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Robust Stabilization of Laminar Flows in Varying Flow Regimes

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Abstract: The stabilization of laminar flows on the base of linearizations and feedback controllers has been the subject of many recent theoretical and computational studies. However, the applicability of the standard approaches is limited due to the inherent fragility of observer based controllers with respect to arbitrary small changes in the system. We show that a slight variation in the *Reynolds* number of a flow setup amounts to a coprime factor perturbation in the associated linear transfer function. Based on these findings, we argue that known concepts from robust control can be exploited to come up with an output feedback law that can stabilize the cylinder wake over the transition period from a stable to a stabilized regime.

Keywords: robust stabilization, flow control, coprime factorization, perturbation theory, nonlinear control systems, linearization

1. INTRODUCTION

The feedback stabilization of laminar flows based on controllers designed to attenuate deviations from a stationary working point has got a substantial theoretical foundation, see Raymond (2005, 2006). The basic idea is to linearize the model about an unstable target steady-state and to define a controller based on the linearization that is capable to damp small perturbations of the target state also in the actual nonlinear model. This approach has been successfully applied in numerical experiments both with state feedback, cf. Bänsch et al. (2015), and with low-dimensional output feedback, cf. Benner and Heiland (2015a), see also Breiten and Kunisch (2014) for an example with a FitzHugh-Nagumo model. However, there remains a major conceptual problem: in applications, the unstable target state, which is the starting point of the stabilization process, may never be attained.

Therefore, we need stabilizing controllers that can operate in, say, two different regimes and in the transition between them so that the system can be safely transferred from a possible state to the desired state.

We opt for the particular scenario of a flow that is stable for low *Reynolds* numbers (Re) and unstable for a medium Re which is the target of the controlled process. Changing the Re in the course of evolution results in a change in the internal dynamics which needs to be modelled as an inherent system uncertainty. There are three commonly used classes of system uncertainties that are best described in terms of the *transfer functions* G and $\tilde{G}$ of the system and its perturbation:

(1) The *additive uncertainty* δG_a that adds to the transfer function: $\tilde{G} = G + \delta G_a$,
(2) the *multiplicative uncertainty* δG_m that multiplies the transfer function: $\tilde{G} = (1 + \delta G_m)G$, and
(3) the *coprime factor uncertainty* $\delta N, \delta M$, that perturbs a coprime factorization $G = NM^{-1}$ to give a coprime factorization of the perturbed system as $\tilde{G} = (N + \delta N)(M + \delta M)^{-1}$.

Roughly speaking, a coprime factorization $M^{-1}N$ of a transfer function G is given through *proper* and real rational stable transfer functions N and M, with M *bounded away from zero* in the right half-plane, that are coprime. These factorizations play an important role in robust control design; cf. Zhou et al. (1996). Similarly, coprime factorizations can be defined for infinite dimensional systems with finite dimensional inputs and outputs (Curtain and Zwart (1995)) and used for the design of robust controllers.

In view of applying the robust control strategies given in Curtain (2003) in the considered flow setup, we argue that

The low-Re regimes can be interpreted as a coprime factor perturbation of the medium-Re system.

In this work, we state a functional analytical framework for the boundary control of incompressible flows that uses a relaxation of the boundary conditions which makes the input of distributional type, cf. the discussion in Benner and Heiland (2015b). We derive the abstract formulation and we extend known results to show that a linearization about a target steady state can be put into the linear systems framework investigated in Curtain and Zwart (1995). We show how the *Reynolds* number affects a *coprime factorization* of the considered system's transfer function and that the linear system meets the necessary conditions for robust stabilization by finite dimensional controls. For the test case of the cylinder wake, we show that a sequence of space discretizations also fulfills sufficient conditions for robust stabilizability. In view of applications we make sure that the numerical tests are suitable for large scale problems. We conclude the paper with summarizing remarks concerning also future research on the topics.

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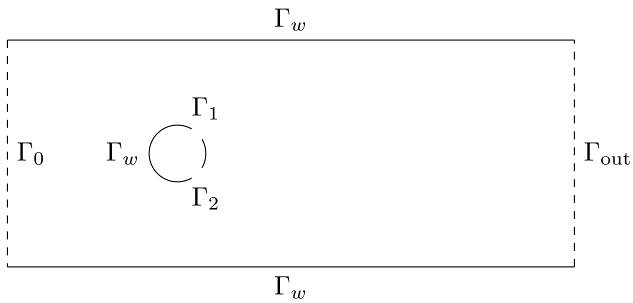

Fig. 1. Computational domain of the cylinder wake.

