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Invariants for minimal conformal supergravity in six dimensions

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Abstract: We develop a new off-shell formulation for six-dimensional conformal supergravity obtained by gauging the 6D $\mathcal{N} = (1, 0)$ superconformal algebra in superspace. This formulation is employed to construct two invariants for 6D $\mathcal{N} = (1, 0)$ conformal supergravity, which contain C^3 and $C\Box C$ terms at the component level. Using a conformal supercurrent analysis, we prove that these exhaust all such invariants in minimal conformal supergravity. Finally, we show how to construct the supersymmetric $F\Box F$ invariant in curved superspace.

Keywords: Conformal Field Theory, Supergravity Models, Superspaces

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1 Introduction

Conformal field theories (CFTs) play a distinguished role among relativistic quantum field theories. It has long been realized that they arise as fixed point theories of renormalization group flows and the study of their properties is clearly of interest. The enlarged symmetry group helps to constrain e.g. the general structure of correlation functions beyond what is already required by Poincaré invariance. Additional symmetries lead to further restrictions.

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One such symmetry which is very powerful in this respect is supersymmetry, in which case one deals with superconformal field theories (SCFTs).

It has been known since the early days of supersymmetry that superconformal theories can only exist in six or lower dimensions [1]. In six dimensions, where $\mathcal{N} = (p, q)$ Poincaré superalgebras exist for any integer $p, q \geq 0$, superconformal algebras only exist for either $p = 0$ or $q = 0$. In fact, the only known non-trivial unitary CFTs in six dimensions are supersymmetric and arise as world-volume theories of appropriate brane configurations in string and M-theory and in F-theory, in the limit where gravity decouples. They realize either $\mathcal{N} = (2, 0)$ or $\mathcal{N} = (1, 0)$ superconformal symmetry. For these theories no Lagrangian description is known but they are believed to obey the axioms of quantum field theories.[1] They should, in particular, have local conserved current operators and among them a local conserved and traceless energy-momentum tensor [3, 4]. Evidence for the existence of $\mathcal{N} = (2, 0)$ theories was first given in [5–7]; for $\mathcal{N} = (1, 0)$ theories we refer to [3, 4, 8–13].

As mentioned before, symmetries in quantum field theories lead to restrictions on correlation functions which have to satisfy Ward identities. In correlation functions of conserved currents one finds, however, that the naive Ward identities which would follow from the symmetries cannot always be satisfied simultaneously. This happens in even dimensions and leads to (super)conformal anomalies which express the fact that imposing conservation and tracelessness of the energy-momentum tensor clashes in certain correlation functions. The general structure of these conformal or Weyl anomalies was analyzed by Deser and Schwimmer [14] who also introduced the classification into two types: type A and type B. In any even dimension there is always one type A anomaly and starting in four dimensions, an increasing number of type B anomalies. The easiest way to discuss them is to couple the conformal field theory to a metric background which serves as a source for the energy-momentum tensor. The anomalies then express the non-invariance of the effective action (generating functional) under a local Weyl rescaling of the metric. The anomalous variation of the non-local effective action results in anomalies which are local diffeomorphism invariant functions of the metric and its derivative, i.e. functions of the curvature and its covariant derivatives. The type A anomaly in any even dimension is given by the Euler density of that dimension; the type B anomalies are Weyl invariant expressions constructed from the curvature tensors and its covariant derivatives [14]. In four dimensions there is one such expression, the square of the Weyl tensor; in six dimensions there are two inequivalent contractions of three Weyl tensors and one Weyl invariant expression which involves two covariant derivatives. If we work in a topologically trivial background, only the type B anomalies contribute if one rescales the metric by a constant factor.

In any dimension the possible Weyl anomalies can be found by imposing the Wess-Zumino consistency condition [15], which expresses the obvious fact that two consecutive Weyl variations of the effective action must commute. Non-supersymmetric CFTs are then characterized by as many anomaly coefficients as there are solutions to the Wess-Zumino consistency condition: one in two, two in four and four in six dimensions, respectively.

[1] Here we are concerned with unitary SCFTs. For an example of a higher-derivative classical SCFT, see [2].

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In SCFTs, the Weyl anomalies are accompanied by superconformal and R-symmetry anomalies; altogether they constitute the so-called super-Weyl anomalies. They are related by supersymmetry and various anomalies in bosonic and fermionic symmetry currents are packaged into anomaly supermultiplets. The most elegant way to exhibit this is using a manifestly supersymmetric formulation, i.e. superspace. In four dimensions, the super-Weyl anomalies were studied in [16, 17] in the $\mathcal{N} = 1$ case and in [18] for $\mathcal{N} = 2$. Furthermore, supersymmetry might also reduce the number of independent anomaly coefficients by packaging several solutions of the Wess-Zumino consistency conditions into one supermultiplet. This is the case for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions where there is only one independent anomaly coefficient.

As Lagrangian descriptions of six-dimensional SCFTs are not known, it is rather difficult to study their dynamics. Interesting non-trivial information can, however, be obtained from their symmetries. One can e.g. show that $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (1, 0)$ SCFTs have neither marginal nor relevant supersymmetry preserving deformations [19, 20]. Another way to approach these theories is via their 't Hooft and Weyl anomalies. This was done in [21–27].

Due to supersymmetry one expects that the two types of anomalies are parametrized by the same coefficients. This is known e.g. for $\mathcal{N} = 1$ SCFT in four dimensions, where the U(1) R-current anomalies are governed by linear combinations of the two independent Weyl anomaly coefficients. It would be useful to know similar relations for SCFTs in six dimensions and furthermore, to know the precise number of independent anomaly coefficients. We consider the analysis of this paper as a first step towards answering these questions for $\mathcal{N} = (1, 0)$ SCFTs. More precisely, we will construct supersymmetry invariants which contain the solutions of the WZ consistency condition for the Weyl anomaly as one of their bosonic components. By supersymmetry, these invariants should contain the solutions to the supersymmetrized version of the WZ condition. Here we content ourselves with the first step, the construction of the supersymmetric invariants and leave a detailed analysis of the anomaly structure for the future. But the results of this paper already show that the number of anomaly coefficients is reduced: while in the non-supersymmetric case there are three independent type B Weyl anomalies, i.e. dimension six combinations of curvature tensors and covariant derivatives which transform homogeneously under Weyl transformations of the metric, there are only two independent superspace invariants which contain them. In addition to their relevance for the anomaly structure, their arbitrary linear combination is the action for minimal conformal supergravity in six dimensions, which will be the main focus of this paper.

To establish these results we develop a new off-shell superspace formulation of this theory. We therefore start with a brief review of six-dimensional (6D) minimal conformal supergravity and conformal superspace methods (see [28] for a review of conformal supergravity theories in 4D). Its superconformal tensor calculus was formulated thirty years ago by Bergshoeff, Sezgin and Van Proeyen [29]. In many respects, it is analogous to the superconformal tensor calculus for 4D $\mathcal{N} = 2$ supergravity [30–35], see [36] for a recent pedagogical review. Soon after the 6D $\mathcal{N} = (1, 0)$ superconformal method [29] appeared, it was applied to construct the off-shell supersymmetric extension of the Riemann curvature

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squared term [37–39]. More recently, the 6D $\mathcal{N} = (1,0)$ superconformal techniques of [29] have been refined [40, 41]. In particular, the complete off-shell action for minimal Poincaré supergravity has been given in [40] (only the bosonic part of this action was explicitly worked out in [29]). Gauged minimal 6D supergravity has been worked out in [41] by coupling the minimal supergravity of [40] to an off-shell vector multiplet. The resulting theory is an off-shell version of the dual formulation [42, 43] of the Salam-Sezgin model [44, 45].

Similar to the 4D $\mathcal{N} = 2$ case, the 6D $\mathcal{N} = (1,0)$ superconformal tensor calculus has two limitations. Firstly, it does not provide tools to describe off-shell hypermultiplets. Only on-shell hypermultiplets were used in [29] as well as in all later developments based on [29]. Secondly, it does not offer insight as to how general higher-derivative supergravity actions can be built, see [46] for a recent discussion. In particular, (off-shell) invariants for 6D $\mathcal{N} = (1,0)$ conformal supergravity have never been constructed. In order to avoid these limitations, one has to resort to superspace techniques. At this point, some comments are in order about the superspace approaches to conformal supergravity in diverse dimensions.

There are two general approaches to describe $\mathcal{N}$-extended conformal supergravity in $D \leq 6$ dimensions[2] using a curved $\mathcal{N}$-extended superspace $\mathcal{M}^{D|\delta}$, where δ denotes the number of fermionic dimensions. One of them, known as $G_R^{[D;\mathcal{N}]}$ superspace, makes use of the superspace structure group $\mathsf{SO}(D-1,1) \times G_R^{[D;\mathcal{N}]}$, where $\mathsf{SO}(D-1,1)$ is the Lorentz group and $G_R^{[D;\mathcal{N}]}$ is the R-symmetry group of the $\mathcal{N}$-extended super-Poincaré algebra in D dimensions.[3] A fundamental requirement on the superspace geometry, which should describe conformal supergravity, is that the constraints on the superspace torsion be invariant under a super-Weyl transformation generated by a real unconstrained superfield parameter. This approach was pioneered in four dimensions by Howe [47, 48] who fully developed the $\mathsf{U}(1)$ and $\mathsf{U}(2)$ superspace geometries [48] corresponding to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, respectively. The superspace formulation for 5D conformal supergravity (5D $\mathsf{SU}(2)$ superspace) was presented in [49], and it was naturally extended to the 6D $\mathcal{N} = (1,0)$ case in [50] where 6D $\mathsf{SU}(2)$ superspace was formulated. In three dimensions, the $\mathsf{SO}(\mathcal{N})$ superspace geometry was developed in [51, 52].

The other superspace approach to conformal supergravity is based on gauging the entire $\mathcal{N}$-extended superconformal group in D dimensions, of which $\mathsf{SO}(D-1,1) \times G_R^{[D;\mathcal{N}]}$ is a subgroup. This approach, known as conformal superspace, was originally developed for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity theories in four dimensions by one of us (DB) [53, 54]. More recently, it has been extended to the cases of 3D $\mathcal{N}$-extended conformal supergravity [55] and 5D conformal supergravity [56]. Conformal superspace is a more general formulation than $G_R^{[D;\mathcal{N}]}$ superspace in the sense that the latter is obtained from the former by partially fixing the gauge freedom, see [53–56] for more details.

Unlike the superconformal tensor calculus, the superspace method offers off-shell formulations for the most general supergravity-matter couplings with eight supercharges in four, five and six dimensions. This includes off-shell formulations for hypermultiplets and

[2] The cases of five and six dimensions are rather special. Conformal supergravity exists only for $\mathcal{N} = 1$ in five dimensions, and only for $\mathcal{N} = (p,0)$ in six dimensions.

[3] The group $G_R^{[D;\mathcal{N}]}$ coincides with $\mathsf{SO}(\mathcal{N})$ for $D = 3$, $\mathsf{U}(\mathcal{N})$ for $D = 4$, and $\mathsf{SU}(2)$ for the cases 5D $\mathcal{N} = 1$ and 6D $\mathcal{N} = (1,0)$.

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their most general locally supersymmetric sigma model couplings. The first such formulations were developed using harmonic superspace [57–59] (see also [60]) and employed explicit supergravity prepotentials (but see [61, 62] for covariant approaches). Later, off-shell geometric formulations were derived for 5D $\mathcal{N}=1$ supergravity-matter systems [49, 63] by putting forward the novel concept of covariant projective multiplets. These supermultiplets are curved-superspace extensions of the 4D $\mathcal{N}=2$ and 5D $\mathcal{N}=1$ superconformal projective multiplets [64, 65]. The latter reduce to the off-shell projective multiplets pioneered by Lindström and Roček [66–68] in the 4D $\mathcal{N}=2$ super-Poincaré case. The 5D off-shell formulations have been generalized to the 4D $\mathcal{N}=2$ [69, 70], 3D $\mathcal{N}=4$ [52] and 6D $\mathcal{N}=(1,0)$ [50] cases.[4] All of these works made use of the appropriate $G_R^{[D;\mathcal{N}]}$ superspace. However, all the results are naturally lifted to conformal superspace.

Conformal superspace is an ideal setting to reduce the locally supersymmetric actions from superspace to components [72, 73]. It also turns out to be an efficient formalism to build general higher-derivative supergravity actions. Recent applications of the conformal superspace approach have involved constructing (i) the $\mathcal{N}$-extended conformal supergravity actions in three dimensions for $3 \leq \mathcal{N} \leq 6$ [74, 75], and (ii) new higher-derivative invariants in 4D $\mathcal{N}=2$ supergravity, including the Gauss-Bonnet term [76]. In the present paper, we develop 6D $\mathcal{N}=(1,0)$ conformal superspace and apply it to construct invariants for conformal supergravity.

Before turning to the details of the six-dimensional case, it is worth recalling the structure of conformal supergravity actions in four dimensions (see for example the reviews [28, 77]). The invariants for $\mathcal{N}<3$ are supersymmetric extensions of the C^2 term and are described by chiral integrals of the form

$$I_{C^2} := \int \mathrm{d}^4x\, \mathrm{d}^{2\mathcal{N}}\theta\, \mathcal{E}\, W^{\alpha_1\dots\alpha_{4-\mathcal{N}}} W_{\alpha_1\dots\alpha_{4-\mathcal{N}}} + \text{c.c.}\,, \qquad \mathcal{N}=1,2, \tag{1.1}$$

where $\mathcal{E}$ is the chiral integration measure. The covariantly chiral tensor superfield $W_{\alpha_1\dots\alpha_{4-\mathcal{N}}} = W_{(\alpha_1\dots\alpha_{4-\mathcal{N}})}$ is the superspace generalization of the Weyl tensor (known as the super-Weyl tensor). Thus the structure of 4D $\mathcal{N}$-extended conformal supergravity is remarkably simple for $\mathcal{N}<3$.

The case of 6D $\mathcal{N}=(1,0)$ conformal supergravity has conceptual differences from its 4D $\mathcal{N}=2$ cousin. First of all, there is no covariantly defined chiral subspace of SU(2) superspace [50], and thus we cannot generalise the 4D $\mathcal{N}=2$ construction to six dimensions. Of course, one could try and construct invariants for conformal supergravity as full superspace integrals of the form

$$S = \int \mathrm{d}^6x\, \mathrm{d}^8\theta\, E\, \mathcal{L}\,, \tag{1.2}$$

where the Lagrangian $\mathcal{L}$ is a real primary superfield of dimension 2 (in the sense of [50]). This Lagrangian should be constructed in terms of the dimension-1 super-Weyl tensor $W^{\alpha\beta} = W^{\beta\alpha}$ [50] and its covariant derivatives. It is obvious that no $\mathcal{L}$ with the required

[4] In the 6D $\mathcal{N}=(1,0)$ super-Poincaré case, the projective-superspace formalism was introduced in [71], where it was used to construct off-shell actions for self-interacting linear multiplets.

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properties exists. In the case of 4D $\mathcal{N}=2$ supergravity, it was shown [78, 79] that the chiral action principle can be reformulated as a special case of the 4D $\mathcal{N}=2$ projective-superspace action [69, 70]. For supergravity theories with eight supercharges in diverse dimensions (including the 3D $\mathcal{N}=4$ [52], 5D $\mathcal{N}=1$ [49] and 6D $\mathcal{N}=(1,0)$ [50] cases), the projective-superspace action principle is known to be universal in the sense that it can be used to realize general off-shell supergravity-matter couplings. The same statement holds for harmonic superspace (see [80] for the 6D $\mathcal{N}=(1,0)$ case in particular). If the goal were to build two-derivative supergravity-matter actions, either approach would suffice. However, if one is interested in realizing the invariants for 6D $\mathcal{N}=(1,0)$ conformal supergravity, it proves to be impossible to construct any projective-superspace Lagrangian $\mathcal{L}^{(2)}$ only in terms of the super-Weyl tensor $W^{\alpha\beta}$, that is without introducing prepotentials for the Weyl multiplet; and while 6D harmonic superspace furnishes such explicit prepotentials, the problem of constructing the necessary higher derivative invariants (while respecting the prepotential gauge transformations) remains a challenge unsolved even in the 4D case, where the covariant actions are known. Therefore, if one is interested in constructing the invariants for 6D $\mathcal{N}=(1,0)$ conformal supergravity solely in terms of the covariant super-Weyl tensor, a new action principle is required. The present paper addresses this problem and demonstrates that there are two action principles which naturally support all the 6D $\mathcal{N}=(1,0)$ Weyl invariants.

This paper is organized as follows. Section 2 is a review on conformal gravity and includes a simple derivation of the 6D Weyl invariants. In section 3 we describe 6D $\mathcal{N}=(1,0)$ conformal superspace. In section 4 an action principle is presented in conformal superspace and it is shown how it can be used to describe a supersymmetric invariant containing a C^3 term. Application to other invariants is also discussed. Section 5 is devoted to deriving another action principle which is used to describe a supersymmetric invariant containing a $C\Box C$ term and a higher derivative action based on the Yang-Mills multiplet in conformal superspace. Concluding comments and a discussion are given in section 6, where it is proved that the 6D $\mathcal{N}=(1,0)$ Weyl invariants constructed exhaust all such invariants in minimal conformal supergravity.

We have included a number of technical appendices. In appendix A we include a summary of our notation and conventions. Appendix B is devoted to a derivation of the superconformal algebra from the algebra of conformal Killing supervector fields of 6D $\mathcal{N}=(1,0)$ Minkowski superspace. Finally, in appendix C we give a description of the Yang-Mills multiplet in conformal superspace.

2 Conformal gravity in six dimensions

The conformal invariants in six dimensions [14, 81, 82] have been constructed previously and are well known. Since we will be concerned with their supersymmetric generalizations, it is natural to first present their bosonic counterparts. In this section, we provide a simple derivation of the conformal invariants. The formulation we use here will be naturally generalized to the supersymmetric case in later sections and will serve as a prelude to the

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conformal superspace formulation in section 3. We begin by reviewing the formulation for conformal gravity in $D > 3$ spacetime dimensions following [55].[5]

2.1 Conformal gravity in $D > 3$ spacetime dimensions

The conformal algebra in $D > 2$ spacetime dimensions, $\mathfrak{so}(D, 2)$, is spanned by the generators $X_{\underline{a}} = \{P_a, M_{ab}, \mathbb{D}, K_a\}$, which obey the commutation relations

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c}\,, \tag{2.1a}$$

$$[M_{ab}, P_c] = 2\eta_{c[a}P_{b]}\,, \qquad [\mathbb{D}, P_a] = P_a\,, \tag{2.1b}$$

$$[M_{ab}, K_c] = 2\eta_{c[a}K_{b]}\,, \qquad [\mathbb{D}, K_a] = -K_a\,, \tag{2.1c}$$

$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab}\,, \tag{2.1d}$$

where P_a is the translation, $M_{ab} = -M_{ba}$ is the Lorentz, $\mathbb{D}$ is the dilatation and K_a is the special conformal generator.

To describe conformal gravity one begins with a D-dimensional manifold $\mathcal{M}^D$ parametrized by local coordinates x^m, $m = 0, 1, \cdots, D-1$. Following the gauging procedure in [55], the covariant derivatives are chosen to have the form

$$\nabla_a = e_a - \frac{1}{2}\omega_a{}^{bc}M_{bc} - b_a\mathbb{D} - \mathfrak{f}_a{}^b K_b\,. \tag{2.2}$$

Here $e_a = e_a{}^m\partial_m$ is the inverse vielbein, while $\omega_a{}^{bc}$ is the Lorentz, b_a is the dilation and $\mathfrak{f}_a{}^b$ is the special conformal connection, respectively. The covariant derivatives may also be cast in the framework of forms

$$\nabla = e^a\nabla_a = \mathrm{d} - \frac{1}{2}\omega^{bc}M_{bc} - b\,\mathbb{D} - \mathfrak{f}^a K_a\,, \tag{2.3}$$

where $e^a := \mathrm{d}x^m e_m{}^a$ is the vielbein, d is the exterior derivative and we have defined $\omega^{bc} := e^a\omega_a{}^{bc}$, $b := e^a b_a$ and $\mathfrak{f}^a := e^b\mathfrak{f}_b{}^a$.

