Prescriptions for measuring and transporting local angular momenta in general relativity

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For observers in curved spacetimes, elements of the dual space of the set of linearized Poincaré transformations from an observer's tangent space to itself can be naturally interpreted as local linear and angular momenta. We present an operational procedure by which observers can measure such quantities using only information about the spacetime curvature at their location. When applied by observers near spacelike or null infinity in stationary, vacuum, asymptotically flat spacetimes, there is a sense in which the procedure yields the well-defined linear and angular momenta of the spacetime.

We also describe a general method by which observers can transport local linear and angular momenta from one point to another, which improves previous prescriptions. This transport is not path independent in general, but becomes path independent for the measured momenta in the same limiting regime. The transport prescription is defined in terms of differential equations, but it can also be interpreted as parallel transport in a particular direct-sum vector bundle. Using the curvature of the connection on this bundle, we compute and discuss the holonomy of the transport law. We anticipate that these measurement and transport definitions may ultimately prove useful for clarifying the physical interpretation of the Bondi-Metzner-Sachs charges of asymptotically flat spacetimes.

I. INTRODUCTION

Asymptotically flat spacetimes in general relativity have an infinite-dimensional group of asymptotic symmetries, rather than the ten translations, rotations, and boosts of flat Minkowski spacetime (see [1] for a review of asymptotically flat spacetimes). This larger symmetry group, the Bondi-Metzner-Sachs (BMS) group [2–4], differs from the Poincaré group of flat spacetime because the BMS group contains an infinite family of "angle-dependent translations" called the supertranslations, rather than the four spacetime translations of the Poincaré group. The quotient of the BMS group by the supertranslations is isomorphic to the Lorentz group, just as the quotient of the Poincaré group by the translations is the Lorentz group.

Associated with each BMS symmetry generator $\vec{\xi}$ is a corresponding Noether-like charge $Q(\vec{\xi})$, which is not conserved, but whose change between different times (or more precisely, cuts of null infinity) is determined by a flux formula [5–7]. These charges can be computed in terms of integrals of the spacetime curvature over cuts of null infinity. They can in principle be measured by families of observers near null infinity who measure the spacetime geometry in their vicinities and who communicate their results to one another so as to evaluate the charge integrals. On the other hand, observers who attempt to measure Poincaré-covariant asymptotic charges will in general recover the BMS charges associated with some observer-dependent Poincaré subgroup of the BMS group. The purpose of this paper is to explore in more detail how such measurements can be made and to understand their observer dependence. One motivation for this exploration is to try to understand more deeply the physical interpretation of the BMS charges themselves.

Consider an observer at an event $\mathcal{P} \in \mathcal{M}$ in the spacetime manifold $\mathcal{M}$. The mathematical space of a linear and angular momentum as measured by that observer is $\mathcal{G}_{\mathcal{P}}^*$, the space dual to linearized Poincaré transformations (affine transformations) from the tangent space $T_{\mathcal{P}}\mathcal{M}$ to itself. This space can be naturally parameterized in terms of pairs of tensors (P^a, J^{ab}) at $\mathcal{P}$, with $J^{(ab)} = 0$, which represent the linear and angular momentum about the observer's location [8]. How can such local linear and angular momenta be defined and measured?

One approach to such definitions is the following: Suppose that a Poincaré subgroup of the BMS group has been specified; for example, it could be the subgroup associated with a stationary region of future null infinity ($\mathscr{I}^+$). Let $\mathfrak{g}$ be the corresponding algebra of generators $\vec{\xi}$, where $\mathfrak{g} \subset \mathfrak{bms}$ is a subalgebra of the BMS algebra, and $\mathfrak{g} \simeq \mathfrak{iso}(3,1)$ is isomorphic to the Poincaré algebra. Suppose also that one has a prescription for extending BMS generators $\vec{\xi}$ (which are vector fields defined on $\mathscr{I}^+$) into the interior of the spacetime. An example of such a prescription associated with the retarded Bondi coordinate conditions is given in Ref. [9]; many other prescriptions exist. We now define tensors P^a, J^{ab} at $\mathcal{P}$ by

$$Q(\vec{\xi}) = P^a \xi_a(\mathcal{P}) + \frac{1}{2} J^{ab} \nabla_{[a}\xi_{b]}(\mathcal{P}) \qquad (1.1)$$

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for any $\vec{\xi}$ in $\mathfrak{g}$. Here the left-hand side is the BMS charge, which is a linear function of $\vec{\xi}$. On the right-hand side, we can identify the Poincaré algebra $\mathfrak{g}$ with the space of values of ξ_a and $\nabla_{[a}\xi_{b]}$ at $\mathcal{P}$, and thereby determine the coefficients P^a and J^{ab}. The definition (1.1) clearly depends on the choice of prescription for extending generators into the interior of the spacetime, but one would expect the leading-order terms in an expansion of the prescription in powers of $1/r$, as $r \to \infty$, would be independent of this choice.

Paper I also defined a prescription for transporting elements of $\mathcal{G}_{\mathcal{P}}^*$ along paths in spacetime. This definition was based on a rule for transporting vectors along curves by a generalization of parallel transport which was called "affine transport." The angular-momentum transport law can also be defined explicitly in the following way, as shown in Appendix A of Paper I: given a curve with tangent k^a, the pair (P^a, J^{ab}) is transported along the curve using the differential equations[1]

$$k^a \nabla_a P^b = 0 \,, \tag{1.2a}$$

$$k^a \nabla_a J^{bc} = 2P^{[b}k^{c]} \,. \tag{1.2b}$$

This transport law allows two observers at different spacetime locations to compare values of angular momentum that they measure, albeit in a curve-dependent fashion. In addition, changes with time of angular momentum can be compared by transporting the angular momentum about a closed curve in spacetime composed of the two observers' worldlines and two spacelike curves connecting their locations. This process amounts to computing a holonomy of the affine transport law. In Paper I, it was shown that such holonomies contain two parts: the normal holonomy associated with parallel transport of tensors (denoted $\Lambda^a{}_b$), and an inhomogeneous part $\Delta \xi^a$ related to a "displacement vector" that depends upon the curve. This displacement vector satisfies $k^a \nabla_a \Delta \xi^b = k^a$ with initial condition $\Delta \xi^a = 0$. The momentum and angular momentum transform under this holonomy by the