2. MODEL OF THE CONTROL PROBLEM

Before introducing the semigroup setting for the considered Navier-Stokes Equations (NSE), we derive the considered equations on an informal level.

For a diffusion parameter ν, we consider a NSE that models the velocity V and the pressure P of an incompressible flow for time $t > 0$ and in a domain Ω with boundary $\Gamma = \Gamma_0 \cup \Gamma_w \cup \Gamma_{\text{out}} \cup \Gamma_1 \cup \Gamma_2$, as illustrated in Figure 1,

$$\dot{V} + (V \cdot \nabla)V + \nabla P - \nu \Delta V = 0, \tag{1a}$$

$$\operatorname{div} V = 0, \quad \text{in } \Omega, \tag{1b}$$

with inflow and outflow boundary conditions

$$V = -ng_0 \cdot \alpha \text{ on } \Gamma_0 \quad \text{and} \quad \nu\tfrac{\partial V}{\partial n} - np = 0 \text{ on } \Gamma_{\text{out}}, \tag{1c}$$

with boundary control

$$V = -ng_1 \cdot u_1 \text{ on } \Gamma_1 \quad \text{and} \quad V = -ng_2 \cdot u_2 \text{ on } \Gamma_2, \tag{1d}$$

where g_0, g_1, and g_2 are shape functions modelling the spatial dimension of the boundary values, where n is the outward normal vector, where $\alpha > 0$ is a scalar, u_1 and u_2 are scalar input functions depending on time t, and with *no-slip* conditions at the walls, i.e. $V = 0$ on Γ_w. By means of α, describing the magnitude of the velocity at the inflow, we will parametrize the Reynolds number and, thus, the varying flow regimes.

If one sets $Re_\alpha = \frac{\alpha r}{\nu}$, where r denotes the cylinder radius, and rescales the domain Ω, the nondimensional equation for the nondimensional velocity $v := \frac{V}{\alpha}$ reads

$$\dot{v} + (v \cdot \nabla)v + \nabla p - \frac{1}{Re_\alpha}\Delta v = 0, \tag{2a}$$

$$\operatorname{div} v = 0, \quad \text{in } \Omega, \tag{2b}$$

with the inflow boundary condition

$$v = -ng_0 \cdot 1 \text{ on } \Gamma_0 \tag{2c}$$

and the remaining boundary conditions adapted accordingly. Here and in what follows, we tacitly redefine the pressure variable p in every step of the derivations.

Let v_{α_*} be the steady-state solution for a given target regime α_* and for $u_1 = u_2 = 0$. Then, with $v =: v_{\alpha_*} + v_\delta$, the system (2) can be rewritten as

$$\dot{v_\delta} + \mathcal{L}(\alpha_*)v_\delta + \nabla p = -(v_\delta \cdot \nabla)v_\delta, \tag{3a}$$

$$\operatorname{div} v_\delta = 0, \tag{3b}$$

with the control boundary conditions

$$v_\delta = -ng_1 \cdot u_1 \text{ on } \Gamma_1, \quad v_\delta = -ng_2 \cdot u_2 \text{ on } \Gamma_2,$$

as well as the inflow and outflow conditions,

$$v_\delta = 0 \text{ on } \Gamma_0, \quad \tfrac{1}{Re_\alpha}\tfrac{\partial v_\delta}{\partial n} - np = 0 \text{ on } \Gamma_{\text{out}},$$

no-slip conditions on Γ_w, and with

$$\mathcal{L}(\alpha_*)w := (v_{\alpha_*} \cdot \nabla)w + (w \cdot \nabla)v_{\alpha_*} - \frac{1}{Re_{\alpha_*}}\Delta w.$$

If one parametrizes the steady state solution over $\alpha(t)$, the equation for the current difference state $v_\delta := v - v_\alpha$ reads

$$\dot{v_\delta} + \mathcal{L}(\alpha)v_\delta + \nabla p = -(v_\delta \cdot \nabla)v_\delta - \dot{v}_\alpha, \tag{4a}$$

$$\operatorname{div} v_\delta = 0, \tag{4b}$$

with boundary conditions as above.