The gravity gauge group is generated by local transformations which can be summarised by[6]

$$\delta_{\mathcal{K}}\nabla_a = [\mathcal{K}, \nabla_a]\,, \quad \mathcal{K} = \xi^a\nabla_a + \Lambda^{\underline{a}}X_{\underline{a}} = \xi^a\nabla_a + \frac{1}{2}\Lambda(M)^{ab}M_{ab} + \sigma\mathbb{D} + \Lambda(K)^a K_a \tag{2.4}$$

provided we interpret

$$\nabla_a\xi^b := e_a\xi^b + \omega_a{}^{\underline{c}}\xi^d f_{d\underline{c}}{}^b\,, \quad \nabla_a\Lambda^{\underline{b}} := e_a\Lambda^{\underline{b}} + \omega_a{}^{\underline{c}}\xi^d f_{d\underline{c}}{}^{\underline{b}} + \omega_a{}^{\underline{c}}\Lambda^{\underline{d}}f_{\underline{dc}}{}^{\underline{b}}\,, \tag{2.5}$$

where the structure constants are defined by

$$[X_{\underline{a}}, P_b] = -f_{\underline{a}b}{}^{\underline{c}}X_{\underline{c}} - f_{\underline{a}b}{}^c P_c\,, \quad [X_{\underline{a}}, X_{\underline{b}}] = -f_{\underline{ab}}{}^{\underline{c}}X_{\underline{c}}\,. \tag{2.6}$$

[5] Conformal gravity has been discussed elsewhere in many places, e.g. [36]. Our review here emphasizes certain points relevant to our paper.

[6] One must take care in applying this formula since one can have $\Lambda^{\underline{a}} = 0$ but $\nabla_a\Lambda^{\underline{b}} \neq 0$.

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The gauging procedure ensures that the generators $X_{\underline{a}}$ act on the covariant derivatives in the same way as they do on P_a, except with P_a replaced by ∇_a, while the covariant derivative algebra obeys commutation relations of the form

$$[\nabla_a, \nabla_b] = -T_{ab}{}^c \nabla_c - \frac{1}{2} R(M)_{ab}{}^{cd} M_{cd} - R(\mathbb{D})_{ab} \mathbb{D} - R(K)_{ab}{}^c K_c \,, \tag{2.7}$$

where the curvatures and torsion are given by the form expressions:

$$T^a = \frac{1}{2} e^c \wedge e^b \, T_{bc}{}^a = \mathrm{d}e^a + e^a \wedge b + e^b \wedge \omega_b{}^a \,, \tag{2.8a}$$

$$R(M)^{cd} = \frac{1}{2} e^b \wedge e^a R(M)_{ab}{}^{cd} = \mathrm{d}\omega^{cd} + \omega^{ce} \wedge \omega_e{}^d - 4 e^{[c} \wedge \mathfrak{f}^{d]} \,, \tag{2.8b}$$

$$R(\mathbb{D}) = \frac{1}{2} e^b \wedge e^a R(\mathbb{D})_{ab} = \mathrm{d}b + 2 e^a \wedge \mathfrak{f}_a \,, \tag{2.8c}$$

$$R(K)^a = \frac{1}{2} e^c \wedge e^b R(K)_{bc}{}^a = \mathrm{d}\mathfrak{f}^a - \mathfrak{f}^a \wedge b + \mathfrak{f}^b \wedge \omega_b{}^a \,. \tag{2.8d}$$

The gravity gauge group acts on a tensor field U (with indices suppressed) as

$$\delta_{\mathcal{K}} U = \mathcal{K} U \,. \tag{2.9}$$

We call a field U satisfying $K_a U = 0$ and $\mathbb{D} U = \Delta U$ a primary field of dimension (or Weyl weight) Δ.

To describe conformal gravity, one must impose some conformal constraints:

$$T_{ab}{}^c = 0 \,, \quad \eta^{bc} R(M)_{abcd} = 0 \,, \quad R(\mathbb{D})_{ab} = 0 \,. \tag{2.10}$$

For $D > 3$, the Bianchi identities constrain the covariant derivative algebra to be of the form

$$[\nabla_a, \nabla_b] = -\frac{1}{2} C_{ab}{}^{cd} M_{cd} - \frac{1}{2(D-3)} \nabla^d C_{abcd} K^c \,, \tag{2.11}$$

where C_{abcd} is the Weyl tensor satisfying[7]

$$C_{abcd} = C_{[ab][cd]} \,, \quad C_{[abc]d} = 0 \tag{2.12}$$

and the Bianchi identity

$$\nabla_{[a} C_{bc]}{}^{de} = -\frac{2}{D-3} \nabla_f C_{[ab}{}^{f[d} \delta_{c]}^{e]} \,. \tag{2.13}$$

The Weyl tensor $C_{ab}{}^{cd}$ proves to be a primary field.[8] This means that when the explicit expression for $\omega_a{}^{bc}$ is used dependence on b_a drops out of the Weyl tensor.

One can always make use of the special conformal gauge freedom to choose a vanishing dilatation connection, $b_a = 0$. The covariant derivatives then take the form

$$\nabla_a = \mathcal{D}_a - \mathfrak{f}_a{}^b K_b \,, \quad \mathcal{D}_a := e_a - \frac{1}{2} \omega_a{}^{bc} M_{bc} \,. \tag{2.14}$$

[7] The symmetry property $C_{abcd} = C_{cdab}$ is not independent and follows from the others.

[8] This follows from considering $[K_a, [\nabla_b, \nabla_c]] = 2[[K_a, \nabla_{[b}], \nabla_{c]}] = 0$.

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In this gauge the Lorentz curvature

$$\mathcal{R}_{ab}{}^{cd} := 2e_{[a}{}^{m}e_{b]}{}^{n}\partial_m\omega_n{}^{cd} - 2\omega_{[a}{}^{cf}\omega_{b]f}{}^{d} \tag{2.15}$$

may be expressed as

$$\mathcal{R}_{ab}{}^{cd} = C_{ab}{}^{cd} - 8\delta^{[c}_{[a}\mathfrak{f}_{b]}{}^{d]}\,. \tag{2.16}$$

One can then solve the special conformal connection in terms of the Lorentz curvature

$$\mathfrak{f}_{ab} = -\frac{1}{2(D-2)}\mathcal{R}_{ab} + \frac{1}{4(D-1)(D-2)}\eta_{ab}\mathcal{R}\,, \tag{2.17}$$

where we have defined

$$\mathcal{R}_{ac} := \eta^{bd}\mathcal{R}_{abcd}\,, \quad \mathcal{R} := \eta^{ab}\mathcal{R}_{ab}\,. \tag{2.18}$$

We will often refer to the procedure of setting $b_a = 0$ and introducing the covariant derivative $\mathcal{D}_a$ as degauging.

It is worth mentioning that one can introduce new covariant derivatives by making use of a compensator ϕ, which we choose to be primary and of dimension 2. One can construct the following covariant derivatives using the compensator

$$\mathscr{D}_a = \phi^{-\frac{1}{2}}\left(\nabla_a + \frac{1}{2}(\nabla^b \ln\phi)M_{ab} - \frac{1}{2}(\nabla_a \ln\phi)\mathbb{D}\right)\,, \tag{2.19}$$

which have the property that if U is some conformally primary tensor field of some dimension then $\mathscr{D}_a U$ is as well. The covariant derivatives annihilate the compensator ϕ, $\mathscr{D}_a\phi = 0$. When acting on primary fields they satisfy the algebra

$$[\mathscr{D}_a, \mathscr{D}_b] = -\frac{1}{2}\mathscr{R}_{ab}{}^{cd}M_{cd}\,, \tag{2.20}$$

where

$$\mathscr{R}_{ab}{}^{cd} := \phi^{-1}C_{ab}{}^{cd} + \frac{4}{D-2}\delta^{[c}_{[a}\mathscr{R}_{b]}{}^{d]} - \frac{2}{(D-1)(D-2)}\delta^{[c}_{[a}\delta^{d]}_{b]}\mathscr{R} \tag{2.21}$$

and

$$\mathscr{R}_{ab} := \frac{1}{2}\phi^{-1/2}\left(\nabla_{(a}\nabla_{b)} - \frac{1}{D}\eta_{ab}\Box\right)\phi^{-1/2} + \frac{D-1}{D(D-2)}\eta_{ab}\phi^{-(D+2)/4}\Box\phi^{(D-2)/4}\,. \tag{2.22}$$

Here we have introduced the conformal d'Alembert operator $\Box := \nabla^a\nabla_a$. Upon degauging and imposing the gauge conditions $b_a = 0$ and $\phi = 1$, one finds $\mathscr{R}_{ab}{}^{cd}$ corresponds to the Lorentz curvature $\mathcal{R}_{ab}{}^{cd}$.

In what follows we will specialize to the six dimensional case. We will find that all conformal gravity invariants can be constructed as

$$I = \int \mathrm{d}^6x\, e\, L\,, \quad K_a L = 0\,, \quad \mathbb{D}L = 6L\,, \tag{2.23}$$

where L is a function of C_{abcd}, its covariant derivatives and possibly a compensator ϕ (but with I possessing no dependence on ϕ).

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2.2 The C^3 invariants

Taking into account the symmetries of the Weyl tensor there are two inequivalent ways of contracting indices in the product of three Weyl tensors. These are as follows:

$$L^{(1)}_{C^3} := C_{abcd}C^{aefd}C_e{}^{bc}{}_f\,, \quad L^{(2)}_{C^3} := C_{abcd}C^{cdef}C_{ef}{}^{ab}\,. \tag{2.24}$$

These lead to two inequivalent invariants $I^{(\mathrm{i})}_{C^3} := \int \mathrm{d}^6x\, e\, L^{(\mathrm{i})}_{C^3}$, $\mathrm{i} = 1, 2$.

It is worth noting that a special combination of the above invariants can be written in the following form:

$$-\frac{1}{8}\varepsilon^{abcdef}\varepsilon_{a'b'c'd'e'f'}C_{ab}{}^{a'b'}C_{cd}{}^{c'd'}C_{ef}{}^{e'f'} = 4L^{(2)}_{C^3} + 8L^{(1)}_{C^3}\,. \tag{2.25}$$

It will turn out that it is precisely this combination that permits a supersymmetric generalization.

2.3 The $C\Box C$ invariant

Considering the product of two Weyl tensors with two covariant derivatives, one finds the following primary

$$L_{C\Box C} := C^{abcd}\Box C_{abcd} + \frac{1}{2}\nabla_e C_{abcd}\nabla^e C^{abcd} + \frac{8}{9}\nabla^d C_{abcd}\nabla_e C^{abce}\,, \tag{2.26}$$

which leads to the corresponding invariant $I_{C\Box C} = \int \mathrm{d}^6x\, e\, L_{C\Box C}$.

Making use of the identity

$$\begin{aligned} &C^{abcd}\Box C_{abcd} + \frac{1}{2}\nabla_e C_{abcd}\nabla^e C^{abcd} + \frac{8}{9}\nabla^d C_{abcd}\nabla_e C^{abce} \\ &= \frac{1}{6}C^{abcd}\Box C_{abcd} + \frac{1}{2}\nabla_e\left(C_{abcd}\nabla^e C^{abcd} + \frac{16}{9}C^{abce}\nabla^d C_{abcd}\right) - \frac{4}{3}L^{(1)}_{C^3} + \frac{1}{3}L^{(2)}_{C^3} \end{aligned} \tag{2.27}$$

and upon degauging (and removing a total derivative) one finds

$$I_{C\Box C} = \frac{1}{6}\int \mathrm{d}^6x\, e\, \left[C^{abce}\left(\delta^f_e\mathcal{D}^2 - 4\mathcal{R}_e{}^f + \frac{6}{5}\delta^f_e\mathcal{R}\right)C_{abcf} - 8L^{(1)}_{C^3} + 2L^{(2)}_{C^3}\right]\,, \tag{2.28}$$

where $\mathcal{D}^2 := \mathcal{D}^a\mathcal{D}_a$.

2.4 The Euler invariant

The Euler invariant may be constructed most easily in the gauge $b_a = 0$. In this gauge we define the Euler invariant as

$$\begin{aligned} \mathcal{E}_6 &:= -\frac{1}{8}\varepsilon^{abcdef}\varepsilon_{a'b'c'd'e'f'}\mathcal{R}_{ab}{}^{a'b'}\mathcal{R}_{cd}{}^{c'd'}\mathcal{R}_{ef}{}^{e'f'} \\ &= 4L^{(2)}_{C^3} + 8L^{(1)}_{C^3} - 6C^{abcd}C_{abce}\mathcal{R}_d{}^e + \frac{6}{5}C^{abcd}C_{abcd}\mathcal{R} \\ &\quad + 3C_{abcd}\mathcal{R}^{bd}\mathcal{R}^{ac} + \frac{3}{2}\mathcal{R}_a{}^b\mathcal{R}_b{}^c\mathcal{R}_c{}^a - \frac{27}{20}\mathcal{R}^{ab}\mathcal{R}_{ab}\mathcal{R} + \frac{21}{100}\mathcal{R}^3\,. \end{aligned} \tag{2.29}$$

Although one can use the above expression, we will instead look for an alternative description for the Euler invariant that is manifestly primary.

To begin with one can show that the following field[9]

$$E_6 := \left(\Box^3 - \frac{8}{3}(\nabla^b\nabla^d C_{abcd})\nabla^a\nabla^c\right)\ln\phi \tag{2.30}$$

is primary. Furthermore, the corresponding invariant

$$I_{\text{Euler}} := \int \mathrm{d}^6x\, e\, E_6 \tag{2.31}$$

does not actually depend on the compensator. To see this we make a reparametrization

$$\phi \to e^{-\sigma}\phi\,, \quad \mathbb{D}\sigma = 0\,, \tag{2.32}$$

which induces the shift

$$E_6 \to E_6 - \left(\Box^3 - \frac{8}{3}(\nabla^b\nabla^d C_{abcd})\nabla^a\nabla^c\right)\sigma\,. \tag{2.33}$$

At this point it is tempting to think that the term involving $\Box^3\sigma$ is a total derivative. However, integration of ∇_a is complicated by the presence of the special conformal connection and it is usually easier to work in the gauge $b_a = 0$ to arrange a total derivative. We now proceed to do this and show that E_6 shifts by a total derivative under the reparametrization (2.32).

In the gauge $b_a = 0$ we find the following results:

$$-\frac{8}{3}(\nabla^b\nabla^d C_{abcd})\nabla^a\nabla^c\sigma = \frac{32}{3}\text{ʃ}^{ac}(\mathcal{D}^b\sigma)\mathcal{D}^d C_{abcd} + \text{total derivative}\,, \tag{2.34a}$$

$$\Box^3\sigma = -\frac{32}{3}\text{ʃ}^{ac}(\mathcal{D}^b\sigma)\mathcal{D}^d C_{abcd} + \text{total derivative}\,, \tag{2.34b}$$

where we made use of the identities

$$\mathcal{D}_{[a}f_{b]c} = \frac{1}{2}R(K)_{abc} = \frac{1}{12}\nabla^d C_{abcd}\,, \quad \mathcal{D}_a\text{ʃ}_b{}^b = \mathcal{D}_b\text{ʃ}_a{}^b\,. \tag{2.35}$$

It is now straightforward to see that the shift in (2.33) is a total derivative and I_{Euler} is invariant under reparametrizations of ϕ.

Since I_{Euler} does not depend on ϕ, we are free to set $\phi = 1$, and since this condition breaks dilatation symmetry it is natural to work in the gauge $b_a = 0$. To do this consistently one must first extract the special conformal connection as in (2.14) before imposing the gauge conditions $\phi = 1$ and $b_a = 0$. Non-trivial terms survive which derive from where the dilatation generator acts on $\ln\phi$. One finds the following:

$$\begin{aligned}-\frac{8}{3}(\nabla^b\nabla^d C_{abcd})\nabla^a\nabla^c\ln\phi = {}& -2L_{C\Box C} - 2C^{abce}C_{abcd}\mathcal{R}^d{}_e + \frac{2}{5}C^{abcd}C_{abcd}\mathcal{R} \\ & + C_{abcd}\mathcal{R}^{bd}\mathcal{R}^{ac} - 4L^{(1)}_{C^3} + L^{(2)}_{C^3} + \text{total derivative}\,,\end{aligned} \tag{2.36}$$

$$\begin{aligned}\Box^3\ln\phi = {}& \frac{1}{2}\mathcal{R}_a{}^b\mathcal{R}_b{}^c\mathcal{R}_c{}^a - \frac{9}{20}\mathcal{R}^{ab}\mathcal{R}_{ab}\mathcal{R} + \frac{7}{100}\mathcal{R}^3 \\ & + \text{total derivative}\,.\end{aligned} \tag{2.37}$$

[9] This can be compared with the result in [83] for primary covariants in six dimensions.

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Finally, it follows from the above that

$$E_6 = \frac{1}{3}\mathcal{E}_6 - \frac{20}{3}L^{(1)}_{C^3} - \frac{1}{3}L^{(2)}_{C^3} - 2L_{C\Box C} + \text{total derivatives}\,. \tag{2.38}$$

Interestingly, we find that besides the construction (2.30) containing the Euler invariant $\mathcal{E}_6$, E_6 also involves the other conformal invariants.

3 $\mathcal{N} = (1,0)$ conformal superspace

Conformal superspace in lower dimensions [53–56] possesses the following key properties: (i) it gauges the entire superconformal algebra; (ii) the curvature and torsion tensors may be expressed in terms of a single primary superfield; and (iii) the algebra obeys the same basic constraints as those of super Yang-Mills theory. In this section, as in the lower dimensional cases, we will make use of these properties to develop the conformal superspace formulation for $\mathcal{N} = (1,0)$ conformal supergravity in six dimensions. We will firstly give the superconformal algebra and describe the geometric setup for conformal superspace. We then constrain the geometry to describe conformal supergravity by constraining its covariant derivative algebra to be expressed in terms of a single primary superfield, the super-Weyl tensor.

3.1 The superconformal algebra

The 6D $\mathcal{N} = (1,0)$ superconformal algebra naturally originates as the algebra of Killing supervector fields of 6D $\mathcal{N} = (1,0)$ Minkowski superspace [84], see appendix B for the technical details. Below we simply summarize the (anti-)commutation relations of generators corresponding to the superconformal algebra.

The bosonic part of the 6D $\mathcal{N} = (1,0)$ superconformal algebra contains the translation (P_a), Lorentz (M_{ab}), special conformal (K_a), dilatation ($\mathbb{D}$) and SU(2) generators (J_{ij}), where $a, b = 0, 1, 2, 3, 4, 5$ and $i, j = \underline{1}, \underline{2}$. Their algebra is

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c}\,, \tag{3.1a}$$

$$[M_{ab}, P_c] = 2\eta_{c[a}P_{b]}\,, \qquad [\mathbb{D}, P_a] = P_a\,, \tag{3.1b}$$

$$[M_{ab}, K_c] = 2\eta_{c[a}K_{b]}\,, \qquad [\mathbb{D}, K_a] = -K_a\,, \tag{3.1c}$$

$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab}\,, \tag{3.1d}$$

$$[J^{ij}, J^{kl}] = \varepsilon^{k(i}J^{j)l} + \varepsilon^{l(i}J^{j)k}\,, \tag{3.1e}$$

with all other commutators vanishing. The $\mathcal{N} = (1,0)$ superconformal algebra is obtained by extending the translation generator to $P_A = (P_a, Q^i_\alpha)$ and the special conformal generator to $K^A = (K^a, S^\alpha_i)$.[10] The fermionic generator Q^i_α obeys the algebra

$$\{Q^i_\alpha, Q^j_\beta\} = -2\mathrm{i}\varepsilon^{ij}(\gamma^c)_{\alpha\beta}P_c\,, \quad [Q^i_\alpha, P_a] = 0\,, \qquad [\mathbb{D}, Q^i_\alpha] = \frac{1}{2}Q^i_\alpha\,, \tag{3.2a}$$

$$[M_{ab}, Q^k_\gamma] = -\frac{1}{2}(\gamma_{ab})_\gamma{}^\delta Q^k_\delta\,, \qquad [J^{ij}, Q^k_\alpha] = \varepsilon^{k(i}Q^{j)}_\alpha\,. \tag{3.2b}$$

[10] For our spinor conventions and notation we refer the reader to appendix A.

while the generator S_i^α obeys the algebra

$$\{S_i^\alpha, S_j^\beta\} = -2\mathrm{i}\varepsilon_{ij}(\tilde{\gamma}^c)^{\alpha\beta}K_c\,, \quad [S_i^\alpha, K_a] = 0\,, \qquad [\mathbb{D}, S_i^\alpha] = -\frac{1}{2}S_i^\alpha\,, \tag{3.3a}$$

$$[M_{ab}, S_k^\gamma] = \frac{1}{2}(\gamma_{ab})_\delta{}^\gamma S_k^\delta\,, \qquad [J^{ij}, S_k^\alpha] = \delta_k^{(i}S_\alpha^{j)}\,. \tag{3.3b}$$

Finally, the (anti-)commutators of K^A with P_A are

$$[K_a, Q_\alpha^i] = -\mathrm{i}(\gamma_a)_{\alpha\beta}S^{\beta i}\,, \quad [S_i^\alpha, P_a] = -\mathrm{i}(\tilde{\gamma}_a)^{\alpha\beta}Q_{\beta i}\,, \tag{3.4a}$$

$$\{S_i^\alpha, Q_\beta^j\} = 2\delta_\beta^\alpha\delta_i^j\mathbb{D} - 4\delta_i^j M_\beta{}^\alpha + 8\delta_\beta^\alpha J_i{}^j\,, \tag{3.4b}$$

where we introduced $M_\alpha{}^\beta = -\frac{1}{4}(\gamma^{ab})_\alpha{}^\beta M_{ab}$. Note that $M_\alpha{}^\beta$ acts on Q_γ^k and S_k^γ as follows

$$[M_\alpha{}^\beta, Q_\gamma^k] = -\delta_\gamma^\beta Q_\alpha^k + \frac{1}{4}\delta_\alpha^\beta Q_\gamma^k\,, \quad [M_\alpha{}^\beta, S_k^\gamma] = \delta_\alpha^\gamma S_k^\beta - \frac{1}{4}\delta_\alpha^\beta S_k^\gamma\,. \tag{3.5}$$

3.2 Gauging the superconformal algebra

To perform the gauging of the superconformal algebra we follow closely the approach given in [53–56]. Below we will give the salient details of the geometry.

