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IFAC PapersOnLine 50-1 (2017) 12296–12300

Convergence of Approximations to Riccati-based Boundary-feedback Stabilization of Laminar Flows [1]

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Abstract: Riccati-based feedback is commonly applied for the stabilization of flows in theory and in simulations. Nonetheless, there are few attempts to show the convergence of numerically computed feedback gains to the feedback defined by the actual model. In this work, we investigate how standard finite-dimensional formulations approximate the system dynamics and provide sufficient conditions for the convergence of the numerical approximations. The sufficient conditions are partially established for the model problem of the cylinder wake.

Keywords: Flow control, Feedback stabilization, Finite element analysis, Riccati equations, Numerical simulation

1. INTRODUCTION

In the last decade, Riccati-based boundary feedback control for the stabilization of laminar flows has been under thorough investigation both in theory, cf. Raymond (2006); Nguyen and Raymond (2015), and in numerical simulations, cf., e.g., Bänsch et al. (2015); Benner and Heiland (2015). Less interest has been shown in proving convergence of the finite-dimensional approximations used in the actual simulations. A positive answer to that is of highest interest since it would confirm that the numerically computed controllers can approximate the controllers of the actual model arbitrarily accurate.

There are two immediate difficulties in tackling convergence. Firstly, the finite dimensional approximates to projected dynamics and the related Riccati equations are so called *external approximations*, cf. Temam (1977), rather than standard Galerkin schemes. Secondly, the numerical incorporation and analysis of Dirichlet boundary conditions is a challenging problem. The other important question of whether the linear models that are used for the controller designs converge will be left to future work.

In this paper, we supplement to the comprehensive work by Badra (2006) in two particular ways. Firstly, we formulate sufficient conditions in a way that resembles the sufficient conditions for the construction of robust controllers as in Curtain (2003) that can account for system uncertainties and, thus, have the chance to work in real-world applications. Secondly, we show that a certain *inf-sup* condition is necessary and sufficient for the stable numerical approximation of the state-space of incompressible flows on which the feedback gains are defined. Furthermore, we establish parts of the conditions for the case of *Robin relaxations* of *Dirichlet* control inputs.

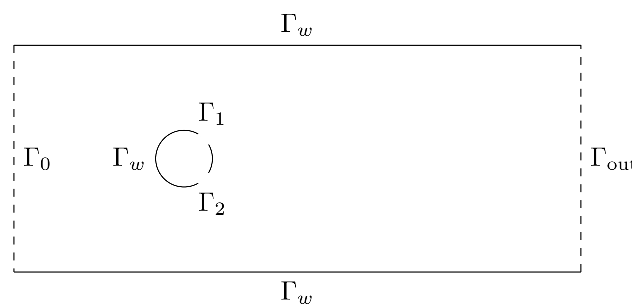

Fig. 1. Computational domain of the cylinder wake.

This manuscript is organized as follows. We will start with introducing a model problem. Then, we will briefly introduce the functional analytical framework for the linearization that is the base for the controller design. After that, we discuss how the controlled dynamics are approximated by the standard approach of *mixed Finite Elements* (FEM). For such discretizations, we then formulate necessary conditions and prove convergence of the Riccati-based feedback gains, which is the main contribution of this paper. Finally, we discuss the validity of the taken assumptions before we conclude the paper by a summary and an outlook.

2. MODEL OF THE CONTROL PROBLEM

The presented theoretical arguments will apply to a general setup with mixed boundary conditions and boundary control that can be parametrized by a finite number of inputs. For illustration, however, we relate all arguments to a model of the well-known 2D cylinder wake.

We consider the stabilization of the cylinder wake described by a Navier–Stokes equation (NSE) that models

[1] The work was supported by the *High-End Foreign Expert Program* (GDT20153100033) of the State Administration of Foreign Experts Affairs (SAFEA), P.R. China.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2017.08.2476

the evolution of the velocity V and the pressure P of an incompressible flow for time $t > 0$ and in a domain Ω with boundary $\Gamma = \Gamma_0 \cup \Gamma_w \cup \Gamma_{\text{out}} \cup \Gamma_1 \cup \Gamma_2$, as illustrated in Figure 1,

$$\dot{V} + (V \cdot \nabla)V + \nabla P - \frac{1}{Re}\Delta V = 0, \tag{1a}$$

$$\operatorname{div} V = 0, \quad \text{in } \Omega, \tag{1b}$$

with inflow and outflow boundary conditions

$$V = -ng_0 \text{ on } \Gamma_0 \quad \text{and} \quad \nu \tfrac{\partial V}{\partial n} - np = 0 \text{ on } \Gamma_{\text{out}}, \tag{1c}$$