The theoretical and numerical treatment of time-depending Dirichlet conditions is a delicate problem (Benner and Heiland (2015b)) which is beyond the scope of this investigation. Thus, for a straight-forward variational formulation, we relax the present Dirichlet boundary conditions to Robin-type conditions:

$$v = -ng_iu_i \quad \to \quad v \approx -ng_iu_i - \gamma(\tfrac{1}{Re_\alpha}\tfrac{\partial v}{\partial n} - pn) \text{ on } \Gamma_i,$$

with $u_0 := \alpha$ and a parameter $0 < \gamma \ll 1$, cf., e.g., Hou and Ravindran (1998) for convergence properties of this relaxation in optimal control of flows.

We introduce the spaces

$$V^1_{\Gamma_w} := \{z \in H^1(\Omega) : \operatorname{div} z = 0 \text{ and } z\big|_{\Gamma_w} = 0\},$$

$$V^0_{n,\Gamma_w} := \{z \in L^2(\Omega) : \operatorname{div} z = 0 \text{ and } z \cdot n\big|_{\Gamma_w} = 0\},$$

and the dual space $V^{-1}_{\Gamma_w}$ with respect to the dense embedding $V^1_{\Gamma_w} \hookrightarrow V^0_{n,\Gamma_w}$, cf. Nguyen and Raymond (2015). For later use, we also define the orthogonal projector

$$\Pi \colon L^2(\Omega) \text{ onto } V^0_{n,\Gamma_w} \subset L^2(\Omega).$$

Here and in what follows, we do not distinguish notationally between scalar and vector valued Sobolev spaces. Also, since the dualities are defined as extensions of the L^2 inner product, we can identify the pivot spaces $L^2(\Omega)$ and V^0_{n,Γ_w} and their duals. We will make use of this by tacitly identifying forms and vectors in $L^2(\Omega)$ and V^0_{n,Γ_w}.

For $i = 0, 1, 2$, let $g_i \in H^{1/2}(\Gamma_i)$ and define the form $b_i \colon H^1(\Omega) \to \mathbb{R}$ via $b_iv := -\frac{1}{\gamma}\int_{\Gamma_i} ng_iv \, \mathrm{d}s$ for $v \in H^1(\Omega)$.

With this, for a given α, we define the steady state solution v_α as the (weak) solution that satisfies

$$c(v, v, w) + a(\alpha; v, w) - \big(p, \operatorname{div} w\big)_{L^2} = b_0w, \tag{5a}$$

$$\big(\operatorname{div} v, q\big)_{L^2} = 0, \tag{5b}$$

for a suitable $p \in L^2(\Omega)$, for all $w \in H^1(\Omega)$ with $w = 0$ on Γ_w and for all $q \in L^2(\Omega)$, and where the forms $a \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $c \colon H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ are defined as

$$a(\alpha; v, w) := \frac{1}{Re_\alpha}\big(\nabla v, \nabla w\big)_{L^2} + \frac{1}{\gamma}\int_{\Gamma_0 \cup \Gamma_1 \cup \Gamma_2} vw \, \mathrm{d}s, \tag{6}$$

and

$$c(u, v, w) := \big((u \cdot \nabla)v, w\big)_{L^2}. \tag{7}$$

Note that (5) is derived from (2) through partial integration considering the Robin relaxation of the nonzero Dirichlet conditions.

For the time being, we make the following assumption:

Assumption 1. Let $0 < \varepsilon < 1/2$ define the degree of regularity of the corresponding Stokes solutions on the considered domain with mixed boundary conditions, cf.

Nguyen and Raymond (2015). Then for all α in a sufficiently small neighborhood around a given $\alpha_* \in \mathbb{R}$, the steady state solution v_α to (5) is in $H^{3/2+\varepsilon}(\Omega) \cap V^1_{\Gamma_w}$ and the map $\alpha \mapsto v_\alpha \in V^1_{\Gamma_w}$ is continuous.

Here, the critical part is the existence of a solution v_α to (5), that has not been proven for the general case and that also depends on the data b_0, cf. also the appendix in Nguyen and Raymond (2015). For small α or in the neighborhood of a known solution, existence can be stated using the arguments in Kučera (1998). In Hou and Ravindran (1998), existence has been proven for the case that there are no more pure Dirichlet boundary conditions. The continuity of $\alpha \mapsto v_\alpha$ is a condition for the existence of a *branch of nonsingular solutions* which is commonly assumed for convergence of space discretizations and which has been established for several setups of Navier-Stokes equations; cf. Ch. 3 in Girault and Raviart (1986).