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[1] Equation (1.2) corrects a sign error in Appendix A of the published version of Paper I.

following laws:

$$P^a \to \Lambda^a{}_b P^b \,, \tag{1.3a}$$

$$J^{ab} \to \Lambda^a{}_c \Lambda^b{}_d (J^{cd} - 2\Delta \xi^{[c} P^{d]}) \,. \tag{1.3b}$$

When the map $\Lambda^a{}_b$ reduces to the identity and $\Delta \xi^a$ is zero (to leading order in M/r, for example), then the asymptotic observers agree on the momentum and angular momentum of the spacetime. However, a nontrivial $\Lambda^a{}_b$ or $\Delta \xi^a$ indicates that spacetime curvature produces an obstacle to observers arriving at a consistent definition of a linear and angular momentum of the spacetime.

One of the primary goals of Paper I was to use the local measurement procedure for angular momentum and the holonomy of the affine transport equation to understand the physics behind what is often called the "supertranslation ambiguity" of angular momentum in general relativity. The ambiguity refers to the fact that while there is a four-parameter translation subgroup of the supertranslations, there is, in general, no preferred Poincaré subgroup of the BMS group. As a result, the charges in general relativity associated with the six-parameter factor group of the BMS group depend, in general, on a smooth function on the 2-sphere rather than a four-parameter origin. Stationary spacetimes are an exception in this regard: they possess a preferred Poincaré subgroup of the BMS group with associated Poincaré charges.

When the measurement and transport procedures are applied to "sandwich-wave" spacetimes (in which a burst of linearized gravitational waves of finite duration with memory pass through Minkowski space), the results of the measurements are observer dependent. Furthermore, Paper I showed that this observer dependence is related to the supertranslation that relates the shear-free cuts in the Minkowski space before the burst to those after the burst (which is, in essence, just the memory effect; see, e.g., [10]). More specifically, the generalized holonomy contains a nontrivial inhomogeneous part which is a function of the aforementioned supertranslation evaluated at the observers' locations and of the separation of the observers. In this context, the measurement procedure gives the linear momentum of the spacetime and an observer-dependent angular momentum of the spacetime that depends on a four-parameter choice of origin of the spacetime. The holonomy gave—in the form of a Poincaré transformation—information about the BMS supertranslation (at the location of the two observers), which creates an obstruction to defining a consistent notion of angular momentum that depends upon a four-parameter origin.

There are two closely related limitations of the measurement and transport procedure of Paper I outside of the context of the sandwich wave spacetimes described above. First, in the measurement procedure, the angular momentum is only defined to a fractional accuracy of order M/r, which, because the angular momentum about the point scales as Mr, implies that there are errors in the angular momentum of order M^2. These errors, however,

are of the same size as the observer dependence arising from the memory effect that is of physical interest. Second, a nontrivial holonomy does not necessarily imply the existence of ambiguities related to BMS transformations. For example, for certain spacelike closed curves in the asymptotic region of a Schwarzschild black hole, the generalized holonomy is nontrivial, even though stationary spacetimes have a preferred Poincaré subgroup of the BMS group (and hence a well-defined angular momentum).

In this paper, we refine the definitions of Paper I of both the local measurement of angular momentum and the transport procedure, in order that the measurement be sufficiently accurate to capture supertranslation/memory effects, and in order that nontrivial asymptotic holonomies only arise because of BMS-type observer dependence. The refined definitions are sufficiently accurate that in vacuum, stationary, asymptotically flat spacetimes, observers near $\mathscr{I}^+$ will agree upon their measured linear and angular momentum (this includes the location of the source's center of mass, which is now measured with an accuracy $\sim M^2/r$).

The paper is organized as follows. In Sec. II, we define the new transport equation for tensors P^a and J^{ab}, and we describe specific path-independent solutions of this transport equation. In Sec. III, we define a prescription for measuring local linear and angular momentum from spacetime curvature and show that it gives the expected answer in appropriate limiting regimes. In Sec. IV, we describe how our transport equation can be understood as a connection on a certain bundle, and we compute the curvature of this bundle (and therefore, also the holonomy of an infinitesimal loop). We conclude in Sec. V. We use throughout geometric units ($G = c = 1$) and the conventions of Wald [11].

II. TRANSPORT EQUATIONS FOR ANGULAR MOMENTUM

As a generalization of (1.2), we will consider transport equations along curves with tangent k^a of the form

$$k^a \nabla_a P^b = -\kappa R^b{}_{acd} J^{cd} k^a \,, \tag{2.1a}$$

$$k^a \nabla_a J^{bc} = 2 P^{[b} k^{c]} \,, \tag{2.1b}$$

where κ is a real constant. From the point of view of the theory of differential equations, these transport equations have nice properties. For all values of κ, these equations are linear in P^a and J^{ab} and reparameterization invariant under changes in the tangent k^a. Solutions of the equations yield a linear map between the spaces of tensors (P^a, J^{ab}) at the initial point and at the final point of the curve. When $\kappa = 0$, this linear map reduces to the standard action of a Poincaré transformation on (P^a, J^{ab}), as discussed in Paper I. For nonzero values of κ, however, the corresponding map has a more involved form.

Transport equations of the form (2.1) arise in several different contexts in the field of general relativity:

(i) The case $\kappa = 0$ is the simplest version of an angular momentum transport law consistent with the properties of angular momentum in special relativity. Its properties were studied in Paper I.

(ii) The $\kappa = 1/2$ transport equations have the same form as the Mathisson-Papapetrou equations [12, 13], when P^a and J^{ab} are taken to be the linear and angular momentum of a particle (rather than of the spacetime) and the curve is the particle's worldline.