where n is the outer normal, where g_0 is a parabola that prescribes the inflow profile, and where Re is a parameter. At the control boundaries, *Robin*-type conditions are applied

$$V = -ng_1 \cdot u_1 - \gamma(\tfrac{1}{Re}\tfrac{\partial V}{\partial n} - Pn) \text{ on } \Gamma_1 \quad \text{and} \tag{1d}$$

$$V = -ng_2 \cdot u_2 - \gamma(\tfrac{1}{Re}\tfrac{\partial V}{\partial n} - Pn) \text{ on } \Gamma_2 \tag{1e}$$

that, for a parameter $0 < \gamma \ll 1$, approximate – see, e.g., Benner and Heiland (2016); Hou and Ravindran (1998) – Dirichlet control boundary conditions with g_1 and g_2 modelling the spatial dimension of the boundary values and with u_1 and u_2 scalar input functions depending on time t. At the remaining boundaries, *no-slip* conditions are applied, i.e. $V = 0$ on Γ_w.

Apart from the general difficulties, the presence of junctions between different types of boundary conditions makes the analysis of well-posedness of (1) challenging already in the steady-state case. Some difficulties were highlighted in Nguyen and Raymond (2015). A general approach might be a suitable extension of recent results on Stokes flow; cf. Fabricius (2017).

3. STABILIZING CONTROLLER

In view of stabilizing a flow described by (1) at a target state v_*, we take the approach of linearizing the system, defining a Riccati-based feedback controller for the linear approximation, and applying it to the actual nonlinear system. This approach has been shown to work in theory, see Raymond (2006), and simulations; cf., e.g., Bänsch et al. (2015); Benner and Heiland (2015). Furthermore, sufficient conditions for the needed stabilizability of the corresponding operators has been analytically investigated in, e.g., Nguyen and Raymond (2015) and numerically tested in Benner and Heiland (2016).

For the mathematical formulation, we set $\Gamma_d = \Gamma_w \cup \Gamma_0$ – the boundary with Dirichlet conditions – and introduce the spaces

$$\mathcal{V}^1_{\Gamma_d;\mathsf{df}} := \{z \in H^1(\Omega) : \operatorname{div} z = 0 \text{ and } z\big|_{\Gamma_d} = 0\},$$

$$\mathcal{H}_{\mathsf{df}} := \{z \in L^2(\Omega) : \operatorname{div} z = 0 \text{ and } z \cdot n\big|_{\Gamma_d} = 0\},$$

and the dual space $V^{-1}_{\Gamma_d}$ with respect to the dense embedding $\mathcal{V}^1_{\Gamma_d;\mathsf{df}} \hookrightarrow \mathcal{H}_{\mathsf{df}}$, cf. Nguyen and Raymond (2015). We also define the orthogonal projector

$$\Pi \colon L^2(\Omega) \text{ onto } \mathcal{H}_{\mathsf{df}} \subset L^2(\Omega).$$

Here and in what follows, we do not distinguish notationally between scalar and vector valued Sobolev spaces. Also, since the dualities are defined as extensions of the L^2 inner product, we can identify the pivot spaces $L^2(\Omega)$ and $\mathcal{H}_{\mathsf{df}}$ and their duals. We will make use of this by tacitly identifying forms and vectors in $L^2(\Omega)$ and $\mathcal{H}_{\mathsf{df}}$. Thus, the linear model for the controller design is given by

$$\dot{v_\delta} + \Pi A_{\alpha_*} v_\delta = \Pi B u \quad \text{in } \mathcal{H}_{\mathsf{df}}, \tag{2}$$

where A_{α_*} is the operator of the *Oseen* linearization at $v = v_*$, cf. Benner and Heiland (2016); Nguyen and Raymond (2015), and the input operator is defined via

$$\langle Bu, w\rangle := -\frac{1}{\gamma}\sum_{i=1}^{2}\Big(\int_{\Gamma_i} ng_i w \,\mathrm{d}s\Big)u_i.$$

It was shown in Benner and Heiland (2016), that in the considered setup, A_{α_*} is a generator of a C^0-semigroup and that for sufficiently smooth g_1, g_2, the projected operator $\Pi B \colon \mathbb{R}^2 \to \mathcal{H}_{\mathsf{df}}$ is bounded. We will consider controller design on the base of an output

$$y = Cv_\delta,$$

with a linear output operator $C \colon L^2(\Omega) \to \mathbb{R}^k$, $k \in \mathbb{N}$.

