- 332S_1^2(\vec{p}_2\cdot\vec{n})^2 - 120\hat{p}_1^2(\vec{S}_1\cdot\vec{n})^2 + 280\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 - 148\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 + 48(\vec{p}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2 \\
&+ 162\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 - 96(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2\Big] - \frac{G^2m_2^2}{16m_1r^4}\Big[138\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 + 139\hat{p}_1^2S_1^2 \\
&- 146\vec{p}_1\cdot\vec{p}_2S_1^2 - 52\hat{p}_2^2S_1^2 - 134(\vec{p}_1\cdot\vec{S}_1)^2 + 60(\vec{p}_2\cdot\vec{S}_1)^2 + 406\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
&- 48\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 404\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 - 128\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 + 368\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}S_1^2 \\
&- 142S_1^2(\vec{p}_1\cdot\vec{n})^2 - 100S_1^2(\vec{p}_2\cdot\vec{n})^2 - 269\hat{p}_1^2(\vec{S}_1\cdot\vec{n})^2 + 224\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 + 136\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 \\
&- 132\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 + 84(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2\Big] - \frac{G^2m_2}{8r^4}\Big[92\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 + 89\hat{p}_1^2S_1^2 \\
&- 72\vec{p}_1\cdot\vec{p}_2S_1^2 - 30\hat{p}_2^2S_1^2 - 95(\vec{p}_1\cdot\vec{S}_1)^2 + 24(\vec{p}_2\cdot\vec{S}_1)^2 + 300\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 \\
&- 84\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 184\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 - 72\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 + 104\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}S_1^2 \\
&- 77S_1^2(\vec{p}_1\cdot\vec{n})^2 + 42S_1^2(\vec{p}_2\cdot\vec{n})^2 - 163\hat{p}_1^2(\vec{S}_1\cdot\vec{n})^2 + 148\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 + 30\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 \\
&- 102(\vec{p}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2 + 12\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 + 18(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2\Big] \\
&+ \frac{G^2C_{1(\mathrm{ES}^2)}m_2}{8r^4}\Big[8\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1 + 49\hat{p}_1^2S_1^2 - 68\vec{p}_1\cdot\vec{p}_2S_1^2 + 20\hat{p}_2^2S_1^2 - 13(\vec{p}_1\cdot\vec{S}_1)^2 + 4(\vec{p}_2\cdot\vec{S}_1)^2 \\
&+ 32\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 16\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1 - 24\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1 + 28\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}S_1^2 \\
&- 38S_1^2(\vec{p}_1\cdot\vec{n})^2 - 2S_1^2(\vec{p}_2\cdot\vec{n})^2 - 102\hat{p}_1^2(\vec{S}_1\cdot\vec{n})^2 + 164\vec{p}_1\cdot\vec{p}_2(\vec{S}_1\cdot\vec{n})^2 - 64\hat{p}_2^2(\vec{S}_1\cdot\vec{n})^2 \\
&- 6(\vec{p}_1\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2 + 12\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{S}_1\cdot\vec{n})^2 + 30(\vec{p}_2\cdot\vec{n})^2(\vec{S}_1\cdot\vec{n})^2\Big] \\
&- \frac{7G^3m_2^3}{2m_1r^5}\Big[S_1^2 - (\vec{S}_1\cdot\vec{n})^2\Big] - \frac{8G^3m_2^2}{r^5}\Big[S_1^2 - 2(\vec{S}_1\cdot\vec{n})^2\Big] - \frac{23G^3C_{1(\mathrm{ES}^2)}m_1m_2}{28r^5}\Big[S_1^2 - 3(\vec{S}_1\cdot\vec{n})^2\Big]
\end{aligned}
$$

$$-\frac{5G^3C_{1(\text{ES}^2)}m_2^2}{4r^5}\Big[13S_1^2-19(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^3C_{1(\text{ES}^2)}m_2^3}{4m_1r^5}\Big[19S_1^2-37(\vec{S}_1\cdot\vec{n})^2\Big]$$
$$-\frac{G^3m_1m_2}{14r^5}\Big[97S_1^2-123(\vec{S}_1\cdot\vec{n})^2\Big]+[1\leftrightarrow 2]. \tag{3.4}$$

This completes the spin-dependent Hamiltonian to 4PN order, together with the LO quartic in spin Hamiltonian at 4PN order from [15], and the Hamiltonian up to 3.5PN order in [14, 17].

It is useful for applications to further specify to the center of mass frame, which considerably simplifies the Hamiltonians. We proceed with the conventions and notations of our previous work [17]. That is, we utilize a triad $\vec{n}$, $\vec{\lambda}$, $\vec{l}$, where $\vec{l}=\vec{L}/L$, $\vec{\lambda}=\vec{l}\times\vec{n}$, and $\vec{L}=r\vec{n}\times\vec{p}$. Furthermore, we employ dimensionless variables denoted by a tilde, with units of length given by the total mass $Gm=G(m_1+m_2)$ and masses given in terms of the reduced mass $\mu=m_1m_2/m$. The result is expressed using the mass ratio $q\equiv m_1/m_2$, and the symmetric mass ratio $\nu\equiv\mu/m$. These conventions lead us to the following spin-dependent Hamiltonians at 4PN order:

$$H^{\text{S}_1\text{S}_2}_{\text{NNLO}}=\bigg[\frac{\tilde{L}^4}{8\tilde{r}^7}(11-23\nu-7\nu^2)+\frac{7\tilde{L}^2}{8\tilde{r}^6}(18+19\nu)-\frac{1}{4\tilde{r}^5}(78+23\nu)-\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(4+125\nu+52\nu^2)$$
$$-\frac{3\tilde{p}_r^2}{4\tilde{r}^4}(25-9\nu)-\frac{\tilde{p}_r^4}{16\tilde{r}^3}(26-161\nu+38\nu^2)\bigg]\nu\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2+\bigg[-\frac{3\tilde{L}^4}{16\tilde{r}^7}(6+7\nu-10\nu^2)$$
$$-\frac{\tilde{L}^2}{8\tilde{r}^6}(86+169\nu)+\frac{3}{4\tilde{r}^5}(42+29\nu)+\frac{3\tilde{p}_r^2}{4\tilde{r}^4}(25-31\nu)+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(4+49\nu+49\nu^2)$$
$$+\frac{\tilde{p}_r^4}{16\tilde{r}^3}(26-193\nu+134\nu^2)\bigg]\nu\vec{n}\cdot\vec{\tilde{S}}_1\vec{n}\cdot\vec{\tilde{S}}_2+\nu^2\bigg[\frac{15\tilde{L}^3\tilde{p}_r}{8\tilde{r}^6}(1-2\nu)+\frac{\tilde{L}\tilde{p}_r}{2\tilde{r}^5}(48+7\nu)$$
$$+\frac{15\tilde{L}\tilde{p}_r^3}{8\tilde{r}^4}(1-8\nu)\bigg]\bigg[q\vec{n}\cdot\vec{\tilde{S}}_2\vec{\tilde{S}}_1\cdot\vec{\lambda}+\frac{\vec{n}\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{\lambda}}{q}\bigg]-\nu\bigg[\frac{\tilde{L}^3\tilde{p}_r}{16\tilde{r}^6}(4-35\nu+82\nu^2)$$
$$+\frac{\tilde{L}\tilde{p}_r}{8\tilde{r}^5}(40-205\nu-28\nu^2)+\frac{\tilde{L}\tilde{p}_r^3}{16\tilde{r}^4}(4-83\nu+316\nu^2)\bigg]\bigg[\vec{n}\cdot\vec{\tilde{S}}_2\vec{\tilde{S}}_1\cdot\vec{\lambda}+\vec{n}\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_2\cdot\vec{\lambda}\bigg]$$
$$-\bigg[\frac{\tilde{L}^4}{8\tilde{r}^7}(11-25\nu-4\nu^2)+\frac{\tilde{L}^2}{8\tilde{r}^6}(124+19\nu)+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(11-139\nu-22\nu^2)\bigg]\nu\vec{\tilde{S}}_1\cdot\vec{\lambda}\vec{\tilde{S}}_2\cdot\vec{\lambda}, \tag{3.5}$$