(iii) The $\kappa = 1/2$ transport equations are also dual to the Killing transport equations, in the following sense: Suppose that A_a and $B_{ab} = B_{[ab]}$ are tensors which satisfy the Killing transport equations along the curve (as would be the case if there were a Killing vector field ξ^a on the spacetime and A_a and B_{ab} were defined by $A_a = \xi_a$ and $B_{ab} = \nabla_a \xi_b = \nabla_{[a} \xi_{b]}$). In addition, suppose that P^a and J^{ab} satisfy the transport equations (2.1) with $\kappa = 1/2$. Then the generalized momentum $P^a A_a + J^{ab} B_{ab}/2$ is conserved along the curve [14].

(iv) In this paper, we will use the $\kappa = -1/4$ transport equations to define a prescription for transporting angular momentum. We will also show that observers who use this prescription will arrive at a mutually consistent definition of angular momentum, near future null infinity in stationary, vacuum, asymptotically flat spacetimes.

While most of this paper focuses on the case $\kappa = -1/4$, our calculations in Sec. IV below are valid for all values of κ, and might prove useful in some of these other contexts.

In this paper, we will be most interested in situations where the transport equations admit solutions that are independent of the path used to transport the tensors P^a and J^{ab} throughout the asymptotically flat region of a stationary, vacuum spacetime. If they admit such path-independent solutions, then there will be a linear and angular momentum of the spacetime that different observers can measure and transport consistently. A sufficient condition for such curve-independent solutions to exist is if there are solutions to the partial differential equations

$$\nabla_a P^b = -\kappa R^b{}_{acd} J^{cd} \,, \tag{2.2a}$$

$$\nabla_a J^{bc} = 2 P^{[b} \delta^{c]}{}_a \tag{2.2b}$$

throughout the region of interest.

In the next two subsections, we will show that solutions to the equations (2.2) do exist. First, in Sec. II A, we show that there is an exact solution in the Kerr spacetime when $\kappa = -1/4$, which is defined throughout the entire spacetime. Next, in Sec. II B we show that approximate solutions exist in general, asymptotically flat, stationary spacetimes near future null infinity (or equivalently, in this case, spacelike infinity), again only when $\kappa = -1/4$. The existence of those approximate solutions

is sufficient to allow asymptotically consistent measurements of angular momentum, as we discuss in more detail in Sec. III B below.

A. Global solution in the Kerr spacetime

The global solution to (2.2) in the Kerr spacetime is a consequence of the relationships between the Killing-Yano (KY) tensor and the timelike Killing field that this spacetime admits. Many of the properties that we use here are well known, and can be found in Floyd's thesis [15] or in more recent review papers [16]. We begin by noting that a second-rank KY tensor is an antisymmetric tensor f_{ab} that satisfies the differential equation

$$\nabla_{(a}f_{b)c} = 0. \tag{2.3}$$

As a consequence of (2.3), a KY tensor also satisfies the integrability condition

$$\nabla_a \nabla_b f_{cd} = -\frac{3}{2} R^e{}_{a[bc}f_{d]e}. \tag{2.4}$$

The dual of the KY tensor will be denoted by

$$^*f^{ab} = \frac{1}{2}\epsilon^{abcd}f_{cd}. \tag{2.5}$$

In the Kerr spacetime, the divergence of the dual of the KY tensor is related to the timelike Killing field by

$$\xi^b = \frac{1}{3}\nabla_a{}^*f^{ab}. \tag{2.6}$$

Using Eqs. (2.3)–(2.6), we can show after some calculation that the gradients of the fields ξ^a and $^*f^{ab}$ satisfy the set of equations

$$\nabla_a \xi^b = -\frac{1}{4} R^b{}_{acd}{}^*f^{cd}, \tag{2.7a}$$

$$\nabla_a{}^*f^{bc} = -2\xi^{[b}\delta^{c]}{}_a. \tag{2.7b}$$

We immediately see that the identification $(P^a, J^{ab}) = (\xi^a, -^*f^{ab})$ exactly solves the transport equations (2.2) with $\kappa = -1/4$. However, this exact solution does not have the physical interpretation we seek. In the limit $r \to \infty$, the tensor $-^*f^{ab}$ has two pieces, one which acts like an intrinsic angular momentum, and one like an orbital angular momentum about the spacetime point. The relative sign of these two pieces is the opposite of what it should be for $-^*f^{ab}$ to be the asymptotic angular momentum, as noted by Floyd [15]. Thus, this exact solution is not directly relevant for our purposes.

B. Asymptotic approximate solutions in stationary asymptotically flat spacetimes

We now show that arbitrary stationary, asymptotically flat spacetimes admit approximate asymptotic solutions to the partial differential equations (2.2) with $\kappa = -1/4$.

We adopt Bondi coordinates (u, r, θ^A), in which the coordinate u foliates $\mathscr{I}^+$ by shear-free cuts, r is an affine parameter along null rays, and θ^A are arbitrary coordinates on the unit 2-sphere. We specialize to a center-of-momentum Bondi coordinate system. It follows (see, e.g., [17]) that the spacetime metric can be written in the form

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{2\mathcal{M}}{r^2}\right)du^2 - 2dudr$$
$$+ r^2 h_{AB}d\theta^A d\theta^B + \frac{4}{3}N_A d\theta^A du + \dots. \tag{2.8}$$

Here M is a constant, the ellipsis denotes higher-order terms in a series in r^{-1}, h_{AB} is a metric on the unit 2-sphere, and D_A will denote a covariant derivative on the 2-sphere. Also $N_A(\theta^A)$ is a function satisfying $(D_B D^B + 1)N_A = 0$ (i.e., it is composed of $\ell = 1$ spherical harmonics), and $\mathcal{M}$ satisfies $6\mathcal{M} = -D^A N_A$, which follows from Einstein's equations.