With the same ε and for given α, we can define the Stokes operator $A_{\alpha;0}\colon D(A_{\alpha;0}) \subset V^0_{n,\Gamma_w} \to V^0_{n,\Gamma_w}$ via

$$\begin{aligned} D(A_{\alpha;0}) := \{ & v \in H^{3/2+\varepsilon}(\Omega) \cap V^1_{\Gamma_w} : \\ & \text{there exists a } p \in H^{1/2+\varepsilon}(\Omega) \text{ such that} \\ & a(\alpha; v, \cdot) + \big(p, \operatorname{div} \cdot\big)_{L^2} \in L^2(\Omega) \text{ and} \\ & \tfrac{1}{Re_\alpha}\tfrac{\partial v}{\partial n} - np = 0 \text{ on } \Gamma_{\text{out}} \} \end{aligned}$$

and $A_{\alpha;0} = \Pi\sigma$.

Using the estimates used in the proof of (Nguyen and Raymond, 2015, Thm. 2.8) and noting that for a smooth bounded domain Ω, the space $L^4(\Omega)$ is embedded in $L^2(\Omega)$, we find that the form $C_\alpha v := c(v_\alpha, v, \cdot) + c(v, v_\alpha, \cdot)$ is an element of $L^2(\Omega)$ and we can define the Oseen operator A_α as in Nguyen and Raymond (2015) via $D(A_\alpha) = D(A_{\alpha;0})$ and $A_\alpha = A_{\alpha;0} + \Pi C_\alpha$. Note that a variation in α is but a scaling in the form a and in the boundary term, so that one can show that the domain of definition $D(A_\alpha) = D(A_{\alpha;0})$ is defined independently of α.

3. OSEEN LINEARIZATION AND ROBUST CONTROLLER DESIGN

As suggested by theoretical, cf. Raymond (2006), and numerical studies, cf. Benner and Heiland (2015a); Bänsch et al. (2015), a stabilizing controller for (3) can be designed on the base of projected linearizations for the difference v_δ from the target state v_{α_*},

$$\dot{v_\delta} + A_{\alpha_*} v_\delta = \Pi B u \quad \text{in } V^0_{n,\Gamma_w}, \tag{8}$$

where

$$Bu(w) := -\frac{1}{\gamma}\sum_{i=1}^{2}\left(\int_{\Gamma_i} n g_i w \,\mathrm{d}s\right) u_i.$$

We will consider controller design on the base of an output

$$y = C v_\delta, \tag{9}$$

with a linear output operator $C\colon V^0_{n,\Gamma_w} \to \mathbb{R}^k$, $k \in \mathbb{N}$. Note that $\Pi B\colon \mathbb{R}^2 \to V^0_{n,\Gamma_w}$, since for $g_i \in H^{1/2}(\Gamma_i)$ and given values of u_1 and u_2, the functional Bu is bounded on V^0_{n,Γ_w}, since $w \in V^0_{n,\Gamma_w} \subset H(\operatorname{div}, \Omega)$ has a well defined trace $wn \in H^{-1/2}(\Gamma)$, cf. Girault and Raviart (1986).

Theorem 2. For a given α, the operator $A_\alpha\colon D(A_\alpha) \subset V^0_{n,\Gamma_w} \to V^0_{n,\Gamma_w}$ is the generator of a C_0-semigroup.

Proof. For a Lipshitz boundary Γ and a part $\Gamma_p \subset \Gamma$ of nonzero measure, the form $H^1(\Omega) \times H^1(\Omega) \ni (v, w) \mapsto \frac{1}{Re_\alpha}\int_\Omega \nabla v \nabla w \,\mathrm{d}x + \frac{1}{\gamma}\int_{\Gamma_p} vw \,\mathrm{d}s$ is coercive, cf. Equation (1.27) in Nečas (2012), provided that γ is sufficiently small, cf. Lemma 3.3 in Hou and Ravindran (1998). Thus, one can show that the Stokes operator (6) with the Robin relaxation is coercive and use the arguments of Theorem 2.8 in Nguyen and Raymond (2015) to conclude that also the related shifted Oseen operator is coercive and therefore a generator of a C_0-semigroup.