$$H^{\text{S}^2}_{\text{NNLO}}=\nu^2\bigg\{\bigg[\frac{\tilde{L}^4}{16\tilde{r}^7}(11-25\nu)+\frac{\tilde{L}^2}{16\tilde{r}^6}(199+33\nu)-\frac{24}{7\tilde{r}^5}+\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(7-74\nu)-\frac{9\tilde{p}_r^2}{16\tilde{r}^4}(3-5\nu)$$
$$-\frac{\tilde{p}_r^4}{4\tilde{r}^3}(1+\nu)\bigg]\tilde{S}_1^2-\bigg[\frac{3\tilde{L}^4}{16\tilde{r}^7}(6-17\nu)+\frac{\tilde{L}^2}{16\tilde{r}^6}(329+61\nu)-\frac{37}{7\tilde{r}^5}+\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(26-145\nu)$$
$$-\frac{\tilde{p}_r^2}{16\tilde{r}^4}(3-85\nu)-\frac{\tilde{p}_r^4}{4\tilde{r}^3}(1+\nu)\bigg](\vec{n}\cdot\vec{\tilde{S}}_1)^2-\bigg[\frac{\tilde{L}^4}{16\tilde{r}^7}(11-25\nu)+\frac{\tilde{L}^2}{8\tilde{r}^6}(91+11\nu)$$
$$-\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(1+85\nu)\bigg](\vec{\tilde{S}}_1\cdot\vec{\lambda})^2-\bigg[\frac{\tilde{L}^3\tilde{p}_r}{16\tilde{r}^6}(1+37\nu)-\frac{\tilde{L}\tilde{p}_r}{8\tilde{r}^5}(93+25\nu)$$
$$-\frac{\tilde{L}\tilde{p}_r^3}{16\tilde{r}^4}(23-67\nu)\bigg]\vec{n}\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_1\cdot\vec{\lambda}\bigg\}+\nu^2C_{1(\text{ES}^2)}\bigg\{\bigg[\frac{\tilde{L}^4}{4\tilde{r}^7}(1+3\nu)-\frac{\tilde{L}^2}{4\tilde{r}^6}(22+5\nu)+\frac{55}{14\tilde{r}^5}$$
$$-\frac{\tilde{L}^2\tilde{p}_r^2}{4\tilde{r}^5}(1-6\nu)+\frac{\tilde{p}_r^2}{4\tilde{r}^4}(9-13\nu)-\frac{\tilde{p}_r^4}{2\tilde{r}^3}(1+6\nu)\bigg]\tilde{S}_1^2+\bigg[\frac{7\tilde{L}^2}{4\tilde{r}^6}(4+\nu)-\frac{95}{14\tilde{r}^5}+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(7+6\nu)$$

$$+\frac{3\tilde{p}_r^2}{4\tilde{r}^4}(9+11\nu)+\frac{\tilde{p}_r^4}{2\tilde{r}^3}(1+6\nu)\Big](\vec{n}\cdot\vec{\tilde{S}}_1)^2-\Big[\frac{\tilde{L}^4}{4\tilde{r}^7}(1+3\nu)-\frac{\nu\tilde{L}^2}{2\tilde{r}^6}+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(5+18\nu)\Big](\vec{\tilde{S}}_1\cdot\vec{\lambda})^2$$
$$+\Big[\frac{5\tilde{L}^3\tilde{p}_r}{8\tilde{r}^6}+\frac{\tilde{L}\tilde{p}_r}{4\tilde{r}^5}(8-11\nu)-\frac{\tilde{L}\tilde{p}_r^3}{8\tilde{r}^4}(1-6\nu)\Big]\vec{n}\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_1\cdot\vec{\lambda}\Big\}+\frac{\nu}{q}\Big\{\Big[-\frac{\tilde{L}^4}{16\tilde{r}^7}(11-59\nu+21\nu^2)$$
$$-\frac{\tilde{L}^2}{16\tilde{r}^6}(139-93\nu-33\nu^2)-\frac{1}{2\tilde{r}^5}(7+2\nu)-\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(7-40\nu+60\nu^2)+\frac{3\tilde{p}_r^2}{16\tilde{r}^4}(1+63\nu+15\nu^2)$$
$$+\frac{\tilde{p}_r^4}{16\tilde{r}^3}(4-4\nu-9\nu^2)\Big]\tilde{S}_1^2+\Big[\frac{3\tilde{L}^4}{16\tilde{r}^7}(6-37\nu+15\nu^2)+\frac{\tilde{L}^2}{16\tilde{r}^6}(269-257\nu-61\nu^2)$$
$$+\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(26-101\nu+126\nu^2)+\frac{1}{2\tilde{r}^5}(7+18\nu)-\frac{\tilde{p}_r^2}{16\tilde{r}^4}(3+93\nu+85\nu^2)$$
$$-\frac{\tilde{p}_r^4}{16\tilde{r}^3}(4-4\nu-9\nu^2)\Big](\vec{n}\cdot\vec{\tilde{S}}_1)^2+\Big[\frac{\tilde{L}^4}{16\tilde{r}^7}(11-59\nu+21\nu^2)+\frac{\tilde{L}^2}{8\tilde{r}^6}(67-37\nu-11\nu^2)$$
$$-\frac{\tilde{L}^2\tilde{p}_r^2}{16\tilde{r}^5}(1+35\nu-69\nu^2)\Big](\vec{\tilde{S}}_1\cdot\vec{\lambda})^2+\Big[\frac{\tilde{L}^3\tilde{p}_r}{16\tilde{r}^6}(1+47\nu-33\nu^2)-\frac{\tilde{L}\tilde{p}_r}{8\tilde{r}^5}(69-9\nu-25\nu^2)$$
$$-\frac{\tilde{L}\tilde{p}_r^3}{16\tilde{r}^4}(23-65\nu+57\nu^2)\Big]\vec{n}\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_1\cdot\vec{\lambda}\Big\}+\frac{\nu}{q}C_{1(\mathrm{ES}^2)}\Big\{\Big[\frac{\tilde{L}^4}{16\tilde{r}^7}(1-28\nu+9\nu^2)$$
$$+\frac{\tilde{L}^2}{8\tilde{r}^6}(64+73\nu-10\nu^2)-\frac{1}{4\tilde{r}^5}(19+27\nu)+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(7-34\nu+6\nu^2)-\frac{\nu\tilde{p}_r^2}{8\tilde{r}^4}(141+26\nu)$$
$$+\frac{\tilde{p}_r^4}{16\tilde{r}^3}(13+20\nu-72\nu^2)\Big]\tilde{S}_1^2-\Big[\frac{\tilde{L}^3\tilde{p}_r}{8\tilde{r}^6}(5+6\nu+9\nu^2)+\frac{\tilde{L}\tilde{p}_r}{4\tilde{r}^5}(16-156\nu+11\nu^2)$$
$$+\frac{\tilde{L}\tilde{p}_r^3}{8\tilde{r}^4}(-1+24\nu+27\nu^2)\Big]\vec{n}\cdot\vec{\tilde{S}}_1\vec{\tilde{S}}_1\cdot\vec{\lambda}-\Big[\frac{3\tilde{L}^4}{16\tilde{r}^7}(5-16\nu-3\nu^2)+\frac{\tilde{L}^2}{4\tilde{r}^6}(60+11\nu-7\nu^2)$$
$$-\frac{1}{4\tilde{r}^5}(37+21\nu)+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(22-66\nu-21\nu^2)+\frac{3\tilde{p}_r^2}{8\tilde{r}^4}(36-25\nu-22\nu^2)$$
$$+\frac{\tilde{p}_r^4}{16\tilde{r}^3}(23-36\nu-120\nu^2)\Big](\vec{n}\cdot\vec{\tilde{S}}_1)^2+\Big[\frac{\tilde{L}^4}{4\tilde{r}^7}(1+3\nu-3\nu^2)+\frac{\tilde{L}^2}{8\tilde{r}^6}(4-97\nu+4\nu^2)$$
$$+\frac{\tilde{L}^2\tilde{p}_r^2}{8\tilde{r}^5}(5+12\nu-15\nu^2)\Big](\vec{\tilde{S}}_1\cdot\vec{\lambda})^2\Big\}+[1\leftrightarrow 2], \tag{3.6}$$

$$H^{\mathrm{S}^4}_{\mathrm{LO}}=\frac{\nu^2}{8\tilde{r}^5}\Big\{\frac{4C_{1(\mathrm{BS}^3)}}{q}\big[15\vec{n}\cdot\vec{\tilde{S}}_1\tilde{S}_1^2\vec{n}\cdot\vec{\tilde{S}}_2-3\tilde{S}_1^2\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-35\vec{n}\cdot\vec{\tilde{S}}_2(\vec{n}\cdot\vec{\tilde{S}}_1)^3+15\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{n}\cdot\vec{\tilde{S}}_1)^2\big]$$
$$-\frac{C_{1(\mathrm{ES}^4)}}{q^2}\big[35(\vec{n}\cdot\vec{\tilde{S}}_1)^4-30\tilde{S}_1^2(\vec{n}\cdot\vec{\tilde{S}}_1)^2+3\tilde{S}_1^4\big]+C_{1(\mathrm{ES}^2)}C_{2(\mathrm{ES}^2)}\big[60\vec{n}\cdot\vec{\tilde{S}}_1\vec{n}\cdot\vec{\tilde{S}}_2\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-3\tilde{S}_1^2\tilde{S}_2^2$$
$$+30\tilde{S}_1^2(\vec{n}\cdot\vec{\tilde{S}}_2)^2-6(\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2)^2-105(\vec{n}\cdot\vec{\tilde{S}}_1)^2(\vec{n}\cdot\vec{\tilde{S}}_2)^2\big]\Big\}+[1\leftrightarrow 2]. \tag{3.7}$$

The spin-dependent center-of-mass Hamiltonians up to 3.5PN order, consistent with our conventions and gauge choices, can be found in [17]. The above expressions complete this knowledge to 4PN order.