If $(\Pi A_{\alpha_*}, \Pi B)$ is stabilizable and $(\Pi A_{\alpha_*}, C)$ is detectable, then there exists a unique non-negative self-adjoint solution X_c to the Riccati equation in $\mathcal{H}_{\mathsf{df}}$:

$$[(A_{\alpha_*}\Pi)^* X_c + X_c \Pi A_{\alpha_*} - X_c \Pi B (B\Pi)^* X_c + C^* C]z = 0 \tag{3}$$

for all $z \in D(A_{\alpha_*})$, and the feedback controller defined via

$$u(t) = -(\Pi B)^* X_c v_\delta(t)$$

stabilizes the system modelled by (2); see, e.g., Ito (1987).

4. DISCRETE APPROXIMATIONS

We consider the standard approach of *mixed Finite Elements* to discretize (1) with the Galerkin sequences

$$\{\mathcal{V}^N\}_{N\in\mathbb{N}} \text{ in } H^1_{\Gamma_d}(\Omega) \quad \text{and} \quad \{\mathcal{Q}^N\}_{N\in\mathbb{N}} \text{ in } H^1(\Omega) \tag{4}$$

for the approximation of the velocity and the pressure.

Remark 1. We require the pressure approximation to be in $H^1(\Omega)$ which is the case for some common mixed FEM and which ensures the pressure gradient to be in $L^2(\Omega)$ such that Π can be applied to it.

In view of defining the discrete operators, let $\mathcal{H}^N$ be the closure of $\mathcal{V}^N$ in $L^2(\Omega)$, let $(\mathcal{V}^N)^*$ be the dual of $\mathcal{V}^N$, and let $(\mathcal{Q}^N)^*$ be the dual of $\mathcal{Q}^N$ – both with respect to $L^2(\Omega)$. Then, the FEM approximation of (1), that defines approximations v^N and p^N of V and P with $v^N(t) \in \mathcal{V}^N$ and $p^N(t) \in \mathcal{Q}^N$, reads

$$\dot{v}^N + \mathcal{N}^N(Re; v^N) + \nabla^N p^N = B^N u + f^N \quad \text{in } (\mathcal{V}^N)^*, \tag{5a}$$

$$\operatorname{div}^N v^N = 0 \quad \text{in } (\mathcal{Q}^N)^*, \tag{5b}$$

where $\operatorname{div}^N \colon \mathcal{H}^N \to (\mathcal{Q}^N)^*$ is defined via

$$\operatorname{div}^N v^N = g \in (\mathcal{Q}^N)^*, \quad \text{if} \quad \int_\Omega \operatorname{div} v^N \cdot q^N \,\mathrm{d}x = \langle g, q^N\rangle$$

for all $q^N \in \mathcal{Q}^N$ and with

$$\nabla^N := (\operatorname{div}^N)^* \colon \mathcal{Q}^N \to (\mathcal{H}^N)^* \cong \mathcal{H}^N.$$

Furthermore, in (5), the term $\mathcal{N}(Re; v^N)$ is the discrete approximation to $(v \cdot \nabla v)v - \frac{1}{Re}\Delta v$, the inhomogeneity $f^N \in \mathcal{H}^N$ accounts for the inflow condition, and $B^N \in \mathcal{H}^N$ is the restriction of B to $\mathcal{H}^N$.

Let $\mathcal{H}^N_{\mathsf{df}} := \ker(\operatorname{div}^N) \subset \mathcal{H}^N$ be the space of discretely divergence-free functions and let Π^N be the $L^2(\Omega)$-orthogonal projector onto $\mathcal{H}^N_{\mathsf{df}}$, cf. also Linke and Merdon

(2016). Then the $\mathcal{V}^N/\mathcal{Q}^N$ finite-dimensional approximation to (2) is given as

$$\dot v^N + \Pi^N A^N v^N = \Pi^N B^N u \quad \text{in } \mathcal{H}^N_{\mathsf{df}} \tag{6a}$$
$$y^N = C v^N, \tag{6b}$$

where A^N is the finite-dimensional approximation to A_{α_*}, and the unique non-negative solution X_c^N to the finite dimensional Riccati equation in $\mathcal{H}^N$

$$(A^N\Pi^N)^* X_c^N + X_c^N \Pi^N A^N - X_c^N \Pi^N B^N (B^N \Pi^N)^* X_c^N + C^*C = 0 \tag{7}$$

stabilizes $\Pi^N A^N$ in (6), provided that $(\Pi^N A^N, \Pi^N B^N)$ and $(\Pi^N A^N, C)$ are stabilizable and detectable, respectively.

