

# Another look at synchronized neutrino oscillations

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## Abstract

In dense neutrino backgrounds present in supernovae and in the early Universe neutrino oscillations may exhibit complex collective phenomena, such as synchronized oscillations, bipolar oscillations and spectral splits and swaps. We consider in detail possible decoherence effects on the simplest of these phenomena – synchronized neutrino oscillations that can occur in a uniform and isotropic neutrino gas. We develop an exact formalism of spectral moments of the flavour spin vectors describing such a system and then apply it to find analytical approaches that allow one to study decoherence effects on its late-time evolution. This turns out to be possible in part due to the existence of the (previously unknown) exact conservation law satisfied by the quantities describing the considered neutrino system. Interpretation of the decoherence effects in terms of neutrino wave packet separation is also given, both in the adiabatic and non-adiabatic regimes of neutrino flavour evolution.

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# 1 Introduction

It is well known that neutrino oscillations in dense neutrino backgrounds existing at certain stages of supernova explosion and in the early Universe may differ drastically from the oscillations in ordinary matter or in vacuum. In particular, synchronized oscillations [1–8], bipolar oscillations [5, 9–14], spectral splits and swaps [15–18] and multiple spectral splits [19] are possible. These phenomena have attracted a great deal of attention recently, see Refs. [20, 21] for reviews and extensive lists of literature.

The simplest system that exhibits collective neutrino oscillations is a dense uniform and isotropic gas consisting of only neutrinos (or only antineutrinos). In such a system, for sufficiently large neutrino density, neutrinos of different energies oscillate with the same frequency (i.e. undergo synchronized oscillations), and therefore even for wide neutrino spectra the oscillations do not average out with time. This is in sharp contrast with what is expected in the case of neutrino oscillations in vacuum, in usual matter or in low-density neutrino backgrounds. In particular, in vacuum neutrinos of different energies oscillate with different frequencies and over the time develop large phase differences, leading to decoherence and averaging out of the oscillations. On the other hand, synchronized neutrino oscillations in a dense neutrino gas mean that no decoherence occurs (or at least that some degree of coherence is maintained) in such a system, since complete decoherence would destroy the synchronization.

In this paper we explore late-time decoherence effects on collective neutrino oscillations. To this end, we concentrate on the simplest possible system where collective oscillations can take place – a uniform and isotropic neutrino gas. Decoherence of neutrino oscillations can be described either in momentum space or in coordinate space. In the momentum space it comes from the dephasing of different neutrino modes at late times and is related to the integration over the neutrino spectrum. In the coordinate space decoherence is related to the spatial separation of the wave packets of different neutrino propagation eigenstates after they have traveled long enough distance. The momentum-space and coordinate-space descriptions are equivalent (see, e.g., [22]).

Since in supernovae and in the early Universe neutrinos are produced at very high densities, their production processes are well localized in space and time and therefore their wave packets are very short in coordinate space [23–25]. As a result, one could expect decoherence by wave packet separation to occur rather quickly and to affect significantly collective neutrino oscillations. In particular, this would destroy synchronized neutrino oscillations at sufficiently late times. Numerical calculations show, however, no trace of such decoherence when the density of the neutrino gas is high enough. One of the main goals of the present study was therefore to understand why no decoherence (and therefore no desynchronization) occurs in high-density neutrino gases, and how in general coherence and decoherence are related to the synchronization of neutrino oscillations or lack thereof. Our study is in a sense complementary to that in [4] where the possibility for a neutrino system to develop a spontaneous synchronization starting with a completely incoherent initial state

was considered.

The paper is organized as follows. In Section 2 we review the standard flavour spin formalism which is especially well suited for describing neutrino oscillations and flavour conversions in dense neutrino backgrounds. We also discuss the conservation law for a quantity  $\mathcal{E}$  which can be interpreted as the total energy of self-interacting magnetic moments in an external magnetic field. This section mainly serves to introduce our framework and notation. Sections 3 - 5 contain our new results. In Section 3 we develop a formalism of spectral moments  $\vec{K}_n$  describing a homogeneous and isotropic gas of neutrinos or neutrinos and antineutrinos. We derive equations of motion for these quantities and relations between the time derivatives of  $\vec{K}_n$  and  $\vec{K}_{n+1}$ . We also establish a new conservation law for this neutrino system, not previously known in the literature. In Section 4 we develop two approximate analytical approaches for describing decoherence effects on synchronized neutrino oscillations. They are based on the formalism of Section 3 augmented by certain assumptions about the late-time behaviour of the neutrino flavour spin vector in the system under consideration. In this section we also compare our approach and its results with those in Ref. [8], where decoherence effects on synchronized neutrino oscillations have also been studied. In Section 5 we give a qualitative interpretation of coherence and partial or full decoherence in terms of wave packet separation based on the consideration in the neutrino propagation eigenstate basis. The roles of adiabaticity and adiabaticity violation for possible decoherence effects is considered. In Section 6 we summarize and discuss our results. Technical details of some derivations related to our analysis in Section 4 are given in the Appendix.