Next, we make the following two ansatzes for the form of the solution. First we assume that the Lie derivative of P^a and J^{ab} with respect to ∂_u vanishes. This is a natural requirement since ∂_u is a Killing vector. Second, we assume the following large-r expansions of the solutions:

$$P^{\hat{\mu}} = P^{\hat{\mu}}_{(0)}(\theta^A) + \frac{M}{r}P^{\hat{\mu}}_{(1)}(\theta^A) + O\left(\frac{M^2}{r^2}\right), \tag{2.9a}$$

$$J^{\hat{\mu}\hat{\nu}} = \frac{r}{M}J^{\hat{\mu}\hat{\nu}}_{(0)}(\theta^A) + J^{\hat{\mu}\hat{\nu}}_{(1)}(\theta^A) + O\left(\frac{M}{r}\right), \tag{2.9b}$$

where the hatted indices refer to components of the tensors on the basis $\vec{e}_{\hat{\mu}}$ given by

$$\vec{e}_{\hat{u}} = \partial_u, \quad \vec{e}_{\hat{r}} = \partial_r, \quad \vec{e}_{\hat{A}} = \frac{1}{r}\partial_A. \tag{2.10}$$

Now substituting the ansatz (2.9) into the differential equations (2.2) and matching order by order in powers of $1/r$, we immediately find several constraints on $P^{\mu}_{(0)}$, $J^{\mu\nu}_{(0)}$, $P^{\mu}_{(1)}$, and $J^{\mu\nu}_{(1)}$. These constraints are that $\kappa = -1/4$, $P^{\mu}_{(1)} = 0$, $J^{ur}_{(0)} = MP^u_{(0)}$, $P^r_{(0)} = 0$, $J^{AB}_{(0)} = 0$, and $J^{rA}_{(0)} = -J^{uA}_{(0)} = 0$. With these conditions imposed, the equations further simplify, and it is then easy to show that $P^A_{(0)} = 0$, that $\partial_\mu P^u_{(0)} = 0$, and $\nabla^{(0)}_\rho J^{\mu\nu}_{(1)} = 0$, where $\nabla^{(0)}_\rho$ is the covariant derivative operator of Minkowski spacetime in the coordinates (u, r, θ^A). It therefore follows that $P^u_{(0)}$ is a constant (which need not coincide with the Bondi mass M of the spacetime though). Because $J^{\mu\nu}_{(1)}$ is antisymmetric and satisfies the same equation as a covariantly constant tensor in Minkowski spacetime, it can be parameterized by six constants.

We have shown, therefore, that for the specific value $\kappa = -1/4$, there exist asymptotic solutions to the equations (2.2) in a stationary spacetime and that their ex-

pansion in Bondi coordinates has the form

$$P^\mu \partial_\mu = P^u_{(0)} \partial_u + \dots , \qquad (2.11a)$$

$$J^{\mu\nu} \partial_\mu \otimes \partial_\nu = r P^u_{(0)} \left[\partial_u \otimes \partial_r - \partial_r \otimes \partial_u \right] \\ + J^{\mu\nu}_{(1)} \partial_\mu \otimes \partial_\nu + \dots , \qquad (2.11b)$$

where $P^u_{(0)}$ is a constant and $J^{\mu\nu}_{(1)}$ is parameterized by six constants. Thus, there is in fact a seven-parameter family of solutions[2] (P^a, J^{ab}) that can be transported by Eq. (2.1) with $\kappa = -1/4$ in an asymptotically path-independent manner in the region of the spacetime described by the metric (2.8).

In the next section, we will discuss a prescription for how observers can measure quantities (P^a, J^{ab}) at their locations, and we will argue that the existence of the approximate solutions (2.11) can be used to demonstrate consistency of such measurements made by different observers.

III. PROCEDURE FOR MEASURING ANGULAR MOMENTUM

In this section, we define a prescription for how an observer at an event $\mathcal{P}$ can measure an element of $\mathcal{G}^*_\mathcal{P}$, the space dual to linearized Poincaré transformations on the tangent space $T_\mathcal{P}\mathcal{M}$. That element can be parameterized as a pair of tensors (P^a, J^{ab}) at $\mathcal{P}$, as discussed in Paper I, which can be interpreted as approximate versions of the linear and angular momentum of the spacetime about the observer's location. The prescription requires several assumptions about the geometry near $\mathcal{P}$ and, consequently, is applicable only in certain situations. The definition of the prescription is given in in Sec. III A, and some of its properties are discussed in Sec. III B.

A. Prescription for measuring angular momentum

The steps of the prescription are as follows:

(i) Measure all the components of the Riemann tensor R_{abcd} and of its gradient $\nabla_a R_{bcde}$ at the event $\mathcal{P}$.

(ii) Compute the curvature invariants

$$K_1 \equiv R_{abcd} R^{abcd}, \qquad (3.1a)$$

$$\mathcal{K}_1 \equiv \nabla_a R_{bcde} \nabla^a R^{bcde}, \qquad (3.1b)$$

which we assume to satisfy $K_1 > 0$ and $\mathcal{K}_1 > 0$. Then compute quantities M and r using (cf. Footnote 8 of Paper I)

$$M = \frac{15\sqrt{5}(K_1)^2}{4\mathcal{K}_1^{3/2}} \left(1 - \frac{15\sqrt{3}K_1^{3/2}}{4\mathcal{K}_1} \right) , \qquad (3.2a)$$

$$r = \sqrt{\frac{15 K_1}{\mathcal{K}_1}} \left(1 - \frac{5\sqrt{3}K_1^{3/2}}{4\mathcal{K}_1} \right) . \qquad (3.2b)$$

(iii) Repeat steps (i) and (ii) at nearby spacetime points, so as to measure the gradient $\nabla_a r$ of the quantity r.

