GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

Fakultät für Physik

**Bachelor's Thesis**

# Projective Spacetime-Geometry of Smectic A Textures

# Projektive Raumzeitgeometrie von smektisch A Texturen


prepared by

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# Abstract

The aim of this work is to get a relation between smectics (SmA) and relativity theory. Through whose spacetime symmetries one obtain new insights about SmA. The properties of SmA are decisive determined by “focal conic domains“ (FCD) respectively the “focal lines“. With FCD one attain to Minkowskispacetime and identified SmA to projections of intersecting light cones. Then generalised that to curved spacetimes: de-Sitter-spacetime and anti-de-Sitter-spacetime. Finally one transform the free energy density between SmA's by using spacetime symmetries.



**Keywords:** liquid crystals, focal conic domains, FCD, smectic, relativity theory, spacetime symmetry, light cones, projection, Minkowski, focal lines, de-Sitter-spacetime, anti-de-Sitter-spacetime, free energy density



# Zusammenfassung

Ziel dieser Arbeit ist es eine Beziehung zwischen smektischen Flüssigkristallen (SmA) und der Relativitätstheorie herzustellen. Über dessen Raumzeitsymmetrien werden neue Erkenntnisse über SmA gewonnen. Eigenschaften von SmA sind maßgeblich durch “focal conic domains“ (FCD) bzw. den “focal lines” bestimmt. Mithilfe der FCD gelangt man zur Minkowskiraumzeit und identifiziert SmA mit Projektionen von sich schneidenen Lichtkegeln. Verallgemeinert wird dieses auf gekrümmte Raumzeiten: de-Sitter-Raumzeit und anti-de-Sitter-Raumzeit. Schließlich transformiert man die freie Energiedichte zwischen SmA's mithilfe von Raumzeitsymmetrien.



**Stichwörter:** Flüssigkristalle, focal conic domains, FCD, smektisch, Relativitätstheorie, Raumzeitsymmetrien, Lichtkegel, Projektion, Minkowski, focal lines, de-Sitter-Raumzeit, anti-de-Sitter-Raumzeit, freie Energiedichte

# Contents

# 1 Introduction

Friedrich Reinitzer first described liquid crystals in 1888 [1]. He observed a further phase between solid and liquid. Liquid crystals distinguish by the fact, that they have fluid properties as well as show anisotropy. The most of them are thermotropic, which means they form different liquid crystalline phases depending on temperature. They are subdivided into nematic, smectic and columnar phases. The molecules of the nematic liquid crystals are rod shaped. The nematic phase is characterized by a preferred orientation, although the barycenters of the molecules are statistical distributed. The smectic phase, which also consist of rod shaped molecules, has additional a layer order. Inside a layer exists also a preferred orientation. Both phases have no postion long range order. The columnar phase is a little bit different. Columnar liquid crystals consist of discoidal or wedge-shaped molecule. To form pillars is characteristic for this phase.

Under the microscope with crossed polarizers every phase has typical textures. Thread-like and schlieren textures are typical for nematics, Schlieren and fan textures are typical for smectics and amongst others mosaic textures for the columnar phase. One constrains on smectic liquid crystals (Sm). They are chronological denoted after their discovery with SmA, SmB, SmC etc.. Precise examination shows that they are only five phases (SmA, SmB, SmC, SmF, SmI).

Today, Liquid crystals play a big role in different areas. Thermotropic liquid crystals, which change the color depending on the temperature, can be use as temperature-dependent sensors. In flat screens one takes advantage of optoelectric effects. The liquid crystals change their polarisation by changing the applied electric tension.

In the context of this report one constrains on SmA phases. One makes a mathematical analogy to relativity theory to use spacetime symmetries to get new insights of the smectic liquid crystals.

The geometry of smectics is leading determined by "*focal conic domains*" (FCD) which are introduced in chapter 2. One use FCD to get an relation to relativity theory (see chapter 3). At first one consider the flat Minkowski spacetime (see section 3.1). Then use the Minkowski symmetries to relates different smectic textures and look what changes by Lorentz boosting (see section 3.1.1). After that, one generalised then the insights to general realtivity (see section 3.2). The description of smectics as projection from a higher dimensional spacetime will be generalised to the de-Sitter and anti-de-Sitter spacetime (see section 3.2.1). How the free energy of the liquid crystals changes by changing textures trough Lorentz boosting is discussed in section 4. In the end one conclused the report by giving a résumé about the main results (see chapter 5).

# 2 Focal Conic Domains

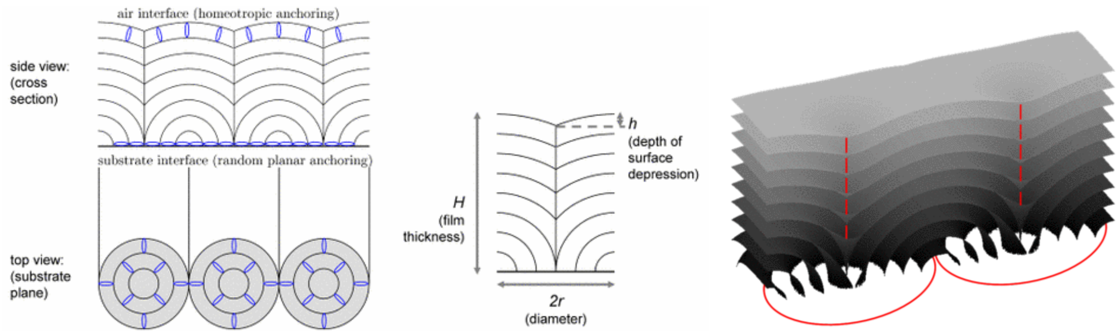


Figure 2.1: Focal Conic Domains: The blue ellipses shows the molecules of the liquid crystals. Left: schematic side and bird's eye view. Center: schematic presentation of one focal conic domain. Right: cross section of focal conic domains. [2].

When one put thin films of smectic liquid crystals on silicon they are laid to two different interfaces. At both the silicon-liquid crystal and liquid crystal-air interface apply different boundary conditions. The liquid crystals align themselves in parallel with the silicon layer, this implies that the layers of the liquid crystals are perpendicular to the silicon layer. On the other hand the crystals align perpendicular to the air layer (the layers of the liquid crystals are in parallel to the air layer). In order to satisfy both boundary conditions, single layers of the smectic liquid crystal are forced to bend. Because of these two opposite boundary conditions "*focal conic*[1] *domains*" (FCDs) form. Figure (2.1) shows a simple sketch of the molecule configurations and the profile of the layers. One use the same definition of FCDs as M. Kléman and O.D. Lavrentovich [3, 4]:

"A focal conic domain (FCD) is the region in space
of the sample which relates to a given pair of conjugate
conics. In effect, in a sample where the only defects present

[1] A conic is an abbreviated version for cone intersection.

are pairs of conjugate conics, any molecule either belongs
to a given domain which can be ascribed to a given pair,
or belongs to none."

In the center of Fig. (2.1) it is illustrates the cross section of one single FCD with film thickness $H$ and diameter $2r$. A characteristic for layer ordering are conics as defects, how one can see by FCDs [5]. What conjugates conics are and what they have do with FCDs will be explained in the next section (2.1).

Textures of liquid crystal can viewed with a petrographic microscope. This is a light microscope with crossed polarizers, that is their optical axes[2] are rotated by 90°. Anisotropic materials rotate the polarization of the propagating light. Thereby, the second polarizer lets pass only a fraction of incoming light. Isotropic constituent parts leaves the direction of vibrations unchanged and generate black areas. Multi colour pictures can originate from interference. For different liquid-crystalline phases (nematic, smectic, comlumnar) exist different characteristic textures under the petrographic microscope [6].

## 2.1 Geometry of Focal Conic Domains

Understanding the geometry of FCDs is important to investigate the behavior of smectics. G. Friedel developed geometrical rules liquid crystals obey [3, 4, 8]. He observed thread-like defects, so called disclinations[3], and deduced the molecules structure of nematics. Namely that the molecules lie parallel but random located. He also discovered the lamellar structure of SmA's by using the petrographic microscope and seeing large scale defects in the shape of conjugate pairs of conics. First one consider some elementary properties of FCDs, especially in the case of parallel layers. Then explain the two laws of Friedel.

The balance of dilation and curvature terms in the free energy $F = \frac{1}{2}\int \mathrm{d}^d x\{C((\nabla\phi)^2 - 1)^2 + S(\nabla^2\phi)^2\}$ is the reason for having preferred elastic distortions, so that the layers maybe curved stays parallel [3, 4]. This dominate the large scale defects [3, 4]. If one talk about a layer $L_i$ one means the mid-surfaces instead of material layers,

[2]An optical axis is an imaginary line in that direction light rays propagates through an optical system like crystals. In our case smectics behaves like crystals in $z$ direction.

[3]Disclination is a topological line defect. It is any defect in rotational symmetry, especially in the orientation of the director, which is equivalent to defects in the orientation of the molecules of liquid crystals [4, 9, 10]. F.C. Frank first used the word "disinclination" and later this word changed to disclination [10].