4 Conserved integrals of motion: The Poincaré algebra

The conservative action is invariant under global Poincaré transformations. This gives rise to conserved integrals of motion, which were constructed for a non-spinning PN Lagrangian containing higher-order time derivatives in [28]. On phase space the integrals of motion form a representation of the Poincaré algebra, which was studied in the PN approximation for the non-spinning [29] and spinning cases [30–33].

Here we construct the conserved quantities on phase space to 4PN order from an ansatz. These conserved quantities generate the Poincaré transformations on phase space and hence must fulfill the Poincaré algebra. It turns out that the Poincaré algebra uniquely fixes the ansatz for the conserved quantities.

The Poincaré algebra reads

$$\{P^i, H\} = 0, \qquad \{J^i, H\} = 0, \qquad \{J^i, P^j\} = \epsilon_{ijk} P^k, \qquad \{J^i, J^j\} = \epsilon_{ijk} J^k, \tag{4.1}$$

$$\{J^i, G^j\} = \epsilon_{ijk} G^k, \qquad \{G^i, H\} = P^i, \qquad \{G^i, P^j\} = \delta_{ij} H, \qquad \{G^i, G^j\} = -\epsilon_{ijk} J^k, \tag{4.2}$$

where $\vec{P}$ is the total linear momentum, $\vec{J}$ is the total angular momentum, $\vec{G}$ is the center of mass, and H is the full Hamiltonian (including rest-mass terms). In order to arrive at this form the explicit time dependence of the boost generator was split as $K^i = G^i - tP^i$. Since $\vec{P}$ and $\vec{J}$ are the generators of translations and rotations, respectively, and our gauge choices do not affect these symmetries, one obtains the simple representations

$$\vec{P} = \vec{p}_1 + \vec{p}_2, \qquad \vec{J} = \vec{y}_1 \times \vec{p}_1 + \vec{y}_2 \times \vec{p}_2 + \vec{S}_1 + \vec{S}_2, \tag{4.3}$$

see [29, 30]. A useful ansatz for the center of mass is obtained by considering

$$\{G^i, P^j\} = \delta_{ij} H, \tag{4.4}$$

which is solved for a $\vec{G}$ of the form [29]

$$\vec{G} = h_1 \vec{y}_1 + h_2 \vec{y}_2 + \vec{Y}, \qquad h_1 + h_2 = H, \tag{4.5}$$

where h_A and $\vec{Y}$ are translation invariant, i.e., $\{h_A, P^j\} = 0$ and $\{Y^i, P^j\} = 0$. Due to the condition $h_1 + h_2 = H$ it is useful to write

$$h_A = \frac{H}{2} + h_A^{\rm PM} + h_A^{\rm SO} + h_A^{\rm S_1S_2} + h_A^{\rm SS}, \tag{4.6}$$

where no cubic and quartic in spin contributions are needed. The condition $h_1 + h_2 = H$ is now equivalent to the antisymmetry of $h_1^{\rm PM}$ etc. under exchange of objects. In contrast, the ansatz for $\vec{Y}$ must be symmetric under exchange of the objects, and we write it in the form

$$\vec{Y} = \vec{Y}^{\rm PM} + \vec{Y}^{\rm SO} + \vec{Y}^{\rm S_1S_2} + \vec{Y}^{\rm SS}, \tag{4.7}$$

where again no cubic and quartic in spin contributions are needed. Finally, the ansatz for $\vec{G}$ is uniquely fixed only by

$$\{G^i, H\} = P^i, \tag{4.8}$$

where the other relations are automatically fulfilled.

The unique solution for h_A in terms of eq. (4.6) is given by

$$h_1^{\rm PM} = \frac{m_1}{2} + \frac{1}{4} m_1 \hat{p}_1^2 - \frac{1}{16} m_1 \hat{p}_1^4 + \frac{G^2 m_1^2 m_2}{4 r^2} - (1 \leftrightarrow 2), \tag{4.9}$$

$$h_1^{\rm SO} = -\frac{G m_2}{4 r^2} \vec{S}_1 \times \vec{n} \cdot \vec{p}_1 - \frac{G m_2}{16 r^2} \Big[\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \big(5 \hat{p}_1^2 - 16 \hat{\vec{p}}_1 \cdot \hat{\vec{p}}_2 + 8 \hat{p}_2^2 - 6 (\hat{\vec{p}}_2 \cdot \vec{n})^2 \big)$$

$$
\begin{aligned}
&- 4\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \big(4\vec{p}_1 \cdot \vec{n} - 3\vec{p}_2 \cdot \vec{n}\big)\Big] + \frac{G^2 m_1 m_2}{4r^3}\Big[31\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 - 4\vec{S}_1 \times \vec{n} \cdot \vec{p}_2\Big] \\
&+ \frac{G^2 m_2^2}{4r^3}\Big[15\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 + 19\vec{S}_1 \times \vec{n} \cdot \vec{p}_2\Big] - (1 \leftrightarrow 2),
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
h_1^{\mathrm{S_1S_2}} =& - \frac{G}{2r^3}\Big[\vec{p}_1 \cdot \vec{S}_1 \vec{p}_1 \cdot \vec{S}_2 - \hat{p}_1^2 \vec{S}_1 \cdot \vec{S}_2 + 3\hat{p}_1^2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} - 3\vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_1 \vec{S}_2 \cdot \vec{n} \\
&- 3\vec{p}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 + 3\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{S}_2\Big] + \frac{G^2 m_2}{r^4}\Big[\vec{S}_1 \cdot \vec{S}_2 - 2\vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n}\Big] \\
&- (1 \leftrightarrow 2),
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
h_1^{\mathrm{SS}} =& \frac{3Gm_2}{4m_1 r^3}\Big[\vec{p}_1 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{S}_1 + \hat{p}_1^2 S_1^2 - \vec{p}_1 \cdot \vec{p}_2 S_1^2 - (\vec{p}_1 \cdot \vec{S}_1)^2 + 3\vec{p}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_1 \\
&- \vec{p}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_1 - 2\vec{p}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{S}_1 + \vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} S_1^2 - S_1^2 (\vec{p}_1 \cdot \vec{n})^2 - 2\hat{p}_1^2 (\vec{S}_1 \cdot \vec{n})^2 \\
&+ 2\vec{p}_1 \cdot \vec{p}_2 (\vec{S}_1 \cdot \vec{n})^2\Big] + \frac{G^2 C_{1(\mathrm{ES}^2)} m_2}{4r^4}\Big[S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2\Big] - \frac{G^2 m_2^2}{m_1 r^4}\Big[S_1^2 - (\vec{S}_1 \cdot \vec{n})^2\Big] \\
&- \frac{G^2 C_{1(\mathrm{ES}^2)} m_2^2}{2m_1 r^4}\Big[5S_1^2 - 6(\vec{S}_1 \cdot \vec{n})^2\Big] + \frac{5G^2 m_2}{2r^4}(\vec{S}_1 \cdot \vec{n})^2 - (1 \leftrightarrow 2),
\end{aligned} \tag{4.12}
$$

and similarly for h_2. The solution for $\vec{Y}$ in terms of eq. (4.7) is given by

$$
\vec{Y}^{\mathrm{PM}} = - \frac{1}{4} G m_1 m_2 \vec{p}_2 \cdot \vec{n} \vec{p}_1 + (1 \leftrightarrow 2), \tag{4.13}
$$