(iv) Assuming that the vector $\nabla_a r$ is spacelike, define the unit vector n^a in the direction of $\nabla_a r$ by $n^a = (N_1)^{-1}\nabla^a r$, where $N_1 = \sqrt{(\nabla^a r)(\nabla_a r)}$. Next, compute the quantity

$$y^a = -(r + M)n^a , \qquad (3.3)$$

which the observer interprets as a perpendicular displacement vector from her location to the position of the center-of-mass worldline of the source.

(v) Construct the symmetric tensor H_{ab} from

$$H_{ab} = R_{acbd} n^c n^d . \qquad (3.4)$$

Compute the eigenvectors ζ^a and eigenvalues λ of H_{ab} from $H_{ab}\zeta^b = \lambda \zeta_a$. From the definition (3.4), one of the eigendirections will be $\zeta^a = n^a$ with corresponding eigenvalue $\lambda = 0$. We will assume that at least one eigenvector has a strictly positive eigenvalue, and we denote the eigendirection corresponding to the largest eigenvalue by t^a. It follows that this vector is orthogonal to n_a (i.e., $t^a n_a = 0$).

(vi) Assuming that the vector t^a is timelike, next define a unit, future-directed timelike vector v^a by $v^a = (N_2)^{-1}t^a$. The normalization $(N_2)^{-1}$ is defined from $(N_2)^2 = -t_a t^a$ and the sign of N_2 is chosen so that v^a is future directed.

(vii) Compute the magnetic part of the Weyl tensor along v^a

$$B_{ae} = -\frac{1}{2}\epsilon_{abcd} C^{cd}_{\ \ ef} v^b v^f . \qquad (3.5)$$

From this construct a spin vector S^a by

$$S^a = \frac{r^4}{2} B^a_{\ b} n^b - \frac{2r^4}{3}(B_{bc} n^b n^c)n^a , \qquad (3.6)$$

and a 4-velocity vector u^a by

$$u^a = \left(1 + \frac{M}{r}\right) v^a + \frac{1}{Mr}\epsilon^a_{\ bcd} v^b S^c n^d . \qquad (3.7)$$

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[2] The family of solutions contains seven parameters rather than the ten parameters associated with the Poincaré group, because the equations (2.2) require the solution P^a to be asymptotically proportional to the Bondi 4-momentum of the spacetime, eliminating the boost freedom.

(viii) Define the angular momentum and linear momentum to be

$$P^a = M u^a \,, \tag{3.8a}$$

$$J^{ab} = \epsilon^{ab}{}_{cd} u^c S^d + 2 y^{[a} P^{b]} \,. \tag{3.8b}$$

Finally from (P^a, J^{ab}) compute an element of $\mathcal{G}^*_{\mathcal{P}}$ using the definition (2.1) of Paper I specialized to $\vec{x}_0 = 0$.

B. Motivation for and properties of the prescription

We now discuss the motivations for the measurement prescription and some of its properties.

First, we note that the prescription refines the prescription given in Paper I, at subleading order in M/r, in a number of ways. First, the expressions (3.2) for M and r contain higher-order correction terms constructed from the curvature invariants (3.1). Second, the spin in (3.6) is constructed using the magnetic part of the Weyl tensor rather than the symmetric tensor H_{ab} of Eq. (3.4) and a pseudoscalar curvature invariant. Finally, the four-velocity of the source is given by Eq. (3.7) rather than being proportional to t^a.

We next discuss some of the motivations for these refinements. Consider the following three properties of algorithms to produce tensor fields P^a and J^{ab} from the local spacetime geometry:

(i) In the context of linearized gravity, the algorithm reproduces the expected answers for stationary, vacuum spacetimes near future null infinity, in the limit $r \to \infty$.

(ii) The specification of the algorithm does not require any preferred lengthscale or a choice of spacetime orientation.

(iii) Consider the tensor fields P^a and J^{ab} obtained by applying the algorithm to a stationary, vacuum region of an asymptotically flat spacetime near future null infinity. When these tensor fields are expanded in powers of $1/r$, the leading and subleading terms yield a solution of the transport equations (2.2) with $\kappa = -1/4$ to the accuracy discussed in Sec. II B. In other words, they yield a specific element of the seven parameter family (2.11) of approximate asymptotic solutions.

The prescription of Paper I satisfies properties (i) and (ii), while the refined algorithm of this paper is designed to additionally satisfy property (iii). This requirement necessitates improving the accuracy of the algorithm, from leading order in M/r to subleading order in M/r. More precisely, when applied to an arbitrary vacuum, stationary, asymptotically flat spacetime, the definitions (3.2) of M and r yield respectively the

Bondi mass and the radial coordinate of the Bondi system (2.8) specialized to the center-of-mass frame condition $D_A N^A = 0$, up to fractional errors of order $\sim M^2/r^2$ and $\sim S^2 M^{-2} r^{-2}$ in both quantities. In particular, when applied to the Kerr spacetime, the algorithm reproduces the ADM mass and the Boyer-Lindquist radial coordinate to an equivalent accuracy. The 4-momentum P^a and angular momentum J^{ab} of the new algorithm have fractional errors[3] of the same order. This implies that components of P^a and J^{ab} in an orthonormal basis have errors that scale as $\sim M^3/r^2$ and $\sim M^3/r$, respectively, as compared to the errors $\sim M^2/r$ and $\sim M^2$ in Paper I. Orthonormal-basis components of the displacement vector y^a to the center of mass have errors of order $\sim M^2/r$, rather than the $\sim M$ errors of Paper I.

Consider now the tensor fields P^a and J^{ab} produced by the algorithm in the stationary, vacuum region of an asymptotically flat spacetime near future null infinity. Since property (iii) is satisfied, when (P^a, J^{ab}) are transported by the transport equations (2.1) with $\kappa = -1/4$, these tensors will be transported in a path-independent way to the above accuracy—the same accuracy with which they are measured. It therefore follows that observers will find consistency between their measured values of linear and angular momentum in the limit $r \to \infty$.