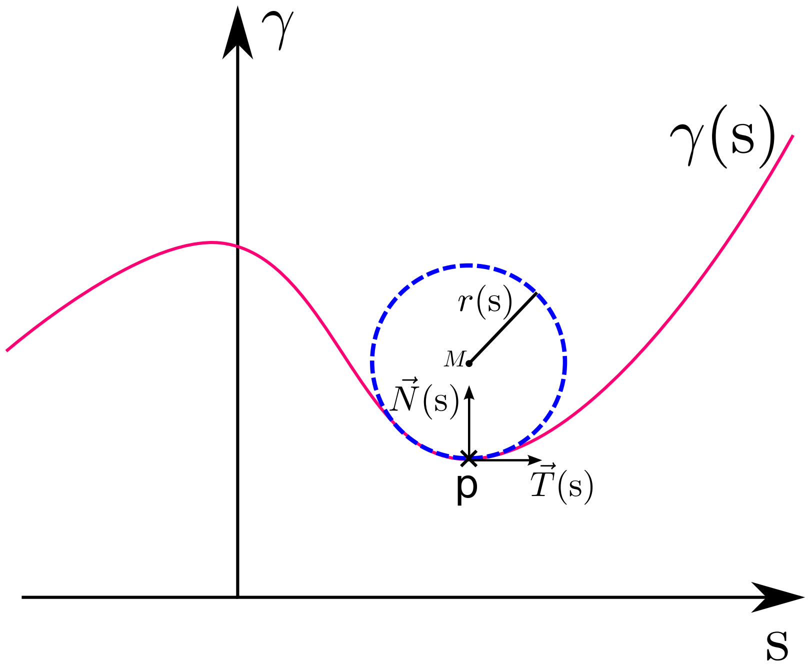


Figure 2.2: Osculating Circle. The parametric plane curve $\gamma(s)$ is regular, in which $s$ is the arclength. The osculating circle (blue circle) at point $p$ with unit tangent vector $\vec{T}(s)$, unit normal vector $\vec{N}(s)$, radius of curvature $r(s)$ and center of curvature $M$ is the circle which approximate the curve in this point at best [7].

which are also parallel [3, 4, 11].

The osculating circle[4] of a curve at point $p$ is the circle which approximate the curve in this point $p$ at best (shown in Fig. (2.2)). Let $\gamma(s)$ be a regular parametric plane curve with $s$ as arclength. Then one can define the unit tangent vector $\vec{T}$, the unit normal vector $\vec{N}$, the signed curvature $\sigma(s)$, the Gaussian curvature $G$ and the

[4]From Latin “circulus osculans“ which stands for “kissing circle“, named by Leibniz

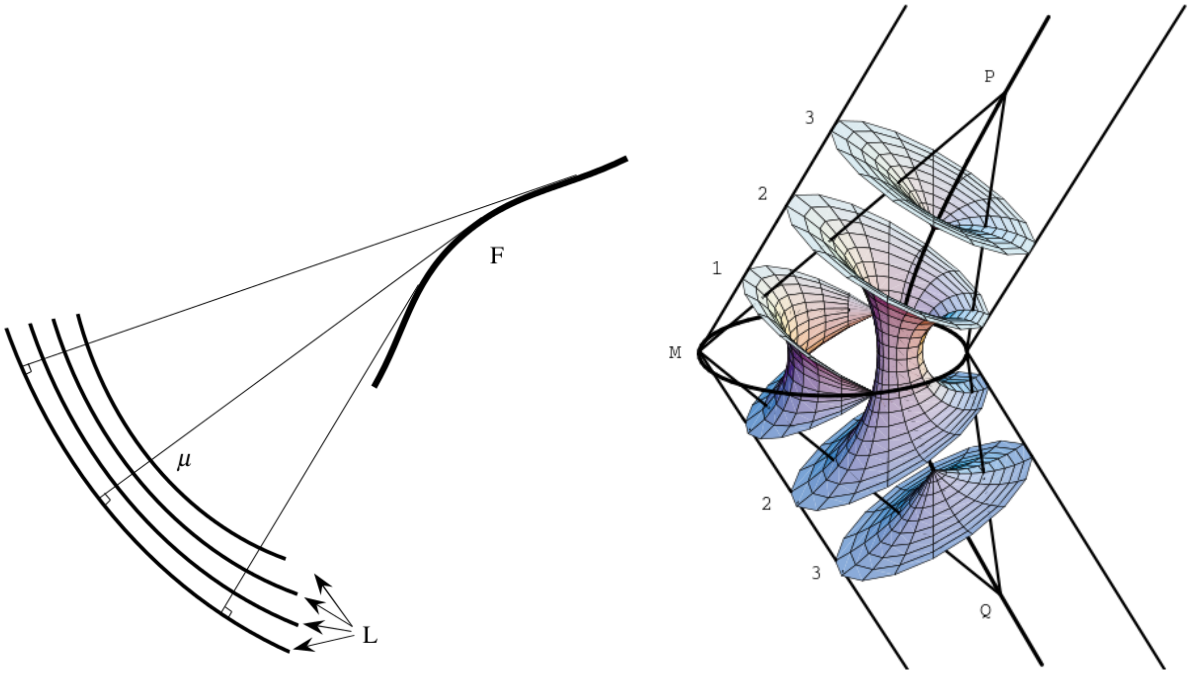


Figure 2.3: Left: Corresponding of the focal surface $F$ and the set of parallel layers $L$. The intersections with the layers $L$ and the normals are describe as $\mu$. It is only shown the one-dimensional case. Right: Focal conic domain with negative Gaussian curvature bounded by cones with apexes at $P$ and $Q$. The ellipse and the hyperbola form a pair of conjugate conics [3, 4].

radius of curvature $r$ as follow [12]:

$$\vec{T}(s) = \gamma'(s)\vec{e}$$
$$\vec{T}'(s) = \sigma(s)\vec{N}(s)$$
$$r(s) = \frac{1}{|\sigma(s)|}$$
$$G = \sigma_M \cdot \sigma_P = \frac{1}{r_M} \cdot \frac{1}{r_P}.$$

If one have instead a regular surface then exist at every point $p$ a minimum and a maximum value of the curvature. This values are the principal curvatures at point $p$ [12]. Connected to these curvatures one have two osculating circles and two principal radii. The center of the osculating circles are called centers of curvature.

An elementary result of the geometry of parallel surfaces (especially layers) is that the same surfaces have the same osculating circles, one for each principal curvature,

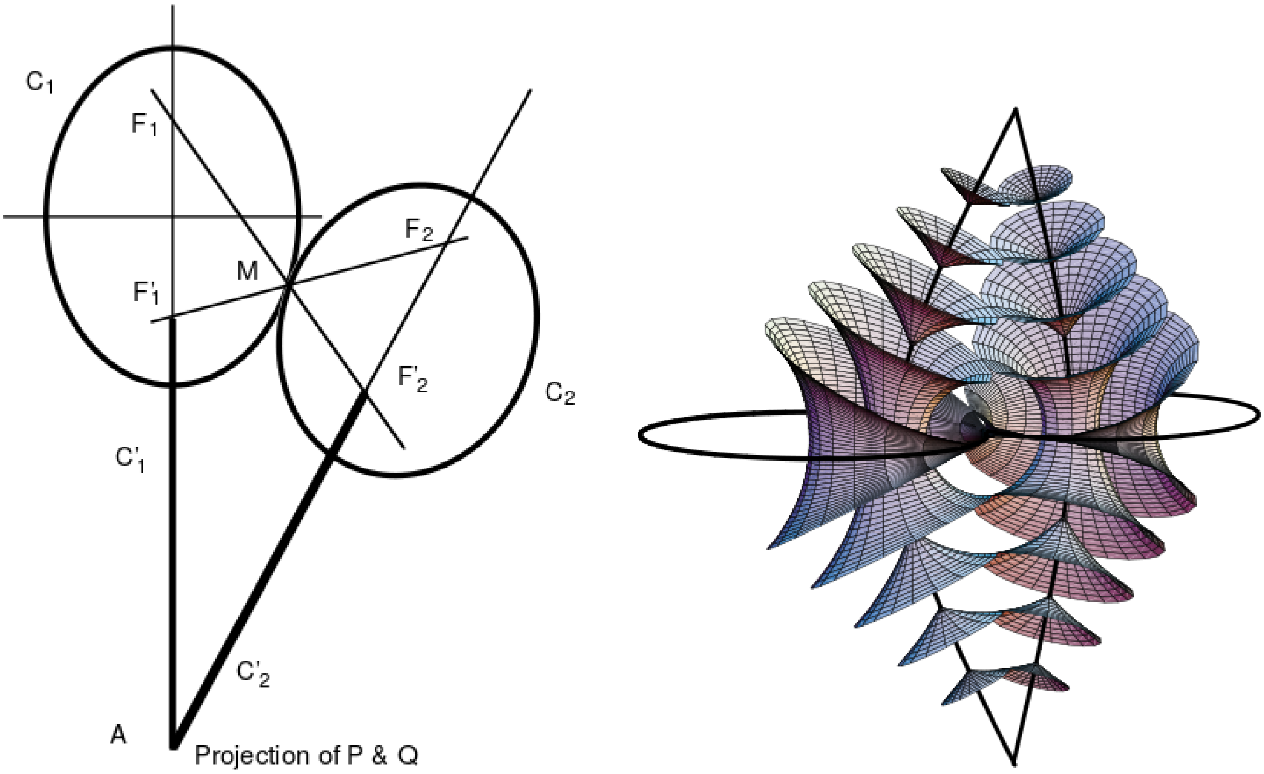


Figure 2.4: Left: schematic 2D view of the law of corresponding cones. Right: Law of corresponding cones in 3D with the FCDs [3, 4]

at every intersection point $\mu$ with the layers and one normal. The centers of curvature $M$ and $P$ are always the same at these intersections but shifted. Because of having the same osculating circles at every $\mu$ one has always the same principal radii of curvature $r_M = \overline{\mu M}$, $r_P = \overline{\mu P}$ and the same principal curvatures $\sigma_M = \frac{1}{r_M} = \frac{1}{\overline{\mu M}}$ and $\sigma_P = \frac{1}{r_P} = \frac{1}{\overline{\mu P}}$ [3]. If one consider all centers of curvature $M$'s and $P$'s at all $\mu$ so one will get two surfaces $F_M$ and $F_P$ (shown in Fig. (2.3)). This surfaces are the envelopes of the normals and are called focal surfaces. One radius of curvature at the contact between one layer $L_i$ and one focal surface is zero, so that the curvature is infinite. For this reason the focal surfaces are on a par with the set of singularities of the parallel layers [3, 4]. To describe the defects one must describe the singularities.

There exist an analogy between this properties of parallel layers and geometrical optics. The normals to the layers $L_i$'s are analogs to light rays, which propagate through the medium. The layers $L_i$'s are analogs to equal phases and focal surfaces are analogs to caustics, because the free energy density (4.4) diverge at the focal surfaces and the light intensity diverge by caustics [3].