4. REGIME CHANGE AS COPRIME FACTOR UNCERTAINTY

We are after an output-based controller that stabilizes (8) and which is robust enough to also stabilize an Oseen linearization in a different regime α in the neighborhood of α_*. We will show that the change in the regime can be interpreted as a change in the coprime factorization of the associated transfer functions.

In this case the robust controllers proposed in Curtain (2003) will be able to stabilize the system between both regimes.

First we show that a small regime change leads to a small change of the semigroup generator.

Theorem 3. Let Assumption 1 hold. Then, for given α, the operator $\delta A_\alpha := A_\alpha - A_{\alpha_*}\colon D(A_\alpha) \subset V^0_{n,\Gamma_w} \to V^0_{n,\Gamma_w}$ is bounded with $\|\delta A_\alpha\|_{\mathcal{L}(V^1_{\Gamma_w}, V^{-1}_{\Gamma_w})} \leq C_\delta$ with $C_\delta \to 0$ as $|\alpha - \alpha_*| \to 0$.

Proof. This claim follows from Assumption 1, the observation that the Reynolds number Re_α is but a factor in the form a, and from the continuity properties of the form c as laid out, e.g., in Nguyen and Raymond (2015).

We point out that, since A_α is not necessarily bounded as an operator in $L^2(\Omega)$, we can not establish a convergence result in the stronger $\|\cdot\|_{\mathcal{L}(L^2(\Omega))}$ norm. The latter would be sufficient (Pandolfi and Zwart (1991)) for the proposition, that if a feedback ΠBF stabilizes A_{α_*}, then for α sufficiently close to α_*, the same feedback also stabilizes A_α which we assume in the following theorem:

Theorem 4. Let the system (8) and (9) with the operators $(A_{\alpha_*}, \Pi B, C)$ be β-exponentially stabilizable and detectable and let $F \in \mathcal{L}(V^1_{\Gamma_w}, \mathbb{R}^2)$ and $L \in \mathcal{L}(\mathbb{R}^k, V^1_{\Gamma_w})$ so that $A_{\alpha_*} + BF$ and $A_{\alpha_*} + LC$ generate β-exponentially stable semigroups. If α is such that $A_\alpha + BF$ and $A_\alpha + LC$ generate β-exponentially stable semigroups, then the associated transfer functions $G_{\alpha_*} \sim (A_{\alpha_*}, \Pi B, C)$ and $G_\alpha \sim (A_\alpha, \Pi B, C)$ have coprime factorizations that differ by a coprime factor perturbation.

Proof. By Theorem 7.3.6 in Curtain and Zwart (1995), we have the coprime factorizations $G_{\alpha_*} = N_{\alpha_*} M^{-1}_{\alpha_*}$ and $G_\alpha = N_\alpha M_\alpha^{-1}$, where

$$\begin{aligned} N_{\alpha_*} &= C(sI - A_{\alpha_*} - \Pi BF)^{-1}\Pi B, \\ M_{\alpha_*} &= I + F(sI - A_{\alpha_*} - LC)^{-1}\Pi B, \end{aligned}$$

and

$$\begin{aligned} N_\alpha &= C(sI - A_\alpha - \Pi BF)^{-1}\Pi B, \\ M_\alpha &= I + F(sI - A_\alpha - LC)^{-1}\Pi B. \end{aligned}$$

With $A_\alpha = A_{\alpha_*} + \delta A_\alpha$, we use the formula (7.27) in Curtain and Zwart (1995) to express the inverse operators in the coprime factors as

$$\begin{aligned}(sI - A_\alpha - \Pi BF)^{-1} &= (sI - A_{\alpha_*} - \Pi BF - \delta A_\alpha)^{-1} \\ &= (sI - A_{\alpha_*} - \Pi BF)^{-1} + \delta R\end{aligned}$$

where

$$\delta R := (sI - A_{\alpha_*} - \Pi BF)^{-1}\delta A_\alpha (sI - A_\alpha - \Pi BF)^{-1}.$$

By stability of $A_{\alpha_*} + \Pi BF$ and $A_\alpha + \Pi BF$, this reformulation holds for all $s \in \mathbb{C}$ with positive real part. Thus, we can write the factors like

$$N_\alpha = N_{\alpha_*} + C\delta R \Pi B$$

and, using the same arguments for M_α, we conclude that there are coprime factorizations of G_{α_*} and G_α that coincide up to an additive component in the factors.