Next, we discuss the extent to which the measurement algorithm is unique. As discussed in Paper I, imposing the requirements (i) and (ii) does not determine a unique algorithm in linearized gravity, since the information about the asymptotic charges is encoded redundantly in the values of the Riemann tensor and its first two derivatives at a point. Nevertheless, in Paper I, the leading-order pieces of P^a and J^{ab} were uniquely determined by the requirement (i).

Similarly, here, imposing the requirements (i), (ii), and (iii) does not yield a unique algorithm, because of redundancy in how information is encoded in the Riemann tensor and its gradients at a point.[4] Nevertheless, the leading and subleading pieces of P^a and J^{ab} are uniquely determined. In other words, all algorithms which satisfy the three properties yield the same solution out of the seven-parameter family of approximate solutions discussed in Sec. II B. This is because the seven free parameters in the solutions (2.11) are fixed by imposing that the algorithm satisfy the requirements (i) and (ii). We

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[3] Here when we refer to the fractional error in the tensor fields we do not mean to imply the existence of "correct" versions to which our prescription can be compared. Instead, we mean that the leading and subleading pieces of P^a and J^{ab} have been carefully specified, but higher-order pieces have not.

[4] For example, one could have chosen the version of the expansion (3.2b) appropriate for the radial coordinate $r' = \sqrt{x^2 + y^2 + z^2}$ of harmonic, quasi-Cartesian, Cook-Scheel coordinates [18] instead of the Boyer-Lindquist radial coordinate. This would require subleading modifications to the prescription for measuring the vectors y^a, S^a, and u^a in order to satisfy property (iii).

note that if the requirement (ii) is dropped, ambiguities in the algorithm do arise at subleading order, of the form $y^a \to y^a + \lambda \epsilon^{abcd} u_b J_{cd}/M$, where λ is a positive dimensionless constant and $y^a \equiv -J^{ab} P_b/M^2$.

IV. TRANSPORT EQUATIONS AS PARALLEL TRANSPORT IN A FIBER BUNDLE

We now turn to studying the transport equation (2.1) from a different point of view. A mathematically equivalent but conceptually different perspective on the transport equation can come from considering a direct-sum (Whitney-sum) bundle over the manifold $\mathcal{M}$. Specifically we will consider the bundle $\mathcal{B} \equiv T\mathcal{M} \oplus \Lambda^2 T\mathcal{M}$, of which the pair $(P^a, J^{[ab]})$ is a section. We will take derivatives of sections of this bundle, which will require us to discuss a number of connections on $\mathcal{B}$ and other bundles.

The Levi-Civita connection ∇ extends from $T\mathcal{M}$ and $T^*\mathcal{M}$ to all tensor-product bundles via the usual definition [11]. We will continue to use the same symbol ∇ for all of the distinct connections acting on different bundles. We can also extend ∇ to a connection on $\mathcal{B}$ in the simplest possible way, by letting ∇ act diagonally on each summand in $\mathcal{B}$.

Now recall that the space of connections is an affine space; thus, if ∇ is a connection, then so is $\tilde{\nabla} = \nabla + \Gamma$, where Γ is a one-form taking values in the space of linear maps. Starting from the transport equations (2.1), we can define a new connection on $\mathcal{B}$ via

$$\tilde{\nabla}_a \begin{pmatrix} P^b \\ J^{bc} \end{pmatrix} \equiv \nabla_a \begin{pmatrix} P^b \\ J^{bc} \end{pmatrix} + \begin{pmatrix} \kappa R^b{}_{acd} J^{cd} \\ -2P^{[b} \delta^{c]}{}_a \end{pmatrix} . \tag{4.1}$$

Because the second term on the right is linear in the vector (P^a, J^{ab}), there is clearly a corresponding linear-map-valued one-form, Γ. Let us introduce an index notation for tensors in the different bundles: specifically, we will use lower-case Latin indices for $T\mathcal{M}$ and $T^*\mathcal{M}$, and capital Latin indices for $\mathcal{B}$ and $\mathcal{B}^*$. A capital index will thus range over the collection of lower-case indices, e.g. $B = (b, [bc])$. In this notation, we may denote the pair as

$$X^B \equiv \begin{pmatrix} P^b \\ J^{bc} \end{pmatrix} \begin{matrix} {}_{\text{B=b}} \\ {}_{\text{B=[bc]}} \end{matrix} , \tag{4.2}$$

and we can rewrite Eq. (4.1) in the form

$$\tilde{\nabla}_a X^B = \nabla_a X^B + \Gamma_a{}^B{}_D X^D . \tag{4.3}$$

Now we would like to read off the form of $\Gamma_a{}^B{}_D$ from Eq. (2.1). By inserting a Kronecker delta and permuting indices, we find

$$\Gamma_a{}^B{}_D = \begin{matrix} {}_{\text{B=b}} \\ {}_{\text{B=[bc]}} \end{matrix} \begin{pmatrix} \overset{\text{D=d}}{0} & \overset{\text{D=[de]}}{\kappa R^b{}_{ade}} \\ 2\delta^{[b}_a \delta^{c]}_d & 0 \end{pmatrix} . \tag{4.4}$$

We can then think of the transport equation (2.1) as simply parallel transport under this new connection $\tilde{\nabla}$ in the direct-sum bundle,

$$\tilde{\nabla}_a X^B = 0 . \tag{4.5}$$

In this bundle viewpoint, the questions related to the existence of solutions to the transport equations (2.1) can be cast in terms of conditions on the curvature of the connection. We thus compute this curvature in the next part, and we give a necessary condition for the existence of solutions in the part thereafter.