We introduce a curved 6D $\mathcal{N} = (1,0)$ superspace $\mathcal{M}^{6|8}$ parametrized by local bosonic (x) and fermionic coordinates (θ_i), $z^M = (x^m,\ \theta_i^\mu)$, where $m = 0,1,2,3,4,5$, $\mu = 1,\cdots,4$ and $i = \underline{1},\underline{2}$. We associate with each generator $X_{\underline{a}} = (M_{ab}, J_{ij}, \mathbb{D}, S_k^\gamma, K^c)$ a connection one-form $\omega^{\underline{a}} = (\Omega^{ab}, \Phi^{ij}, B, \mathfrak{F}_\gamma^k, \mathfrak{F}_c) = \mathrm{d}z^M\omega_M{}^{\underline{a}}$ and with P_A the vielbein $E^A = (E_i^\alpha, E^a)$. They are used to construct the covariant derivatives, which have the form

$$\nabla_A = E_A - \frac{1}{2}\Omega_A{}^{ab}M_{ab} - \Phi_A{}^{kl}J_{kl} - B_A\mathbb{D} - \mathfrak{F}_{AB}K^B\,. \tag{3.6}$$

Here $E_A = E_A{}^M\partial_M$ is the inverse vielbein. The action of the generators on the covariant derivatives resembles that for the P_A generators given in (3.2).

The supergravity gauge group is generated by local transformations of the form

$$\delta_\mathcal{K}\nabla_A = [\mathcal{K}, \nabla_A]\,, \tag{3.7}$$

where $\mathcal{K} = \xi^C\nabla_C + \frac{1}{2}\Lambda^{cd}M_{cd} + \Lambda^{kl}J_{kl} + \sigma\mathbb{D} + \Lambda_A K^A$, and the gauge parameters satisfy natural reality conditions. In applying eq. (3.7), one interprets the following

$$\nabla_A\xi^B := E_A\xi^B + \omega_A{}^{\underline{c}}\xi^D f_{D\underline{c}}{}^B\,, \quad \nabla_A\Lambda^{\underline{b}} := E_A\Lambda^{\underline{b}} + \omega_A{}^{\underline{c}}\xi^D f_{D\underline{c}}{}^{\underline{b}} + \omega_A{}^{\underline{c}}\Lambda^{\underline{d}}f_{\underline{dc}}{}^{\underline{b}}\,, \tag{3.8}$$

where the structure constants are defined as

$$[X_{\underline{a}}, X_{\underline{b}}\} = -f_{\underline{ab}}{}^{\underline{c}}X_{\underline{c}}\,, \quad [X_{\underline{a}}, \nabla_B\} = -f_{\underline{a}B}{}^C\nabla_C - f_{\underline{a}B}{}^{\underline{c}}X_{\underline{c}}\,. \tag{3.9}$$

The covariant derivatives satisfy the (anti-)commutation relations

$$\begin{aligned}[\nabla_A, \nabla_B\} = &-T_{AB}{}^C\nabla_C - \frac{1}{2}R(M)_{AB}{}^{cd}M_{cd} - R(J)_{AB}{}^{kl}J_{kl}\\ &- R(\mathbb{D})_{AB}\mathbb{D} - R(S)_{AB}{}^k_\gamma S_k^\gamma - R(K)_{ABc}K^c\,,\end{aligned} \tag{3.10}$$

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where the torsion and curvature tensors are given by

$$T^a = \mathrm{d}E^a + E^b \wedge \Omega_b{}^a + E^a \wedge B\,, \tag{3.11a}$$

$$T_i^\alpha = \mathrm{d}E_i^\alpha + E_i^\beta \wedge \Omega_\beta{}^\alpha + \frac{1}{2}E_i^\alpha \wedge B - E^{\alpha j} \wedge \Phi_{ji} - \mathrm{i}\, E^c \wedge \mathfrak{F}_{\beta i}(\tilde{\gamma}_c)^{\alpha\beta}\,, \tag{3.11b}$$

$$R(\mathbb{D}) = \mathrm{d}B + 2E^a \wedge \mathfrak{F}_a + 2E_i^\alpha \wedge \mathfrak{F}_\alpha^i\,, \tag{3.11c}$$

$$R(M)^{ab} = \mathrm{d}\Omega^{ab} + \Omega^{ac} \wedge \Omega_c{}^b - 4E^{[a} \wedge \mathfrak{F}^{b]} + 2E_j^\alpha \wedge \mathfrak{F}_\beta^j(\gamma^{ab})_\alpha{}^\beta\,, \tag{3.11d}$$

$$R(J)^{ij} = \mathrm{d}\Phi^{ij} - \Phi^{k(i} \wedge \Phi^{j)}{}_k - 8E^{\alpha(i} \wedge \mathfrak{F}_\alpha^{j)}\,, \tag{3.11e}$$

$$R(K)^a = \mathrm{d}\mathfrak{F}^a + \mathfrak{F}^b \wedge \Omega_b{}^a - \mathfrak{F}^a \wedge B - \mathrm{i}\mathfrak{F}_\alpha^k \wedge \mathfrak{F}_{\beta k}(\tilde{\gamma}^a)^{\alpha\beta}\,, \tag{3.11f}$$

$$R(S)_\alpha^i = \mathrm{d}\mathfrak{F}_\alpha^i - \mathfrak{F}_\beta^i \wedge \Omega_\alpha{}^\beta - \frac{1}{2}\mathfrak{F}_\alpha^i \wedge B - \mathfrak{F}_\alpha^j \wedge \Phi_j{}^i - \mathrm{i}E^{\beta i} \wedge \mathfrak{F}^c(\gamma_c)_{\alpha\beta}\,. \tag{3.11g}$$

The covariant derivatives satisfy the Bianchi identities

$$0 = [\nabla_A, [\nabla_B, \nabla_C\}\} + (\text{graded cyclic permutations})\,. \tag{3.12}$$

A superfield U is said to be *primary* if it is annihilated by the special conformal generators, $K^A U = 0$. From the algebra (3.3), we see that if a superfield is annihilated by S-supersymmetry it is necessarily primary. The superfield U is said to have dimension (or Weyl weight) Δ if $\mathbb{D}U = \Delta U$.

3.3 Conformal supergravity

In the conformal superspace approach to supergravity in four [53, 54], three [55] and five dimensions [56], the entire covariant derivative algebra may be expressed in terms of a single primary superfield: the super-Weyl tensor for $D > 3$ and the super Cotton tensor for $D = 3$. In six dimensions we will look for a similar solution in terms of a single primary superfield, the super-Weyl tensor [50].

In the lower dimensional cases the appropriate constraints to describe conformal supergravity were such that the covariant derivative algebra obeyed the same constraints as the super Yang-Mills theory. Guided by the structure of 6D $\mathcal{N} = (1,0)$ super Yang-Mills theory [85–88], we constrain the covariant derivative algebra as

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = -2\mathrm{i}\varepsilon^{ij}(\gamma^a)_{\alpha\beta}\nabla_a\,, \tag{3.13a}$$

$$\left[\nabla_a, \nabla_\alpha^i\right] = (\gamma_a)_{\alpha\beta}\mathcal{W}^{\beta i}\,, \tag{3.13b}$$

where $\mathcal{W}^{\alpha i}$ is some primary dimension 3/2 operator taking values in the superconformal algebra. The Bianchi identities give the commutator

$$[\nabla_a, \nabla_b] = -\frac{\mathrm{i}}{8}(\gamma_{ab})_\alpha{}^\beta\{\nabla_\beta^k, \mathcal{W}_k^\alpha\} \tag{3.14}$$

and the additional constraints

$$\{\nabla_\alpha^{(i}, \mathcal{W}^{\beta j)}\} = \frac{1}{4}\delta_\alpha^\beta\{\nabla_\gamma^{(i}, \mathcal{W}^{\gamma j)}\}\,, \quad \{\nabla_\gamma^k, \mathcal{W}_k^\gamma\} = 0\,. \tag{3.15}$$

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We constrain the form of the operator $\mathcal{W}^{\alpha i}$ to be

$$\mathcal{W}^{\alpha i} = W^{\alpha\beta}\nabla^i_\beta + \frac{1}{2}\mathcal{W}(M)^{\alpha i a b}M_{ab} + \mathcal{W}(J)^{\alpha i j k}J_{jk} + \mathcal{W}(\mathbb{D})^{\alpha i}\mathbb{D} + \mathcal{W}(K)^{\alpha i}{}_B K^B\,, \quad (3.16)$$

where $W^{\alpha\beta}$ is the super-Weyl tensor [50] which is a symmetric primary superfield of dimension 1. One can show that the Bianchi identities (3.15) are identically satisfied for

$$\begin{aligned}\mathcal{W}^{\alpha i} = {}& W^{\alpha\beta}\nabla^i_\beta + \nabla^i_\gamma W^{\alpha\beta}M_\beta{}^\gamma - \frac{1}{4}\nabla^i_\gamma W^{\beta\gamma}M_\beta{}^\alpha + \frac{1}{2}\nabla_{\beta j}W^{\alpha\beta}J^{ij} + \frac{1}{8}\nabla^i_\beta W^{\alpha\beta}\mathbb{D}\\ & - \frac{1}{16}\nabla^j_\beta\nabla^i_\gamma W^{\alpha\gamma}S^\beta_j + \frac{\mathrm{i}}{2}\nabla_{\beta\gamma}W^{\gamma\alpha}S^{\beta i}\\ & - \frac{1}{12}(\gamma^{ab})_\beta{}^\gamma\nabla_b\left(\nabla^i_\gamma W^{\beta\alpha} - \frac{1}{2}\delta^\alpha_\gamma\nabla^i_\delta W^{\beta\delta}\right)K_a \end{aligned} \quad (3.17)$$

provided $W^{\alpha\beta}$ satisfies

$$\nabla^{(i}_\alpha\nabla^{j)}_\beta W^{\gamma\delta} = -\delta^{(\gamma}_{[\alpha}\nabla^{(i}_{\beta]}\nabla^{j)}_\rho W^{\delta)\rho}\,, \quad (3.18a)$$

$$\nabla^k_\alpha\nabla_{\gamma k}W^{\beta\gamma} - \frac{1}{4}\delta^\beta_\alpha\nabla^k_\gamma\nabla_{\delta k}W^{\gamma\delta} = 8\mathrm{i}\nabla_{\alpha\gamma}W^{\gamma\beta}\,. \quad (3.18b)$$

It will be useful to introduce the dimension 3/2 superfields

$$X^{k\alpha\beta}_\gamma = -\frac{\mathrm{i}}{4}\nabla^k_\gamma W^{\alpha\beta} - \delta^{(\alpha}_\gamma X^{\beta)k}\,, \quad X^{\alpha i} := -\frac{\mathrm{i}}{10}\nabla^i_\beta W^{\alpha\beta}\,, \quad (3.19)$$

and the following higher dimension descendant superfields constructed from spinor derivatives of $W^{\alpha\beta}$:

$$Y_\alpha{}^{\beta ij} := -\frac{5}{2}\left(\nabla^{(i}_\alpha X^{\beta j)} - \frac{1}{4}\delta^\beta_\alpha\nabla^{(i}_\gamma X^{\gamma j)}\right) = -\frac{5}{2}\nabla^{(i}_\alpha X^{\beta j)}\,, \quad (3.20a)$$

$$Y := \frac{1}{4}\nabla^k_\gamma X^\gamma_k\,, \quad (3.20b)$$

$$Y_{\alpha\beta}{}^{\gamma\delta} := \nabla^k_{(\alpha}X_{\beta)k}{}^{\gamma\delta} - \frac{1}{6}\delta^{(\gamma}_\beta\nabla^k_\rho X_{\alpha k}{}^{\delta)\rho} - \frac{1}{6}\delta^{(\gamma}_\alpha\nabla^k_\rho X_{\beta k}{}^{\delta)\rho}\,. \quad (3.20c)$$

Note that $X^{k\alpha\beta}_\gamma$ is traceless, $Y_\alpha{}^{\beta\,ij}$ is symmetric in its SU(2) indices and traceless in its spinor indices, and $Y_{\alpha\beta}{}^{\gamma\delta}$ is separately symmetric in its upper and lower spinor indices and traceless.

One can check that only the superfields (3.20) together with (3.19) and their vector derivatives appear upon taking successive spinor derivatives of $W^{\alpha\beta}$. Specific relations we

will need later are given below:

$$\nabla^i_\alpha X^{\beta j} = -\frac{2}{5} Y_\alpha{}^{\beta ij} - \frac{2}{5}\varepsilon^{ij}\nabla_{\alpha\gamma} W^{\gamma\beta} - \frac{1}{2}\varepsilon^{ij}\delta^\beta_\alpha Y\,, \tag{3.21a}$$

$$\begin{aligned}\nabla^i_\alpha X^{j\gamma\delta}_\beta = {}& \frac{1}{2}\delta^{(\gamma}_\alpha Y_\beta{}^{\delta)ij} - \frac{1}{10}\delta^{(\gamma}_\beta Y_\alpha{}^{\delta)ij} - \frac{1}{2}\varepsilon^{ij} Y_{\alpha\beta}{}^{\gamma\delta} - \frac{1}{4}\varepsilon^{ij}\nabla_{\alpha\beta} W^{\gamma\delta} \\ & + \frac{3}{20}\varepsilon^{ij}\delta^{(\gamma}_\beta \nabla_{\alpha\rho} W^{\delta)\rho} - \frac{1}{4}\varepsilon^{ij}\delta^{(\gamma}_\alpha \nabla_{\beta\rho} W^{\delta)\rho}\,,\end{aligned} \tag{3.21b}$$

$$\nabla^i_\alpha Y = -2\mathrm{i}\nabla_{\alpha\beta} X^{\beta i}\,, \tag{3.21c}$$

$$\begin{aligned}\nabla^k_\gamma Y_\alpha{}^{\beta ij} = {}& \frac{2}{3}\varepsilon^{k(i}\Big(- 8\mathrm{i}\nabla_{\gamma\delta} X^{j)\delta\beta}_\alpha - 4\mathrm{i}\nabla_{\alpha\delta} X^{j)\delta\beta}_\gamma + 3\mathrm{i}\nabla_{\gamma\alpha} X^{\beta j)} \\ & + 3\mathrm{i}\delta^\beta_\gamma \nabla_{\alpha\delta} X^{\delta j)} - \frac{3\mathrm{i}}{2}\delta^\beta_\alpha \nabla_{\gamma\delta} X^{\delta j)}\Big)\,,\end{aligned} \tag{3.21d}$$

$$\begin{aligned}\nabla^i_\epsilon Y_{\alpha\beta}{}^{\gamma\delta} = {}& -4\mathrm{i}\nabla_{\epsilon(\alpha} X^l_{\beta)}{}^{\gamma\delta} + \frac{4\mathrm{i}}{3}\delta^{(\gamma}_{(\alpha}\nabla_{\beta)\rho} X^{l\delta)\rho}_\epsilon + \frac{8\mathrm{i}}{3}\delta^{(\gamma}_{(\alpha}\nabla_{|\epsilon\rho|} X^l_{\beta)}{}^{\delta)\rho} \\ & + 8\mathrm{i}\delta^{(\gamma}_\epsilon \nabla_{\rho(\alpha} X^l_{\beta)}{}^{\delta)\rho}\,.\end{aligned} \tag{3.21e}$$

These equations guarantee that any number of spinor derivatives of $W^{\alpha\beta}$ can always be rewritten in terms of $W^{\alpha\beta}$, the superfields defined in (3.19) and (3.20), and their vector derivatives. The descendant superfields transform under S-supersymmetry as follows:

$$S^\alpha_i X^{j\gamma\delta}_\beta = -\mathrm{i}\,\delta^j_i\delta^\alpha_\beta W^{\gamma\delta} + \frac{2\mathrm{i}}{5}\delta^j_i\delta^{(\gamma}_\beta W^{\delta)\alpha}, \qquad S^\alpha_i X^{\beta j} = \frac{8\mathrm{i}}{5}\delta^j_i W^{\alpha\beta}\,, \tag{3.22a}$$

$$S^\gamma_k Y_\alpha{}^{\beta ij} = \delta^{(i}_k\Big(- 16 X^{j)\gamma\beta}_\alpha + 2\delta^\beta_\alpha X^{\gamma j)} - 8\delta^\gamma_\alpha X^{\beta j)}\Big)\,, \tag{3.22b}$$

$$S^\rho_j Y_{\alpha\beta}{}^{\gamma\delta} = 24\Big(\delta^\rho_{(\alpha} X_{\beta)j}{}^{\gamma\delta} - \frac{1}{3}\delta^{(\gamma}_{(\alpha} X_{\beta)j}{}^{\delta)\rho}\Big)\,, \quad S^\alpha_i Y = -4X^\alpha_i\,. \tag{3.22c}$$

Expressing the covariant derivative algebra in terms of the descendant fields gives

$$\{\nabla^i_\alpha, \nabla^j_\beta\} = -2\mathrm{i}\varepsilon^{ij}(\gamma^a)_{\alpha\beta}\nabla_a\,, \tag{3.23a}$$

$$\begin{aligned}\big[\nabla_a, \nabla^i_\alpha\big] = {}& (\gamma_a)_{\alpha\beta}\Big(W^{\beta\gamma}\nabla^i_\gamma + 4\mathrm{i}\, X^{i\beta\gamma}_\delta M_\gamma{}^\delta - \frac{\mathrm{i}}{2} X^{\gamma i} M_\gamma{}^\beta - 5\mathrm{i}\, X^\beta_j J^{ij} + \frac{5\mathrm{i}}{4} X^{\beta i}\mathbb{D} \\ & + \frac{\mathrm{i}}{4} Y_\gamma{}^{\beta ij} S^\gamma_j + \frac{\mathrm{i}}{4}\nabla_{\gamma\delta} W^{\delta\beta} S^{\gamma i} - \frac{5\mathrm{i}}{16} Y S^{\beta i} \\ & + \frac{\mathrm{i}}{3}(\gamma^{bc})_\delta{}^\gamma\left(\nabla_b X^{i\,\delta\beta}_\gamma - \frac{3}{4}\delta^\beta_\gamma \nabla_b X^{\delta i}\right) K_c\Big)\,.\end{aligned} \tag{3.23b}$$

An explicit expression for the remaining commutator

$$\begin{aligned}[\nabla_a, \nabla_b] = {}& -T_{ab}{}^\gamma_k \nabla^k_\gamma - T_{ab}{}^c\nabla_c - \frac{1}{2} R(M)_{ab}{}^{cd} M_{cd} - R(J)_{ab}{}^{kl} J_{kl} - R(\mathbb{D})_{ab}\mathbb{D} \\ & - R(S)_{ab}{}^k_\gamma S^\gamma_k - R(K)_{abc} K^c\end{aligned} \tag{3.24}$$

follows from the Bianchi identities. For completeness, we provide the torsion and curvature