The incentive of this section is to show that for small changes in the regime, also the corresponding system operators only change slightly (Theorem 3) and that for small differences in the operators, the corresponding transfer functions differ only by coprime factor perturbations (Theorem 4). However, there is a gap between the theorems since the convergence in the operators has been established only in a weaker norm. To close this gap one could try to adapt perturbation results for unbounded perturbations provided, e.g., in Pandolfi and Zwart (1991).

Also note, that the *state* feedback F from the assumptions in Theorem 4, that is capable to stabilize the system for varying regimes, is not the solution to the problem. It is known from finite dimensional theory that a state feedback has a certain robustness while an observer based controller, the actual subject of this investigation, has no guaranteed robustness margins, cf. Doyle (1978).

5. CONDITIONS FOR ROBUST STABILIZATION AND NUMERICAL APPROXIMATIONS

Another assumption in Theorem 4 was the stabilizability and detectability of the considered system. In this section, we review known sufficient conditions and propose a numerically accessible test for these properties. We will restrict our considerations to the stability issue.

In the considered case, where the input operator ΠB is of finite-rank, a necessary and sufficient condition for exponential stability, cf. (Curtain and Zwart, 1995, Thm. 5.2.6), is that the spectrum of A_{α_*} can be decomposed at β so that the system can be split into

a.) a subsystem that is β-exponentially stable and
b.) a finite-dimensional subsystem that is controllable.

One can show that the operator A_{α_*}, also with the boundary conditions considered here, fulfills condition a.) since it allows for a spectrum decomposition into a β-exponentially stable subsystem and a finite-dimensional *remainder* system, cf. (Nguyen and Raymond, 2015, Sec. 3.1).

As for the condition b.), Nguyen and Raymond (2015) investigate controllability of the finite-dimensional subsystem via a *Hautus*-type criterion due to Badra and Takahashi (2011) and the theoretical result that, under certain conditions, there are finitely many control shape functions which can stabilize the system. In the present situation, however, where the shape functions g_1 and g_2 are defined by the setup, we will assess controllability numerically for the model problem and for a sequence of spatial discretizations.

Concretely, we consider the cylinder wake setup as in Benner and Heiland (2015a); cf. also Figure 1. As the computational domain we define the rectangle $[0, 2.2] \times [0, 0.41]$ with the cylinder of radius 0.05 centered at $(0.2, 0.2)$. The two control outlets Γ_1, Γ_2 are centered at the cylinder periphery at $\pm\pi/3$ occupying $\pi/6$ of the circumference. The two shape functions g_1 and g_2 are defined as parabolas that are zero at the edges and 1 at the center of the outlets.

We set $Re_{\alpha_*} := 100$, calculated with the peak inflow velocity and the cylinder diameter. For the Robin-type relaxation of the boundary conditions, we use $\gamma = 10^{-5}$.

Let N be a mesh parameter, namely the number of velocity nodes of the discretization, and let

$$\mathbf{M}_N \dot{\mathbf{v}}_N = \mathbf{A}_N \mathbf{v}_N + \mathbf{B}_N u \tag{10}$$

be a corresponding finite element approximation to (8) with $\mathbf{M}_N$, $\mathbf{A}_N \in \mathbb{R}^{N,N}$, and $\mathbf{B}_N \in \mathbb{R}^{N,2}$ denoting the mass matrix and the discrete approximation to the system and input operators A_{α_*} and ΠB. System (10) is obtained via a discretization of the associated (v,p)-formulation, cf. (3), with *LBB*-stable *Taylor-Hood* mixed finite elements and a subsequent projection onto the discrete divergence free subsystem as described, e.g., in Benner and Heiland (2015a) and Heinkenschloss et al. (2008).

The finite dimensional system (10) is exponentially stabilizable if the unstable modes are controllable, i.e., if for all left eigenvectors η_N of the pencil $(\mathbf{M}_N, \mathbf{A}_N)$ associated with an eigenvalue λ with real part $\Re\lambda > 0$, the product $\eta_N^T \mathbf{B}_N$ is not zero.

Thus, for a given discretization N, we compute the unstable modes $\eta_{N,1}, \eta_{N,2}, \dots, \eta_{N,d_u}$ and estimate the angle between these modes and the input directions via the quantity θ_N that we define as

$$\theta_N := \min_{i=1,\dots,d_u} \Big\{ \big[\sum_{j=1,2} \frac{|\eta_{N,i}^T \mathbf{B}_{N,j}|}{\eta_{N,i}^T \mathbf{M}_N \eta_{N,i} \cdot \mathbf{B}_{N,j}^T \mathbf{M}_N^{-1} \mathbf{B}_{N,j}} \big]^{\frac{1}{2}} \Big\},$$

where $\mathbf{B}_{N,j}$ denotes the j-th column of $\mathbf{B}_N$.