$$
\begin{aligned}
\vec{Y}^{\mathrm{SO}} =& - \frac{1}{2}\vec{S}_1 \times \vec{p}_1 + \frac{1}{8}\hat{p}_1^2 \vec{S}_1 \times \vec{p}_1 - \frac{Gm_2}{4r}\Big[\vec{S}_1 \times \vec{n} \vec{p}_2 \cdot \vec{n} - 10\vec{S}_1 \times \vec{p}_1 + 11\vec{S}_1 \times \vec{p}_2\Big] \\
&- \frac{1}{16}\vec{S}_1 \times \vec{p}_1 \hat{p}_1^4 + \frac{Gm_2}{16r}\Big[8\vec{S}_1 \times \vec{n} \cdot \vec{p}_1 \big(\vec{p}_2 \cdot \vec{n} \vec{p}_1 - \vec{p}_1 \cdot \vec{n} \vec{p}_2\big) - 8\vec{S}_1 \times \vec{n} \cdot \vec{p}_2 \big(\vec{p}_2 \cdot \vec{n} \vec{p}_1 \\
&- \vec{p}_1 \cdot \vec{n} \vec{p}_2\big) - 8\vec{S}_1 \times \vec{p}_1 \cdot \vec{p}_2 \vec{p}_2 - \vec{S}_1 \times \vec{n}\big(5\hat{p}_1^2 \vec{p}_2 \cdot \vec{n} - 14\vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_1 \cdot \vec{n} \hat{p}_2^2 + 7\vec{p}_2 \cdot \vec{n} \hat{p}_2^2 \\
&- 3\vec{p}_1 \cdot \vec{n} (\vec{p}_2 \cdot \vec{n})^2 - 3(\vec{p}_2 \cdot \vec{n})^3\big) - \vec{S}_1 \times \vec{p}_1 \big(2\hat{p}_1^2 + 34\vec{p}_1 \cdot \vec{p}_2 - 25\hat{p}_2^2 + 6\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} \\
&+ 13(\vec{p}_2 \cdot \vec{n})^2\big) - \vec{S}_1 \times \vec{p}_2 \big(3\hat{p}_1^2 - 26\vec{p}_1 \cdot \vec{p}_2 + 9\hat{p}_2^2 - 6\vec{p}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{n} - 13(\vec{p}_2 \cdot \vec{n})^2\big)\Big] \\
&+ \frac{G^2 m_2^2}{8r^2}\Big[\vec{S}_1 \times \vec{n}\big(12\vec{p}_1 \cdot \vec{n} + 59\vec{p}_2 \cdot \vec{n}\big) - 20\vec{S}_1 \times \vec{p}_1 + 30\vec{S}_1 \times \vec{p}_2\Big] \\
&+ \frac{G^2 m_1 m_2}{8r^2}\Big[3\vec{S}_1 \times \vec{n}\big(14\vec{p}_1 \cdot \vec{n} + 3\vec{p}_2 \cdot \vec{n}\big) - 43\vec{S}_1 \times \vec{p}_1 + 34\vec{S}_1 \times \vec{p}_2\Big] + (1 \leftrightarrow 2),
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
\vec{Y}^{\mathrm{S_1S_2}} =& \frac{3G}{2r^2}\Big[\vec{S}_1 \cdot \vec{S}_2 \vec{n} - \vec{S}_1 \cdot \vec{n} \vec{S}_2\Big] + \frac{G}{8r^2}\Big[4\vec{p}_1 \cdot \vec{S}_1 \vec{p}_1 \cdot \vec{S}_2 \vec{n} + 16\vec{p}_2 \cdot \vec{S}_1 \vec{p}_1 \cdot \vec{S}_2 \vec{n} + 8\vec{p}_1 \cdot \vec{S}_1 \vec{p}_2 \cdot \vec{S}_2 \vec{n} \\
&- 6\hat{p}_1^2 \vec{S}_1 \cdot \vec{S}_2 \vec{n} - 34\vec{p}_1 \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{S}_2 \vec{n} - 14\vec{p}_1 \cdot \vec{S}_1 \vec{S}_2 \cdot \vec{n} \vec{p}_1 + 11\vec{p}_2 \cdot \vec{S}_1 \vec{S}_2 \cdot \vec{n} \vec{p}_1 \\
&+ 12\vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 \vec{p}_1 - 8\vec{S}_1 \cdot \vec{n} \vec{p}_2 \cdot \vec{S}_2 \vec{p}_1 - 4\vec{p}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{S}_2 \vec{p}_1 + 4\vec{p}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{S}_2 \vec{p}_1 \\
&- 9\vec{p}_1 \cdot \vec{S}_1 \vec{S}_2 \cdot \vec{n} \vec{p}_2 - 14\vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 \vec{p}_2 + 26\vec{p}_1 \cdot \vec{n} \vec{S}_1 \cdot \vec{S}_2 \vec{p}_2 - 12\vec{p}_2 \cdot \vec{n} \vec{S}_1 \cdot \vec{S}_2 \vec{p}_2 \\
&+ 18\hat{p}_1^2 \vec{S}_2 \cdot \vec{n} \vec{S}_1 - 8\vec{p}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 \vec{S}_1 + \vec{p}_2 \cdot \vec{n} \vec{p}_1 \cdot \vec{S}_2 \vec{S}_1 - 12\hat{p}_1^2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 + 34\vec{p}_1 \cdot \vec{p}_2 \vec{S}_1 \cdot \vec{n} \vec{S}_2
\end{aligned}
$$

$$
\begin{aligned}
&+ 10\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2 - 7\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2 - 30\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2 + 12\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2 \\
&+ 12\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{n} + 3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{n} - 12\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{n} \\
&- 21\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{S}_2\vec{n} - 6\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{p}_1 - 3\vec{p}_1\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{p}_2 \\
&+ 12\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{p}_2 + 21\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{S}_2 + 12\vec{S}_1\cdot\vec{S}_2\vec{n}(\vec{p}_1\cdot\vec{n})^2 \\
&- 12\vec{S}_2\cdot\vec{n}\vec{S}_1(\vec{p}_1\cdot\vec{n})^2 + 12\vec{S}_1\cdot\vec{S}_2\vec{n}(\vec{p}_2\cdot\vec{n})^2 - 12\vec{S}_1\cdot\vec{n}\vec{S}_2(\vec{p}_2\cdot\vec{n})^2\Big] \\
&- \frac{G^2 m_2}{8r^3}\Big[\vec{S}_1\cdot\vec{S}_2\vec{n} + 39\vec{S}_2\cdot\vec{n}\vec{S}_1 - 40\vec{S}_1\cdot\vec{n}\vec{S}_2\Big] + (1\leftrightarrow 2),
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
\vec{Y}^{\rm SS} =& \frac{Gm_2}{2m_1 r^2}\Big[S_1^2\vec{n} - \vec{S}_1\cdot\vec{n}\vec{S}_1\Big] + \frac{GC_{1(\mathrm{ES}^2)}m_2}{2m_1 r^2}\Big[S_1^2\vec{n} - \vec{S}_1\cdot\vec{n}\vec{S}_1\Big] + \frac{GC_{1(\mathrm{ES}^2)}m_2}{8m_1 r^2}\Big[6\hat{p}_1^2 S_1^2\vec{n} \\
&- 14\vec{p}_1\cdot\vec{p}_2 S_1^2\vec{n} + 6\hat{p}_2^2 S_1^2\vec{n} + 2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1 - 2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1 - 4\vec{p}_1\cdot\vec{n}S_1^2\vec{p}_1 \\
&+ 3\vec{p}_2\cdot\vec{n}S_1^2\vec{p}_1 + 2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2 + \vec{p}_1\cdot\vec{n}S_1^2\vec{p}_2 - 6\hat{p}_1^2\vec{S}_1\cdot\vec{n}\vec{S}_1 + 14\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{S}_1 \\
&- 6\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{S}_1 + 2\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_1 - 2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_1 - 2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_1 - 6\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}S_1^2\vec{n} \\
&+ 6\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{S}_1 + 3\vec{p}_2\cdot\vec{n}\vec{p}_1(\vec{S}_1\cdot\vec{n})^2 - 3\vec{p}_1\cdot\vec{n}\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big] \\
&+ \frac{Gm_2}{16m_1 r^2}\Big[6\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\vec{n} + \hat{p}_1^2 S_1^2\vec{n} - 6\vec{p}_1\cdot\vec{p}_2 S_1^2\vec{n} + 8\hat{p}_2^2 S_1^2\vec{n} - 28\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1 \\
&+ 4\vec{p}_1\cdot\vec{n}S_1^2\vec{p}_1 + 40\vec{p}_2\cdot\vec{n}S_1^2\vec{p}_1 + 22\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2 + 8\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2 - 22\vec{p}_1\cdot\vec{n}S_1^2\vec{p}_2 \\
&- 16\vec{p}_2\cdot\vec{n}S_1^2\vec{p}_2 - \hat{p}_1^2\vec{S}_1\cdot\vec{n}\vec{S}_1 + 6\vec{p}_1\cdot\vec{p}_2\vec{S}_1\cdot\vec{n}\vec{S}_1 - 8\hat{p}_2^2\vec{S}_1\cdot\vec{n}\vec{S}_1 - 2\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_1 \\
&- 40\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{S}_1 + 22\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_1 + 16\vec{p}_2\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{S}_1 - 2\vec{n}(\vec{p}_1\cdot\vec{S}_1)^2 - 8\vec{n}(\vec{p}_2\cdot\vec{S}_1)^2 \\
&+ 30\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{n} - 24\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{n} - 30\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}S_1^2\vec{n} \\
&+ 30\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{S}_1\cdot\vec{n}\vec{S}_1 + 12S_1^2\vec{n}(\vec{p}_2\cdot\vec{n})^2 - 12\vec{S}_1\cdot\vec{n}\vec{S}_1(\vec{p}_2\cdot\vec{n})^2 - 30\vec{p}_2\cdot\vec{n}\vec{p}_1(\vec{S}_1\cdot\vec{n})^2 \\
&+ 24\vec{p}_2\cdot\vec{n}\vec{p}_2(\vec{S}_1\cdot\vec{n})^2\Big] - \frac{3G^2 m_2}{2r^3}\Big[S_1^2\vec{n} - \vec{S}_1\cdot\vec{n}\vec{S}_1\Big] - \frac{G^2 C_{1(\mathrm{ES}^2)}m_2}{2r^3}\Big[S_1^2\vec{n} - \vec{S}_1\cdot\vec{n}\vec{S}_1\Big] \\
&- \frac{4G^2 m_2^2}{m_1 r^3}\Big[S_1^2\vec{n} - \vec{S}_1\cdot\vec{n}\vec{S}_1\Big] + \frac{23G^2 C_{1(\mathrm{ES}^2)}m_2^2}{2m_1 r^3}\Big[S_1^2\vec{n} - \vec{S}_1\cdot\vec{n}\vec{S}_1\Big] + (1\leftrightarrow 2).
\end{aligned}
\tag{4.16}
$$