Remark 2. For the numerical realization and solution of the finite dimensional approximation (6) and (7), one assembles matrices that represent the linear operators like

$$\mathbf{G} \leftrightarrow \operatorname{div}^N \text{ – the discrete divergence,}$$
$$\mathbf{G}^T \leftrightarrow \nabla^N \text{ – the discrete gradient,}$$
$$\mathbf{M} \leftrightarrow j \text{ – the discrete } \textit{Riesz} \text{ isomorphism,}$$

and solves for coefficient vectors $\mathbf{v}^N(t)$, $\mathbf{p}^N(t)$ representing the discrete velocity and pressure approximations. Then, the discrete realization of Π^N is given as

$$\mathbf{\Pi} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{G}^T(\mathbf{G}\mathbf{M}^{-1}\mathbf{G}^T)^{-1}\mathbf{G},$$

as it is frequently considered in the literature concerned with simulation, model reduction, and control of incompressible flows Benner and Heiland (2015); Heinkenschloss et al. (2008) where also its implicit realization for efficient numerical treatment is discussed.

The question now is whether the stabilizing feedback gain $-(\Pi^N B^N)^* X_c^N$ defined via (7) will approach $-(\Pi B)^* X_c$ which stabilizes the actual model (2) as $N \to \infty$, i.e. as $\mathcal{V}^N \to H^1_{\Gamma_d}(\Omega)$ and $\mathcal{Q}^N \to H^1(\Omega)$.

5. CONVERGENCE OF THE DISCRETE CONTROLLERS

In this section, we discuss the convergence of the finite-dimensional approximations. The presented results are standard, see, e.g., Ito (1987); Banks and Kunisch (1984), for Galerkin approximations of ordinary evolution equations. We show how the results extend to mixed-FEM approximation of the projected Oseen equation (2) and confirm the validity of the assumptions in the present case.

The first assumption is that for the scheme of approximation via (6), that is induced by $\mathcal{V}^N/\mathcal{Q}^N$, for the projected equation (2), it holds that:

(A0) With P^N being the orthogonal projector onto $\mathcal{H}^N$, the operator

$$R^N := \Pi^N P^N \colon \mathcal{H}_{\mathsf{df}} \to \mathcal{H}^N_{\mathsf{df}}$$

is bounded independent of N and

$$R^N z \to z, \text{ for any } z \in \mathcal{H}_{\mathsf{df}} \text{ as } N \to \infty.$$

For standard Galerkin schemes, where basically $R^N = P^N$, this assumption is generally valid. In the presented case, where in general the discrete spaces are not nested and where, in particular, $\mathcal{H}^N_{\mathsf{df}} \not\subset \mathcal{H}_{\mathsf{df}}$, assumption **(A0)** needs to be checked for every choice of $\mathcal{V}^N$ and $\mathcal{Q}^N$.

The following assumptions are then standard and are formulated as in Ito (1987).

(A1) Let $S^N(t) = e^{\Pi^N A^N t}$, $t > 0$ and let S denote the C^0-semigroup generated by A_{α_*}; cf., e.g., Benner and Heiland (2016); Nguyen and Raymond (2015). Then, for each $z \in \mathcal{H}_{\mathsf{df}}$, it holds that

$$S^N(t)R^N z \to S(t)z \quad \text{and} \quad (S^N)^*(t)R^N z \to S^*(t)z,$$

where the convergences are uniform in z in t on bounded subsets of $[0, \infty)$.

(A2) For each $u \in \mathbb{R}^2$, $\Pi^N B^N u \to \Pi B u$.