By its definition, the quantity θ_N serves the following purposes. Firstly, θ_N is nonzero if and only if there is not a single unstable mode such that $\eta_{N,i}^T \mathbf{B}_N = 0$, i.e., if and only if the discretized system is stabilizable. Secondly, because the chosen metrics are consistent with the corresponding norms in the infinite dimensions, if the discrete unstable modes $\eta_{N,1}, \eta_{N,2}, \dots$ converge to some modes $\eta_1, \eta_2, \dots \in V^1_{\Gamma_w}$ of the infinite dimensional system (8), then θ_N converges to

$$\theta := \min_{i=1,\dots,d_u} \Big\{ \big[\sum_{j=1,2} \frac{|b_j \eta_i|}{\|\eta_i\|_{L^2(\Omega)} \|b_j\|_{V^{-1}_{\Gamma_w}}} \big]^{\frac{1}{2}} \Big\}.$$

Thus, provided these modes $\eta_1, \eta_2, \dots, \eta_{d_u}$ are exactly the unstable modes of the infinite dimensional system (8), the resulting θ is nonzero if and only if (8) is exponentially stabilizable.

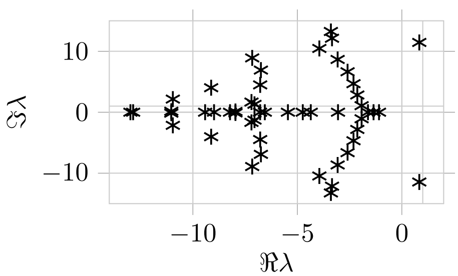

Fig. 2. Distribution of 50 eigenvalues λ of $(\mathbf{M}_N, \mathbf{A}_N)$ for $N = 68048$ and $\gamma = 10^{-5}$ in the vicinity of the origin including the two unstable ones.

Remark 5. The computation of all unstable eigenmodes is a difficult task in terms of computational time and memory requirements. To make the approach feasible for large scale systems we proceed as follows. Firstly, we will make use of the known facts that the eigenvalues of the projected matrices can be computed via a saddle point formulation, see Bänsch et al. (2015), and that the infinite eigenvalues can be shifted to an arbitrary location, cf. Cliffe et al. (1994). Secondly, we use the *shift and invert* algorithm to iteratively compute some eigenvalues around the origin. Finally, we rely on the common observation that at moderate *Re*-numbers, the cylinder wake possesses two unstable modes so that we stop the iteration when we have found them, cf. Figure 2.

In the considered setup, over a large range of discretization grades N, the quantity θ_N is nonzero, cf. Table 1, meaning that the corresponding discrete systems (10) are exponentially stabilizable. Moreover, since θ_N stays more or less constant, stabilizability seems inherent for ever finer discretization which indicates stabilizability also for the associated infinite dimensional system (8).

The code used for the computation of θ_N is available from the author's public git repository (Heiland (2015)).

Table 1. The value of θ_N (scaled by 10^7) for varying discretization levels N.

N	19512	28970	38588	48040	58180	68048
$\theta_N \cdot 10^7$	0.4	1.9	1.5	1.2	0.6	1.4

6. CONCLUSIONS

We have presented the idea of interpreting a regime change in a flow simulation as a coprime factor perturbation to conclude that a certain class of robust controllers would allow for the stabilization of flows over a small range of Reynolds numbers. Therefor, we have provided a functional analytic framework that models the control problem through an infinite-dimensional linear system and investigated how a regime change affects the underlying linear operator.

In view of applications, we have described a numerical approach to assess the stabilizability of the system which is feasible for large-scale discretizations.

However, more analytical insight into the infinite dimensional system would be desirable. This concerns stabilizability, like the convergence of the discrete unstable modes, as much as the crucial assumption for Theorem 4 that a feedback F can be stabilizable for a certain range of *Re*-numbers. The validity of the latter assumption has been observed, at least in finite dimensional approximations, cf. Benner and Heiland (2015a) where feedbacks that were stabilizing for low *Re*-numbers proved to be stabilizing initial guesses for the design of controllers for higher *Re*-numbers.

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