This completes the construction of the integrals of motion and the Poincaré algebra, which provides a strong check for our new NNLO spin-squared Hamiltonian.

5 Complete gauge invariant relations to 4PN order with spins

We proceed with a completion of the gauge invariant relations to 4PN order with spins following section 8 in [14], and section 5 in [17]. These relations express the binding energy in terms of the orbital angular momentum or the orbital frequency. The relations hold for circular orbits and spins aligned with the orbital angular momentum. The spin-dependent relations to 3.5PN order can be found in [14, 17].

In order to obtain the gauge invariant binding energy the gauge dependent radial coordinate must be substituted, where its expansion for a circular orbit reads

$$
\frac{1}{\tilde{r}} = \cdots + \frac{1}{\tilde{L}^{10}}\left\{\left[(100403 - 6755\nu)\frac{\nu^2}{112} + \left(\frac{538}{7}\nu^2 - 3\nu^3\right)C_{1(\mathrm{ES}^2)}\right]\tilde{S}_1^2\right.
$$

$$+\left[10747+9751\nu-909\nu^2+(5064+7\nu-48\nu^2)C_{1(\mathrm{ES}^2)}\right]\frac{\nu\tilde{S}_1^2}{16q}$$
$$+\left[\frac{11349}{4}-\frac{665}{8}\nu-7\nu^2\right]\frac{\nu}{2}\tilde{S}_1\tilde{S}_2+\frac{9\nu^2}{8}(8+9C_{1(\mathrm{ES}^2)}C_{2(\mathrm{ES}^2)})\tilde{S}_1^2\tilde{S}_2^2$$
$$+\frac{3\nu^2}{8q^2}(5C_{1(\mathrm{ES}^4)}+12C_{1(\mathrm{ES}^2)}^2)\tilde{S}_1^4+\frac{3\nu^2}{2q}(5C_{1(\mathrm{BS}^3)}+12C_{1(\mathrm{ES}^2)})\tilde{S}_1^3\tilde{S}_2+[1\leftrightarrow 2]\Bigg\}, \quad (5.1)$$

and the dots denote the terms up to 3.5PN order given in eq. (5.1) of [17]. Similarly, extending eq. (5.2) of [17] to 4PN order leads to

$$\frac{1}{\tilde{L}^2}=\cdots+x^5\Bigg\{\left[\frac{65}{24}(549-98\nu)+(117+518\nu)C_{1(\mathrm{ES}^2)}\right]\frac{\nu^2\tilde{S}_1^2}{63}$$
$$-\left[\frac{1}{12}(6777+2964\nu+419\nu^2)+(29-44\nu)\nu C_{1(\mathrm{ES}^2)}\right]\frac{\nu\tilde{S}_1^2}{6q}$$
$$+\left[\frac{155\nu^2}{108}-\frac{8177\nu}{72}-\frac{313}{2}\right]\frac{\nu}{2}\tilde{S}_1\tilde{S}_2+\frac{\nu^2}{2}(22-4C_{1(\mathrm{ES}^2)}C_{2(\mathrm{ES}^2)})\tilde{S}_1^2\tilde{S}_2^2$$
$$+(11C_{1(\mathrm{ES}^2)}^2-5C_{1(\mathrm{ES}^4)})\frac{\nu^2\tilde{S}_1^4}{2q^2}+(22C_{1(\mathrm{ES}^2)}-10C_{1(\mathrm{BS}^3)})\frac{\nu^2}{q}\tilde{S}_1^3\tilde{S}_2+[1\leftrightarrow 2]\Bigg\}, \quad (5.2)$$

where $x=\tilde{\omega}^{2/3}$. This relation between the orbital angular momentum and the orbital frequency is in fact also gauge invariant for the configuration considered. The gauge invariant relations for the binding energy then read

$$e_{\mathrm{spin}}^{4\mathrm{PN}}(\tilde{L})=\frac{1}{\tilde{L}^{10}}\Bigg\{-\frac{\nu^2\tilde{S}_1^2}{28}\left[\frac{13}{4}(1380+7\nu)+(429+14\nu)C_{1(\mathrm{ES}^2)}\right]$$
$$+\frac{\nu\tilde{S}_1^2}{16q}\left[3(-672-637\nu+3\nu^2)-(819+165\nu+11\nu^2)C_{1(\mathrm{ES}^2)}\right]$$
$$-\nu\tilde{S}_1\tilde{S}_2\left[\frac{25\nu^2}{16}+\frac{507\nu}{32}+\frac{4041}{16}\right]-\frac{9\nu^2}{4}\tilde{S}_1^2\tilde{S}_2^2(1+C_{1(\mathrm{ES}^2)}C_{2(\mathrm{ES}^2)})$$
$$-\frac{3\nu^2}{8q^2}\tilde{S}_1^4(C_{1(\mathrm{ES}^4)}+3C_{1(\mathrm{ES}^2)}^2)-\frac{3\nu^2}{2q}\tilde{S}_1^3\tilde{S}_2(C_{1(\mathrm{BS}^3)}+3C_{1(\mathrm{ES}^2)})+[1\leftrightarrow 2]\Bigg\}, \quad (5.3)$$

$$e_{\mathrm{spin}}^{4\mathrm{PN}}(x)=x^5\Bigg\{\frac{\nu^2\tilde{S}_1^2}{36}\left[\frac{1}{3}(360-749\nu)+(279-70\nu)C_{1(\mathrm{ES}^2)}\right]$$
$$-\frac{7\nu\tilde{S}_1^2}{12q}\left[\frac{1}{3}(27+6\nu+31\nu^2)+\frac{1}{4}(-27-11\nu+13\nu^2)C_{1(\mathrm{ES}^2)}\right]$$
$$+\frac{7\nu}{432}\tilde{S}_1\tilde{S}_2(135-429\nu-53\nu^2)+\frac{7\nu^2}{4}\tilde{S}_1^2\tilde{S}_2^2(C_{1(\mathrm{ES}^2)}C_{2(\mathrm{ES}^2)}-1)$$
$$+\frac{7\nu^2}{8q^2}\tilde{S}_1^4(C_{1(\mathrm{ES}^4)}-C_{1(\mathrm{ES}^2)}^2)+\frac{7\nu^2}{2q}\tilde{S}_1^3\tilde{S}_2(C_{1(\mathrm{BS}^3)}-C_{1(\mathrm{ES}^2)})+[1\leftrightarrow 2]\Bigg\}, \quad (5.4)$$

with the spin-dependent lower orders given in eqs. (5.3) and (5.4) of [17], and the spin independent part given in [10]. The relation $e(x)$ can be directly applied to improve the phasing of gravitational waveforms for the GW detectors.

6 Conclusions

In this paper we completed the spin dependent conservative dynamics of inspiralling compact binaries at the 4PN order, and in particular the recent derivation of the NNLO spin-squared interaction potential [18]. These high PN orders, in particular taking into account the spins of the binary constituents, will enable to gain more accurate information from even more sensitive GW detections to come.

We have derived the physical EOMs of the position and the spin, and the quadratic in spin Hamiltonians, as well as their expressions in the center of mass frame. We have constructed the conserved integrals of motion, which form the Poincaré algebra. As there is currently no other independent derivation of the NNLO spin-squared interaction other than [18], this construction actually provided a crucial consistency check for the validity of our result. Finally, we provide complete gauge invariant relations among the binding energy, angular momentum, and orbital frequency of an inspiralling binary with generic compact spinning components to the 4PN order.