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components below:

$$T_{ab}{}^{c} = -4W_{ab}{}^{c}\,, \tag{3.25a}$$

$$T_{ab}{}_{k}^{\gamma} = (\gamma_{ab})_{\beta}{}^{\alpha}\Big(X_{\alpha k}{}^{\beta\gamma} - \frac{3}{4}\delta_{\alpha}{}^{\gamma}X_{k}^{\beta}\Big)\,, \tag{3.25b}$$

$$R(M)_{ab}{}^{cd} = Y_{ab}{}^{cd} - \delta^{c}_{[a}\delta^{d}_{b]}Y - 4\nabla_{[a}W_{b]}{}^{cd} - 4\nabla^{f}W_{f[b}{}^{[d}\delta_{a]}{}^{c]}\,, \tag{3.25c}$$

$$R(J)_{ab}{}^{ij} = Y_{ab}{}^{ij}\,, \tag{3.25d}$$

$$R(\mathbb{D})_{ab} = -2\nabla^{c}W_{cab}\,, \tag{3.25e}$$

$$R(S)_{ab}{}_{\gamma}^{i} = \mathrm{i}(\gamma_{ab})_{\beta}{}^{\alpha}\left(\frac{3}{16}\nabla_{\alpha\gamma}X^{\beta i} - \frac{3}{16}\delta^{\beta}_{\gamma}\nabla_{\alpha\delta}X^{\delta i} - \frac{1}{6}\nabla_{\alpha\delta}X_{\gamma}^{i\,\beta\delta} - \frac{1}{3}\nabla_{\gamma\delta}X_{\alpha}^{i\,\beta\delta}\right), \tag{3.25f}$$

$$\begin{aligned}
R(K)_{abc} = {}& \frac{1}{8}(\gamma_{cd[a})_{\alpha\beta}\nabla_{b]}\nabla^{d}W^{\alpha\beta} + \frac{1}{4}\eta_{c[a}\nabla_{b]}Y + \frac{1}{24}(\gamma_{ab})_{\alpha}{}^{\beta}(\gamma_{cd})_{\gamma}{}^{\delta}\nabla^{d}Y_{\beta\delta}{}^{\alpha\gamma} \\
& + \frac{1}{24}W^{\alpha\beta}Y_{\alpha\beta}{}^{\gamma\delta}(\gamma_{abc})_{\gamma\delta} - \frac{1}{8}W^{\alpha\beta}Y_{\alpha\gamma}{}^{\delta\epsilon}(\gamma_{c})_{\beta\delta}(\gamma_{ab})_{\epsilon}{}^{\gamma} \\
& + \frac{15\mathrm{i}}{32}X^{\alpha k}X_{k}^{\beta}(\gamma_{abc})_{\alpha\beta} + \frac{5\mathrm{i}}{8}X^{\alpha k}X_{\alpha k}{}^{\beta\gamma}(\gamma_{abc})_{\beta\gamma} - \frac{5\mathrm{i}}{4}X^{\alpha k}X_{\beta k}{}^{\gamma\delta}(\gamma_{c})_{\alpha\gamma}(\gamma_{ab})_{\delta}{}^{\beta} \\
& + \frac{\mathrm{i}}{3}X_{\alpha}^{k\beta\gamma}X_{\beta k}{}^{\alpha\delta}(\gamma_{abc})_{\gamma\delta} + \mathrm{i}X_{\alpha k}{}^{\beta\gamma}X_{\beta k}{}^{\delta\epsilon}(\gamma_{c})_{\gamma\delta}(\gamma_{ab})_{\epsilon}{}^{\alpha} \\
& + \frac{5}{16}W^{\alpha\beta}(\gamma_{c})_{\beta\delta}(\gamma_{[a})_{\alpha\gamma}\nabla_{b]}W^{\gamma\delta} - \frac{1}{32}W^{\alpha\beta}(\gamma_{c})_{\alpha\gamma}(\gamma_{abd})_{\beta\delta}\nabla^{d}W^{\gamma\delta} \\
& + \frac{3}{32}W^{\alpha\beta}(\gamma_{abc})_{\beta\delta}\nabla_{\alpha\gamma}W^{\gamma\delta} + \frac{1}{16}W^{\alpha\beta}\eta_{c[a}(\gamma_{b]})_{\alpha\gamma}\nabla_{\beta\delta}W^{\gamma\delta}\,.
\end{aligned} \tag{3.25g}$$

The component structure of the supergravity multiplet described by this superspace geometry can be identified with the standard Weyl multiplet of $6D$ $\mathcal{N} = (1,0)$ conformal supergravity [29]. The details of this will be presented in a future paper. Here we mainly point out that the independent one-forms $e_{m}{}^{a}$, $\psi_{m}{}^{\alpha i}$, b_{m}, and $V_{m}{}^{ij}$ in that approach coincide (up to conventions) with the $\theta = 0$ parts of the superspace one-forms $E_{m}{}^{a}$, $E_{m}{}^{\alpha i}$, B_{m} and $\Phi_{m}{}^{ij}$, respectively. Similarly, the independent covariant fields T^{-}_{abc}, $\chi^{\alpha i}$, and D are given by the $\theta = 0$ parts of $W_{abc} = \frac{1}{8}(\gamma_{abc})_{\alpha\beta}W^{\alpha\beta}$, $X^{\alpha i}$, and Y. The other components of the super-Weyl tensor $W^{\alpha\beta}$ correspond to covariant curvatures; for example, the $\theta = 0$ part of $Y_{ab}{}^{cd}$ is the traceless part of $R(M)_{ab}{}^{cd}$, which is the supercovariant Weyl tensor.

3.4 Introducing a compensator

An alternative formulation of conformal supergravity was given in [50], which we will refer to as SU(2) superpace. The formulation does not gauge the entire superconformal algebra and instead may be thought of as a gauge fixed version of the formulation introduced in the previous sections. Instead of applying the method of degauging used in [54–56] we will make contact with SU(2) superspace by utilizing a compensator. Here we will develop the alternative approach advocated in lower dimensions in [72, 89], which makes clear how SU(2) superspace may be understood as conformal supergravity coupled to some compensator at the superspace level.

We introduce a primary superfield X of dimension 2,

$$\mathbb{D}X = 2X\,, \quad S_{i}^{\alpha}X = 0\,. \tag{3.26}$$

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The superfield can be used to furnish new spinor covariant derivatives,

$$\mathscr{D}^i_\alpha = X^{-\frac{1}{4}}\Big(\nabla^i_\alpha + \big(\nabla^i_\beta \ln X\big)\, M_\alpha{}^\beta - 2(\nabla^j_\alpha \ln X) J_j{}^i - \frac{1}{2}(\nabla^i_\alpha \ln X)\mathbb{D}\Big)\,. \tag{3.27}$$

The covariant derivatives have been constructed to take a primary superfield to another primary superfield of the same dimension. Note also that X is annihilated by $\mathscr{D}^i_\alpha$, $\mathscr{D}^i_\alpha X = 0$.

When acting on a primary superfield, the algebra of the covariant derivatives becomes[11]

$$\begin{aligned}\{\mathscr{D}^i_\alpha, \mathscr{D}^j_\beta\} \;=\; & -2\mathrm{i}\varepsilon^{ij}\mathscr{D}_{\alpha\beta} - 4\mathrm{i}\varepsilon^{ij}\mathscr{W}^{abc}(\gamma_a)_{\alpha\beta}M_{bc} - 4\mathrm{i}\varepsilon^{ij}\mathscr{N}^{abc}(\gamma_a)_{\alpha\beta}M_{bc} + 6\mathrm{i}\varepsilon^{ij}\mathscr{C}_{\alpha\beta}{}^{kl}J_{kl} \\ & +2\mathrm{i}\mathscr{C}_a{}^{ij}(\gamma^{abc})_{\alpha\beta}M_{bc} - 16\mathrm{i}\mathscr{N}_{\alpha\beta}J^{ij}\,,\end{aligned} \tag{3.28}$$

where

$$\mathscr{D}_{\alpha\beta} = -\frac{\mathrm{i}}{4}\{\mathscr{D}^k_\alpha, \mathscr{D}_{\beta k}\} - 2\mathscr{N}^{bcd}(\gamma_b)_{\alpha\beta}M_{cd} - 2\mathscr{W}^{bcd}(\gamma_b)_{\alpha\beta}M_{cd} + 3\mathscr{C}_{\alpha\beta}{}^{kl}J_{kl} \tag{3.29}$$

and we have introduced

$$\mathscr{C}_{\alpha\beta}{}^{ij} := -\frac{\mathrm{i}}{4}X^{-\frac{3}{2}}\nabla^{(i}_\alpha\nabla^{j)}_\beta X\,, \tag{3.30a}$$

$$\mathscr{N}_{\alpha\beta} := -\frac{\mathrm{i}}{16}X^{\frac{3}{2}}\nabla^k_{(\alpha}\nabla_{\beta)k}X^{-2}\,, \tag{3.30b}$$

$$\mathscr{W}^{\alpha\beta} := X^{-\frac{1}{2}}W^{\alpha\beta}\,. \tag{3.30c}$$

Here we have introduced $\mathscr{W}^{\alpha\beta}$ which is a rescaling of $W^{\alpha\beta}$ so that it is inert under dilatations. The superfields $\mathscr{C}_{\alpha\beta}{}^{ij}$ and $\mathscr{N}_{\alpha\beta}$ are the only dimensionless primary combinations involving two spinor derivatives acting on X. The super-Weyl transformations of [50] correspond to a reparametrization of the compensator superfield, $X \to X\, e^{-2\sigma}$.

4 An action principle for the supersymmetric C^3 invariant

Having developed conformal superspace in the previous section we are now in a position to address the problem of constructing conformal supergravity invariants. This will require an action principle capable of supporting such an invariant. In this section we expound such an action principle and show that it may be used to construct a supersymmetric C^3 invariant.

4.1 Flat superspace actions and their generalization

Before discussing curved superspace actions, it is useful to briefly review action principles with manifest $\mathcal{N} = (1,0)$ Poincaré supersymmetry. The simplest is the full superspace integral

$$S = \int \mathrm{d}^6x\, \mathrm{d}^8\theta\, \mathcal{L}\,, \tag{4.1}$$

where $\mathcal{L}$ is an unconstrained real superfield. Because the Grassmann coordinates $\theta^{\alpha i}$ are irreducible under the Lorentz and R-symmetry groups, there is no separate notion of chiral

[11] This agrees with the dimension 1 anticommutation relations of the covariant derivative algebra in [50].

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superspace as in four dimensions. To construct smaller superspaces involving a reduced set of θ's, additional structure is needed. The most well-known example is 6D $\mathcal{N} = (1,0)$ harmonic superspace [90, 91], $\mathbb{R}^{6|8} \times S^2$, where additional bosonic coordinates $u^{i\pm}$ are introduced to describe the coset space $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$.[12] Introducing a new basis for the Grassmann coordinates as $\theta^{\alpha\pm} := u_i^{\pm}\theta^{\alpha i}$, one may construct an invariant action

$$S = \int \mathrm{d}^6x\, \mathrm{d}u\, \mathrm{d}^4\theta^+\, \mathcal{L}^{+4} = \int \mathrm{d}^6x\, \mathrm{d}u\, (D^-)^4\mathcal{L}^{+4}|_{\theta=0}\,, \qquad D^+_\alpha \mathcal{L}^{+4} = 0\,, \tag{4.2}$$

where $D^\pm_\alpha := u^\pm_i D^i_\alpha$ and $\mathrm{d}u$ is the invariant measure for SU(2). A special case is when $\mathcal{L}^{+4}$ is an $\mathcal{O}(4)$ multiplet C^{+4} with simple quartic dependence on the harmonics, $C^{+4} \equiv u^+_i u^+_j u^+_k u^+_l C^{ijkl}$. Its component action is given by[13]

$$\begin{aligned} S &= \int \mathrm{d}^6x\, \mathrm{d}u\, (D^-)^4 C^{+4}|_{\theta=0} = \frac{1}{5}\int \mathrm{d}^6x\, (D^4)_{ijkl}C^{ijkl}|_{\theta=0}\,, \\ (D^4)^{ijkl} &:= -\frac{1}{96}\varepsilon^{\alpha\beta\gamma\delta} D^{(i}_\alpha D^j_\beta D^k_\gamma D^{l)}_\delta\,, \qquad D^{(i}_\alpha C^{jklp)} = 0\,. \end{aligned} \tag{4.3}$$

For the similar case of 4D $\mathcal{N} = 2$ supersymmetry, the $\mathcal{O}(4)$ multiplet and associated action were introduced by [94]. It is clear that any full superspace action can be rewritten in this way using $C^{+4} = (D^+)^4\mathcal{L}$. The converse is not always true within the family of local and gauge-invariant operators. More specifically, given an $\mathcal{O}(4)$ multiplet C^{+4}, there always exists a harmonic-independent potential $\mathcal{L}$ such that $C^{+4} = (D^+)^4\mathcal{L}$, as proved in appendix G of [56] in the 5D $\mathcal{N} = 1$ case. However, such a potential $\mathcal{L}$ cannot always be defined as a local gauge-invariant operator. A simple example is when the $\mathcal{O}(4)$ multiplet is the product of two $\mathcal{O}(2)$ multiplets.

Our task is to construct the conformal supergravity invariants, so a natural step would be to generalize the above actions to curved superspace and to choose the appropriate Lagrangians. Both in SU(2) superspace [50] and in conformal superspace, it is straightforward to generalize eq. (4.1) to

$$S = \int \mathrm{d}^6x\, \mathrm{d}^8\theta\, E\, \mathcal{L}\,, \tag{4.4}$$

where E is the Berezinian (or superdeterminant) of the supervielbein. In order to be invariant under the supergravity gauge transformations, $\mathcal{L}$ must be a conformal primary scalar superfield of dimension two. Unfortunately, there is no suitable Lagrangian that can be built directly from the covariant fields of the Weyl multiplet. Furthermore, there is no obvious way to generalize (4.3) without introducing a compensator field. The reason is C^{+4} should clearly have dimension four, but the analyticity condition $\nabla^+_\alpha C^{+4} = 0$ cannot be conformally invariant, assuming C^{+4} is a primary, unless C^{+4} has dimension eight. A natural step here would be to relax the assumption that C^{+4} is itself primary and instead consider it as a descendant of some other primary superfield. One could imagine a number

[12] This superspace is a natural extension of the 4D $\mathcal{N} = 2$ harmonic superspace [60, 92].

[13] One can also have an action principle with C^{ijkl} obeying the weaker condition $D^i_{(\alpha}D^j_{\beta)}C^{klpq} = 0$. This leads to the action discussed in eq. (4.72) of [93].

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of ways of doing this. In fact, we will discover an action principle in section 5 involving such a non-primary C^{+4}.

For the moment, however, we will follow a different approach and attempt to construct the actions as six-forms directly rather than as superspace integrals.

4.2 Primary closed six-forms in superspace

While supersymmetric actions are frequently realized as integrals over the full superspace or its invariant subspaces, there is an alternative construction involving the use of closed super D-forms [95–97].[14] For 6D $\mathcal{N} = (1,0)$ superspace, we introduce a closed six-form J

$$J = \frac{1}{6!}\mathrm{d}z^{M_6} \wedge \cdots \wedge \mathrm{d}z^{M_1}\, J_{M_1\cdots M_6}\,, \qquad \mathrm{d}J = 0\,. \tag{4.5}$$

(The closure condition is trivial on the spacetime $\mathcal{M}^6$ since there a six-form is a top form, but there are no top forms on the supermanifold $\mathcal{M}^{6|8}$ since $\mathrm{d}\theta_i^\mu$ commutes with itself.) Such a closed superform leads immediately to the action principle

$$S = \int_{\mathcal{M}^6} i^* J = \int \mathrm{d}^6x\, e\, {}^*J|_{\theta=0}\,, \qquad {}^*J := \frac{1}{6!}\varepsilon^{mnpqrs} J_{mnpqrs}\,, \tag{4.6}$$

where $i : \mathcal{M}^6 \to \mathcal{M}^{6|8}$ is the inclusion map and i^* is its pullback, the effect of which is to project $\theta_i^\mu = \mathrm{d}\theta_i^\mu = 0$. Closure of J guarantees that the action is invariant under general coordinate transformations of superspace.[15] In addition, the action must be invariant under all gauge transformations: for conformal supergravity, this includes the standard superconformal transformations, which form the subgroup $\mathcal{H}$. This implies that J must transform into an exact form

$$\delta_{\mathcal{H}} J = \mathrm{d}\Theta(\Lambda^{\underline{a}})\,, \quad \Lambda = \Lambda^{\underline{a}} X_{\underline{a}}\,. \tag{4.7}$$

A special case is when the closed six-form is itself invariant, $\delta_{\mathcal{H}} J = 0$. This implies that if one instead decomposes J in the tangent frame,

$$J = \frac{1}{6!} E^{A_6} \wedge \cdots \wedge E^{A_1} J_{A_1\cdots A_6}\,, \tag{4.8}$$

the components $J_{A_1\cdots A_6}$ transform covariantly and obey the covariant constraints

$$\nabla_{[A_1} J_{A_2\cdots A_7\}} + 3T_{[A_1A_2}{}^B J_{|B|A_3\cdots A_7\}} = 0\,. \tag{4.9}$$

In particular, their S and K transformations are given by

$$S_j^\beta J_{a_1\cdots a_n}{}^{i_1}_{\alpha_1}\cdots{}^{i_{6-n}}_{\alpha_{6-n}} = -\mathrm{i}\, n(\tilde\gamma_{[a_1})^{\beta\gamma} J_{\gamma j a_2\cdots a_n]}{}^{i_1}_{\alpha_1}\cdots{}^{i_{6-n}}_{\alpha_{6-n}}\,, \qquad K^b J_{A_1\cdots A_6} = 0\,. \tag{4.10}$$

Such superforms are called *primary*.

It follows from eq. (4.10) that the component of a primary superform with lowest dimension is a primary superfield, so it is natural to ask what primary constraints are

[14] The approach proves equivalent to the rheonomic formalism [98].

[15] Here we assume the general coordinate transformations are generated by a vector field $\xi = \xi^A E_A = \xi^M \partial_M$ which vanishes at the boundary of $\mathcal{M}^6$.

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compatible with the closure conditions (4.9). This general question was addressed by Arias et al. [93] using 6D SU(2) superspace [50], and we will arrive at similar results to theirs. First observe that the component of the superform J with lowest dimension (which we will refer to as the lowest component of the superform) cannot be a scalar without either that scalar being covariantly constant (which is forbidden by the superconformal algebra due to its non-vanishing dimension) or the superform being exact.[16] This means we have to allow for the possibility that the lowest component carries some Lorentz and SU(2) indices. We let the lowest component of the superform be directly constructed in terms of the primary superfield

$$A_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_m k_1\cdots k_p} = A_{(\alpha_1\cdots\alpha_n)}{}^{\beta_1\cdots\beta_m (k_1\cdots k_p)} \tag{4.11}$$

with dimension Δ. In analogy to the chiral action principle in 4D, we seek a primary constraint involving one spinor derivative with totally symmetrized SU(2) indices, $\nabla_\delta^{(l} A_{\alpha_1\cdots\alpha_n}{}^{\beta_1\cdots\beta_m k_1\cdots k_p)}$. Such constraints are natural: they appear in solving the first non-trivial Bianchi identity (if it is not identically satisfied) since the part symmetric in SU(2) indices cannot be countered by the term proportional to the superspace torsion. We will suppose further that

$$\nabla^{(l}_{(\alpha_1} A_{\alpha_2\cdots\alpha_{n+1})}{}^{\beta_1\cdots\beta_m k_1\cdots k_p)} - \text{traces} = 0\,, \tag{4.12}$$

where we subtract out all possible traces to render the result traceless in its spinor indices. Requiring the constraint to be primary implies

$$2\Delta + 3n + m - 4p = 0\,, \tag{4.13}$$

which can only have solutions for $2p \geq \Delta$. Notice that the upper Lorentz indices are not assumed to be symmetric, which generalizes some of the corresponding results of [93]. Remarkably, apart from the one degenerate case of the tensor multiplet, all known closed primary superforms have underlying primary superfields satisfying a constraint of the form (4.12) with the condition (4.13).

We now seek to find a primary closed superform to act as an action principle supporting a supersymmetric C^3 invariant. Since we will want to set the superfield to be cubic in $W^{\alpha\beta}$ and its spinor derivatives, the underlying superfield should satisfy $\Delta \geq 3 + \frac{p}{2}$. Considering all the possible ways of embedding such a superfield into a (non-exact) closed six form leads one to consider a primary dimension 9/2 superfield of the form $A_\alpha{}^{ijk}$ satisfying the constraint

$$\nabla^{(i}_{(\alpha} A_{\beta)}{}^{jkl)} = 0\,. \tag{4.14}$$

In fact, a superfield obeying this constraint was already used to construct a closed six-form in [93] in the context of 6D SU(2) superspace [50]; such a superfield also appeared in the

[16] This is unlike what happens in four dimensions, where one can construct the chiral action principle.