A. Curvature of the connection and holonomy of the transport equation

Any connection $\mathcal{D}$ on a vector bundle has an associated curvature tensor, by virtue of the linearity of the map

$$(\mathcal{D}_u \mathcal{D}_v - \mathcal{D}_v \mathcal{D}_u - \mathcal{D}_{[u,v]})X = R^{(\mathcal{D})}(u, v)X , \tag{4.6}$$

where u, v are two arbitrary tangent vectors on the base manifold (they are *not* related to the vectors u^a and v^a of Sec. III A), and $R^{(\mathcal{D})}(-, -)$ is a two-form taking values in linear transformations on the fiber space. If we work in indices and in a holonomic frame then we can write

$$(\mathcal{D}_a \mathcal{D}_b - \mathcal{D}_b \mathcal{D}_a)X^C = R_{ab}^{(\mathcal{D})C}{}_E X^E . \tag{4.7}$$

Let us start by presenting the curvature of the connection ∇ on $\mathcal{B}$. This can be derived by working with the basic connection ∇ on $T\mathcal{M}$ that acts diagonally on the two summands of $\mathcal{B}$. As a shorthand we will simply write $R_{ab}{}^C{}_E$ in place of $R_{ab}^{(\nabla)C}{}_E$. In our index notation, we find

$$R_{ab}{}^C{}_E = \begin{matrix} {}_{\text{C=c}} \\ {}_{\text{C=[cd]}} \end{matrix} \begin{pmatrix} \overset{\text{E=e}}{R_{ab}{}^c{}_e} & \overset{\text{E=[ef]}}{0} \\ 0 & 2R_{ab}{}^{[c}{}_{[e} \delta^{d]}_{f]} \end{pmatrix} . \tag{4.8}$$

Now we may compute the curvature, $\tilde{R}$, of our new connection, $\tilde{\nabla}$. We will make use of the difference between the two connections, $\tilde{\nabla}_a X^B = \nabla_a X^B + \Gamma_a{}^B{}_D X^D$. From the connection coefficients, we can find $\tilde{R}$ in terms of R. A straightforward calculation gives (again in a holonomic frame),

$$\tilde{R}_{ab}{}^C{}_E = R_{ab}{}^C{}_E + \nabla_a \Gamma_b{}^C{}_E - \nabla_b \Gamma_a{}^C{}_E$$
$$+ \Gamma_a{}^C{}_G \Gamma_b{}^G{}_E - \Gamma_b{}^C{}_G \Gamma_a{}^G{}_E . \tag{4.9}$$

Now we will combine Eq. (4.4) and Eq. (4.8) to compute the coefficients of $\tilde{R}$ in indices. First, "squaring" the matrix in Eq. (4.4) gives

$$\Gamma_a{}^C{}_G \Gamma_b{}^G{}_E = \begin{matrix} {}_{\text{C=c}} \\ {}_{\text{C=[cd]}} \end{matrix} \begin{pmatrix} \overset{\text{E=e}}{2\kappa R^c{}_{abe}} & \overset{\text{E=[ef]}}{0} \\ 0 & 2\kappa \delta^{[c}_a R^{d]}{}_{bef} \end{pmatrix} . \tag{4.10}$$
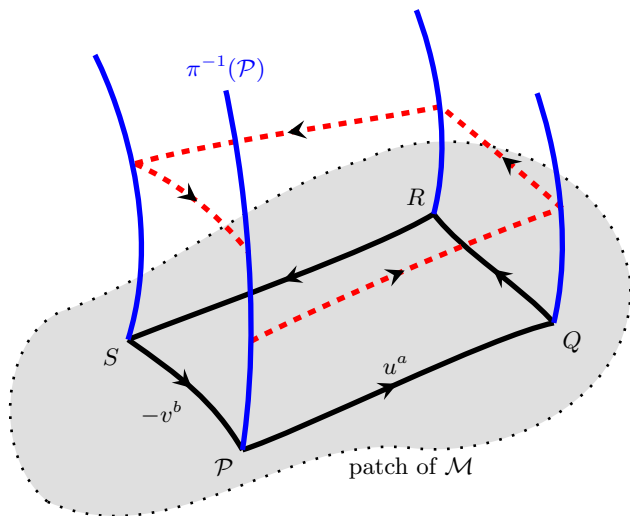
FIG. 1. (Color online) Transport about a coordinate rectangle $\mathcal{PQRS}$ in a small patch of the base manifold $\mathcal{M}$. The coordinate rectangle is defined by the vector u^a, which points from $\mathcal{P}$ to Q, and v^b, which points from $\mathcal{P}$ to S. Over each point $p \in \mathcal{M}$ is a 10-dimensional fiber $\pi^{-1}(p)$ (depicted in blue) in the bundle $\mathcal{B}$. Transport around $\mathcal{PQRS}$ on the base manifold (depicted in black) with the new connection $\tilde{\nabla}$ gives transport through $\mathcal{B}$ (depicted in red, dashed). This induces a linear transformation acting on $\pi^{-1}(\mathcal{P})$, as expressed in Eq. (4.13).

Combining Eq. (4.8), Eq. (4.10), taking the gradient of Eq. (4.4), antisymmetrizing, and using Bianchi identities, we find

$$\tilde{R}_{ab}{}^{C}{}_{E} =$$

$$
\begin{array}{c}
 \\
\text{C=c} \\
\text{C=[cd]}
\end{array}
\begin{pmatrix}
\overset{\text{E=e}}{(1-2\kappa)R_{ab}{}^{c}{}_{e}} & \overset{\text{E=[ef]}}{\kappa\nabla^{c}R_{abef}} \\
0 & 2R_{ab}{}^{[c}{}_{[e}\delta_{f]}^{d]} + 4\kappa\delta_{[a}^{[c}R^{d]}{}_{b]ef}
\end{pmatrix}.
$$
(4.11)

From this calculation, we can directly read off the holonomy that the vector $X^E = (P^e, J^{[ef]})$ acquires when transported (via the new connection $\tilde{\nabla}$) around a "coordinate rectangle." We construct such a rectangle based at point $\mathcal{P}$, first going in direction u^a, then v^b, then back along $-u^a$, and finally back along $-v^b$ (see Fig. 1). Starting with vector X^E, after the transport we will have the new vector

$$X^E \xrightarrow{\circlearrowleft_{uv}} X^E + \delta X^E,$$
(4.12)

where the deviation vector is given by

$$\delta X^C = -u^a v^b \tilde{R}_{ab}{}^{C}{}_{E} X^E.$$
(4.13)

Combining Eqs. (4.11) and (4.13), we find

$$
\begin{pmatrix} \delta P^c \\ \delta J^{cd} \end{pmatrix} = \begin{pmatrix} -u^a v^b \left[(1-2\kappa)R_{ab}{}^{c}{}_{e}P^e + \kappa\nabla^c R_{abef}J^{ef} \right] \\ -u^a v^b \left(2R_{ab}{}^{[c}{}_{[e}\delta_{f]}^{d]} + 4\kappa\delta_{[a}^{[c}R^{d]}{}_{b]ef} \right) J^{ef} \end{pmatrix}.
$$
(4.14)

Using the geometric bitensor approach in [19] reproduces this more algebraic calculation [20].