Because of the analogy to the caustics in geometrical optics the focal surfaces, two-dimensional defects, degenerate in focal lines, one-dimensional defects for SmA[5] [3]. A famous theorem by Dupin says [3, 4, 11]:

"If the focal manifolds are both lines, these lines are necessarily a pair of conjugate conics."

Two conics which lie in mutually orthogonal planes in such a way that the foci of one conic conform with the apexes of the another and the other way round are by definition conjugated. A common situation is when ellipse and hyperbola form a pair of conjugate conics (so the foci of the ellipse are the apexes of the hyperbola)[6]. Then the layers adopts the shape of Dupin cyclides[7] [3, 4, 11]. The easiest case is a circle instead of an ellipse, then the hyperbola is a straight line and the Duplin cyclides are parallel tori [3, 4].

A torus splits into two parts, one part with positive and one part with negative Gaussian curvature. In a physical experiment an observer can only see one part of the torus [3]. Therefore one restrict our treatment to the part with negative Gaussian curvature[8]. A FCD of negative Gaussian curvature with an ellipse and a hyperbola as a pair of conjugate conics is shown right in Fig. (2.3). At all intersections of the line $\overline{MP}$ with the layers $L_i$'s ($i = 1, 2, 3$) $M$ and $P$ are the centers of curvature [3, 4].

Now one will focus our treatment on the work of Friedel and his laws about the geometrical and physical behavior of FCDs. This laws allow a correct analysis of FCDs at large scales [3, 4]. Both make a point about the contact of two FCDs. The first, the law of impenetrability, says that parts of two FCDs can not occupy the same space at the same time. Or in Friedel's words [3, 4, 8]:

"*Law of impenetrability*: two FCDs cannot penetrate each other; if they are in tan-

[5]Also for SmC

[6]There also exist FCDs with other conjugate conics as focal lines, like parabolic FCDs, in which the two conics are parabolae. This kind of FCD was proofed theoretically and considered experimentally [13].

[7]A Dupin cyclide is any geometric inversion of a cylinder, double cone or a standart torus. Here one consider with that an inversion of a standart torus. This geometrical object is named after Charles Dupin a french mathematician.

[8]By interest in FCDs with positive Gaussian curvature look at [14].

gential contact at a point $M$, they are tangent to each other along at least one generatrix common to two of the bounding cones."

This is a experimental insight. Physically that means FCDs behave like solid objects. The second law is the law of corresponding cones (shown in Fig. (2.4)) [3, 4, 8]:

"*Law of corresponding cones (l.c.c.)*: when two conics $C_1$ and $C_2$ belonging to two different FCDs (FCD$_1$ and FCD$_2$ ) are in contact at a point $M$ , the two cones of revolution with common vertex $M$ , which rest on the two other conics $C_1'$ and $C_2'$ of the two FCDs, coincide. Therefore $C_1'$ and $C_2'$ have two points of intersection $P$ and $Q$ on the common cone, and the straight lines $\overline{PM}$ and $\overline{QM}$ are two generatrices along which the two FCDs are in contact."

This law is true in general for arbitrary conics $C_1, C_2, C_1'$ and $C_2'$ but one will simplify this and consider only the case that $C_1$ and $C_2$ are ellipses and $C_1'$ and $C_2'$ are two hyperbolae. The first part describe the following situation. One have two FCDs (FCD$_1$ and FCD$_2$) which are in contact at point $M$. FCD$_1$ has the conics $C_1$ (ellipse) and $C_1'$ (hyperbolae). Similarly FCD$_2$ has $C_2$ (ellipse) and $C_2'$ (hyperbolae). Every conic lies by definition on a cone, so one have four conics and therefore four cones. The two ellipses are in contact at point $M$ (shown in Fig.(2.4)). Then the l.c.c. says that the two hyperbolae lie on a single cone, which means the two cones belonging to $C_2$ and $C_2'$ are identical. The reason for this is that they have the same symmetry axis. The tangent between $C_1$ and $C_2$ at point $M$ is the symmmetry axis of the cone of revolution belonging $C_2$ and $C_2'$. The second part elucidated that the two hyperbolae have two common points, two intersection points $P$ and $Q$, because of they are on the same cone (shown in Fig. (2.3) and Fig. (2.4)). Moreover $M$ is the common apex of the cone and the lines $\overline{PM}$ and $\overline{PQ}$ are generatrices of the same cone. So if two FCDs have two contacts on their generating cones and have one common generatrix then they obey l.c.c. [3, 4].

The left of Fig. (2.4) shows the case of coplanar ellipses. The foci of the ellipses are $F_1$, $F_1'$, $F_2$ and $F_2'$. The three points $F_1$, $M$ and $F_2'$ are collinear[9] exactly like $F_1'$, $M$ and $F_2$ [3, 4]. The point $A$ is the projection of $P$ to the plane. It is also the projection of $Q$ [3, 4].

[9]that means the three points are on one line.

# 3 Relation to Relativity Theory

## 3.1 Flat Spacetime

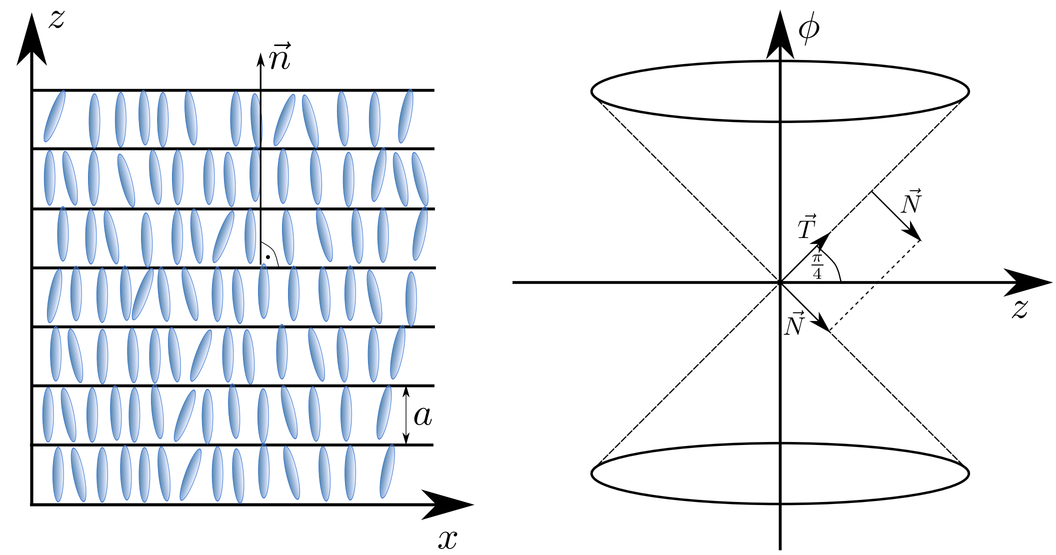


Figure 3.1: Left: Schematic of SmA liquid crystal. The blue ellipses shows the molecules of the liquid crystals. The constant layer thickness is $a$ and the director $\vec{n}$ is perpendicular to the layers [7]. Right: The tangent vector $\vec{T}$ and normal vector $\vec{N}$ are both on the light cone. In respect to the Minkowskimetric they are both null [7].

Now one want to get a relation between smectic liquid crystals and the relativity theory. An undisturbed smectic liquid crystal consist of equidistant layers, that is the layer thickness $a$ is constant; without confinement of generality in $z$-direction (shown in Fig. (3.1)). A one-dimensional density wave

$$\rho(\vec{r}) = \rho_0 + \rho_1 \cos[2\pi \frac{\phi(\vec{r})}{a}] \tag{3.1}$$

characterize the smectic phase [10, 15, 16]. The function $\phi(\vec{r})$ is called smectic phase field and is implicit defined by (3.1). The director $\vec{n}$ of a smectic is defined as the optical axis. The normal $\vec{N} = \nabla\phi$ coincides with the director $\vec{n}$[1]. Hence the phase field $\phi(\vec{r})$ could also defined implicit by the director [17]

$$\vec{n} = \nabla\phi(\vec{r}).$$

In the ground state the smectic consist of equidistant flat layers [15]. The free energy (see Equation (4.4) in Chapt. (4)) is invariant under following transformations [15]

$$\begin{aligned} \vec{n} &\to -\vec{n} \\ \phi &\to \phi + constant \\ \nabla\phi \to -\nabla\phi &\Leftrightarrow \phi \to -\phi. \end{aligned}$$

For this reason $\phi \in \mathbb{S}^1/\mathbb{Z}_2$, where $\mathbb{S}^1$ is the unit circle and $\mathbb{S}^1/\mathbb{Z}_2$ means that the antipodal points are identified [15]. To work with completely $\mathbb{R}$, one assign $\phi$ for every point $\vec{r} \in \mathbb{R}$ (in the material) another point in $\mathbb{R}$ [15].

To define the layers and describe smectic textures consider level sets

$$\phi(\vec{r}) = na \tag{3.2}$$

with $n \in \mathbb{Z}$ the set of integers. By introducing $\phi(\vec{r})$ as a timelike additional coordinate, beside the local coordinates $\vec{r}$, one will get a relation to relativity theory respectively to the Minkowski $\mathbb{M}$ spacetime [15, 17, 18]. The focus will be on configurations with vanishing compression (see Chapt. (4)).