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context of the anomalous current multiplet [99, 100]. The resulting closed six-form is

$$J = \frac{1}{6!}E^{A_6}\wedge\cdots\wedge E^{A_1}J_{A_1\cdots A_6}\,, \tag{4.15a}$$

$$J_{abc\alpha^i_\beta{}^j_\gamma{}^k} = 3(\gamma_{abc})_{(\alpha\beta}A_{\gamma)}{}^{ijk}\,, \tag{4.15b}$$

$$J_{abcd\alpha^i_\beta{}^j} = -\frac{\mathrm{i}}{6}\varepsilon_{abcdef}(\gamma^{ef})_{(\alpha}{}^{\gamma}S_{\beta)\gamma}{}^{ij} - \frac{\mathrm{i}}{12}\varepsilon_{abcdef}(\gamma^{efg})_{\alpha\beta}(\tilde{\gamma}_g)^{\rho\eta}E_{\rho\eta}{}^{ij}\,, \tag{4.15c}$$

$$J_{abcde\alpha^i} = \frac{\mathrm{i}}{2}\varepsilon_{abcdef}(\tilde{\gamma}^f)^{\beta\gamma}(\Omega_{\beta\gamma,\alpha}{}^{i} + \Omega_{\alpha\beta,\gamma}{}^{i})\,, \tag{4.15d}$$

$$J_{abcdef} = -\varepsilon_{abcdef}F\,, \tag{4.15e}$$

and all other components vanish. Here we have introduced the descendant superfields

$$S_{\alpha\beta}{}^{ij} := \frac{3}{4}\nabla_{(\alpha k}A_{\beta)}{}^{ijk}\,, \quad E_{\alpha\beta}{}^{ij} := \frac{3}{4}\nabla_{[\alpha k}A_{\beta]}{}^{ijk}\,, \tag{4.16a}$$

$$\Omega_{\alpha\beta,\gamma}{}^{i} := \frac{\mathrm{i}}{16}\nabla_{[\alpha j}\nabla_{\beta]k}A_{\gamma}{}^{ijk} = \frac{\mathrm{i}}{16}\nabla_{\alpha j}\nabla_{\beta k}A_{\gamma}{}^{ijk}\,, \tag{4.16b}$$

$$F := \frac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta}\nabla_{\alpha i}\Omega_{\beta\gamma,\delta}{}^{i} = \frac{\mathrm{i}}{2^4 4!}\varepsilon^{\alpha\beta\gamma\delta}\nabla_{\alpha i}\nabla_{\beta j}\nabla_{\gamma k}A_{\delta}{}^{ijk}\,. \tag{4.16c}$$

Reality of the action implies that $\overline{A_\alpha{}^{ijk}} = A_{\alpha\,ijk}$, and similarly for its descendants, $\overline{E_{\alpha\beta}{}^{ij}} = E_{\alpha\beta\,ij}$, $\overline{S_{\alpha\beta}{}^{ij}} = S_{\alpha\beta\,ij}$, $\overline{\Omega_{\alpha\beta,\gamma}{}^{i}} = \Omega_{\alpha\beta,\gamma\,i}$, and $\overline{F} = F$. These transform under S-supersymmetry as follows:

$$S^\epsilon_m S_{\alpha\beta}{}^{ij} = -24\,\delta^\epsilon_{(\alpha}A_{\beta)}{}^{ij}{}_m\,, \tag{4.17a}$$

$$S^\epsilon_m E_{\alpha\beta}{}^{ij} = -18\,\delta^\epsilon_{[\alpha}A_{\beta]}{}^{ij}{}_m\,, \tag{4.17b}$$

$$S^\delta_l \Omega_{\alpha\beta,\gamma}{}^{k} = -4\mathrm{i}\,\delta^\delta_{[\alpha}S_{\beta]\gamma}{}^{i}{}_l - 4\mathrm{i}\,\delta^\delta_{[\alpha}E_{\beta]\gamma}{}^{i}{}_l + \frac{2\mathrm{i}}{3}\delta^\delta_\gamma E_{\alpha\beta}{}^{i}{}_l\,, \tag{4.17c}$$

$$S^\alpha_i F = -2\,\varepsilon^{\alpha\beta\gamma\delta}\Omega_{\beta\gamma,\delta i}\,. \tag{4.17d}$$

Making use of these results one can check that the superform (4.15) is primary.

It is worth mentioning that the closed six-form (4.15) may be derived by analogy with the construction of the closed four-form [101] which describes the chiral action in 4D $\mathcal{N}=2$ supergravity [102]. Ref. [101] considered the closed four-form $\omega = F\wedge F$, where F is the two-form field strength of an on-shell U(1) vector multiplet. Under certain assumptions on the vector multiplet, it was shown that all components of ω are expressed in terms of a single chiral $\mathcal{N}=2$ superfield W^2, with W the chiral field strength of the vector multiplet. In the 6D $\mathcal{N}=(1,0)$ case, one can consider the topological term $\mathrm{Tr}(\boldsymbol{F}\wedge\boldsymbol{F}\wedge\boldsymbol{F})$, where $\boldsymbol{F}$ is the two-form field strength of a YM multiplet, see appendix C. Rewriting the superform in terms of $A_\alpha{}^{ijk} \propto \varepsilon_{\alpha\beta\gamma\delta}\,\mathrm{Tr}\left(\boldsymbol{W}^{\beta(i}\boldsymbol{W}^{\gamma j}\boldsymbol{W}^{\delta k)}\right)$, where $\boldsymbol{W}^{\alpha i}$ is the field strength of the Yang-Mills supermultiplet, and throwing away a covariantly exact piece one uncovers the structure of the superform J.

4.3 The supersymmetric C^3 invariant

In order to describe the supersymmetric C^3 invariant it is now necessary to construct a composite $A_\alpha{}^{ijk}$ out of the super-Weyl tensor. Since the invariant must contain a C^3 term

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and since the Weyl tensor is directly constructed out of the spacetime projection of the superfield $Y_{\alpha\beta}{}^{\gamma\delta}$, the composite $A_\alpha{}^{ijk}$ must be at least cubic in $W^{\alpha\beta}$ and its descendants. Taking into account the constraints on $A_\alpha{}^{ijk}$ gives the following unique solution:

$$\begin{aligned} A_\alpha{}^{ijk} &= 5\mathrm{i}\varepsilon_{\alpha\beta\gamma\delta}X^{\beta(i}X^{\gamma j}X^{\delta k)} - 8\mathrm{i}\varepsilon_{\alpha\beta\gamma\delta}X^{\beta(i}X_{\alpha'}^{j}{}^{\gamma\beta'}X_{\beta'}^{k)\delta\alpha'} + \frac{64\mathrm{i}}{3}\varepsilon_{\alpha\beta\gamma\delta}X_{\alpha'}^{(i}{}^{\beta\beta'}X_{\beta'}^{j}{}^{\gamma\gamma'}X_{\gamma'}^{k)\delta\alpha'} \\ &\quad +4\varepsilon_{\alpha\beta\gamma\delta}Y_\rho{}^{\beta(ij}X_\eta^{k)\rho\gamma}W^{\eta\delta} - 3\varepsilon_{\alpha\beta\gamma\delta}Y_\rho{}^{\beta(ij}X^{\gamma k)}W^{\rho\delta}\,. \end{aligned} \tag{4.18}$$

In particular, one can check that the above superfield is primary and satisfies the constraint (4.14).

The component reduction (although tedious) is straightforward and may be carried out similarly as in [72]. Furthermore, one can readily verify that the action contains a C^3 term proportional to the combination (2.25). We leave the detailed analysis of the component action to a forthcoming paper.

4.4 Other invariants

A natural question that one may ask is whether other invariants may be constructed using the same action principle. Specifically, can we construct another primary composite $A_\alpha{}^{ijk}$ that is (for example) quadratic in the super-Weyl tensor $W^{\alpha\beta}$? Unfortunately, enumerating the possibilities it turns out that only the cubic solution (4.18) is possible. There are however certain composite primary superfields that one can construct at dimension 3. These are

$$H^{\alpha\beta\,ij} = W^{\gamma[\alpha}Y_\gamma{}^{\beta]ij} + 8\mathrm{i}\,X_\gamma{}^{\delta[\alpha(i}X_\delta{}^{\beta]\gamma j)} - \frac{5\mathrm{i}}{2}X^{[\alpha(i}X^{\beta]j)}\,, \tag{4.19a}$$

$$H^{\alpha\beta} = YW^{\alpha\beta} + \frac{2}{7}Y_{\gamma\delta}{}^{\alpha\beta}W^{\gamma\delta} - \frac{\mathrm{i}}{2}X^{\alpha k}X_k^\beta + \frac{8\mathrm{i}}{7}X_\gamma^{k\delta(\alpha}X_{\delta k}{}^{\beta)\gamma} + 4\mathrm{i}X^{\gamma k}X_{\gamma k}{}^{\alpha\beta}\,. \tag{4.19b}$$

It turns out that the first may be used to generate another action, which will be discussed in detail in the next section. Before moving on to the discussion there, it is worth illustrating the existence of the other action principle using the primary superform construction of this section.

The important property of eq. (4.19a) (besides being primary) is that it satisfies the differential constraint[17]

$$\nabla_\alpha^{(i}B^{\beta\gamma\,jk)} = -2\mathrm{i}\,\delta_\alpha^{[\beta}\Lambda^{\gamma]\,ijk}\,, \tag{4.20}$$

with $B^{\alpha\beta ij} = H^{\alpha\beta ij}$ for some non-primary $\Lambda^{\alpha ijk}$. One can check that it is not possible to construct a primary composite $A_\alpha{}^{ijk}$ directly from $B^{\alpha\beta ij}$ with various covariant derivatives only. Despite this one can construct a composite $A_\alpha{}^{ijk}$ out of $B^{\alpha\beta ij}$ with the use of a compensating supermultiplet. To demonstrate this we choose a compensating tensor multiplet Φ, which satisfies the constraint

$$\nabla_\alpha^{(i}\nabla_\beta^{j)}\Phi = 0\,. \tag{4.21}$$

[17] Notice that this constraint is a special case of eq. (4.12).

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Then using the results of section 3.4 (with $X = \Phi$), one can construct the following composite

$$A_\alpha{}^{ijk} = -\frac{1}{60}\Phi^{\frac{3}{4}}\mathscr{D}_{\alpha l}\mathscr{D}_\beta^{(i}\mathscr{D}_\gamma^{j}B^{\beta\gamma kl)} + 2\mathrm{i}\Phi^{\frac{3}{4}}\mathscr{N}_{\alpha\beta}\mathscr{D}_\gamma^{(i}B^{\beta\gamma jk)} - \frac{8\mathrm{i}}{3}\Phi^{\frac{3}{4}}(\mathscr{D}_\beta^{(i}\mathscr{N}_{\gamma\alpha})B^{\beta\gamma jk)} + \frac{2\mathrm{i}}{3}\Phi^{\frac{1}{4}}(\mathscr{D}_\gamma^{(i}W^{\beta\gamma})B_{\alpha\beta}{}^{jk)} + a\Phi^{\frac{3}{4}}\mathscr{D}_\alpha^{(i}\mathscr{D}_{\beta\gamma}B^{\beta\gamma jk)}\,. \tag{4.22}$$

The last term involves a free parameter a and generates an exact six-form, which may be removed. The composite $A_\alpha{}^{ijk}$ is primary and satisfies the differential constraint (4.14). As a result we can associate an action with any primary superfield satisfying eq. (4.20), and we therefore have an action principle based on $B^{\alpha\beta ij}$.

The action principle based on $B^{\alpha\beta ij}$, eq. (4.22), can be used immediately to describe certain invariants. If we take $B^{\alpha\beta ij} = H^{\alpha\beta ij}$, the component action will contain a $C\Box C$ term. One can also construct a unique higher-derivative $F\Box F$ action for a non-abelian gauge theory by taking

$$B^{\alpha\beta\,ij} = \mathrm{i}\,\mathrm{Tr}(\boldsymbol{W}^{\alpha(i}\boldsymbol{W}^{\beta j)})\,, \tag{4.23}$$

where $\boldsymbol{W}^{\alpha i}$ is the field strength of the 6D $\mathcal{N} = (1,0)$ Yang-Mills multiplet [85–88], see appendix C for details. The corresponding component action will contain a term of the form $\mathrm{Tr}(\boldsymbol{F}_{ab}\Box\boldsymbol{F}^{ab})$ upon integrating by parts.

It should be mentioned that in the rigid supersymmetric case the supersymmetric $F\Box F$ action was constructed in [2] within the harmonic superspace approach. Their result can also be recast as the $\mathcal{O}(4)$ multiplet action (4.3) with

$$C^{ijkl} \propto \mathrm{Tr}(\boldsymbol{X}^{(ij}\boldsymbol{X}^{kl)})\,, \tag{4.24}$$

where $\boldsymbol{X}^{ij}$ denotes the flat-superspace limit of the descendant (C.8). The interesting feature of the model proposed in [2] is that the operator $\boldsymbol{X}^{ij}$ is not a primary superfield, but the action (4.3) based on (4.24) is superconformal.

It is important to point out that the action principle based on $B^{\alpha\beta ij}$ may contain dependence on Φ. Although we do not explicitly show this here, we expect that the action principle will be independent of the compensator. In the the next section we show that such an action principle based on $B^{\alpha\beta ij}$ exists without the need to introduce any compensator.

Before moving on we would like to mention one more application of the action principle based on a composite $A_\alpha{}^{ijk}$. Let $V^{\alpha i}$ be a prepotential for the tensor multiplet,[18]

$$\Phi = \nabla_{\alpha i}V^{\alpha i}\,, \quad \nabla_\alpha^{(i}V^{\beta j)} = \frac{1}{4}\delta_\alpha^\beta\nabla_\delta^{(i}V^{\delta j)}\,, \quad K^A V^{\alpha i} = 0\,. \tag{4.25}$$

It is defined modulo gauge transformations of the form

$$V^{\alpha i} \to V^{\alpha i} + W^{\alpha i}\,, \tag{4.26}$$

[18]The prepotential for the tensor multiplet was introduced by Sokatchev in the framework of his harmonic-superspace formulation for 6D $\mathcal{N} = (1,0)$ supergravity [80]. More recently this prepotential has been described in SU(2) superspace in [50].

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where $W^{\alpha i}$ is the field strength of an abelian vector multiplet, see appendix C. Using $V^{\alpha i}$ one can construct the following primary composite

$$A_\alpha{}^{ijk} = \varepsilon_{\alpha\beta\gamma\delta} V^{\beta(i} B^{\gamma\delta jk)} \,. \tag{4.27}$$

It is simple to verify the differential constraint (4.14) by making use of (4.20) and (4.25). The action corresponding to the composite (4.27) is invariant under arbitrary gauge transformations (4.26) when $B^{\alpha\beta ij}$ is further constrained as

$$[\nabla_\alpha^{(i}, \nabla_{\beta k}] B^{\alpha\beta j)k} = -8\mathrm{i} \nabla_{\alpha\beta} B^{\alpha\beta ij} \,, \tag{4.28}$$

which imposes a constraint on $B^{\alpha\beta ij}$ to describe a closed 4-form [93]. Below we give two examples of gauge-invariant actions.

Our first example of a gauge-invariant action corresponds to the choice (4.23). In this case it is rather simple to see that a gauge transformation (4.26) shifts the invariant by a topological term and the invariant contains the term $\Phi \operatorname{Tr}(\boldsymbol{F}_{ab}\boldsymbol{F}^{ab})$. Thus the action describes the non-Abelian vector multiplet coupled to the dilaton Weyl multiplet. In the flat-superspace limit, the prepotential of the tensor compensator may be chosen as $V^{\alpha i} \propto \theta^{\alpha i}$. Then the top component (4.16c) of the closed six-form (4.15) becomes

$$F \propto D_{\alpha i} D_{\beta j} \operatorname{Tr}(\boldsymbol{W}^{\alpha(i} \boldsymbol{W}^{\beta j)}) \,, \tag{4.29}$$

which is the Lagrangian for the 6D $\mathcal{N} = (1,0)$ super Yang-Mills theory postulated in [88]. Here we derived this Lagrangian from a more general action principle.

Our second example, derives from the fact that the constraint (4.28) is satisfied for the composite (4.19a). In the case where $B^{\alpha\beta ij} = H^{\alpha\beta ij}$, eq. (4.27) may be seen to describe a supersymmetric Riemann curvature squared term [37, 39].

5 An action principle for the supersymmetric $C\Box C$ invariant

Although we have shown in the previous section that one can construct a supersymmetric $C\Box C$ action with an explicit compensator field, this has an obvious disadvantage. One would have to show that terms involving the compensator could be eliminated by integrating by parts in order for it to be an invariant for minimal conformal supergravity. Due to the complexity involved in doing this, it would be better to have a compensator-independent approach, but as we have already discussed, it seems impossible to generate an appropriate primary closed six-form. This suggests that we should consider non-primary six-forms instead; however, since these are rather more difficult to deal with, it would be helpful to know where to start looking.

Let us return to a point we raised earlier. The full superspace action (4.4) is always a possible action principle, and it must correspond to some general six-form action involving $\mathcal{L}$ and its derivatives. It turns out that its six-form *cannot* be primary. The reason is that if it were, then the lowest dimensional component would be S-invariant and at least of dimension 3. Now it is straightforward to investigate the S-transformation properties of

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all higher components of $\mathcal{L}$: the only primary aside from $\mathcal{L}$ itself appears at the θ^2 level,

$$B_a{}^{ij} = -\frac{\mathrm{i}}{16}(\tilde{\gamma}_a)^{\alpha\beta}\nabla_\alpha^{(i}\nabla_\beta^{j)}\mathcal{L}\,. \tag{5.1}$$

(In particular, there is no primary at dimension 9/2 corresponding to $A_\alpha{}^{ijk}$ without introducing a compensator.) We have denoted this descendant as $B_a{}^{ij}$ as it obeys the same constraint (4.20) as the superfield $B^{\alpha\beta\,ij} \equiv (\tilde{\gamma}^a)^{\alpha\beta}B_a{}^{ij}$ introduced in the previous section. Note however that it cannot be the bottom component of an invariant six-form: it would have to be multiplied by six E_i^α to balance its dimension, but the Lorentz and SU(2) indices cannot be contracted appropriately. This means that *no corresponding primary six-form exists*. Of course, it is not possible to construct an invariant scalar $\mathcal{L}$ from the superfields of the Weyl multiplet, so what purpose does this observation serve? It turns out that one can build an action principle upon a primary superfield $B_a{}^{ij}$ obeying certain properties consistent with (but not implying) its derivation from a scalar superfield $\mathcal{L}$. In this way, $B_a{}^{ij}$ will lead to something analogous to the chiral action principle of four dimensions.

The argument goes as follows. Suppose we choose $\mathcal{L}$ to be a tensor multiplet Φ subject to the constraint (4.21). Its superspace integral must vanish,

$$S = \int \mathrm{d}^6x\,\mathrm{d}^8\theta\,E\,\Phi = 0 \tag{5.2}$$

since one can introduce the prepotential $V^{\alpha i}$ for the tensor multiplet, as in eq. (4.25), and then integrate by parts. Now the descendant $B_a{}^{ij}$ precisely vanishes for a tensor multiplet, so it must be that that the six-form associated with a general $\mathcal{L}$ can be written purely in terms of the superfield $B_a{}^{ij}$ and its derivatives. This is analogous to the situation in four dimensions, where a full $N \leq 2$ conformal superspace action can always be converted first to a chiral superspace action using the chiral projection operator. The converse is not true — there are chiral Lagrangians that do not come from any full superspace Lagrangian (at least not without introducing compensators). Taking this analogy seriously, we conjecture that any primary superfield $B_a{}^{ij}$ obeying the S-invariant constraint (4.20), which is consistent with (5.1), must lead to an invariant action.

This proves to be precisely the action principle we need to describe the supersymmetric $C\Box C$ invariant. As a consequence of (4.20), one can show that

$$\nabla_\alpha^{(i}\Lambda^{\beta jkl)} = \delta_\alpha{}^\beta C^{ijkl}\,, \qquad \nabla_\alpha^{(i}C^{jklp)} = 0\,. \tag{5.3}$$

for non-primary superfields $\Lambda^{\alpha\,ijk}$ and C^{ijkl}. The superfield C^{ijkl} is a non-primary version of the $\mathcal{O}(4)$ multiplet that we have already discussed in section 4.1, and its S-transformation is exactly as needed to permit the second condition of (5.3) to hold. This suggests that the six-form action principle should begin with a term

$$J = \frac{1}{6!}\,E^{a_1}\wedge\cdots\wedge E^{a_6}\,\varepsilon_{a_1\cdots a_6}\,F + \cdots\,, \qquad F = \frac{1}{5}(\nabla^4)_{ijkl}C^{ijkl}\,, \tag{5.4}$$

providing a covariant version of the action principle (4.3). As already mentioned, we should not expect that the full six-form is primary. Nevertheless, starting from the top component,

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one can iteratively reconstruct the full six-form in a straightforward (albeit laborious) way. The result turns out to include explicit S and K connections, which makes J transform into an exact form under those respective gauge transformations.