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A Equivalent NNLO spin1-spin2 potential

The NNLO spin1-spin2 sector was already completed in earlier work [13, 23]. Yet, we want to complete here the line of work from [16–18], which follows specific different conventions and gauge choices. For completeness we therefore present in this appendix the NNLO spin1-spin2 potential obtained within the formulation and gauge choices of [16]. This is needed to complete all spin potentials to 4PN order in a consistent manner.

Due to the length of the potential we split it according to the number of higher-order time derivatives as follows:

$$V^{\rm S_1S_2}_{\rm NNLO} = \overset{(0)}{V} + \overset{(1)}{V} + \overset{(2)}{V} + \overset{(3)}{V} + \overset{(4)}{V}, \tag{A.1}$$

and

$$\overset{(1)}{V} = V_a + V_{\dot{S}}. \tag{A.2}$$

These specific parts are given then by

$$\begin{aligned}\overset{(0)}{V} = \frac{G}{8r^3}\Big[&15\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_1 v_1^2 - 14\vec{\tilde{S}}_2\cdot\vec{v}_1 v_1^2\vec{\tilde{S}}_1\cdot\vec{v}_2 - 2v_1^2\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2 - 4\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2\\
&+ 18\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2 v_1^2\vec{v}_1\cdot\vec{v}_2 + 18\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2 - 8\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2 - 4\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2\\
&- 2\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_1 v_2^2 + 3\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2 v_1^2 v_2^2 - 14\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{v}_2 v_2^2 + 15\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2 v_2^2 + 18\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{v}_2 v_2^2\\
&- 8\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{v}_1\cdot\vec{v}_2)^2 - 15\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2 v_1^4 - 15\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2 v_2^4 - 21\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1 v_1^2\end{aligned}$$

$$
\begin{aligned}
&- 24\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1 v_1^2 - 60\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + 42\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} \\
&+ 18\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1 v_1^2\vec{v}_2 \cdot \vec{n} + 42\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}v_1^2\vec{\tilde{S}}_1 \cdot \vec{v}_2 + 102\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 \\
&+ 12\vec{\tilde{S}}_2 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 + 6\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}v_1^2\vec{\tilde{S}}_2 \cdot \vec{v}_2 + 36\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 \\
&+ 18\vec{\tilde{S}}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 - 60\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2\vec{\tilde{S}}_2 \cdot \vec{v}_2 - 12\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_1 \cdot \vec{v}_2 \\
&- 30\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_1^2\vec{v}_1 \cdot \vec{v}_2 - 120\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 + 6\vec{\tilde{S}}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
&- 42\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 42\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2\vec{v}_1 \cdot \vec{v}_2 + 6\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2\vec{v}_1 \cdot \vec{v}_2 \\
&- 12\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2\vec{v}_1 \cdot \vec{v}_2 + 18\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1 v_2^2 + 12\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1 v_2^2 \\
&+ 3\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_1^2 v_2^2 + 42\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 + 6\vec{\tilde{S}}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}v_2^2 + 42\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}v_2^2 \\
&+ 18\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 v_2^2 - 24\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 v_2^2 - 21\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 v_2^2 \\
&- 30\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 v_2^2 + 24\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_1^2(\vec{v}_1 \cdot \vec{n})^2 - 24\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{\tilde{S}}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 \\
&- 6\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 + 24\vec{\tilde{S}}_1 \cdot \vec{v}_2\vec{\tilde{S}}_2 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 + 24\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 \\
&- 27\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_2^2(\vec{v}_1 \cdot \vec{n})^2 + 24\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_1(\vec{v}_2 \cdot \vec{n})^2 - 27\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_1^2(\vec{v}_2 \cdot \vec{n})^2 \\
&- 24\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{\tilde{S}}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 - 6\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 + 24\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 \\
&+ 12\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{v}_2)^2 + 24\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_2^2(\vec{v}_2 \cdot \vec{n})^2 + 21\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_1^4 + 21\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_2^4 \\
&- 90\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}v_1^2\vec{v}_2 \cdot \vec{n} + 180\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 - 90\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}v_2^2 \\
&- 120\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^3 + 90\vec{\tilde{S}}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 + 120\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2 \\
&- 150\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 - 90\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2(\vec{v}_1 \cdot \vec{n})^2 - 15\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_2^2(\vec{v}_1 \cdot \vec{n})^2 \\
&+ 225\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2 - 90\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1(\vec{v}_2 \cdot \vec{n})^2 - 150\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1(\vec{v}_2 \cdot \vec{n})^2 \\
&- 15\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_1^2(\vec{v}_2 \cdot \vec{n})^2 - 120\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{n}(\vec{v}_2 \cdot \vec{n})^3 + 120\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 \\
&+ 90\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2(\vec{v}_2 \cdot \vec{n})^2 + 105\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2(\vec{v}_2 \cdot \vec{n})^2\Big] \\
&+ \frac{G^2 m_1}{r^4}\Big[5\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_1 - 4\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_1^2 - 8\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{\tilde{S}}_1 \cdot \vec{v}_2 - 2\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_2 + 5\vec{\tilde{S}}_1 \cdot \vec{v}_2\vec{\tilde{S}}_2 \cdot \vec{v}_2 \\
&+ 8\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{v}_2 - 5\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_2^2 - 12\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1 - 16\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1 + 2\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_1^2 \\
&- 16\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 6\vec{\tilde{S}}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + 16\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + 10\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 \\
&- 8\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 + 2\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 - 6\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 - 4\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2 \\
&+ 6\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_2^2 + 4\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2(\vec{v}_1 \cdot \vec{n})^2 + 8\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2(\vec{v}_2 \cdot \vec{n})^2 + 24\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}(\vec{v}_1 \cdot \vec{n})^2\Big] \\
&+ \frac{G^2 m_2}{r^4}\Big[5\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_1 - 5\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_1^2 - 8\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{\tilde{S}}_1 \cdot \vec{v}_2 - 2\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{\tilde{S}}_2 \cdot \vec{v}_2 + 5\vec{\tilde{S}}_1 \cdot \vec{v}_2\vec{\tilde{S}}_2 \cdot \vec{v}_2 \\
&+ 8\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{v}_2 - 4\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2 v_2^2 - 6\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1 - 8\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1 + 6\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}v_1^2 \\
&- 16\vec{\tilde{S}}_1 \cdot \vec{\tilde{S}}_2\vec{v}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n} + 2\vec{\tilde{S}}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + 10\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_1\vec{v}_2 \cdot \vec{n} + 16\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 \\
&- 16\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_1 \cdot \vec{v}_2 + 6\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 - 12\vec{\tilde{S}}_1 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{v}_2 - 4\vec{\tilde{S}}_1 \cdot \vec{n}\vec{\tilde{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{v}_2
\end{aligned}
$$