(A3) The family of pairs $(\Pi^N A^N, \Pi^N B^N)$ is uniformly stabilizable, i.e. there exists a sequence of operators $K^N \in \mathcal{L}(\mathcal{H}^N_{\mathsf{df}}, \mathbb{R}^2)$ such that $\sup_N \|K^N\| < \infty$ and

$$\|e^{(\Pi^N A^N - \Pi^N B^N K^N)t} R^N\| \le M_1 e^{-\omega_1 t}, \quad t \ge 0,$$

and the family of pairs $(\Pi^N A^N, C)$ is uniformly detectable, i.e., there exists a sequence of operators $L^N \in \mathcal{L}(\mathbb{R}^k, \mathcal{H}^N_{\mathsf{df}})$ such that $\sup_N \|L^N\| < \infty$ and

$$\|e^{(\Pi^N A^N - L^N C)t} R^N\| \le M_2 e^{-\omega_2 t}, \quad t \ge 0,$$

for some positive constants M_1, M_2, ω_1, and ω_2.

Remark 3. Note that **(A2)** does not include assumptions on $(\Pi^N B^N)^*$ or C, since with **(A2)**, with B and C finite-dimensional, and with $C^N = C$, the assumptions in (Ito, 1987, (H2)) are readily fulfilled.

If not indicated otherwise, the norms correspond to the $L^2(\Omega)$ norm which is the norm for $\mathcal{H}_{\mathsf{df}}$ too, and the convergence is measured in $L^2(\Omega)$.

With these assumptions, we can state that the solutions X_c^N of the finite dimensional Riccati equations (7) uniformly stabilize the finite dimensional model and that they converge to the solution of the infinite-dimensional Riccati equation.

Theorem 4. Let assumptions **(A0)**–**(A3)** be satisfied. Then, for each N, the finite-dimensional Riccati equation (3) admits a unique nonnegative solution X_c^N with $\sup_N \|X_c^N\|_{\mathcal{L}(\mathcal{H}^N_{\mathsf{df}})} < \infty$ and $X_c^N R^N z \to X_c z$, for each $z \in \mathcal{H}_{\mathsf{df}}$, and there exist positive constants M_3 and ω_3 independent of N such that

$$\|e^{(\Pi^N A^N - (\Pi^N B^N)(\Pi^N B^N)^* X_c^N)t} R^N\| \le M_3 e^{-\omega_3 t}, \quad t \ge 0.$$

Proof. The proof of the uniform bounds on X_c^N and the uniform stabilization property is given for a standard Galerkin discretization in (Ito, 1987, (Thm. 2.1)) and, by virtue of **(A0)**, readily extends to the considered case. The convergence of the discrete Riccati solutions is stated in (Banks and Kunisch, 1984, (Thm. 2.2, Rem. 2.1)).

Next, we show that under an *inf-sup* condition on the discrete divergence div^N and, thus, on $\mathcal{V}^N$ and $\mathcal{Q}^N$, assumptions **(A0)** and **(A3)** hold.

Lemma 5. If the Galerkin sequences $\mathcal{V}^N$ and $\mathcal{Q}^N$ in (4) fulfill the condition that

$$\inf_{0 \ne q^N \in \mathcal{Q}^N} \sup_{0 \ne h^N \in \mathcal{H}^N} \frac{\int_\Omega q^N \cdot \operatorname{div}^N h^N \,\mathrm{d}x}{\|q^N\|_{H^1(\Omega)} \|h^N\|_{L^2(\Omega)}} \ge \beta > 0, \tag{8}$$

with a constant β independent of N, then **(A0)** holds.

Remark 6. Condition (8) is often referred to as *LBB*-condition; cf. Girault and Raviart (1986). Note, however, that the formulation above is with respect to

$L^2(\Omega)/H^{-1}(\Omega)$ as opposed to the standard case concerning velocity/pressure approximations in $H^1(\Omega)/L^2(\Omega)$. Since its difference to the standard form is only a shift in the spaces, condition (8) is probably readily established in a similar fashion for given mixed-finite elements. Anyways, condition (8) is closely related to the uniform boundedness of Π^N, cf. (Heiland, 2014, (Rem. 4.16)), and, in fact, also a necessary condition for the uniform boundedness of R^N; cf. (Girault and Raviart, 1986, Lem. II.1.1).