We give the complete structure of this six-form in section 5.2. However, in order to better explain certain features of its construction, it helps to describe the general properties of non-primary forms, especially if one wishes to verify gauge invariance of the action. Section 5.1 is a self-contained discussion of this topic.

5.1 Non-primary closed forms in superspace

Let us begin with the following observation. It has become apparent that superforms that are not invariant under certain gauge symmetries nevertheless play an important role in constructing invariant actions. These frequently involve Chern-Simons terms with bare connections: recent examples have included the 4D and 5D linear multiplets [56, 103, 104], 3D $\mathcal{N} \leq 6$ conformal supergravity [74, 75], and non-abelian $\mathcal{N} \leq 4$ gauge theories [105]. However, such a geometric structure does not seem to be a necessary requirement. For example, in the context of 4D $\mathcal{N} = 2$ conformal superspace, bare S and K connections were recently observed when constructing actions involving projective [106] and harmonic superfields [107]. These were associated with closed four-forms J that transformed into exact forms under S and K transformations. In this subsection, we will establish some general properties of such non-primary closed forms in six dimensions.

Let J be a closed super p-form. We assume it is invariant under Lorentz, Weyl, and SU(2) transformations, but that it transforms under $K^A = (S_i^\alpha, K^a)$ transformations into an exact form. It is possible to expand J in terms of the vielbein E^A and the K-connection $\mathfrak{F}_A$,

$$\begin{aligned} J = \frac{1}{p!} E^{A_1} \wedge \cdots \wedge E^{A_p} J_{A_p \cdots A_1} + \frac{1}{(p-1)!} \mathfrak{F}_{A_1} \wedge E^{A_2} \wedge \cdots \wedge E^{A_p} J_{A_p \cdots A_2}{}^{A_1} \\ + \cdots + \frac{1}{p!} \mathfrak{F}_{A_1} \wedge \cdots \wedge \mathfrak{F}_{A_p} J^{A_p \cdots A_1} \,, \end{aligned} \tag{5.5}$$

so that the coefficient functions $J_{A_p \cdots A_{n+1}}{}^{A_n \cdots A_1}$ are covariant superfields. Let us derive the conditions on these superfields so that $\mathrm{d}J = 0$.

Because J is assumed to be invariant under Lorentz, Weyl, and SU(2) transformations, it is equivalent to analyze $\mathcal{D}J = 0$ where

$$\mathcal{D} := \mathrm{d} - \frac{1}{2}\Omega^{ab} M_{ab} - B\mathbb{D} - \Phi^{ij} J_{ij} \tag{5.6}$$

is covariant with respect to those symmetries. Using the definitions (3.11) of the torsion tensor T^A and K-curvature $R(K)_A$, one verifies that

$$\mathcal{D}E^A = \frac{1}{2} E^B \wedge E^C T_{CB}{}^A + E^B \wedge \mathfrak{F}_C f^C{}_B{}^A \,, \tag{5.7a}$$

$$\mathcal{D}\mathfrak{F}_A = \frac{1}{2} E^B \wedge E^C R(K)_{CBA} + E^B \wedge \mathfrak{F}_C f^C{}_{BA} + \frac{1}{2} \mathfrak{F}_B \wedge \mathfrak{F}_C f^{CB}{}_A \,, \tag{5.7b}$$

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where the constants f are the relevant structure constants appearing in the algebra

$$\begin{aligned}[K^A, \nabla_B] &= -f^A{}_B{}^C \nabla_C - f^A{}_{BC} K^C + \text{other generators}\,, \\ [K^A, K^B] &= -f^{AB}{}_C K^C\,. \end{aligned} \tag{5.8}$$

From the definition of ∇_A one also has

$$\mathcal{D} J_{A_p \cdots A_{n+1}}{}^{A_n \cdots A_1} = E^B \nabla_B J_{A_p \cdots A_{n+1}}{}^{A_n \cdots A_1} + \mathfrak{F}_B K^B J_{A_p \cdots A_{n+1}}{}^{A_n \cdots A_1}\,. \tag{5.9}$$

Now it is straightforward to analyze the conditions for closure on J. These will be somewhat involved, so it is helpful to give a shorthand approach that will allow us to compactly consider all equations at once. We can introduce a generalized frame one-form $\mathcal{E}^{\mathcal{A}} = (E^A, \mathfrak{F}_A)$ and rewrite (5.5) as[19]

$$J = \frac{1}{p!} \mathcal{E}^{\mathcal{A}_1} \wedge \cdots \wedge \mathcal{E}^{\mathcal{A}_p} J_{\mathcal{A}_p \cdots \mathcal{A}_1}\,, \tag{5.10}$$

with the superfields $J_{\mathcal{A}_p \cdots \mathcal{A}_1}$ encapsulating those appearing in (5.5) in the obvious way. This expansion formally treats the one-forms E^A and $\mathfrak{F}_A$ on the same footing. Imposing this democracy in the relations (5.7) and (5.9) leads respectively to

$$\mathcal{D}\mathcal{E}^{\mathcal{A}} = \frac{1}{2} \mathcal{E}^{\mathcal{B}} \wedge \mathcal{E}^{\mathcal{C}} \mathcal{T}_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}}\,, \qquad \mathcal{D} J_{A_p \cdots A_1} = \mathcal{E}^{\mathcal{B}} \nabla_{\mathcal{B}} J_{\mathcal{A}_p \cdots A_1}\,, \tag{5.11}$$

where we have introduced $\nabla_{\mathcal{A}} := (\nabla_A, K^A)$ and a tensor $\mathcal{T}_{\mathcal{C}\mathcal{B}}{}^{\mathcal{A}}$ defined as

$$\begin{aligned} &\mathcal{T}_{AB}{}^C = T_{AB}{}^C\,, && \mathcal{T}_A{}^{BC} = f_A{}^{BC}\,, && \mathcal{T}^{ABC} = 0\,, \\ &\mathcal{T}_{ABC} = R(K)_{ABC}\,, && \mathcal{T}_A{}^B{}_C = f_A{}^B{}_C\,, && \mathcal{T}^{AB}{}_C = f^{AB}{}_C\,. \end{aligned} \tag{5.12}$$

Now it is immediately apparent that the condition for closure on J becomes

$$\nabla_{[\mathcal{A}_{p+1}} J_{\mathcal{A}_p \cdots \mathcal{A}_1\}} + \frac{p}{2} \mathcal{T}_{[\mathcal{A}_{p+1} \mathcal{A}_p}{}^{\mathcal{B}} J_{|\mathcal{B}| \mathcal{A}_{p-1} \cdots \mathcal{A}_1\}} = 0\,. \tag{5.13}$$

The above structure suggests the interpretation that we are enlarging the superspace and introducing new coordinates associated with K^A so that $\mathcal{E}^{\mathcal{A}}$ becomes the new vielbein. From our perspective, this analogy is purely a formal one — we are *not* introducing any new coordinates. However, because the structure of the transformations is consistent with such a possibility,[20] many useful properties follow. For example, the tensor $\mathcal{T}$ can be interpreted as the generalized torsion tensor of $\nabla_{\mathcal{A}}$, that is

$$[\nabla_{\mathcal{A}}, \nabla_{\mathcal{B}}] = -\mathcal{T}_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} \nabla_{\mathcal{C}} + \text{other generators}\,. \tag{5.14}$$

[19] The notion of a generalized frame appeared naturally in the context of multiplets with central charge coupled to $\mathcal{N} = 2$ supergravity. There it facilitates the description of vector-tensor multiplets [108, 109] and the construction of the linear multiplet action [103].

[20] In more formal language, we could choose to work on the total (super)space of the fiber bundle associated with K-transformations.

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Similarly, the δ_K transformations of the connections $\mathcal{E}^{\mathcal{A}} = (E^A, \mathfrak{F}_A)$ and the covariant components $J_{\mathcal{A}_p \cdots \mathcal{A}_1}$ precisely satisfy a covariant form of Cartan's formula,

$$\delta_K(\Lambda) = \mathcal{D}\imath_\Lambda + \imath_\Lambda \mathcal{D}\,, \tag{5.15}$$

where $\imath_\Lambda$ is an antiderivation defined to act as

$$\imath_\Lambda \mathfrak{F}_A = \Lambda_A\,, \qquad \imath_\Lambda E^A = \imath_\Lambda(\nabla_{\mathcal{A}_n} \cdots \nabla_{\mathcal{A}_{p+1}} J_{\mathcal{A}_p \cdots \mathcal{A}_1}) = 0\,. \tag{5.16}$$

From these results, it is immediate to see that for a closed p-form (5.5)

$$\delta_K(\Lambda) J = \mathcal{D}\imath_\Lambda J = \mathrm{d}\imath_\Lambda J \tag{5.17}$$

which establishes that J transforms as an exact form.

It is obvious that the class of primary superforms, discussed in section 4.2, is simply one for which no $\mathfrak{F}_A$ appears within the decomposition (5.5). Then the closure condition (5.13) amounts to two conditions:

$$\nabla_{[A_{p+1}} J_{A_p \cdots A_1\}} + \frac{p}{2} T_{[A_{p+1} A_p}{}^B J_{|B| A_{p-1} \cdots A_1\}} = 0\,, \tag{5.18a}$$

$$K^C J_{A_p \cdots A_1} + p\, f^C{}_{[A_p}{}^B J_{|B| A_{p-1} \cdots A_1\}} = 0\,. \tag{5.18b}$$

The first is the usual covariant closure condition, and the second is the condition for S and K-invariance (compare to eq. (4.10)). This illustrates how the single condition (5.13) concisely encodes both the conditions for closure and for gauge invariance modulo an exact piece.

5.2 A non-primary six-form action principle

Now we turn to our specific goal of finding a non-primary six-form that begins with the term (5.4). Taking into account the closure conditions, one can deduce the structure of the remaining terms. We use the definitions

$$\Lambda^{\alpha ijk} := \frac{\mathrm{i}}{3} \nabla_\beta{}^{(i} B^{\beta \alpha jk)}\,, \qquad \Lambda_{\alpha b}{}^i := \frac{2\mathrm{i}}{3} \nabla_{\alpha j} B_b{}^{ij}\,, \tag{5.19a}$$

$$C^{ijkl} := \frac{1}{4} \nabla_\alpha^{(i} \Lambda^{\alpha jkl)}\,, \qquad C_\alpha{}^{\beta ij} := \frac{3}{4} \nabla_{\alpha k} \Lambda^{\beta ijk}\,, \qquad C_{ab} := \frac{1}{8} (\tilde{\gamma}_a)^{\alpha\beta} \nabla_{\alpha k} \Lambda_{\beta b}{}^k\,, \tag{5.19b}$$

$$\rho_\alpha{}^{ijk} := -\frac{4\mathrm{i}}{5} \nabla_{\alpha l} C^{ijkl}\,, \qquad \rho_{\alpha\beta}{}^{\gamma i} := -\frac{2\mathrm{i}}{3} \nabla_{[\alpha j} C_{\beta]}{}^{\gamma ij}\,, \tag{5.19c}$$

$$E_a{}^{ij} := \frac{3}{16} (\tilde{\gamma}_a)^{\alpha\beta} \nabla_{\alpha k} \rho_\beta{}^{ijk}\,, \tag{5.19d}$$

$$\Omega^{\alpha i} := \frac{\mathrm{i}}{18} \nabla_{\beta j} E^{\beta \alpha ij}\,, \qquad F := \frac{1}{8} \nabla_{\alpha j} \Omega^{\alpha j} = \frac{1}{5} (\nabla^4)_{ijkl} C^{ijkl}\,, \tag{5.19e}$$

with factors of i chosen so that all fields obey $\overline{\Psi^{ij\cdots}} = \Psi_{ij\ldots}$ where Ψ carries any number of spinor indices. In terms of these components, the action six-form may concisely be factorized as

$$J = J_0 + \mathfrak{F}^i_\alpha \wedge J_{S_i^\alpha} + \mathfrak{F}_a \wedge J_K{}^a\,, \tag{5.20}$$

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where the six-form J_0 and the five-forms $J_{S_i}^{\alpha}$ and $J_K{}^a$ involve only the supervielbein one-forms E^A. The non-vanishing tangent-space components of J_0 are

$$
\begin{aligned}
J_{0\,abc}{}^{i\,j\,k}_{\alpha\beta\gamma} &= -3(\gamma_{abc})_{(\alpha\beta}\rho_{\gamma)}{}^{ijk}\,, \\
J_{0\,abcd}{}^{i\,j}_{\alpha\beta} &= -\frac{8\mathrm{i}}{3}(\gamma_{[abc})_{\alpha\beta}E_{d]}{}^{ij}\,, \\
J_{0\,a_1a_2a_3a_4a_5}{}^{i}_{\alpha} &= -\varepsilon_{a_1a_2a_3a_4a_5c}\,(\gamma^c)_{\alpha\beta}\Big(\mathrm{i}\Omega^{\beta i} - 8\mathrm{i}\,B_a{}^{ij}\nabla_b X_j^{\gamma}(\gamma^{ab})_{\gamma}{}^{\beta} \\
&\qquad + \frac{32\mathrm{i}}{3}B_a{}^{ij}\,(\gamma^{ab})_{\gamma}{}^{\delta}\,\nabla_b X_{\delta j}{}^{\gamma\beta} - 3\mathrm{i}\,\Lambda^{\gamma ijk}\,Y_{\gamma}{}^{\beta}{}_{jk}\Big)\,, \\
J_{0\,a_1a_2a_3a_4a_5a_6} &= -\varepsilon_{a_1a_2a_3a_4a_5a_6}\,\Big(F + 4\mathrm{i}\,\Lambda_{\alpha b}{}^{k}(\gamma^{bc})_{\beta}{}^{\alpha}\nabla_c X_k^{\beta} - \frac{16\mathrm{i}}{3}\Lambda_{\alpha b}{}^{k}(\gamma^{bc})_{\beta}{}^{\gamma}\nabla_c X_{\gamma k}{}^{\beta\alpha} \\
&\qquad + 2B_{b\,ij}(\gamma^{bc})_{\alpha}{}^{\beta}\nabla_c Y_{\beta}{}^{\alpha ij} - \frac{4}{3}C_{\beta}{}^{\alpha}{}_{ij}Y_{\alpha}{}^{\beta ij}\Big)\,.
\end{aligned}
\tag{5.21}
$$

Note that there are some similarities between components of J_0 and those of the $A_{\alpha}{}^{ijk}$ six-form (4.15). In particular, the lowest dimensional component $\rho_{\alpha}{}^{ijk}$ of J_0 obeys the same differential constraint (4.14) as $A_{\alpha}{}^{ijk}$; the difference is that $\rho_{\alpha}{}^{ijk}$ is not primary but transforms into C^{ijkl} under S-supersymmetry. The non-vanishing components of the five-forms $J_{S_i}^{\alpha}$ and $J_K{}^a$ are simpler in structure and given by

$$
J_{S\,abc}{}^{j\,k\,\alpha}_{\beta\gamma\,i} = 24\mathrm{i}\,(\gamma_{abc})_{\beta\gamma}\Lambda^{\alpha\,jk}{}_{i}\,, \tag{5.22a}
$$

$$
J_{S\,abcd}{}^{j\,\alpha}_{\beta\,i} = \frac{8}{3}\varepsilon_{abcdef}(\gamma^{ef})_{\beta}{}^{\gamma}\,C_{\gamma}{}^{\alpha j}{}_{i}\,, \tag{5.22b}
$$

$$
J_{S\,abcde}{}^{\alpha}_{i} = \varepsilon_{abcdef}(\tilde{\gamma}^f)^{\beta\gamma}\rho_{\beta\gamma i}{}^{\alpha}\,, \tag{5.22c}
$$

and

$$
J_{K\,bcd}{}^{i\,j\,a}_{\alpha\beta} = -64\mathrm{i}(\gamma_{bcd})_{\alpha\beta}\,B^{a\,ij}\,, \tag{5.23a}
$$

$$
J_{K\,bcde}{}^{i\,a}_{\alpha} = 8\mathrm{i}\,\varepsilon_{bcdefg}\,(\gamma^{fg})_{\alpha}{}^{\beta}\,\Lambda_{\beta}{}^{a\,i}\,, \tag{5.23b}
$$

$$
J_{K\,bcdef}{}^{a} = \varepsilon_{bcdefg}(\tilde{\gamma}^g)^{\gamma\delta}C_{\gamma\delta}{}^{\alpha\beta}\,(\gamma^a)_{\alpha\beta}\,. \tag{5.23c}
$$

They are essentially determined by the requirement that the full six-form J should transform as

$$
\delta_S J = -\mathrm{d}(\Lambda_{S\,\alpha}^{i}J_{S\,i}^{\alpha})\,, \qquad \delta_K J = -\mathrm{d}(\Lambda_{Ka}J_K{}^a)\,, \tag{5.24}
$$

under S and K transformations, consistent with (5.17). Note that since J is not primary, we may freely add any exact form we choose to it. In particular, some of the terms in J_S and J_K can be removed by choosing such a form appropriately; however, since it does not seem possible to eliminate either J_S or J_K completely, we have not tried to simplify J any further.

Using this non-primary six-form, we can immediately construct the invariants corresponding respectively to the supersymmetric $C\Box C$ invariant and the supersymmetric $F\Box F$ actions. The first, as already mentioned, involves choosing $B^{\alpha\beta\,ij} = H^{\alpha\beta\,ij}$ in (4.19a). The

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leading components of the action can be deduced by observing that the non-primary descendant $\mathcal{O}(4)$ superfield is simply

$$C^{ijkl} = -\frac{1}{2} Y_\alpha{}^{\beta(ij} Y_\beta{}^{\alpha\, kl)} \tag{5.25}$$

from which the leading contributions to $F = \frac{1}{5}(\nabla^4)_{ijkl} C^{ijkl}$ may be determined. The term associated with the Weyl tensor is straightforward to derive:

$$F = \frac{2}{9}(\nabla^d Y_{abcd})^2 + \cdots = \frac{2}{9}(\nabla^d R(M)_{abcd})^2 + \cdots . \tag{5.26}$$

Note that even this leading term is not K-invariant, as one must include the explicit K-connection terms in the six-form. Removing a total derivative and higher order terms in the Weyl tensor leads to

$$F = -\frac{1}{12} R(M)_{abcd} \Box R(M)^{abcd} + \cdots . \tag{5.27}$$

The second case, the supersymmetric $F\Box F$ action, involves the composite (4.23). Here one finds the non-primary $\mathcal{O}(4)$ descendant superfield is $C^{ijkl} = \mathrm{Tr}(\boldsymbol{X}^{(ij} \boldsymbol{X}^{kl)})$. As we have already noted, this is precisely the harmonic superspace Lagrangian used in [2] to construct this invariant in flat space. At leading order, one finds the top component of the multiplet is

$$F = 2\,\mathrm{Tr}(\nabla^b \boldsymbol{F}_{ba} \nabla_c \boldsymbol{F}^{ca}) + \cdots = -\,\mathrm{Tr}(\boldsymbol{F}_{ab} \Box \boldsymbol{F}^{ab}) + \cdots , \tag{5.28}$$

where we have discarded a total derivative and higher order terms.

The details of the component action corresponding to the supersymmetric $C\Box C$ and $F\Box F$ invariants will appear in a forthcoming paper.

6 Discussion

In this paper we have constructed two invariants for minimal conformal supergravity in six dimensions. These include the supersymmetric C^3 invariant described by the composite (4.18) together with the action principle (4.15), as well as the supersymmeric $C\Box C$ invariant described by the composite (4.19a) together with the action principle (5.20). The number of invariants constructed is consistent with the expectation that there should only be two in the case of $\mathcal{N} = (1,0)$ local supersymmetry, see e.g. [110]. However, it would be good to confirm that there does not remain another invariant. A rather simple way to answer this question is to consider possible supercurrents of the Weyl multiplet.

In supersymmetric field theory, the supercurrent is a supermultiplet containing the energy-momentum tensor and the supersymmetry current(s), along with some additional components such as the R-symmetry current. In the case of 6D $\mathcal{N} = (1,0)$ superconformal field theory, the supercurrent was described in [88] in Minkowski superspace. Its generalization to the curved case is described by a scalar primary superfield $\mathcal{J}$ of dimension 4 satisfying the differential constraint[21]

$$\nabla^{(i}_{[\alpha} \nabla^j_\beta \nabla^{k)}_{\gamma]} \mathcal{J} = 0 . \tag{6.1}$$

[21] This is the only possible curved extension of the flat case description in [88] provided $\mathcal{J}$ is primary.