$$
\begin{aligned}
&+2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}v_2^2+8\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{v}_1\cdot\vec{n})^2+4\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2(\vec{v}_2\cdot\vec{n})^2+24\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\Big]\\
&-\frac{G^3m_1^2}{2r^5}\Big[7\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-31\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\Big]-\frac{G^3m_2^2}{2r^5}\Big[7\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-31\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\Big]\\
&-\frac{6G^3m_1m_2}{r^5}\Big[3\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2-13\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\Big],
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
V_a=&\frac{G}{8r^2}\Big[16\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}-24\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2v_1^2\vec{a}_1\cdot\vec{n}-16\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{a}_1+24\vec{\tilde{S}}_1\cdot\vec{n}v_1^2\vec{\tilde{S}}_2\cdot\vec{a}_1\\
&+10\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+26\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}-36\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}-8\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\\
&+32\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{\tilde{S}}_1\cdot\vec{v}_2-12\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{a}_1\vec{\tilde{S}}_1\cdot\vec{v}_2-42\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2-10\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\\
&-2\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{\tilde{S}}_2\cdot\vec{v}_2+12\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_1\vec{\tilde{S}}_2\cdot\vec{v}_2-6\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2+10\vec{a}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2\\
&+24\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+6\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{v}_1\cdot\vec{v}_2-32\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{v}_1\cdot\vec{v}_2-24\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2\\
&+6\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{a}_1\cdot\vec{v}_2+8\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_1\cdot\vec{v}_2+46\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2+6\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{a}_1\cdot\vec{v}_2\\
&-14\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{a}_1\cdot\vec{v}_2-9\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}v_2^2-5\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1v_2^2+15\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1v_2^2\\
&-10\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_2\cdot\vec{n}+9\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2v_1^2\vec{a}_2\cdot\vec{n}+8\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}+10\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}\\
&-16\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}-24\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}+24\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2v_2^2\vec{a}_2\cdot\vec{n}+42\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{a}_2\\
&-15\vec{\tilde{S}}_2\cdot\vec{n}v_1^2\vec{\tilde{S}}_1\cdot\vec{a}_2-32\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2-26\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{\tilde{S}}_1\cdot\vec{a}_2+16\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{\tilde{S}}_1\cdot\vec{a}_2\\
&+32\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\vec{\tilde{S}}_1\cdot\vec{a}_2-24\vec{\tilde{S}}_2\cdot\vec{n}v_2^2\vec{\tilde{S}}_1\cdot\vec{a}_2+6\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{a}_2+5\vec{\tilde{S}}_1\cdot\vec{n}v_1^2\vec{\tilde{S}}_2\cdot\vec{a}_2\\
&+2\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2-10\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{a}_2-6\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{a}_2-46\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2\\
&+14\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{a}_2-6\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_1\cdot\vec{a}_2+24\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2-8\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{a}_2\\
&-6\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{a}_2+36\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2-12\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{a}_2+12\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{a}_2\\
&+72\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-18\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-24\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-18\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}-48\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+36\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}\\
&+30\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2+18\vec{\tilde{S}}_1\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2-42\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2\\
&+3\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}v_2^2-18\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{a}_2\cdot\vec{n}-30\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_2\cdot\vec{n}\\
&-3\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}v_1^2\vec{a}_2\cdot\vec{n}-72\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{a}_2\cdot\vec{n}+24\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}\\
&+18\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}+48\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2+18\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2\\
&+42\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2-36\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2+45\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&-27\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2(\vec{v}_1\cdot\vec{n})^2-15\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2(\vec{v}_1\cdot\vec{n})^2-45\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\\
&+15\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1(\vec{v}_2\cdot\vec{n})^2+27\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1(\vec{v}_2\cdot\vec{n})^2+15\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&-15\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\Big]\\
&-\frac{2G^2m_1}{r^3}\Big[\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}-2\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1+\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1-\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_2\cdot\vec{n}+\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2\Big]
\end{aligned}
$$

$$
-\frac{2G^2m_2}{r^3}\Big[\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}-\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1-\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_2\cdot\vec{n}-\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2+2\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2\Big], \quad \text{(A.4)}
$$

$$
\begin{aligned}
V_{\dot{S}} = &-\frac{G}{4r^2}\Big[8\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_2\cdot\vec{v}_1+4\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}v_1^2-9\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}v_1^2-4\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1v_1^2\\
&+\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1v_1^2-11\dot{\vec{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+4\vec{\tilde{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+11\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2v_1^2\vec{v}_2\cdot\vec{n}\\
&-3\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2v_1^2\vec{v}_2\cdot\vec{n}-22\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{v}_2+7\dot{\vec{S}}_2\cdot\vec{n}v_1^2\vec{\tilde{S}}_1\cdot\vec{v}_2+8\dot{\vec{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\\
&-8\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\dot{\vec{S}}_1\cdot\vec{v}_2+3\vec{\tilde{S}}_2\cdot\vec{n}v_1^2\dot{\vec{S}}_1\cdot\vec{v}_2+22\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2+3\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{v}_2\\
&-3\dot{\vec{S}}_1\cdot\vec{n}v_1^2\vec{\tilde{S}}_2\cdot\vec{v}_2+4\dot{\vec{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2-4\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2-8\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2\\
&-4\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_2\cdot\vec{v}_2-3\vec{\tilde{S}}_1\cdot\vec{n}v_1^2\dot{\vec{S}}_2\cdot\vec{v}_2-3\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2+11\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\dot{\vec{S}}_2\cdot\vec{v}_2\\
&+4\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+26\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-8\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2-3\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2\\
&+4\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2+2\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1\vec{v}_1\cdot\vec{v}_2-26\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-4\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-4\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2-2\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2+8\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2+3\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{v}_2\\
&+3\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}v_2^2-11\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}v_2^2+3\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1v_2^2+3\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1v_2^2-7\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1v_2^2\\
&-3\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1v_2^2+9\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}v_2^2-4\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_2\cdot\vec{n}v_2^2+4\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2v_2^2-\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2v_2^2\\
&+9\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}v_1^2+12\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+9\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}\\
&+24\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+18\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}-9\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\\
&+3\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-24\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2-18\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\\
&-12\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2-9\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2-18\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+18\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}v_2^2+9\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}v_2^2\\
&-9\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2+12\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2(\vec{v}_1\cdot\vec{n})^3-9\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n})^2-12\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1(\vec{v}_1\cdot\vec{n})^2\\
&-24\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-27\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+15\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+9\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+27\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2+24\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\\
&-9\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1(\vec{v}_2\cdot\vec{n})^2-15\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1(\vec{v}_2\cdot\vec{n})^2-12\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2(\vec{v}_2\cdot\vec{n})^3+12\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&+9\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-15\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+15\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\Big]\\
&+\frac{G^2m_1}{r^3}\Big[6\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}-\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}+\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1+4\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1+6\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\\
&+2\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1-4\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}+2\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_2\cdot\vec{n}-2\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2-4\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\\
&-4\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2-24\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}+16\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\Big]\\
&-\frac{G^2m_2}{r^3}\Big[2\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}-4\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}-4\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1-2\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1-4\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1\\
&-\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}+6\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_2\cdot\vec{n}+6\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2+2\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2+\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2+4\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2\\
&+16\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}-24\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\Big], \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
\overset{(2)}{V} = &-\frac{G}{8r}\Big[8\big(\vec{\tilde{S}}_2\cdot\dot{\vec{a}}_1\vec{\tilde{S}}_1\cdot\vec{v}_2-\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\dot{\vec{a}}_1\cdot\vec{v}_2+\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\dot{\vec{a}}_2-\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\dot{\vec{a}}_2+\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\dot{\vec{a}}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\dot{\vec{a}}_1\vec{v}_2\cdot\vec{n}+\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\dot{\vec{a}}_2\cdot\vec{n}-\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\dot{\vec{a}}_2\big)-\big(2\vec{\tilde{S}}_1\cdot\vec{v}_1\ddot{\vec{S}}_2\cdot\vec{v}_1-3\vec{\tilde{S}}_1\cdot\ddot{\vec{S}}_2v_1^2\\
&-8\ddot{\vec{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{v}_2-8\vec{\tilde{S}}_2\cdot\vec{v}_1\ddot{\vec{S}}_1\cdot\vec{v}_2+2\ddot{\vec{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{v}_2+8\ddot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{v}_2+8\vec{\tilde{S}}_1\cdot\ddot{\vec{S}}_2\vec{v}_1\cdot\vec{v}_2\\
&-3\ddot{\vec{S}}_1\cdot\vec{\tilde{S}}_2v_2^2-2\ddot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1-2\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\ddot{\vec{S}}_2\cdot\vec{v}_1-\vec{\tilde{S}}_1\cdot\vec{n}\ddot{\vec{S}}_2\cdot\vec{n}v_1^2-8\ddot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-8\vec{\tilde{S}}_1\cdot\ddot{\vec{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+8\ddot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+8\ddot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2-2\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\ddot{\vec{S}}_1\cdot\vec{v}_2\\
&-2\ddot{\vec{S}}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2-\ddot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}v_2^2+3\vec{\tilde{S}}_1\cdot\ddot{\vec{S}}_2(\vec{v}_1\cdot\vec{n})^2+3\ddot{\vec{S}}_1\cdot\vec{\tilde{S}}_2(\vec{v}_2\cdot\vec{n})^2\\
&+3\vec{\tilde{S}}_1\cdot\vec{n}\ddot{\vec{S}}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+3\ddot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)+\big(33\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{\tilde{S}}_1\cdot\vec{a}_2+\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{\tilde{S}}_2\cdot\vec{a}_2\\
&-31\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{a}_2+15\vec{\tilde{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}-5\vec{\tilde{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{a}_2\cdot\vec{n}-9\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{a}_2\cdot\vec{n}\\
&-9\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2-5\vec{\tilde{S}}_1\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2+15\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{a}_2+3\vec{\tilde{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)\\
&-2\big(\dot{\vec{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{a}_1+5\vec{\tilde{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_2\cdot\vec{a}_1-6\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{a}_1-17\dot{\vec{S}}_2\cdot\vec{a}_1\vec{\tilde{S}}_1\cdot\vec{v}_2-8\vec{\tilde{S}}_2\cdot\vec{a}_1\dot{\vec{S}}_1\cdot\vec{v}_2\\
&-\vec{\tilde{S}}_1\cdot\vec{a}_1\dot{\vec{S}}_2\cdot\vec{v}_2+8\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{v}_2+17\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{a}_1\cdot\vec{v}_2-8\dot{\vec{S}}_2\cdot\vec{v}_1\vec{\tilde{S}}_1\cdot\vec{a}_2-17\vec{\tilde{S}}_2\cdot\vec{v}_1\dot{\vec{S}}_1\cdot\vec{a}_2\\
&+5\vec{\tilde{S}}_2\cdot\vec{v}_2\dot{\vec{S}}_1\cdot\vec{a}_2-\dot{\vec{S}}_1\cdot\vec{v}_1\vec{\tilde{S}}_2\cdot\vec{a}_2+\dot{\vec{S}}_1\cdot\vec{v}_2\vec{\tilde{S}}_2\cdot\vec{a}_2+17\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{a}_2+8\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{a}_2\\
&-6\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{a}_2+12\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}\vec{a}_1\cdot\vec{n}-3\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}-4\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1\vec{a}_1\cdot\vec{n}\\
&-3\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1-8\vec{\tilde{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{a}_1+6\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{a}_1-8\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-9\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+3\dot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+8\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}+5\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{a}_1\vec{v}_2\cdot\vec{n}\\
&+5\dot{\vec{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{v}_2+3\vec{\tilde{S}}_1\cdot\vec{n}\vec{a}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2-7\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{v}_2-9\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\\
&-8\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}+3\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1\vec{a}_2\cdot\vec{n}+5\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_1\vec{a}_2\cdot\vec{n}+12\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{v}_2\cdot\vec{n}\vec{a}_2\cdot\vec{n}\\
&-4\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}-3\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{v}_2\vec{a}_2\cdot\vec{n}+8\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_2+5\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{a}_2\\
&-8\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{a}_2+3\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2-3\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2-7\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{a}_2\\
&+6\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{a}_2-3\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-3\dot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)-2\big(5\dot{\vec{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_2\cdot\vec{v}_1\\
&-5\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2v_1^2-18\dot{\vec{S}}_2\cdot\vec{v}_1\dot{\vec{S}}_1\cdot\vec{v}_2-2\dot{\vec{S}}_1\cdot\vec{v}_1\dot{\vec{S}}_2\cdot\vec{v}_2+5\dot{\vec{S}}_1\cdot\vec{v}_2\dot{\vec{S}}_2\cdot\vec{v}_2+20\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{v}_2\\
&-5\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2v_2^2-3\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1-8\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1+3\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}v_1^2\\
&-12\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+4\dot{\vec{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+6\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_1\vec{v}_2\cdot\vec{n}+6\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2\\
&-8\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_2+4\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2-3\dot{\vec{S}}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{v}_2-6\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+3\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}v_2^2+8\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2(\vec{v}_1\cdot\vec{n})^2+8\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2(\vec{v}_2\cdot\vec{n})^2-6\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\big)\Big]\\
&-\frac{G^2m_1}{r^2}\Big[2\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2-5\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\Big]-\frac{G^2m_2}{r^2}\Big[2\dot{\vec{S}}_1\cdot\dot{\vec{S}}_2-5\dot{\vec{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\vec{n}\Big],
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
\overset{(3)}{V} = &-\frac{1}{8}G\Big[8\big(\vec{\tilde{S}}_1\cdot\dot{\vec{S}}_2\dot{\vec{a}}_1\cdot\vec{n}-\vec{\tilde{S}}_1\cdot\vec{n}\dot{\vec{S}}_2\cdot\dot{\vec{a}}_1-\dot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\dot{\vec{a}}_2\cdot\vec{n}+\vec{\tilde{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\dot{\vec{a}}_2\big)+\big(3\vec{\tilde{S}}_1\cdot\ddot{\vec{S}}_2\vec{a}_1\cdot\vec{n}\\
&-\ddot{\vec{S}}_2\cdot\vec{n}\vec{\tilde{S}}_1\cdot\vec{a}_1-\vec{\tilde{S}}_1\cdot\vec{n}\ddot{\vec{S}}_2\cdot\vec{a}_1-3\ddot{\vec{S}}_1\cdot\vec{\tilde{S}}_2\vec{a}_2\cdot\vec{n}+\vec{\tilde{S}}_2\cdot\vec{n}\ddot{\vec{S}}_1\cdot\vec{a}_2+\ddot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{a}_2\\
&+\vec{\tilde{S}}_1\cdot\vec{n}\ddot{\vec{S}}_2\cdot\vec{n}\vec{a}_1\cdot\vec{n}-\ddot{\vec{S}}_1\cdot\vec{n}\vec{\tilde{S}}_2\cdot\vec{n}\vec{a}_2\cdot\vec{n}\big)+2\big(4\ddot{\vec{S}}_1\cdot\dot{\vec{S}}_2\vec{v}_1\cdot\vec{n}+3\dot{\vec{S}}_1\cdot\ddot{\vec{S}}_2\vec{v}_1\cdot\vec{n}-\ddot{\vec{S}}_2\cdot\vec{n}\dot{\vec{S}}_1\cdot\vec{v}_1
\end{aligned}
$$