The ground state of equidistant smectics layers is $\phi = r$ [15], so that the surface $S = \{(\vec{r}, \phi(\vec{r}))^T | \vec{r} \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ is a plane. One other possibly important example of smectics states includes the point defect [15]

$$\phi = |\vec{r}|. \tag{3.3}$$

Equation (3.3) is equivalent to $|\vec{r}|^2 - |\phi|^2 = 0$, which is an equation for a right circular cone. If one adopt the convention that the light velocity is $c = 1$, Eq. (3.3) defines

[1]One only consider SmA. By this phase of smectic liquid crystals the molecules are perpendicular to the layers, so it valids $\vec{N} = \vec{n}$. In general the director is not perpendicular to the layer, so that the normal of the layers $\vec{N}$ and the director $\vec{n}$ include an abritrary angle unequal 0; such as SmC

the surface $S$ of a light cone. One can interpretate the apex of the cone as an event in the Minkowskispacetime $\mathbb{M}$, in which the value of the apex is the value of the quasi timelike phase field $\phi$ [15, 17]. Smectic textures are identified to projections of level sets (Eq. (3.2)) of nullhypersurfaces like light cones in a one dimensional higher spacetime onto the plane [15, 17, 18]. One have found a relation between smectic textures and nullhypersurfaces in Minkowskispacetime. This is not the only way to get a corresponding between spacetimes, nullhypersufaces and smectics.

Another possibility is to consider Eq. (3.2) and use the layer ordered structure of smectics. The constraint imposed by Eq. (3.2) means to cut the graph $S$ with planes spaced by the distance of the layer thickness $a$ perpendicular to the $\phi(\vec{r})$-direction. At first one only consider flat spacetime. Equidistant planar layers are the only possibility to get no curvature and no compression [19, 20]. Then the surface $S$ assumes a constant angle with the $\phi$-direction [17, 21]. Because in the $xz$ plane the layer thickness is $a$, the projections of the graph to the $xz$ plane is unique determined. If one split the tangent vector $\vec{T}$ of $S$ in a horizontal $\Delta z$ and in a vertical $\Delta\phi(\vec{r})$ component, one will obtain a ratio of the two components $\frac{\Delta\phi(\vec{r})}{\Delta z} = \frac{a}{a} = 1$, which leads to a unique angle $\Theta$ between the tangent vector $\vec{T}$ and the $z$-direction, or alternatively between the normal vector $\vec{N}$ and the $z$-direction:

$$\Theta = \arctan(\frac{\Delta\phi(\vec{x})}{\Delta z}) = \arctan(1) = \frac{\pi}{4}.$$

The Euclidean space $\mathbb{R}^{d+1}$ is isomorphic to the Minkowski space $\mathbb{M} = \mathbb{R}^{d,1}$. Because of the Minkowski metric is not positive definite, but has a signature $(-,-,-,+)$, the local coordinates $\vec{r}$ will mirrored onto the $\phi$-axis after bijection. Thereby the Minkowski metric $\mathrm{d}s^2 = \mathrm{d}\phi^2 - \mathrm{d}\vec{r}^2$ denotes the square of the spacetime generalized distance function[2]. The tangent and normal vectors of the graph $S$ form because of the constant angle $\Theta = \frac{\pi}{4}$ by the convention, that the light velocity is set $c = 1$, again a nullhypersurface. Then respective to the Minkowski metric $\mathrm{d}s^2 = \mathrm{d}\phi^2 - \mathrm{d}\vec{r}^2$ the tangent vectors as well as the normal vectors are *null* (also called lightlike), because they are lying on a light cone (shown in Fig. (3.1)).

Moreover different possible configurations of the layers of the liquid crystals can

[2] In contravariant notation it reeds: $\mathrm{d}s^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. The diagonal matrix $\eta_{\mu\nu}$=diag(-1,-1,-1,+1) is often called Minkowski metric in physics. Strictly speaking this distance square represents the squared Minkowski norm [22, 23]. In this report, if not specify otherwise, one define the Minkowski metric $g$ depending on context as the infinitesimal or not-infinitesimale distance square $(x-y)\cdot(x-y) = (\Delta s)^2 = \eta_{\mu\nu}\Delta x^\mu \Delta y^\nu$; $x, y \in \mathbb{M}$; in which one use the Einstein notation for repeated indeces.

be related which each other by means of the symmetry of the Minkowski spacetime $\mathbb{M}$ [17]. A isometry is a transformation, which leaves the metric invariant[3]. The symmetry group respectively the isometry group of the Minkowski spacetime is the Poincaré group [22, 24]

$$\mathbb{T} \rtimes \text{O(3,1)},$$

which consist of the semidirect product of the group of translations $\mathbb{T}$ and the Lorentz group O(3,1). The Lorentz group consist thereby of all linear automorphism, which leave the metric invariant[4].

The semidirect product combines two groups $\text{G}, \text{F}$ into a new one. Let $\psi : \text{F} \to \text{Aut(G)}$ be a homomorphism. Aut(G) is the group of all automorphism of G. At first constitute the Cartesian product of G and F. With the aid of this the following operation

$$(g_1, f_1) \cdot (g_2, f_2) = (g_1 \cdot \psi(f_1)(g_2), f_1 \cdot f_2), \quad \forall g_i \in \text{G}, \forall f_i \in \text{F} \quad i \in \{1, 2\}$$

generate a new group, which is denoted as $\text{G} \rtimes \text{F}$. Let $t_x \in \mathbb{T}$ be a translation in direction $x \in \mathbb{R}^{3,1}$ and $\Lambda$ a Lorentz transformation. Set $\text{G} = \mathbb{T}$, $\text{F} = \text{O(3,1)}$ and choose $\psi(\Lambda)(t_x) = t_{\Lambda x}$ as a homomorphism, then the semidirect product $\text{G} \rtimes \text{F}$ yields the Poincaré group.

Consider the intersection of two light cones, whose projection onto $\mathbb{R}^2$ is visible under light microscope. Because a Lorentz boost leaves the metric invariant, but changes the intersection of the light cones, mixes the $\phi$-coordinate with the components of $\vec{x}$ and therefore also changes the projection, one realize a hidden symmetry between equidistant ground states on different substrates [17].

One can interpret the situation also in such a way that different observers see different projections and therefore different smectic textures from the same nullhypersurface [15, 17]. So the common nullhypersurface is an underlying structure [15].

Every nullhypersurfaces has points of singularity, e.g. the apex of the light cone. These singularities corresponds to focal sets and one can use them to describe smectic textures. Because of ill-defined normals at these points the projection of these points is also ill-defined and this leads to topological defects like disclinations and

[3] Isometries have a group structure

[4] $\text{O(3,1)} = \left\{\Lambda \in \mathcal{M}(4, \mathbb{R}) : (\Lambda x) \cdot (\Lambda y) = \eta_{\mu\nu} \Lambda^{\mu}{}_{\acute{\mu}} x^{\acute{\mu}} \Lambda^{\nu}{}_{\acute{\nu}} y^{\acute{\nu}} = \eta_{\mu\nu} x^{\mu} y^{\nu} = x \cdot y, \quad \forall x, y \in \mathbb{R}^{3,1}\right\}$ with $\mathcal{M}(4, \mathbb{R})$ as the set of all $4 \times 4$ matrices over $\mathbb{R}$

kinks in the director [15].

One need all kinds of focal sets to get all nullhypersurfaces corresponding to any focal conic texture [15, 17]. To specify all focal sets of the nullhypersurface one reduce the problem from two dimension (surface) to one dimension (focal lines). To characterize the focal sets one first consider nullhypersurfaces with focal sets, which build a dual pair. A dual pair is a 3-tuple $(X, Y, \langle\cdot,\cdot\rangle)$ [25], in which $X$ and $Y$ are vector fields over the same field $\mathbb{K}$ and $\langle\cdot,\cdot\rangle$ is a bilinear operation

$$\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{K}.$$

With the following properties

$$\begin{aligned} \forall x \in X \setminus \{0\} \quad & \exists y \in Y : \langle x, y\rangle \neq 0 \\ \forall y \in Y \setminus \{0\} \quad & \exists x \in X : \langle x, y\rangle \neq 0. \end{aligned}$$

The bilinear operation $\langle\cdot,\cdot\rangle$ establishes a duality between the vector fields $X$ and $Y$. Two elements are orthogonal if

$$\langle x, y\rangle = 0, \quad \text{for} \quad x \in X \quad \text{and} \quad y \in Y. \tag{3.4}$$

Two sets $X' \subset X$ and $Y' \subset Y$ are orthogonal if equation (3.4) is obtained for all $x' \in X'$ and for all $y' \in Y'$. To consider dual pairs is useful, because one can generalize that from dual to multiple domain configurations, which are seen in the experiments [15].

### 3.1.1 Curves of Intersection

Depending on the sign of the invariant interval $I = \Delta x_\mu \Delta x^\mu$ between two events in spacetime, their separation is said to be spacelike, $I < 0$, timelike, $I > 0$, and lightlike or null, $I = 0$. At first one consider the case of spacelike separated events. These two events generates two light cones. Without less of generality the events have the coordinates $(\pm r, 0, 0)$ in the rest frame and in general (after a Lorentz boost in the $x$-direction[5]) $(\pm\gamma r, 0, \mp\gamma\beta r)$ where $r \in \mathbb{R}$. The curve of intersection of the

[5]For two events at $(0, \pm r, 0)$ one consider the Lorentz boost in $y$-direction, which produce the coordinates $(0, \pm\gamma r, \mp\gamma\beta r)$.

two light cones lies in the $x = 0$ plane (shown in Fig. 3.2), thereby follows that

$$\phi' = \gamma(\phi - \beta x)|_{x=0} = \gamma\phi \Rightarrow \phi = \frac{1}{\gamma}\phi' \tag{3.5}$$

$$x' = \gamma(x - \beta\phi)|_{x=0} = -\gamma\beta\phi = -\beta\phi'. \tag{3.6}$$

The structure of a nullhypersurface does not change after a Lorentz boost, because Lorentz boosts let the metric invariant, so in general nullhypersurfaces stay null after boosting [15, 17]. Focal conics inherit this symmetry [15]. For every constant timelike coordinate $\phi'$, a point of the spacetime $\mathbb{M} = \mathbb{R}^{2,1}$ is on one of the two light cones if and only if the coordinates after Lorentz boost $x', y', \phi'$ satisfy the equation of a circle. Light cones are cones of revolution by rotating the angle bisector about the $\phi$-axis. The special property of the angle bisector is that for every point on it the $x$- and the $\phi$-coordinate have the same value. Therefore the $\phi$ coordinate, the radius $R$ of the circle and the light cone form a isosceles triangle. The value of the radius $R$ and absolute value of the $\phi$ coordinate are the same (shown in Fig. (3.2)).