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When the superconformal theory is coupled to conformal supergravity, the lowest component of $\mathcal{J}$ matches the variational derivative of the action with respect to the highest dimension independent field of the Weyl multiplet, which is the scalar auxiliary field D as mentioned in section 3.3.

We may now ask the following question: how many possible supercurrents can be built purely from the super-Weyl tensor and its covariant derivatives? The most general possible ansatz is

$$\begin{aligned}\mathcal{J} = {} & c_1 \nabla^a \nabla_a Y + c_2 Y^2 + \mathrm{i}\, c_3\, X^{\alpha i} \nabla_{\alpha\beta} X_i^{\beta} + \mathrm{i}\, c_4\, X_\alpha^{i\,\beta\gamma} \nabla_{\gamma\delta} X_{\beta i}{}^{\alpha\delta} + c_5\, Y_\alpha{}^{\beta ij} Y_\beta{}^{\alpha}{}_{ij} \\ & + c_6\, Y_{\alpha\beta}{}^{\gamma\delta} Y_{\gamma\delta}{}^{\alpha\beta} + c_7\, W^{\alpha\gamma} \nabla_{\alpha\beta} \nabla_{\gamma\delta} W^{\delta\beta} + c_8 \nabla_{\beta\alpha} W^{\alpha\gamma} \nabla_{\gamma\delta} W^{\delta\beta} \\ & + c_9 \varepsilon_{\alpha_1 \cdots \alpha_4} \varepsilon_{\beta_1 \cdots \beta_4} W^{\alpha_1\beta_1} \cdots W^{\alpha_4\beta_4}\,, \end{aligned} \tag{6.2}$$

where c_n, $n = 1, \cdots 9$, are real coefficients. Requiring that $\mathcal{J}$ be primary and satisfy the constraint (6.1) yields a two-parameter family of possibilities,

$$\begin{aligned} & c_3 = -\frac{8}{3} c_2 - 5 c_1\,, \qquad c_4 = -\frac{32}{15} c_2 - 16 c_1\,, \quad c_5 = \frac{2}{15} c_2 + \frac{6}{5} c_1\,, \\ & c_6 = \frac{2}{45} c_2 + \frac{1}{3} c_1\,, \qquad c_7 = -\frac{2}{15} c_2 - \frac{1}{5} c_1\,, \quad c_8 = \frac{1}{2} c_7 = -\frac{1}{15} c_2 - \frac{1}{10} c_1\,, \\ & c_9 = 0\,, \end{aligned} \tag{6.3}$$

given here in terms of the coefficients c_1 and c_2. The family with $c_1 = 0$ corresponds to a supercurrent built from the cubic Weyl invariant, whereas a combination with nonzero c_1 must correspond to the quadratic Weyl invariant. There are no other possibilities, so the two invariants we have constructed are the only ones.

In section 2.4 we discussed the Euler invariant, eq. (2.29). Here we briefly comment on its extension to the supersymmetric case. It can naturally be introduced by first using the special conformal (and S-supersymmetry) transformations to gauge away the dilatation connection entirely, $B_A = 0$. It is now natural to perform the degauging procedure as in [53–56], and extract the special conformal connection $\mathfrak{F}_A$ by introducing the *degauged* covariant derivatives $\mathcal{D}_A := \nabla_A + \mathfrak{F}_{AB} K^B$, with $\mathrm{SO}(5,1) \times \mathrm{SU}(2)$ being the corresponding structure group. They satisfy (anti-)commutation relations of the form[22]

$$[\mathcal{D}_A, \mathcal{D}_B\} = -\mathcal{T}_{AB}{}^C \mathcal{D}_C - \frac{1}{2} \mathcal{R}_{AB}{}^{cd} M_{cd} - \mathcal{R}_{AB}{}^{kl} J_{kl}\,, \tag{6.4}$$

where $\mathcal{T}_{AB}{}^C$ is the torsion, and $\mathcal{R}_{AB}{}^{cd}$ and $\mathcal{R}_{AB}{}^{kl}$ are the Lorentz and SU(2) curvatures, respectively. A detailed analysis of the torsion and curvature tensors will be given elsewhere. The Euler invariant is defined to be the closed six-form

$$\mathcal{E}_6 = \frac{1}{8} \mathcal{R}^{ab} \wedge \mathcal{R}^{cd} \wedge \mathcal{R}^{ef} \varepsilon_{abcdef}\,, \qquad \mathrm{d}\mathcal{E}_6 = 0\,, \tag{6.5}$$

where $\mathcal{R}^{cd} = \frac{1}{2} E^B \wedge E^A \mathcal{R}_{AB}{}^{cd}$.

[22] The reader should be made aware that our notation for the curvature tensor coincides with the bosonic case, but they are not to be confused.

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It may be seen that $\mathcal{E}_6$ contains the same C^3 combination (2.25) (modulo an overall coefficient) which originates in the closed six-form J_{C^3} describing the supersymmetric C^3 invariant, eq. (4.15). As a result, the closed six-form

$$\mathcal{E}_6 + 12 J_{C^3} \,, \tag{6.6}$$

does not contain any term involving only the Weyl tensor. All bosonic structures in the above invariant involve the Ricci tensor. However, it is not actually an independent invariant since we have only added a total derivative.

It was shown in section 2 that there exists a primary construction in terms of the logarithm of a compensator. Upon degauging the compensator it contains a linear combination of the conformal invariants. Although outside of the scope of this work it would be interesting to construct its supersymmetric extension.

A detailed analysis of the component structure of the supergravity multiplet, as well as of the invariants for 6D $\mathcal{N} = (1,0)$ conformal supergravity constructed, will be given in a forthcoming publication [111].

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A Notation and conventions

We follow similar 6D notations and conventions as [50], with a few minor modifications. All relevant details are summarized here.

The Lorentzian metric is $\eta_{ab} = \text{diag}(-1,1,1,1,1,1)$, the Levi-Civita tensor ε_{abcdef} obeys $\varepsilon_{012345} = -\varepsilon^{012345} = 1$, and the Levi-Civita tensor with world indices is given by $\varepsilon^{mnpqrs} := \varepsilon^{abcdef} e_a{}^m e_b{}^n e_c{}^p e_d{}^q e_e{}^r e_f{}^s$.

We exclusively use four component spinors in the body of the paper, but it is useful to link these to eight component spinor conventions. Our 8×8 Dirac matrices Γ^a and charge conjugation matrix C obey

$$\begin{aligned} \{\Gamma_a, \Gamma_b\} &= -2\eta_{ab}\mathbf{1}\,, \quad (\Gamma^a)^\dagger = -\Gamma_a\,, \quad C\Gamma_a C^{-1} = -\Gamma_a^T\,, \\ C^\dagger C &= \mathbf{1}\,, \qquad\qquad C = C^T = C^*\,. \end{aligned} \tag{A.1}$$

In particular, $\Gamma_a C^{-1}$ is antisymmetric. The chirality matrix Γ_* is defined by

$$\Gamma_{[a}\Gamma_b\Gamma_c\Gamma_d\Gamma_e\Gamma_{f]} = \varepsilon_{abcdef}\Gamma_*\,. \tag{A.2}$$

As a consequence of the above conditions, one can show that

$$\Gamma^a = B(\Gamma^a)^* B^{-1}\,, \qquad B = \Gamma_*\Gamma_0 C^{-1}\,. \tag{A.3}$$

The charge conjugate Ψ^c of a Dirac spinor is conventionally defined by

$$\bar\Psi \equiv \Psi^\dagger\Gamma_0 =: (\Psi^c)^T C \qquad \Longrightarrow \qquad \Psi^c = -\Gamma_0 C^{-1}\Psi^* = -\Gamma_* B\Psi^*\,. \tag{A.4}$$

Because $B^*B = -1$, charge conjugation is an involution only for objects with an even number of spinor indices, so it is not possible to have Majorana spinors in six dimensions. One can instead have a symplectic Majorana condition when the spinors possess an SU(2) index. Conventionally this is denoted

$$(\Psi_i)^c = \Psi^i \qquad \Longrightarrow \qquad \Psi^i = -\Gamma_0 C^{-1}(\Psi_i)^* = -\Gamma_* B(\Psi_i)^* \tag{A.5}$$

for a spinor of either chirality. We raise and lower SU(2) indices $i = \underline{1}, \underline{2}$ using the conventions

$$\Psi^i = \varepsilon^{ij}\Psi_j\,, \qquad \Psi_i = \varepsilon_{ij}\Psi^j\,, \qquad \varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1\,. \tag{A.6}$$

We employ a Weyl basis for the gamma matrices so that an eight-component Dirac spinor Ψ decomposes into a four-component left-handed Weyl spinor ψ^α and a four-component right-handed spinor χ_α so that

$$\Psi = \begin{pmatrix} \psi^\alpha \\ \chi_\alpha \end{pmatrix}\,, \qquad \Gamma_* = \begin{pmatrix} \delta^\alpha{}_\beta & 0 \\ 0 & -\delta_\alpha{}^\beta \end{pmatrix}\,, \qquad \alpha = 1, \cdots, 4\,. \tag{A.7}$$

The spinors ψ^α and χ_α are valued in the two inequivalent fundamental representations of $\mathfrak{su}^*(4) \cong \mathfrak{so}(5,1)$. We further take

$$\Gamma^a = \begin{pmatrix} 0 & (\tilde\gamma^a)^{\alpha\beta} \\ (\gamma^a)_{\alpha\beta} & 0 \end{pmatrix}\,, \qquad C = \begin{pmatrix} 0 & \delta_\alpha{}^\beta \\ \delta^\alpha{}_\beta & 0 \end{pmatrix}\,. \tag{A.8}$$

The Pauli-type 4×4 matrices $(\gamma^a)_{\alpha\beta}$ and $(\tilde\gamma^a)^{\alpha\beta}$ are antisymmetric and related by

$$(\tilde\gamma^a)^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}(\gamma^a)_{\gamma\delta}\,, \qquad (\gamma^a)^* = \tilde\gamma_a\,, \tag{A.9}$$

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where $\varepsilon^{\alpha\beta\gamma\delta}$ is the canonical antisymmetric symbol of $\mathfrak{su}^*(4)$. They obey

$$(\gamma^a)_{\alpha\beta}(\tilde{\gamma}^b)^{\beta\gamma} + (\gamma^b)_{\alpha\beta}(\tilde{\gamma}^a)^{\beta\gamma} = -2\eta^{ab}\delta_\alpha^\gamma\,, \tag{A.10a}$$

$$(\tilde{\gamma}^a)^{\alpha\beta}(\gamma^b)_{\beta\gamma} + (\tilde{\gamma}^b)^{\alpha\beta}(\gamma^a)_{\beta\gamma} = -2\eta^{ab}\delta_\gamma^\alpha\,, \tag{A.10b}$$

and as a consequence of (A.3),

$$(\gamma^a)_{\alpha\beta} = B_\alpha{}^{\dot\gamma}B_\beta{}^{\dot\delta}\big((\gamma^a)_{\gamma\delta}\big)^*\,, \quad (\tilde{\gamma}^a)^{\alpha\beta} = B^\alpha{}_{\dot\gamma}B^\beta{}_{\dot\delta}\big((\tilde{\gamma}^a)^{\gamma\delta}\big)^*\,, \quad B = \begin{pmatrix} B^\alpha{}_{\dot\beta} & 0 \\ 0 & B_\alpha{}^{\dot\beta} \end{pmatrix}. \tag{A.11}$$

A dotted index denotes the complex conjugate representation in $\mathfrak{su}^*(4)$. It is natural to use the B matrix to define bar conjugation on a four component spinor via

$$\bar{\psi}^\alpha = B^\alpha{}_{\dot\beta}(\psi^\beta)^*\,, \qquad \bar{\chi}_\alpha = B_\alpha{}^{\dot\beta}(\chi_\beta)^*\,, \tag{A.12}$$

with the obvious extension to any object with multiple spinor indices. For example, $\overline{(\gamma^a)_{\alpha\beta}} = (\gamma^a)_{\alpha\beta}$ using (A.11) and similarly for $\tilde{\gamma}^a$. Note that $\overline{\overline{\psi^\alpha}} = -\psi^\alpha$ and similarly for any object with an odd number of spinor indices as a consequence of $B^*B = -\mathbf{1}$. A symplectic Majorana spinor Ψ_i, decomposed as in (A.7) and obeying (A.5), has Weyl components that obey

$$\overline{\psi^{\alpha i}} = \psi^\alpha_i\,, \qquad \overline{\chi_{\alpha i}} = \chi^i_\alpha\,. \tag{A.13}$$

The Grassmann coordinates θ^α_i and the parameters η^i_α of S-supersymmetry are both symplectic Majorana-Weyl using this definition.

We define the antisymmetric products of two or three Pauli-type matrices as

$$\gamma_{ab} := \gamma_{[a}\tilde{\gamma}_{b]} := \frac{1}{2}(\gamma_a\tilde{\gamma}_b - \gamma_b\tilde{\gamma}_a)\,, \quad \tilde{\gamma}_{ab} := \tilde{\gamma}_{[a}\gamma_{b]} = -(\gamma_{ab})^T\,, \tag{A.14a}$$

$$\gamma_{abc} := \gamma_{[a}\tilde{\gamma}_b\gamma_{c]}\,, \qquad \tilde{\gamma}_{abc} := \tilde{\gamma}_{[a}\gamma_b\tilde{\gamma}_{c]}\,. \tag{A.14b}$$

Note that γ_{ab} and $\tilde{\gamma}_{ab}$ are traceless, whereas γ_{abc} and $\tilde{\gamma}_{abc}$ are symmetric. Further antisymmetric products obey

$$\gamma_{abc} = -\frac{1}{3!}\varepsilon_{abcdef}\gamma^{def}\,, \qquad \tilde{\gamma}_{abc} = \frac{1}{3!}\varepsilon_{abcdef}\tilde{\gamma}^{def}\,, \tag{A.15a}$$

$$\gamma_{abcd} = \frac{1}{2}\varepsilon_{abcdef}\gamma^{ef}\,, \qquad \tilde{\gamma}_{abcd} = -\frac{1}{2}\varepsilon_{abcdef}\tilde{\gamma}^{ef}\,, \tag{A.15b}$$

$$\gamma_{abcde} = \varepsilon_{abcdef}\gamma^f\,, \qquad \tilde{\gamma}_{abcde} = -\varepsilon_{abcdef}\tilde{\gamma}^f\,, \tag{A.15c}$$

$$\gamma_{abcdef} = -\varepsilon_{abcdef}\,, \qquad \tilde{\gamma}_{abcdef} = \varepsilon_{abcdef}\,. \tag{A.15d}$$

Making use of the completeness relations

$$(\gamma^a)_{\alpha\beta}(\tilde{\gamma}_a)^{\gamma\delta} = 4\,\delta_{[\alpha}{}^\gamma\delta_{\beta]}{}^\delta\,, \tag{A.16a}$$

$$(\gamma^{ab})_\alpha{}^\beta(\gamma_{ab})_\gamma{}^\delta = -8\,\delta_\alpha{}^\delta\delta_\gamma{}^\beta + 2\,\delta_\alpha{}^\beta\delta_\gamma{}^\delta\,, \tag{A.16b}$$

$$(\gamma^{abc})_{\alpha\beta}(\tilde{\gamma}_{abc})^{\gamma\delta} = 48\,\delta_{(\alpha}{}^\gamma\delta_{\beta)}{}^\delta\,, \tag{A.16c}$$

$$(\gamma^{abc})_{\alpha\beta}(\tilde{\gamma}_{abc})_{\gamma\delta} = (\gamma^{abc})^{\alpha\beta}(\tilde{\gamma}_{abc})^{\gamma\delta} = 0\,, \tag{A.16d}$$

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it is straightforward to establish natural isomorphisms between tensors of $\mathfrak{so}(5,1)$ and matrix representations of $\mathfrak{su}^*(4)$. Vectors V^a and antisymmetric matrices $V_{\alpha\beta} = -V_{\beta\alpha}$ are related by

$$V_{\alpha\beta} := (\gamma^a)_{\alpha\beta} V_a \quad \Longleftrightarrow \quad V_a = \frac{1}{4}(\tilde{\gamma}_a)^{\alpha\beta} V_{\alpha\beta} \,. \tag{A.17}$$

Antisymmetric rank-two tensors F_{ab} are related to traceless matrices $F_\alpha{}^\beta$ via

$$F_\alpha{}^\beta := -\frac{1}{4}(\gamma^{ab})_\alpha{}^\beta F_{ab} \,, \quad F_\alpha{}^\alpha = 0 \quad \Longleftrightarrow \quad F_{ab} = \frac{1}{2}(\gamma_{ab})_\beta{}^\alpha F_\alpha{}^\beta = -F_{ba} \,. \tag{A.18}$$

Self-dual and anti-self-dual rank-three antisymmetric tensors $T^{(\pm)}_{abc}$,

$$\frac{1}{3!}\varepsilon^{abcdef} T^{(\pm)}_{def} = \pm T^{(\pm)abc} \,, \tag{A.19}$$

are related to symmetric matrices $T_{\alpha\beta}$ and $T^{\alpha\beta}$ via

$$T_{\alpha\beta} := \frac{1}{3!}(\gamma^{abc})_{\alpha\beta} T_{abc} = T_{\beta\alpha} \quad \Longleftrightarrow \quad T^{(+)}_{abc} = \frac{1}{8}(\tilde{\gamma}_{abc})^{\alpha\beta} T_{\alpha\beta} \,, \tag{A.20a}$$

$$T^{\alpha\beta} := \frac{1}{3!}(\tilde{\gamma}^{abc})^{\alpha\beta} T_{abc} = T^{\beta\alpha} \quad \Longleftrightarrow \quad T^{(-)}_{abc} = \frac{1}{8}(\gamma_{abc})_{\alpha\beta} T^{\alpha\beta} \,. \tag{A.20b}$$

Further irreducible representations of the Lorentz group take particularly simple forms when written with spinor indices. For example, a gamma-traceless left-handed spinor two-form $\Psi_{ab}{}^\gamma$ is related to a symmetric traceless $\Psi_\alpha{}^{\beta\gamma}$,

$$\begin{aligned} &\Psi_\alpha{}^{\beta\gamma} := -\frac{1}{4}(\gamma^{ab})_\alpha{}^\beta \Psi_{ab}{}^\gamma = \Psi_\alpha{}^{\gamma\beta} \,, \qquad \Psi_\alpha{}^{\alpha\gamma} = 0 \quad \Longleftrightarrow \\ &\Psi_{ab}{}^\gamma = \frac{1}{2}(\gamma_{ab})_\beta{}^\alpha \Psi_\alpha{}^{\beta\gamma} \,, \qquad (\gamma^b)_{\delta\gamma} \Psi_{ab}{}^\gamma = 0 \,, \end{aligned} \tag{A.21}$$

and a rank-four tensor C_{abcd} with the symmetries of the Weyl tensor is related to a symmetric traceless $C_{\alpha\gamma}{}^{\beta\delta}$ via

$$\begin{aligned} &C_{\alpha\gamma}{}^{\beta\delta} := \frac{1}{16}(\gamma^{ab})_\alpha{}^\beta (\gamma^{cd})_\gamma{}^\delta \, C_{abcd} = C_{(\alpha\gamma)}{}^{(\beta\delta)} \,, \qquad C_{\alpha\gamma}{}^{\beta\gamma} = 0 \quad \Longleftrightarrow \\ &C_{abcd} = \frac{1}{4}(\gamma_{ab})_\beta{}^\alpha (\gamma_{cd})_\delta{}^\gamma \, C_{\alpha\gamma}{}^{\beta\delta} = C_{[cd][ab]} \,, \qquad C_{[abc]d} = 0 \,. \end{aligned} \tag{A.22}$$

B The conformal Killing supervector fields of $\mathbb{R}^{6|8}$

Simple Minkowski superspace in six dimensions, $\mathbb{R}^{6|8}$, is parametrized by coordinates $z^A = (x^a, \theta^\alpha_i)$. The flat covariant derivatives $D_A = (\partial_a, D^i_\alpha)$

$$\partial_a := \frac{\partial}{\partial x^a} \,, \qquad D^i_\alpha := \frac{\partial}{\partial \theta^\alpha_i} - \mathrm{i}(\gamma^a)_{\alpha\beta}\theta^{\beta i}\partial_a \,, \tag{B.1}$$

satisfy the algebra:

$$\{D^i_\alpha, D^j_\beta\} = -2\mathrm{i}\varepsilon^{ij}(\gamma^a)_{\alpha\beta}\partial_a \,, \qquad [\partial_a, D^j_\beta] = 0 \,, \qquad [\partial_a, \partial_b] = 0 \,. \tag{B.2}$$