Proof. The convergence of $R^N v$ to v for any $v \in \mathcal{H}_{\mathsf{df}}$, can be inferred from the application of (Heiland, 2014, (Lem. 4.25)) which's proof can be extended to show also the uniform boundedness:

Let $v \in \mathcal{H}_{\mathsf{df}}$, then $P^N v = v^N_{\mathsf{df}} + v^N_{\mathsf{c}}$, where $v^N_{\mathsf{df}} = R^N v \in \mathcal{H}^N_{\mathsf{df}}$ and $v^N_{\mathsf{c}} \in (\mathcal{H}^N_{\mathsf{df}})_\perp$, i.e., in the orthogonal complement of $\mathcal{H}^N_{\mathsf{df}}$. Accordingly, $R^N v = P^N v - v^N_{\mathsf{c}}$. Since with $\operatorname{div} v = 0$, the restriction $\operatorname{div}^N v = 0$, and since $\operatorname{div}^N(R^N v) = 0$, it holds that

$$\operatorname{div}^N(v - P^N v) = \operatorname{div}^N(v - R^N v - v^N_{\mathsf{c}}) = -\operatorname{div}^N v^N_{\mathsf{c}}.$$

Assumption (8) implies that $\operatorname{div}^N : \mathcal{H}^N \to (\mathcal{Q}^N)^*$ has a uniformly bounded right inverse (Girault and Raviart, 1986, Lem. I.4.1) or, equivalently, that $\operatorname{div}^N\big|_{(\mathcal{H}^N_{\mathsf{df}})_\perp}$ is uniformly bounded from below by a constant η. Thus, we find that

$$\begin{aligned}\|v^N_{\mathsf{c}}\| \leq & \frac{1}{\eta}\|\operatorname{div}^N v^N_{\mathsf{c}}\| = \frac{1}{\eta}\|\operatorname{div}^N(v - P^N v)\| \\ \leq & \frac{\|\operatorname{div}\|_{\mathcal{L}(L^2(\Omega),H^{-1}(\Omega))}}{\eta}\|I - P^N\|\|v\|.\end{aligned}$$

With this and with $\|P^N\|$ being bounded independently of N, we can infer that

$$\|R^N v\| \leq \|P^N v\| + \|v^N_{\mathsf{c}}\| \leq \zeta\|v\|,$$

with ζ independent of N, which implies that $R^N : \mathcal{H}_{\mathsf{df}} \to \mathcal{H}^N_{\mathsf{df}}$ is bounded independent of N.

Finally, we show that if (**A0**) holds and if Bu is uniformly bounded in the discrete spaces, then, in the presented setup (2) and (6), the projected discrete approximations $\Pi^N B^N$ converge to $\Pi^N B^N$. Again, for standard Galerkin schemes, where $(\Pi B)^N$ is the restriction to the subspace, this result is readily established.

Lemma 7. Let (**A0**) hold and assume that Bu is uniformly bounded on $R^N\mathcal{H}_{\mathsf{df}}$. Then, for each $u \in \mathbb{R}^2$, $\Pi^N B^N u \to \Pi B u$ in $L^2(\Omega)$.

Proof. By the definition of B it holds that

$$\begin{aligned}\langle \Pi^N B^N u, w\rangle &= -\frac{1}{\gamma}\sum_{i=1}^{2}\left(\int_{\Gamma_i} n g_i R^N w \,\mathrm{d}s\right)u_i \\ &= \langle Bu, R^N w\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\|\Pi^N B^N u - \Pi B u\| &= \sup_{w\in\mathcal{H}_{\mathsf{df}}, \|w\|=1}\|Bu(R^N w - \Pi w)\| \\ &\leq \|Bu\|\|R^N w - w\|\end{aligned}$$

which goes to zero by the assumed boundedness of Bu and by Lemma 5.

Remark 8. Establishing validity of assumption (**A1**) and (**A3**) is not discussed here. A particular problem with (**A1**) is that already the existence of A_{α_*} is not clear, since its definition involves the associated steady-state solution. As discussed in the appendix of Nguyen and Raymond (2015) the existence of such a steady state solution is not guaranteed in general. The existence of a uniformly stabilizing controller as required by (**A3**) may be obtained by suitable extensions of the results provided in (Lasiecka and Triggiani, 2000, Ch. 4).

6. CONCLUSION

The presented theoretical results establish sufficient conditions for the convergence of finite-dimensional approximations of Riccati-based feedback stabilization for flow equations. We also partially proved the validity of the basic assumptions in the example set up of the cylinder wake. The presented theory justifies the approach taken in simulations that have been reported recently. Moreover, the conditions (**A0**)–(**A3**) also theoretically undermine the synthesis of robust controllers as proposed in (Curtain, 2003, (Thm. 4.10)).

The validity of assumptions (**A1**) and (**A3**) is still open. First insights into this issues may be obtained through numerical investigations. Similarly, the confirmation of (**A0**) that needs to be obtained for any given choice of finite-element pairs might be supported by known numerical approaches for calculating such *inf-sup* constants.

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