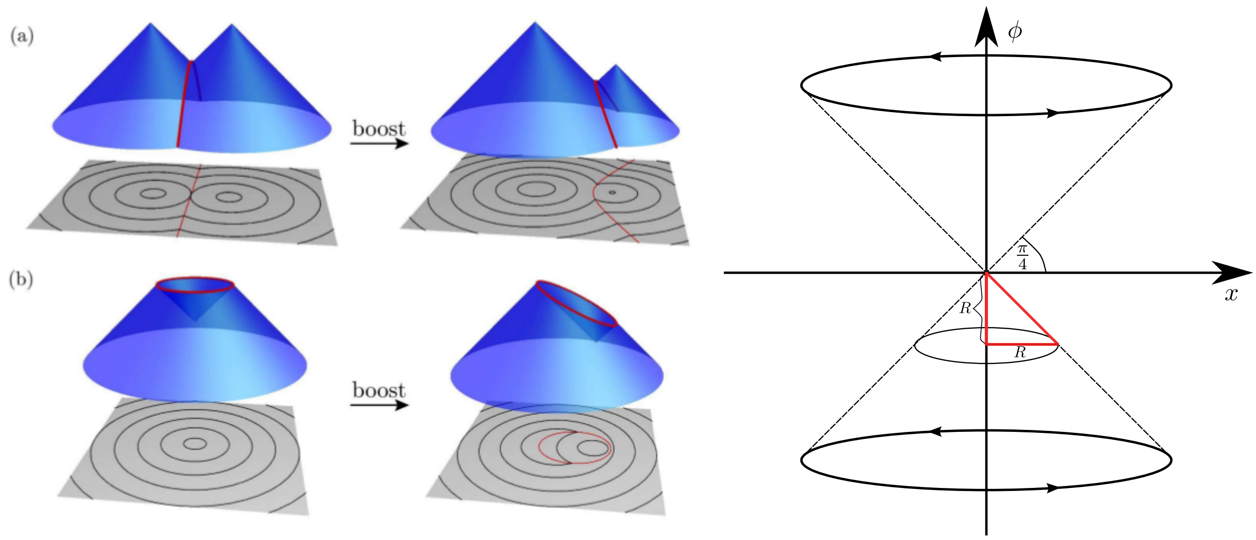


Figure 3.2: Schematic representation of the lightcones. Left: The intersection of two light cones of two spacelike separated events in (a) and two timelike separated events in (b). Left in their rest frame and right in a general frame. The corresponding smectic textures are below each surface. The red lines denotes the focal lines [15]. Right: A cone of revolution by rotating the angle bisector about the $\phi$-axis. The $\phi$ coordinate, the Radius $R$ and the light cone form a isosceles triangle. So the absolute value of $\phi$ and the radius are the same [7].

From this it follows that the points of the curve of intersection of two light cones, in this case, are given by the solution of the two equations:

$$\begin{cases} (x' + \gamma r)^2 + y'^2 = (\phi' - \gamma\beta r)^2 \\ (x' - \gamma r)^2 + y'^2 = (\phi' + \gamma\beta r)^2. \end{cases}$$

By using Equation (3.6) and expanding the brackets one will get:

$$\begin{cases} \beta^2\phi'^2 - 2\gamma\beta\phi' r + \gamma^2 r^2 + y'^2 = \phi'^2 - 2\gamma\beta\phi' r + \gamma^2\beta^2 r^2 \\ \beta^2\phi'^2 + 2\gamma\beta\phi' r + \gamma^2 r^2 + y'^2 = \phi'^2 + 2\gamma\beta\phi' r + \gamma^2\beta^2 r^2. \end{cases}$$

Substract the second from the first equation and cancel the term $2\gamma\beta\phi' r$:

$$\begin{cases} -4\gamma\beta\phi' r = -4\gamma\beta\phi' r \\ \beta^2\phi'^2 + \gamma^2 r^2 + y'^2 = \phi'^2 + \gamma^2\beta^2 r^2. \end{cases}$$

The first equation is trivial, i.e. it is equivalent to $1 = 1$. The equation of the curve of intersection is given by the second one:

$$\begin{aligned} \beta^2\phi'^2 + \gamma^2 r^2 + y'^2 &= \phi'^2 + \gamma^2\beta^2 r^2 \\ \Rightarrow y'^2 &= -\gamma^2 r^2 + \gamma^2\beta^2 r^2 + \phi'^2 - \beta^2\phi'^2 \\ &= -\gamma^2 r^2 + \gamma^2(1 - \frac{1}{\gamma^2})r^2 + \phi'^2 - (1 - \frac{1}{\gamma^2})\phi'^2 \\ &= -\gamma^2 r^2 + \gamma^2 r^2 - r^2 + \phi'^2 - \phi'^2 + \left(\frac{\phi'}{\gamma}\right)^2 \\ &= -r^2 + \left(\frac{\phi'}{\gamma}\right)^2, \end{aligned}$$

where one used $\beta^2 = 1 - \frac{1}{\gamma^2}$. Then

$$y'^2 - \left(\frac{\phi'}{\gamma}\right)^2 = -r^2 \tag{3.7}$$

$$\Rightarrow y'^2 - \left(\frac{x'}{\gamma\beta}\right)^2 = -r^2, \tag{3.8}$$

where one used the equation (3.6). This is the equation of a hyperbola. It results that in general the curve of intersection of two spacelike separated light cones is a hyperbola. Equation (3.8) is the projected curve onto the $x'y'$ plane.

In the rest frame, as a special case, the equation of the intersection is:

$$(x \pm r)|_{x=0}^2 + y^2 = r^2 + y^2 = \phi^2 \Rightarrow y^2 - \phi^2 = -r^2.$$

In the rest frame $\gamma$ is 1 and $\beta$ is 0. So one will get the same equation for the rest frame by setting $\gamma = 1$ in the general equation (3.7). From (3.8) follows:

$$x = \sqrt{r^2 + y^2} \cdot \gamma\beta = \sqrt{r^2 + y^2} \cdot 1 \cdot 0 = 0.$$

The projected curve to the $xy$ plane in the rest frame is the $y$-axis.

The case of timelike separated light cones are very similar. The steps of calculation are technically the same. The two events have the coordinates $(0, 0, \pm r)$ in their rest frame and $(\mp\gamma\beta r, 0, \pm\gamma r)$ after Lorentz boosting. The intersection now lies on the $\phi = 0$ plane (shown in Fig. 3.2):

$$\begin{aligned} x' &= \gamma(x - \beta\phi)|_{\phi=0} = \gamma x \Rightarrow x = \frac{1}{\gamma}x' \\ \phi' &= \gamma(\phi - \beta x)|_{\phi=0} = -\gamma\beta x = -\beta x'. \end{aligned} \tag{3.9}$$

The equations of circle are now

$$\begin{cases} (x' - \gamma\beta r)^2 + y'^2 = (\phi' + \gamma r)^2 \\ (x' + \gamma\beta r)^2 + y'^2 = (\phi' - \gamma r)^2. \end{cases}$$

By using equation (3.9) and expanding the brackets:

$$\begin{cases} x'^2 - 2\gamma\beta x'r + \gamma^2\beta^2 r^2 + y'^2 = \beta^2 x'^2 - 2\gamma\beta x'r + \gamma^2 r^2 \\ x'^2 + 2\gamma\beta x'r + \gamma^2\beta^2 r^2 + y'^2 = \beta^2 x'^2 + 2\gamma\beta x'r + \gamma^2 r^2. \end{cases}$$

After substracting the second equation from the first and canceling the term $2\gamma\beta x'r$ one will get:

$$\begin{cases} -4\gamma\beta x'r = -4\gamma\beta x'r \\ x'^2 + \gamma^2\beta^2 r^2 + y'^2 = \beta^2 x'^2 + \gamma^2 r^2. \end{cases}$$

The first equation is again trivial. If one consider the second one, one obtain the equation of an ellipse:

$$
\begin{aligned}
x'^2 + \gamma^2\beta^2 r^2 + y'^2 &= \beta^2 x'^2 + \gamma^2 r^2 \\
\Rightarrow x'^2 - \beta^2 x'^2 + y'^2 &= \gamma^2 r^2 - \gamma^2\beta^2 r^2 \\
\Rightarrow x'^2 - (1 - \frac{1}{\gamma^2})x'^2 + y'^2 &= \gamma^2 r^2 - \gamma^2(1 - \frac{1}{\gamma^2})r^2 \\
\Rightarrow x'^2 - x'^2 + \left(\frac{x'}{\gamma}\right)^2 + y'^2 &= \gamma^2 r^2 - \gamma^2 r^2 + r^2 \\
\Rightarrow \left(\frac{x'}{\gamma}\right)^2 + y'^2 &= r^2. \qquad (3.10)
\end{aligned}
$$

The intersection of two timelike light cones is in general an ellipse. The projected curve is now equation (3.10).

The rest frame shows a special case in which the curve of intersection is a circle:

$$x^2 + y^2 = (\phi \pm r)|^2_{\phi=0} = r^2.$$

This equation also follows from equation (3.10) by setting $\gamma = 1$.

Moreover, one recognises that the vertical projections of the two apexes (= the two events) are the foci of one branch of the projected hyperbola (3.8) in the case of spacelike separated light cones [15]. The same is valid for the timelike separated case; the apexes are projected to the foci of the projected ellipse (3.10) [15].