The conformal Killing supervector fields

$$\xi = \bar{\xi} = \xi^a \partial_a + \xi_i^\alpha D_\alpha^i \tag{B.3}$$

may be defined to satisfy

$$[\xi, D_\alpha^i] = -(D_\alpha^i \xi_j^\beta) D_\beta^j \,, \tag{B.4}$$

which implies the fundamental equation

$$D_\alpha^i \xi_a = -2\mathrm{i}(\gamma_a)_{\alpha\beta} \xi^{\beta i} \,. \tag{B.5}$$

From eq. (B.5) one finds

$$\varepsilon^{ij} (\gamma^b)_{\alpha\beta} \partial_b \xi_a = (\gamma_a)_{\alpha\gamma} D_\beta^j \xi^{\gamma i} + (\gamma_a)_{\beta\gamma} D_\alpha^i \xi^{\gamma j} \,, \tag{B.6}$$

which gives us the equation for a conformal Killing vector field,

$$\partial_{(a} \xi_{b)} = \frac{1}{6} \eta_{ab} \partial^c \xi_c \,, \tag{B.7}$$

as well as the following useful identities:

$$D_\alpha^{(i} \xi^{\beta j)} = \frac{1}{4} \delta_\alpha^\beta D_\gamma^{(i} \xi^{\gamma j)} \,, \tag{B.8a}$$

$$D_\gamma^k \xi_k^\gamma = \frac{2}{3} \partial^a \xi_a \,, \tag{B.8b}$$

$$D_\alpha^k \xi_k^\beta - \frac{1}{4} \delta_\alpha^\beta D_\gamma^k \xi_k^\gamma = -\frac{1}{2} (\gamma^{ab})_\alpha{}^\beta \partial_a \xi_b \,. \tag{B.8c}$$

The conformal Killing supervector field acts on the spinor covariant derivatives as follows

$$[\xi, D_\alpha^i] = -\omega_\alpha{}^\beta D_\beta^i + \Lambda^{ij} D_{\alpha j} - \frac{1}{2} \sigma D_\alpha^i \,, \tag{B.9}$$

where the parameters $\omega_\alpha{}^\beta$, σ and Λ^{ij} are given by the following expressions:

$$\omega_\alpha{}^\beta := -\frac{1}{4} (\gamma^{ab})_\alpha{}^\beta \partial_a \xi_b \,, \tag{B.10a}$$

$$\sigma := \frac{1}{4} D_\gamma^k \xi_k^\gamma = -\frac{1}{6} \partial^a \xi_a \,, \tag{B.10b}$$

$$\Lambda^{ij} := \frac{1}{4} D_\gamma^{(i} \xi^{\gamma j)} \,. \tag{B.10c}$$

Using eq. (B.7) one finds that the parameters (B.10) satisfy

$$\partial_a \omega_{bc} = -2 \eta_{a[b} \partial_{c]} \sigma \,, \tag{B.11a}$$

$$\partial_a \partial_b \xi_c = \eta_{ab} \partial_c \sigma - 2 \eta_{c(a} \partial_{b)} \sigma \,, \tag{B.11b}$$

while using eq. (B.5) one finds

$$D_\gamma^k \omega_\alpha{}^\beta = 2 \delta_\gamma^\beta D_\alpha^k \sigma - \frac{1}{2} \delta_\alpha^\beta D_\gamma^k \sigma \,, \tag{B.12a}$$

$$D_\alpha^i \Lambda^{jk} = -4 \varepsilon^{i(j} D_\alpha^{k)} \sigma \,, \tag{B.12b}$$

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where σ obeys

$$D^i_\alpha D^j_\beta \sigma = -\mathrm{i}\varepsilon^{ij}\partial_{\alpha\beta}\sigma\,, \quad \partial_a D^j_\beta \sigma = 0\,. \tag{B.13}$$

Finally, one can verify that the following holds

$$\partial_a \xi^\gamma_k = \frac{\mathrm{i}}{2}(\tilde{\gamma}_a)^{\beta\gamma} D_{\beta k}\sigma\,. \tag{B.14}$$

The above results tell us that we can parametrize superconformal Killing vectors as

$$\xi \equiv \xi(\lambda(P)^a, \lambda(Q)^\alpha_i\,, \lambda(M)_{ab}, \lambda(J)^{ij}, \lambda(\mathbb{D}), \lambda(K)_a, \lambda(S)^i_\alpha)\,, \tag{B.15}$$

where we have defined the parameters

$$\lambda(P)^a := \xi^a|_{x=\theta=0}\,, \qquad \lambda(Q)^\alpha_i = \xi^\alpha_i|_{x=\theta=0}\,, \tag{B.16a}$$

$$\lambda(M)_{ab} := \omega_{ab}|_{x=\theta=0}\,, \qquad \lambda(\mathbb{D}) := \sigma|_{x=\theta=0}\,, \quad \lambda(J)^{ij} = \Lambda^{ij}|_{x=\theta=0}\,, \tag{B.16b}$$

$$\lambda(K)_a := \frac{1}{2}\partial_a\sigma|_{x=\theta=0}\,, \quad \lambda(S)^i_\alpha := \eta^i_\alpha|_{x=\theta=0}\,, \tag{B.16c}$$

and we have introduced

$$\eta^i_\alpha := \frac{1}{2} D^i_\alpha \sigma\,. \tag{B.17}$$

The commutator of two superconformal Killing vectors,

$$\xi = \xi(\lambda(P)^a, \lambda(Q)^\alpha_i\,, \lambda(M)_{ab}, \lambda(J)^{ij}, \lambda(\mathbb{D}), \lambda(K)^a, \lambda(S)^i_\alpha) \tag{B.18}$$

and

$$\tilde{\xi} = \xi(\tilde{\lambda}(P)^a, \tilde{\lambda}(Q)^\alpha_i, \tilde{\lambda}(M)_{ab}, \tilde{\lambda}(J)^{ij}, \tilde{\lambda}(\mathbb{D}), \tilde{\lambda}(K)^a, \tilde{\lambda}(S)^i_\alpha)\,, \tag{B.19}$$

is another superconformal Killing vector given by

$$\begin{aligned}
[\xi, \tilde{\xi}] &= (\xi^a\partial_a\tilde{\xi}^b - \tilde{\xi}^a\partial_a\xi^b + \xi^\alpha_i D^i_\alpha\tilde{\xi}^b - \tilde{\xi}^\alpha_i D^i_\alpha\xi^b + 2\mathrm{i}\xi^\alpha_k\tilde{\xi}^{\beta k}(\gamma^b)_{\alpha\beta})\partial_b \\
&\quad + (\xi^a\partial_a\tilde{\xi}^\beta_j - \tilde{\xi}^a\partial_a\xi^\beta_j + \xi^\alpha_i D^i_\alpha\tilde{\xi}^\beta_j - \tilde{\xi}^\alpha_i D^i_\alpha\xi^\beta_j)D^j_\beta \\
&= \Big(\xi^a\tilde{\omega}_a{}^b + \xi^b\tilde{\sigma} - \tilde{\xi}^a\omega_a{}^b - \tilde{\xi}^b\sigma - 2\mathrm{i}\xi^\alpha_k\tilde{\xi}^{\beta k}(\gamma^b)_{\alpha\beta}\Big)\partial_b \\
&\quad + \Big(-\mathrm{i}\xi^a(\tilde{\gamma}_a)^{\beta\gamma}\tilde{\eta}_{\gamma j} + \frac{1}{2}\xi^\beta_j\tilde{\sigma} - \xi^\alpha_j\tilde{\omega}_\alpha{}^\beta + \xi^\beta_i\tilde{\Lambda}^i{}_j \\
&\qquad + \mathrm{i}\tilde{\xi}^a(\tilde{\gamma}_a)^{\beta\gamma}\eta_{\gamma j} - \frac{1}{2}\tilde{\xi}^\beta_j\sigma + \tilde{\xi}^\alpha_j\omega_\alpha{}^\beta - \tilde{\xi}^\beta_i\Lambda^i{}_j\Big)D^j_\beta \\
&\equiv \xi(\hat{\lambda}(P)^a, \hat{\lambda}(Q)^\alpha_i, \hat{\lambda}(M)_{ab}, \hat{\lambda}(J)^{ij}, \hat{\lambda}(\mathbb{D}), \hat{\lambda}(K)^a, \hat{\lambda}(S)^i_\alpha)\,,
\end{aligned} \tag{B.20}$$

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where

$$\hat{\lambda}^a(P) := \lambda(P)^b\tilde{\lambda}(M)_b{}^a+\lambda(P)^a\tilde{\lambda}(\mathbb{D})-2\mathrm{i}\lambda(Q)_k^\alpha\tilde{\lambda}(Q)^{\beta k}(\gamma^a)_{\alpha\beta} - \tilde{\lambda}(P)^b\lambda(M)_b{}^a-\tilde{\lambda}(P)^a\lambda(\mathbb{D})\,, \tag{B.21a}$$

$$\hat{\lambda}_i^\alpha(Q) := -\mathrm{i}(\tilde{\gamma}_a)^{\alpha\beta}\lambda(P)^a\tilde{\lambda}(S)_{\beta i}-\lambda(Q)_i^\beta\tilde{\lambda}(M)_\beta{}^\alpha+\frac{1}{2}\lambda(Q)_i^\alpha\tilde{\lambda}(\mathbb{D})+\lambda(Q)_j^\alpha\tilde{\lambda}(J)^j{}_i + \mathrm{i}(\tilde{\gamma}_a)^{\alpha\beta}\tilde{\lambda}(P)^a\lambda(S)_{\beta i}+\tilde{\lambda}(Q)_i^\beta\lambda(M)_\beta{}^\alpha-\frac{1}{2}\tilde{\lambda}(Q)_i^\alpha\lambda(\mathbb{D})-\tilde{\lambda}(Q)_j^\alpha\lambda(J)^j{}_i\,, \tag{B.21b}$$

$$\hat{\lambda}(M)_{ab} := 2\lambda(M)_{[a}{}^c\tilde{\lambda}(M)_{b]c}-4\lambda(P)_{[\hat{a}}\tilde{\lambda}(K)_{\hat{b}]}+4\tilde{\lambda}(P)_{[\hat{a}}\lambda(K)_{\hat{b}]} + 2(\gamma_{ab})_\alpha{}^\beta\lambda(Q)_k^\alpha\tilde{\lambda}(S)_\beta^k-2(\gamma_{ab})_\alpha{}^\beta\tilde{\lambda}(Q)_k^\alpha\lambda(S)_\beta^k\,, \tag{B.21c}$$

$$\hat{\lambda}(J)^{ij} := 2\lambda(J)_k{}^{(i}\tilde{\lambda}(J)^{j)k}-8\lambda(Q)^{\gamma(i}\tilde{\lambda}(S)_\gamma^{j)}+8\tilde{\lambda}(Q)^{\gamma(i}\lambda(S)_\gamma^{j)}\,, \tag{B.21d}$$

$$\hat{\lambda}(\mathbb{D}) := 2\lambda(P)^a\tilde{\lambda}(K)_a-2\tilde{\lambda}(P)^a\lambda(K)_a+2\lambda(S)_\alpha^i\tilde{\lambda}(Q)_i^\alpha-2\tilde{\lambda}(S)_\alpha^i\lambda(Q)_i^\alpha\,, \tag{B.21e}$$

$$\hat{\lambda}(K)^a := \lambda(M)^{ab}\tilde{\lambda}(K)_b+\lambda(\mathbb{D})\tilde{\lambda}(K)^a+2\mathrm{i}(\tilde{\gamma}_a)^{\alpha\beta}\tilde{\lambda}(S)_\alpha^k\lambda(S)_{\beta k} - \tilde{\lambda}(M)^{ab}\lambda(K)_{\dot{b}}-\tilde{\lambda}(\mathbb{D})\lambda(K)^a\,, \tag{B.21f}$$

$$\hat{\lambda}(S)_\alpha^i := \mathrm{i}(\gamma_a)_{\alpha\beta}\lambda(K)^a\tilde{\lambda}(Q)^{\beta i}+\lambda(S)_\beta^i\tilde{\lambda}(M)_\alpha{}^\beta-\frac{1}{2}\lambda(S)_\alpha^i\tilde{\lambda}(\mathbb{D})-\lambda(S)_\alpha^j\tilde{\lambda}(J)^i{}_j - \mathrm{i}(\gamma_a)_{\alpha\beta}\tilde{\lambda}(K)^a\lambda(Q)^{\beta i}-\tilde{\lambda}(S)_\beta^i\lambda(M)_\alpha{}^\beta+\frac{1}{2}\tilde{\lambda}(S)_\alpha^i\lambda(\mathbb{D})+\tilde{\lambda}(S)_\alpha^j\lambda(J)^i{}_j\,. \tag{B.21g}$$

Representing the superconformal Killing vectors as

$$\xi = \lambda(P)^aP_a+\lambda(Q)_i^\alpha Q_\alpha^i+\frac{1}{2}\lambda(M)^{ab}M_{ab}+\lambda(J)^{ij}J_{ij}+\lambda(\mathbb{D})\mathbb{D} + \lambda(K)^aK_a+\lambda(S)_\alpha^iS_i^\alpha \tag{B.22}$$

and comparing eq. (B.21) to the commutator

$$[\xi,\tilde{\xi}] = -\tilde{\lambda}^{\underline{b}}\lambda^{\underline{a}}[X_{\underline{a}},X_{\underline{b}}\} \tag{B.23}$$

gives the superconformal algebra.

C The Yang-Mills multiplet in conformal superspace

To describe a non-abelian vector multiplet, the covariant derivative $\nabla = E^A\nabla_A$ has to be replaced with a gauge covariant one,

$$\boldsymbol{\nabla} = E^A\boldsymbol{\nabla}_A\,,\quad \boldsymbol{\nabla}_A := \nabla_A-\mathrm{i}\boldsymbol{V}_A\,. \tag{C.1}$$

Here the gauge connection one-form $\boldsymbol{V} = E^A\boldsymbol{V}_A$ takes its values in the Lie algebra of the (unitary) Yang-Mills gauge group, G_{YM}, with its (Hermitian) generators commuting with all the generators of the superconformal algebra. The algebra of the gauge covariant derivatives is

$$[\boldsymbol{\nabla}_A,\boldsymbol{\nabla}_B\} = -T_{AB}{}^C\boldsymbol{\nabla}_C-\frac{1}{2}R(M)_{AB}{}^{cd}M_{cd}-R(J)_{AB}{}^{kl}J_{kl}-R(\mathbb{D})_{AB}\mathbb{D} - R(S)_{AB}{}_k^\gamma S_\gamma^k-R(K)_{AB}{}^cK_c-\mathrm{i}\boldsymbol{F}_{AB}\,, \tag{C.2}$$

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where the torsion and curvatures are those of conformal superspace but with $\boldsymbol{F}_{AB}$ corresponding to the gauge covariant field strength two-form $\boldsymbol{F} = \frac{1}{2}E^B \wedge E^A \boldsymbol{F}_{AB}$. The field strength $\boldsymbol{F}_{AB}$ satisfies the Bianchi identity

$$\boldsymbol{\nabla}\boldsymbol{F} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\nabla}_{[A}\boldsymbol{F}_{BC\}} + T_{[AB}{}^{D}\boldsymbol{F}_{|D|C\}} = 0\,. \tag{C.3}$$

The Yang-Mills gauge transformation acts on the gauge covariant derivatives $\boldsymbol{\nabla}_A$ and a matter superfield U (transforming in some representation of the gauge group) as

$$\boldsymbol{\nabla}_A \;\to\; \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}}\boldsymbol{\nabla}_A\mathrm{e}^{-\mathrm{i}\boldsymbol{\tau}}, \qquad U \;\to\; U' = \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}}U\,, \qquad \boldsymbol{\tau}^\dagger = \boldsymbol{\tau}\,, \tag{C.4}$$

where the Hermitian gauge parameter $\boldsymbol{\tau}(z)$ takes its values in the Lie algebra of G_{YM}. This implies that the gauge one-form and the field strength transform as follows:

$$\boldsymbol{V} \;\to\; \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}}\boldsymbol{V}\mathrm{e}^{-\mathrm{i}\boldsymbol{\tau}} + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\tau}}\mathrm{d}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\tau}}\,, \qquad \boldsymbol{F} \;\to\; \mathrm{e}^{\mathrm{i}\boldsymbol{\tau}}\boldsymbol{F}\mathrm{e}^{-\mathrm{i}\boldsymbol{\tau}}\,. \tag{C.5}$$

Some components of the field strength have to be constrained in order to describe an irreducible multiplet. The constraints are (see e.g. [88])

$$\boldsymbol{F}_{\alpha\beta}^{i\;j} = 0\,, \quad \boldsymbol{F}_{a\beta}^{\;\;j} = (\gamma_a)_{\alpha\beta}\boldsymbol{W}^{\beta i}\,, \tag{C.6a}$$

where $\boldsymbol{W}^{\alpha i}$ is a conformal primary of dimension 3/2, $S_k^\gamma \boldsymbol{W}^{\alpha i} = 0$ and $\mathbb{D}\boldsymbol{W}^{\alpha i} = \frac{3}{2}\boldsymbol{W}^{\alpha i}$. The Bianchi identity (C.3) together with the constraints (C.6a) fix the remaining component of the field strength to be

$$\boldsymbol{F}_{ab} = -\frac{\mathrm{i}}{8}(\gamma_{ab})_\beta{}^\alpha \boldsymbol{\nabla}_\alpha^k \boldsymbol{W}_k^\beta \tag{C.6b}$$

and constrain $\boldsymbol{W}^{\alpha i}$ to obey the differential constraints

$$\boldsymbol{\nabla}_\gamma^k \boldsymbol{W}_k^\gamma = 0\,, \quad \boldsymbol{\nabla}_\alpha^{(i}\boldsymbol{W}^{\beta j)} = \frac{1}{4}\delta_\alpha^\beta \boldsymbol{\nabla}_\gamma^{(i}\boldsymbol{W}^{\gamma j)}\,. \tag{C.7}$$

It is helpful to introduce the following descendant superfield:

$$\boldsymbol{X}^{ij} := \frac{\mathrm{i}}{4}\boldsymbol{\nabla}_\gamma^{(i}\boldsymbol{W}^{\gamma j)}\,. \tag{C.8}$$

The superfield $\boldsymbol{W}^{\alpha i}$ and $\boldsymbol{X}^{ij}$, together with

$$\boldsymbol{F}_\alpha{}^\beta = -\frac{\mathrm{i}}{4}\Big(\boldsymbol{\nabla}_\alpha^k \boldsymbol{W}_k^\beta - \frac{1}{4}\delta_\alpha^\beta \boldsymbol{\nabla}_\gamma^k \boldsymbol{W}_k^\gamma\Big) = -\frac{\mathrm{i}}{4}\boldsymbol{\nabla}_\alpha^k \boldsymbol{W}_k^\beta\,, \tag{C.9}$$

satisfy the following useful identities:

$$\boldsymbol{\nabla}_\alpha^i \boldsymbol{W}^{\beta j} = -\mathrm{i}\delta_\alpha^\beta \boldsymbol{X}^{ij} - 2\mathrm{i}\varepsilon^{ij}\boldsymbol{F}_\alpha{}^\beta\,, \tag{C.10a}$$

$$\boldsymbol{\nabla}_\alpha^i \boldsymbol{F}_\beta{}^\gamma = -\boldsymbol{\nabla}_{\alpha\beta}\boldsymbol{W}^{\gamma i} - \delta_\alpha^\gamma \boldsymbol{\nabla}_{\beta\delta}\boldsymbol{W}^{\delta i} + \frac{1}{2}\delta_\beta^\gamma \boldsymbol{\nabla}_{\alpha\delta}\boldsymbol{W}^{\delta i}\,, \tag{C.10b}$$

$$\boldsymbol{\nabla}_\alpha^i \boldsymbol{X}^{jk} = 2\varepsilon^{i(j}\boldsymbol{\nabla}_{\alpha\beta}\boldsymbol{W}^{\beta k)}\,. \tag{C.10c}$$

The S-supersymmetry generator acts on these descendants as

$$S_k^\gamma \boldsymbol{F}_\alpha{}^\beta = -4\mathrm{i}\delta_\alpha^\gamma \boldsymbol{W}_k^\beta + \mathrm{i}\delta_\alpha^\beta \boldsymbol{W}_k^\gamma\,, \qquad S_k^\gamma \boldsymbol{X}^{ij} = -4\mathrm{i}\delta_k^{(i}\boldsymbol{W}^{\gamma j)}\,. \tag{C.11}$$

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