$$
\begin{aligned}
&- 4\ddot{\vec{S}}_1 \cdot \vec{n}\dot{\vec{S}}_2 \cdot \vec{v}_1 - \dot{\vec{S}}_1 \cdot \vec{n}\ddot{\vec{S}}_2 \cdot \vec{v}_1 - 3\ddot{\vec{S}}_1 \cdot \dot{\vec{S}}_2\vec{v}_2 \cdot \vec{n} - 4\dot{\vec{S}}_1 \cdot \ddot{\vec{S}}_2\vec{v}_2 \cdot \vec{n} + 4\ddot{\vec{S}}_2 \cdot \vec{n}\dot{\vec{S}}_1 \cdot \vec{v}_2 + \dot{\vec{S}}_2 \cdot \vec{n}\ddot{\vec{S}}_1 \cdot \vec{v}_2 \\
&+ \ddot{\vec{S}}_1 \cdot \vec{n}\dot{\vec{S}}_2 \cdot \vec{v}_2 + \dot{\vec{S}}_1 \cdot \vec{n}\ddot{\vec{S}}_2 \cdot \vec{n}\vec{v}_1 \cdot \vec{n} - \ddot{\vec{S}}_1 \cdot \vec{n}\dot{\vec{S}}_2 \cdot \vec{n}\vec{v}_2 \cdot \vec{n}\big) + 16\big(\dot{\vec{S}}_1 \cdot \dot{\vec{S}}_2\vec{a}_1 \cdot \vec{n} - \dot{\vec{S}}_1 \cdot \vec{n}\dot{\vec{S}}_2 \cdot \vec{a}_1 \\
&- \dot{\vec{S}}_1 \cdot \dot{\vec{S}}_2\vec{a}_2 \cdot \vec{n} + \dot{\vec{S}}_2 \cdot \vec{n}\dot{\vec{S}}_1 \cdot \vec{a}_2\big)\Big],
\end{aligned}
\tag{A.7}
$$

$$
\overset{(4)}{V} = -\frac{1}{8}Gr\Big[3\ddot{\vec{S}}_1 \cdot \ddot{\vec{S}}_2 - \ddot{\vec{S}}_1 \cdot \vec{n}\ddot{\vec{S}}_2 \cdot \vec{n}\Big]. \tag{A.8}
$$

The Hamiltonian resulting from this potential is given by eq. (3.3). It is canonically equivalent to the NNLO spin1-spin2 Hamiltonians in [14, 23]. The generator of the canonical transformation, which connects it to the ADM-gauge Hamiltonian [23] is given by eq. (7.5) of [14], with the coefficients being equal to

$$
\begin{array}{lllllll}
g_1 = 0, & g_2 = -\frac{7}{4}, & g_3 = \frac{9}{12}, & g_4 = 0, & g_5 = \frac{9}{4}, & g_6 = 0, & g_7 = \frac{9}{4}, \\
g_8 = -2, & g_9 = \frac{1}{4}, & g_{10} = 0, & g_{11} = -\frac{9}{2}, & g_{12} = 0, & g_{13} = \frac{1}{4}, & g_{14} = \frac{1}{2}, \\
g_{15} = -\frac{1}{4}, & g_{16} = 0, & g_{17} = -3, & g_{18} = 0, & g_{19} = -\frac{3}{2}, & g_{20} = \frac{9}{4}, & g_{21} = 0, \\
g_{22} = 0, & g_{23} = 0, & g_{24} = -\frac{3}{4}, & g_{25} = \frac{3}{4}, & g_{26} = \frac{5}{2}, & g_{27} = 0, & g_{28} = -\frac{3}{2}, \\
g_{29} = -2, & g_{30} = -\frac{11}{4}, & g_{31} = -2, & g_{32} = -\frac{13}{4}, & g_{33} = 0, & g_{34} = 4. &
\end{array}
\tag{A.9}
$$

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