## 3.2 Curved Spacetime

Because the FCD have curved layers it is useful to consider curved spacetimes too. The correspondence between the nullhypersurfaces and smectic liquid crystals is generally valid also for curved spaces [15, 17]. Equidistant smectic textures can be described by level sets of nullhypersurfaces with the metric $\mathrm{d}s^2 = \mathrm{d}\phi^2 - \mathrm{d}l^2$, in which $\mathrm{d}l^2 = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ denotes the spatial section of the metric. The null geodesics, which describe the surface $S$, will project onto geodesics of the spatial section $\mathrm{d}l^2$ of the metric onto the spatial section $\mathbb{U}$ of the spacetime. The crucial point is that in general $\mathrm{d}s^2 = \mathrm{d}\phi^2 - \mathrm{d}l^2$ is not invariant under Poincaré transformations. Therefore a general spacetime exhibits other symmetries then Minkowski $\mathbb{M}$. In general there exist no symmetries, which mix the time, respectively the $\phi$ coordinate, with the spatial coordinates. Consequently the whole curvature lies in the spatial section $\mathbb{U}$ of the spacetime and it can described as $\mathbb{U} \times \mathbb{R}$. A Lorentz boost would mix the curved and flat sections and produce a new spacetime. In this spacetime the metric has not the hidden symmetry [17].

The structure of nullhypersurfaces will be conserved by conformal[6] rescaling of the metric [17, 26]:

$$\mathrm{d}s^2 = \Omega^2[\mathrm{d}\phi^2 - h_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j],$$

in which $\Omega(\phi, \vec{x})$ is an arbitrary conformal factor. In particular, the structure of the nullhypersurface is preserved, if $\mathbb{U} \times \mathbb{R}$ is conformal to Minkowski spacetime $\mathbb{M}$. All of these metrics can used to describe equidistant textures on $\mathbb{U}$ [17]. The system of coordinates should be chosen in such a way that the hidden symmetry belonging to the spacetime is preserved.

Before one turn to the choice of the coordinates and the conformal factor, one consider the isometries of $\mathbb{M}$ and their consequences on $\mathbb{U}$.

### 3.2.1 Maximally Symmetric Spaces

To examine the symmetries of $\mathbb{U}$, one needs a few other ones mathematical tools, like the Lie derivation. The Lie derivation is a generalisation of the directional derivative. Let $f : M \to \mathbb{R}$ be a smooth function and $X$ a smooth vector field on $M$. The Lie derivative $\mathcal{L}_X f(m)$ at point $m \in M$ is then defined as the directional derivative of $f$ at point $X(m)$ in the direction $m$ [12, 27]:

$$\mathcal{L}_X f(m) = \mathrm{d}_m f(X(m)).$$

The Lie derivative enables the definition of Killing vector fields, which can be used to make statements about the symmetries of manifolds. Killing vector fields are the infinitesimal generators of isometries [22, 24]. That is, the metric is left unchanged

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}$ is the Lie derivative, $X$ the Killing vector field and $g$ is as always the metric. The Noether-Theorem states that every conserved quantity belongs to a

[6]conform means that the angle does not change by this kind of transformation

symmetry, every Killing vector field $X$ belongs to a symmetry. A vector space with an additional bilinear operation, the Lie bracket

$$[\cdot,\cdot] : V \times V \to V, \quad (x,y) \mapsto [x,y],$$

satisfying the following properties

$$\begin{aligned} [x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= 0 \quad \forall x,y,z \in V \\ [x,x] &= 0 \quad \forall x \in V, \end{aligned} \tag{3.11}$$

is called a Lie algebra [12, 27]. The first constraint (3.11) is the Jacobi identity. The set of all Killing vector fields on one manifold generates the Lie algebra of the isometry group. One can use the Lie algebra to describe symmetries. The Lie algebra of the isometry group of one manifold of the dimension $d$ can at most be generated by $\frac{d(d+1)}{2}$ Killing vector fields. Spaces which symmetry group is generated from the maximal number of Killing vector fields $\frac{d(d+1)}{2}$ are called maximally symmetric [22, 24].

The Minkowski sapcetime $\mathbb{M}$ is trival local homeomorphic to $\mathbb{R}^{d+1}$ and the dimension is therefore $d+1$. Furthermore one consider the Poincaré group $\mathbb{T} \rtimes \mathrm{O}(d,1)$. The group of translations $\mathbb{T}$ in $\mathbb{R}^{d,1}$ is generated from $d+1$, and the Lorentz group $\mathrm{O}(d,1)$ is generated from $\frac{d(d+1)}{2}$ Killing vector fields[7]. A short calculation shows, that the Minkowski spacetime is maximally symmetric in every dimension:

$$\dim(\mathbb{T}) + \dim(\mathrm{O}(d,1)) = (d+1) + \frac{d(d+1)}{2} = \frac{(d+1)(d+2)}{2}.$$

It follows that $\mathbb{U} \times \mathbb{R}$ must also show maximal symmetry. With it this is warranted $\mathbb{U}$ must be maximally symmetric [17, 28]. This is an enormous constraint for the spatial section of the spacetime. For $\mathbb{U}$ there exist only the following three possibilities: $\mathbb{R}^d$, $\mathbb{S}^d$ and $\mathbb{H}^d$ [17, 22, 24], where $\mathbb{S}^d$ is the $d$ dimensional sphere and $\mathbb{H}^d$ is the $d$ dimensional hyperbolic plane. With a similar short calculating, one can show that they are all maximally symmetric.

One can also visualise that there can be only three possibilities for $\mathbb{U}$. With the help of differential geometry one can show that from maximal symmetry a constant curvature of the spacetime follows [24]. A constant curvature can be either positive, negative or zero. $\mathbb{R}^d$ has curvature zero. Whereas $\mathbb{S}^d$ has constant positive and

[7] It valid $\dim(\mathrm{O}(d,1))=\dim(\mathrm{O}(d+1))=\frac{d(d+1)}{2}$ [12]

$\mathbb{H}^d$ constant negative curvature. Therefore there are only these three possibilities for $\mathbb{U}$. To every maximally symmetric spatial section exists a maximally symmetric spacetime. Therefore, there are three spacetimes which receive the hidden symmetry: $\mathbb{M}$, dS (de-Sitter-spacetime) and AdS (anti-de-Sitter-spacetime). Moreover one can show that the maximal symmetry is equivalent to the isotropy of the space and isotropy in every point implies homogeneity [24]. Consequently the three spacetimes are all isotropic and homogeneous.

### 3.2.2 Change of Coordinates

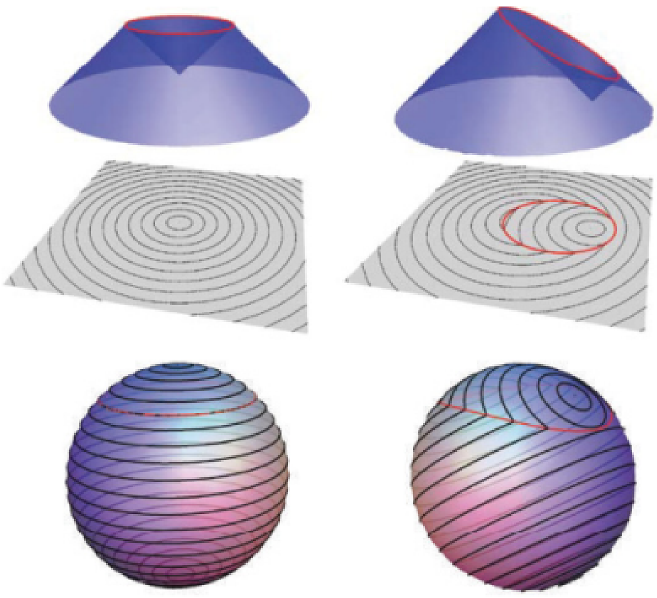

Figure 3.3: The same nullhypersurfaces, such as the intersection of two light cones, in Minkowski spacetime $\mathbb{M}$ can be used to describe equidistant smectic textures on the plane $\mathbb{R}^2$, or the sphere $\mathbb{S}^2$, or the hyperbolic plane $\mathbb{H}^2$. Here one can see only the plane $\mathbb{R}^2$ and the sphere $\mathbb{S}^2$. The left cases are in the rest frame and the right cases are the general cases after Lorentz boosting. The red line is the focal line. [17]

The coordinates and the conformal factor $\Omega$ must be chosen in such a way that the hidden symmetry is preserved with respect to the spacetime. $\mathbb{U}$ must be of one of three spaces.

For simplicity, from now on one will use $t$ instead of $\phi$ as the timelike coordinate. The mathematical statements are valid in all dimensions,however, here one limit ourselves to the important (2+1)-dimensional case. One look at the (2+1)-

dimensional Minkowski spacetime $\mathbb{M} = \mathbb{R}^{2,1}$ and carry out the following coordinate transformations:

$$\begin{aligned} t &= \Omega_P \sin(t') \\ x + iy &= \Omega_P \sin(\alpha) e^{i\beta} \\ \text{with} \quad \Omega_P &= \frac{1}{\cos(t') + \cos(\alpha)} \end{aligned}$$

Thereby the Cartesian coordinates $(x, y, t)$ were transformed into Carter-Penrose-coordinates $(\alpha, \beta, t')$ [17]. This induced the metric [17, 26]:

$$\mathrm{d}s^2 = \Omega_P^2 [\mathrm{d}t'^2 - \mathrm{d}\alpha^2 - \sin(\alpha)^2 \mathrm{d}\beta^2].$$

By this choice of coordinates the Minkowski spacetime is conformal to $\mathbb{S}^2 \times \mathbb{R}$ (shown in Fig.3.3). Nullhypersurfaces in $\mathbb{M}$ correspond to equidistant level sets on $\mathbb{U} = \mathbb{S}^2$ [17]. If one choose instead

$$\begin{aligned} t &= e^{t'} \cosh(\alpha) \\ x + iy &= e^{t'} \sinh(\alpha) e^{i\beta}, \end{aligned}$$

as coordinates, then $\mathbb{M}$ is conform to $\mathbb{H}^2 \times \mathbb{R}$ [17].