B. Existence of solutions in an extended region

From the calculation of $\tilde{R}_{ab}{}^{C}{}_{E}$, we can now state a necessary condition for the existence of solutions to Eq. (4.5) in an extended region. Suppose a solution $\mathcal{X}^C$ exists in an extended region that includes the point $\mathcal{P}$, and it takes the value $\mathcal{X}^C(\mathcal{P})$. Then, a necessary condition for the solution's existence is that under transport about an arbitrary coordinate rectangle determined by u^a, v^b, the value $\mathcal{X}^C(\mathcal{P})$ returns to itself (i.e., there is a vanishing deviation vector $\delta\mathcal{X}^C = 0$). From Eq. (4.13), we have

$$u^a v^b \tilde{R}_{ab}{}^{C}{}_{E}\mathcal{X}^E = 0 \quad \forall u, v \qquad (4.15)$$

$$\therefore \quad \tilde{R}_{ab}{}^{C}{}_{E}\mathcal{X}^E = 0. \qquad (4.16)$$

To interpret this condition, let us treat $\tilde{R}_{ab}{}^{C}{}_{E}$ as a linear map $\tilde{R} : \mathcal{B} \to \Lambda^2 T^*\mathcal{M} \times \mathcal{B}$. Then, this necessary condition is that the map $\tilde{R}$ has a nontrivial kernel.

We have checked through an explicit coordinate-component calculation in the Kerr spacetime, in Boyer-Lindquist coordinates, that $\tilde{R}$ with $\kappa = -1/4$ has a one-dimensional kernel. Thus, the space of solutions to Eq. (4.5) is a one-dimensional linear space. This is equivalent to the timelike Killing field and dual KY tensor of Sec. II being unique solutions to (2.1) up to an overall multiplicative constant.

It is important to note, though, that this condition is only a necessary condition, and not a *sufficient* condition, for the existence of solutions in an extended region. It is easy to see why this is true by looking at the case $\kappa = 1/2$ in Eq. (4.11). We see that the 4-dimensional subspace $T\mathcal{M} \subset \mathcal{B}$ is automatically within the kernel of $\tilde{R}$ for $\kappa = 1/2$. In fact, we have verified in Kerr that this is the entirety of the kernel. However, the system (4.5) with $\kappa = 1/2$ does not have solutions in an extended region. If one starts at a point $\mathcal{P}$ with data $\mathcal{X}^C(\mathcal{P}) = (P^c, 0)$ with $P^c \neq 0$, then at some nearby point $Q \neq \mathcal{P}$, the transported data will have rotated out of the kernel, such that $J^{[cd]}(Q) \neq 0$.

A stronger condition is required for sufficiency. This condition comes from Frobenius' theorem for a tangent distribution to be integrable into a submanifold (i.e., for the distribution to be involutive). We leave an investigation of this sufficient condition for future work.

V. CONCLUSIONS

In this paper, we have introduced a method for observers to measure a kind of linear and angular momentum at a spacetime point from the Riemann tensor and its derivatives, and we also proposed a method to transport these momenta. These measurement and transport procedures are in the same spirit as those of Paper I, but they also contain some important refinements.

The refinements are designed so that observers who use both these procedures in stationary, vacuum regions of asymptotically flat spacetimes will find that their measurements are consistent with one another, asymptotically as $r \to \infty$. Thus, the procedures give a simple operational meaning to the linear and angular momentum of the spacetime in stationary regions of $\mathscr{I}^+$.

In this paper, the transport and measurement procedures are much more closely coupled to one another than they were in Paper I. We introduced a one-parameter family of transport equations, and we found that for a unique value of this parameter ($\kappa = -1/4$), there is a seven-parameter family of approximate solutions that can be transported independently of path in the $r \to \infty$ limit for stationary, vacuum spacetimes. Our measurement procedure was designed to reproduce one of these solutions in the appropriate limit.

We also explained how the transport equation for linear and angular momentum could be understood as parallel transport for a specific connection on a certain direct-sum vector bundle. We computed the curvature of this connection and used it to find the holonomy for an infinitesimal quadrilateral loop. From the curvature, we could also formulate a necessary (though not sufficient) condition for the existence of global sections of the bundle.

A similar procedure is not possible, we conjecture, at higher order in powers of $1/r$, in a general stationary, asymptotically-flat, vacuum spacetime. That is, the order to which we have worked in this article is the highest possible order where a procedure of path-independent transport is possible for (P^a, J^{ab}). This is because to the present order, all stationary, asymptotically-flat, vacuum spacetimes can be matched with an expansion of the Kerr spacetime. However, if we were to specialize to the Kerr spacetime and to continue to expand to higher orders in $1/r$, we conjecture that the transport equations (2.2) would constrain $J^{\mu\nu}_{(1)}, J^{\mu\nu}_{(2)}, \ldots$ to have the form of the dual to the Killing-Yano tensor, expanded to the appropriate order. We leave investigation of this conjecture to future work.

Other possible future directions include an exploration of sufficient conditions for global sections to exist using Frobenius' theorem and an exploration of BMS-type ambiguities in angular momentum in intermittently stationary spacetimes by the measurement and transport procedures developed here.

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