Although $\mathbb{S}^2$ and $\mathbb{H}^2$ are curved, nevertheless, it concerns the Minkowski spacetime $\mathbb{M}$. The metric remains with conformal transformations those of the Minkowski spacetime, completely regardless of the choice of the coordinates [17, 29]. The spatial section is only flat in Cartesian coordinates. By using the other coordinates it stays the Minkowski spacetime, but with curved spatial section.

One summarise: the same nullhypersurface in $\mathbb{M}$ is able, according to coordinate choice, to describe equidistant smectic textures on $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{H}^2$ (the cases $\mathbb{R}^2$ and $\mathbb{S}^2$ are shown in Fig.3.3). In addition, one have the freedom to view smectic liquid crystals with one and the same spacetime, but to different spatial sections $\mathbb{U}$, or with one and the same $\mathbb{U}$, but with different spacetimes ($\mathbb{M}$, dS, AdS) [17].

# 4 Frank Free Energy

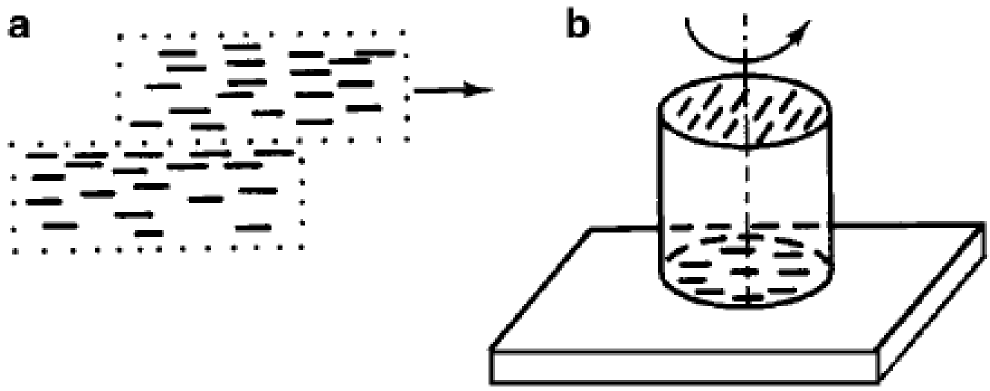


Figure 4.1: Schematic of translational and rotational distortion. (a): By a translation distortion there is no elastic modulus of moving layer. (b): One needs a twist modulus to describe the twist of the upper layer with respect to the bottom one [16].

This chapter will explain how the Frank free energy of smectics will change under Lorentz boosting. But before considering the correspondence between different focal lines and the energy one introduces the elementary basic ideas to get a formula for the energy. The energy depends on the distortions. Several people were involved in the development of a continuum theory of elasticity for liquid crystals, at first for nematics. Oseen, Ericksen, Leslie and Frank have given the largest contribution [9, 10, 16, 30]. Oseen had the idea to use the vector field of the director $\vec{n}$ to describe the elasticity of liquid crystals [16, 31]. Ericksen was the to who use the asymmetry of the stress tensor for hydrostatics of nematic liquid crystals. He developed the basics for a general theory of continuum theory of liquid crystals, based on the conservation laws for mass, linear and angular momentum [16, 32]. But this were all about a static continuum theory of elasticity. The dynamics were developed by Leslie [16]. Nowaday the theory is known as the Ericksen-Leslie theory [9, 10, 16]. And finally Frank helped to understand the hydrostatic parts of the Ericksen-Leslie theory and defects like disclinations in liquid crystals [16, 33]. The energy formula is named by him.

## 4.1 Distortions

For a fixed direction of the director the shear modulus is vanishing. Figure (4.1) shows the difference between translational and rotational distortions in a liquid crystal. In solids a change in the distance of the next neighbor points causes the stress, but in liquid crystals the curvature of the director field $n_{i,j} \equiv \frac{\partial n_i}{\partial x_j}$[1] causes the stress [16]. Different boundary conditions can be the cause of stress like electric or magnetic fields, or mechanical bending etc. For a more general case one consider also the quadratic terms of the distortion free energy density $f_{dist}$ [10, 16]:

$$f_{dist} = K_{ij} \frac{\partial n_i}{\partial x_j} + \frac{1}{2} K_{ijlm} \frac{\partial n_i}{\partial x_j} \cdot \frac{\partial n_l}{\partial x_m}, \tag{4.1}$$

where $K_{ij}$ and $K_{ijlm}$ are elasticity tensors. Now one consider the elementary distortions of an undistorted director $n = (0,0,1)^T$ aligned along the $z$-axis (that is $n_z$=const $\Rightarrow \frac{\partial n_z}{\partial x_j} = 0 \;\; \forall j \in$\{1,2,3\}):

$$\frac{\partial n_x}{\partial x} = a_1, \quad \frac{\partial n_x}{\partial y} = a_2, \quad \frac{\partial n_x}{\partial z} = a_3, \quad \frac{\partial n_y}{\partial x} = -a_4, \quad \frac{\partial n_y}{\partial y} = a_5, \quad \frac{\partial n_y}{\partial z} = a_6,$$

where $a_1$ and $a_5$ are the elementary splay distortions, $a_2$ and $a_4$ the elementary twist distortions, $a_3$ and $a_6$ are the elementary bend distortions [16]. The minus sign in front of $a_4$ appears because of having different signs in the variations $\delta n_y$ and $\delta n_x$ [16]. Then the components of the director $n = (n_x(x,y,z), n_y(x,y,z), n_z(x,y,z))^T$ are [16]

$$\begin{aligned} n_x &= a_1 x + a_2 y + a_3 z + \mathcal{O}(r^2) \\ n_y &= a_4 x + a_5 y + a_6 z + \mathcal{O}(r^2) \\ n_z &= const + \mathcal{O}(r^2), \end{aligned}$$

[1] One identified $x_1 = x$, $x_2 = y$ and $x_3 = z$.

where $\mathcal{O}(r^2)^2$ are higher terms in $r^2 = x^2 + y^2 + z^2$. In general the director is not parallel to the $z$-axis and one has a more general curvature distortion tensor:

$$n_{i,j} = \frac{\partial n_i}{\partial x_j} = \begin{pmatrix} \frac{\partial n_x}{\partial x} & \frac{\partial n_x}{\partial y} & \frac{\partial n_x}{\partial z} \\ \frac{\partial n_y}{\partial x} & \frac{\partial n_y}{\partial y} & \frac{\partial n_y}{\partial z} \\ \frac{\partial n_z}{\partial x} & \frac{\partial n_z}{\partial y} & \frac{\partial n_z}{\partial z} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix},$$

where $a_7$, $a_8$ and $a_9$ are the missing elementary distortions if the director is not parallel to the $z$-axis.

### 4.1.1 Elasticity Tensors

At first one considers the elasticity tensor $K_{ij}$. This tensor has nine components, because of $i, j \in \{1, 2, 3\} \Rightarrow 3^2 = 9$. One has uniaxial (cylindric) symmetry and therefore Eq.(4.1) is invariant under the rotation about the $z$-axis [10, 16]:

$$x' = y,$$
$$y' = x,$$
$$z' = z.$$

Then $K_{ij}$ reduces to [10, 16]

$$K_{ij} = \begin{pmatrix} K_1 & K_2 & 0 \\ -K_2 & K_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By molecules with mirror symmetry (achiral molecules) the distortion free energy density Eq. (4.1) get invariant under [16]

$$n \mapsto -n.$$

Then $K_2 = 0$.

Now one considers the other elasticity tensor. The tensor $K_{ijlm}$ has 81 components, because of $i, j, l, m \in \{1, 2, 3\} \Rightarrow 3^4 = 81$. By considering SmA the director is again parallel to the $z$-axis and $a_7, a_8$ and $a_9$ are vanishing [10, 16]. So this reduces $K_{ijlm}$ to 36 components (one has six elementary distortions $a_1$ to $a_6$. Squaring

[2] The Landau notation $\mathcal{O}(r^2)$ means that the terms growth slower as $r^2$.

this to have a square matrix yields $6^2 = 36$ components). The uniaxial symmetry reduces the number of components to 18 [10, 16]. Only the five components $K_{11}, K_{12}, K_{22}, K_{24}$ and $K_{33}$ are independent [10, 16]. The head-to-tail symmetry implies that Eq. (4.1) is invariant under the transformation:

$$x' = x,$$
$$y' = y,$$
$$z' = -z.$$

Therefore the component $K_{12}$ vanishes [10, 16]. Finally there remains only the four components $K_{11}, K_{22}, K_{33}$ and $K_{24}$ [10, 16, 30, 31].

### 4.1.2 Elastic Energy

For the distortion free energy $f_{dist}$ of SmA phases one obtains the linear combinations of the elementary distortions corresponding to splay, twist and bend:

$$f_{dist} = \frac{1}{2}[K_{11} \underbrace{(a_1 + a_5)^2}_{splay} + K_{22} \underbrace{(a_2 + a_4)^2}_{twist} + K_{33} \underbrace{(a_3 + a_4)^2}_{bend}], \tag{4.2}$$

in which one has only the quadratic terms of Eq. (4.1) [15–17]. The term with $K_{24}$[3] only appears, if distortions have two or three dimensional structures [16]. Next, we will deduce the Frank formula for the distortion free energy density. The divergence of the director $n$ gives the splay term [10, 16]:

$$\nabla \cdot n = \frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} = (a_1 + a_5),$$

where one used $dn_z = 0$, because $n \parallel z$. For the twist term one writes [10, 16]

$$n \cdot \nabla \times n = -n_x \frac{\partial n_y}{\partial z} + n_y \frac{\partial n_x}{\partial z} + n_z \left( \frac{\partial n_y}{\partial x} - \frac{\partial n_x}{\partial y} \right) \approx \left( \frac{\partial n_y}{\partial x} - \frac{\partial n_x}{\partial y} \right) = (a_2 + a_4).$$

[3] The so called saddle-play modulus.

Since $n = (0,0,1)^T$ the first two terms can be neglected. By the conditions $n_x, n_y \ll n_z \approx 1$ one finally gets this expression for the bend term [10, 16]

$$n \times \nabla \times n = -\left(\frac{\partial n_x}{\partial z}\vec{e_x} + \frac{\partial n_y}{\partial z}\vec{e_y}\right).$$

By squaring these terms we obtain (compare with Eq.(4.2))

$$\begin{aligned} f_{dist} &= \frac{1}{2}\left[K_{11}\left(\frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y}\right)^2 + K_{22}\left(\frac{\partial n_x}{\partial y} - \frac{\partial n_y}{\partial x}\right)^2 + K_{33}\left[\left(\frac{\partial n_x}{\partial z}\right)^2 + \left(\frac{\partial n_y}{\partial z}\right)^2\right]\right] \\ &= \frac{1}{2}\left[K_{11}\left(\nabla \cdot n\right)^2 + K_{22}\left(n \cdot \nabla \times n\right)^2 + K_{33}\left(n \times \nabla \times n\right)^2\right]. \end{aligned} \tag{4.3}$$

For SmA the twist and the bend term vanish, so that the splay term is the only one, which remains [10, 15, 17]. The layer in SmA are flexible and easily distorted, but preserve the layer thickness [10]. In addition one has a term for compression [9, 10, 15, 17] and after integrating Eq. (4.3) over the volume we obtain the Frank free energy:

$$\begin{aligned} F &= \frac{1}{2}\int \mathrm{d}^d x\{C(n^2-1)^2 + S(\nabla \cdot n)^2\} \\ &= \frac{1}{2}\int \mathrm{d}^d x\{C((\nabla\phi)^2-1)^2 + S(\nabla^2\phi)^2\}, \end{aligned} \tag{4.4}$$

where one replaces $K_{11}$ with $S$ for the splay term and $n = \nabla\phi$. The compression modulus is $C$.

#### Dependence on Focal Curves

In Sect. (3.1.1) we considered two cases: spacelike and timelike separated events which produce two different curves of intersection. After Lorentz boosting the coordinates $(x, y, \phi(x, y))$ transform to $(x', y', \phi'(x', y'))$. The Frank free energy (Eq. (4.4)) depends indirect on the focal lines (curves of intersection) about the $\phi$ coordinate. Lorentz boosting transfers $F(\phi(x, y))$ to $F(\phi'(x', y'))$. Then one can expresses the Frank free energy with the origin coordinates: $F(\phi'(x', y')) = \widetilde{F}(\phi(x, y))$, where $\widetilde{F}$ is the Lorentz boosted Frank free energy in dependence of coordinates $(x, y, \phi)$. So we need to distinguish the two different cases:

For spacelike separated events:

$$\begin{aligned}F(\phi'(x',y')) &= \frac{1}{2}\int \mathrm{d}^d x' \{C((\nabla\phi')^2-1)^2+S(\nabla^2\phi')^2\} \\ &= \underbrace{\frac{1}{2}\int \mathrm{d}^d x \{C((\nabla\gamma\phi)^2-1)^2+S(\nabla^2\gamma\phi)^2\}}_{\widetilde{F}_1(\phi(x,y))} \\ &= \underbrace{\frac{1}{2}\int \mathrm{d}^d x \{C\left(\left(\nabla\gamma\sqrt{r^2+y^2}\right)^2-1\right)^2+S\left(\nabla^2\gamma\sqrt{r^2+y^2}\right)^2\}}_{\widetilde{F}_2(\phi(x,y))}\end{aligned} \tag{4.5}$$

For timelike separated events :

$$\begin{aligned}F(\phi'(x',y')) &= \frac{1}{2}\int \mathrm{d}^d x' \{C((\nabla\phi')^2-1)^2+S(\nabla^2\phi')^2\} \\ &= \underbrace{\frac{1}{2}\int \mathrm{d}^d x \{C((-\nabla\gamma\beta x)^2-1)^2+S(-\nabla^2\gamma\beta x)^2\}}_{\widetilde{F}_3(\phi(x,y))}\end{aligned} \tag{4.6}$$

In Eq.(4.5) $r$ is the half distance between the two apexes of the intersecting lightcones and $\beta$ and $\gamma$ are factors from the Lorentz boosts. By using Eq. (3.5) one obtains $\widetilde{F}_1(\phi(x,y))$ and using Eq. (3.7) gives $\widetilde{F}_2(\phi(x,y))$. In the timelike separated case the curve of intersection lies on the $\phi = 0$ plane, so there is no dependence on $\phi$ instead, by using (3.9), $\widetilde{F}_3(\phi(x,y))$ (Eq. (4.6)) depends only on $x$.

# 5 Conclusion

We have studied the geometry of liquid crystals in the specific situation of focal conic domains. We described an elegant mathematical relation between focal conic domains and relativity theory, largely based on [15, 17]. The references of the figures are declared or they are drawn with Inkscape version 0.48.5. The main results of this report are:

- The law of impenetrability: FCD behave like physically solid objects. Overlapping is not possible as e.g. by waves (superposition principle). Two FCD can only be tangent to each other.

- The law of corresponding cones: Two tangent FCD have two common points, in such a way that two conics belonging to two different FCD are on a single cone.

- It is a new interpretation to see SmA's identified to projections of a one-dimensional higher spacetime onto the plane.

- About the spacetime symmetry one get a hidden symmetry of SmA's. Different configurations, textures corresponds to each other by using spacetime symmetries.

- One can describe multiple-conic domain configurations by using dual pairs, specifically dual conic domains. So it is useful to consider the intersections of a pair of lightcones.

- There are only three possibilities ($\mathbb{R}^d, \mathbb{S}^d, \mathbb{H}^d$) for the spatial section to restore the hidden symmetry. These are the only three maximally symmetric spaces.

- A consequence of the hidden symmetry is that smectics with different configurations and hence also different Frank free energy correspond to each other by Lorentz boosting.

# Bibliography


[1] F. Reinitzer, Monatshefte für Chemie **9**, 421 (1888).

[2] V. Designolle, S. Herminghaus, T. Pfohl, and C. Bahr, Langmuir **22**, 363 (2006).

[3] M. Kléman and O. D. Lavrentovich, The European Physical Journal E **2**, 47 (2000).

[4] M. Kléman and O. Lavrentovich, *Soft Matter Physics: An introduction* (Springer Verlag, Berlin, 2003), Vol. 2.

[5] G. Friedel and F. Grandjean, Bulletin de la Société Française de Minéralogie et de Crystallographie **33**, 409 (1910).

[6] I. Dierking, Physik Journal **8**, 4 (2009).

[7] Drawn with Inkscape version 0.48.5.

[8] G. Friedel, Annals of Physics **18**, 273 (1922).

[9] P. G. de Gennes, *The Physics Of Liquid Crystals* (Oxford University Press, Oxford, 2002), Vol. 2.

[10] S. Chandrasekhar, *Liquid Crystals* (Cambridge University Press, Cambridge, 1992), Vol. 2.

[11] D. Hilbert and S. C. Vossen, *Geometry and the Imagination* (Chelsea Publishing Cy, New York, 1952), Vol. 1.

[12] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups* (Springer Verlag, Berlin, 1983).

[13] C. Rosenblatt, R. Pindak, N. A. Clark, and R. B. Meyer, Journal de Physique **38**, 1105 (1977).

[14] P. Boltenhagen, M. Kléman, and O.Lavrentovich, Comptes Rendus de l'Académie des Sciences **315**, 931 (1992).

[15] G. P. Alexander, B. Chen, E. Matsumoto, and R. D. Kamien, Physical Review Letters **104**, 257802 (2010).

[16] L. M. Blinov, *Structure and Properties of Liquid Crystals* (Springer Verlag, Berlin, 2011), Vol. 1.

[17] G. P. Alexander, R. D. Kamien, and R. A. Mosna, Physical Review E **85**, 050701 (2012).

[18] B. Chen, G. Alexander, and R. D. Kamien, Proceeding of the National Academy of Sciences **106**, 15577 (2009).

[19] M. Archard, M. Kléman, Y. Natishin, and H. Nguyen, European Physical Journal E **16**, 37 (2005).

[20] B. A. DiDonna and R. D. Kamien, Physical Review Letters **89**, 215504 (2002).

[21] P. Cermelliand and A. D. Scala, Philosophical Magazine **87**, 1871 (2007).

[22] S. Carrol, *Spacetime and Geometry: An Introduction to General Relativity* (Person Addison Wesley, Bonn, 2003), Vol. 1.

[23] G. F. R. Ellis and R. M. Williams, *flat and curved space-times* (Oxford University Press, Oxford, 2001), Vol. 2.

[24] E. Rebhan, *Theoretische Physik: Relativitätstheorie und Kosmologie* (Spektrum Akademischer Verlag, Heidelberg, 2011).

[25] H. Jarchow, *Locally Convex Spaces* (Vieweg+Teubner Verlag, Wiesbaden, 2014), Vol. 2.

[26] S. Hawking, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).

[27] J. Serre, *Lie Algebras and Lie Groups* (Springer Verlag, Berlin, 2008).

[28] A. G. Walker, Journal of the London Mathematical Society **19**, 219 (1944).

[29] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, 2005).

[30] T. Sluckin, D. Dunmur, and H. Segemeyer, *Crystals That Flow: Classic Papers from the History of Liquid Crystals* (Routledge Chapman and Hall, London, 2004), Vol. 1.

[31] C. W. Oseen, Transactions of the Faraday Society **29**, 883 (1933).

[32] J. L. Ericksen, Archive for Rational Mechanics and Analysis **9**, 371 (1961).

[33] F. C. Frank, Discussions of the Faraday Society **25**, 19 (1958).

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**Erklärung** nach §13(8) der Prüfungsordnung für den Bachelor-Studiengang Physik und den Master-Studiengang Physik an der Universität Göttingen:

Hiermit erkläre ich, dass ich diese Abschlussarbeit selbständig verfasst habe, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe und alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen wurden, als solche kenntlich gemacht habe.
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Göttingen, den September 5, 2014

(Florian Hollemann)