## 2 The flavour spin formalism

Flavor mixing and evolution in a neutrino gas can be described by time-dependent density matrices  $\varrho_{\mathbf{p}}$ , which for each momentum mode  $\mathbf{p}$  are matrices in flavour space [26–28]. Their diagonal elements are actually occupation numbers for neutrinos of given flavour, while the off-diagonal elements contain information about coherence properties of the system. The evolution of these matrices is governed by the Liouville equation [28–31].

We will consider a homogeneous and isotropic neutrino gas evolving in time. It has been recently realized that even very small initial deviations from space-time symmetries of a system of self-interacting neutrinos may be strongly enhanced in the course of its evolution [32–34]. Such effects could profoundly influence the flavor evolution of the system and are currently under active investigation. Here we ignore such complications and assume that the uniformity and isotropy of the neutrino gas are exact and are preserved during its evolution.

For simplicity, we confine ourselves to 2-flavour neutrino oscillations  $\nu_e \leftrightarrow \nu_x$ , where  $\nu_x = \nu_\mu, \nu_\tau$  or a superposition thereof. For an isotropic system one can use the absolute value of the neutrino momentum  $p \equiv |\mathbf{p}|$  rather than the momentum itself to label the neutrino kinematic characteristics. However, it is more convenient to use instead the vacuum

oscillation frequency

$$\omega = \frac{\Delta m^2}{2p}, \quad (1)$$

with  $\Delta m^2$  being the mass squared difference of neutrino mass eigenstates. In the 2-flavour case one can decompose the density matrices and the Hamiltonian in terms of the Pauli matrices  $\sigma_i$ . The flavour evolution of each  $\omega$ -mode can then be described by the equation of motion (EoM) of the corresponding flavour spin vector [28]:

$$\dot{\vec{P}}_\omega = \vec{H}_\omega \times \vec{P}_\omega. \quad (2)$$

Here  $\vec{H}_\omega$  is the Hamiltonian (or “effective magnetic field”) vector:

$$\vec{H}_\omega = \omega \vec{B} + \lambda \vec{L} + \mu \vec{P} \quad (3)$$

with

$$\vec{B} = (s_{20}, 0, -c_{20}), \quad \vec{L} = \vec{n}_z \equiv (0, 0, 1), \quad \lambda = \sqrt{2}G_F n_e, \quad \mu = \sqrt{2}G_F n_\nu. \quad (4)$$

Here  $s_{20} \equiv \sin 2\theta_0$ ,  $c_{20} \equiv \cos 2\theta_0$  with  $\theta_0$  being the leptonic mixing angle in vacuum,  $G_F$  is the Fermi constant, and  $n_e$  and  $n_\nu$  are the net electron and neutrino number densities, respectively (i.e. the differences of number densities of the corresponding particles and antiparticles). The quantity  $\vec{P}$  is the global flavour spin vector of the neutrino system:

$$\vec{P} = \int_{-\infty}^{\infty} d\omega \vec{P}_\omega. \quad (5)$$

We use the convention according to which positive values of  $\omega$  correspond to neutrinos and negative values to antineutrinos [20, 28]. The terms in eq. (3) proportional to  $\omega$ ,  $\lambda$  and  $\mu$  correspond, respectively, to the vacuum contribution to the neutrino Hamiltonian and to the contributions coming from coherent forward scattering of a test neutrino on the particles of ordinary matter and on the neutrino background. If the density of ordinary matter is constant or nearly constant in the region where collective neutrino oscillations are expected to take place, effects of ordinary matter can be removed by going into a frame rotating around  $\vec{L}$  and replacing  $\theta_0$  by an effective mixing angle [10, 12]. In what follows we will be assuming that this has already been done (or that the effects of ordinary matter are negligible), and we will keep the notation  $\theta_0$  for the mixing angle defining the vector  $\vec{B}$ . The vector  $\vec{H}_\omega$  can then be written as

$$\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}. \quad (6)$$

Equations of motion (EoMs) (2) describe the precession of the flavour spin vectors  $\vec{P}_\omega$  of the individual neutrino modes around their corresponding “magnetic field” vectors  $\vec{H}_\omega$ . Obviously, they conserve the lengths of  $\vec{P}_\omega$ :

$$|\vec{P}_\omega| \equiv P_0 g_\omega = \text{const}. \quad (7)$$

The function  $g_\omega$  is just the spectrum of neutrinos in the variable  $\omega$ , which we will assume to be normalized according to

$$\int g_\omega d\omega = 1 \quad (8)$$

and to have the effective width  $\sigma_\omega$ . For example, for the Gaussian spectrum

$$g_\omega = \frac{1}{\sqrt{2\pi}\sigma_\omega} e^{-\frac{(\omega-\omega_0)^2}{2\sigma_\omega^2}}. \quad (9)$$

In our study we will be assuming that  $g_\omega$  corresponds to the  $\omega$ -spectrum of the wave packets of individual neutrinos, i.e. we deal with an ensemble of neutrinos described by identical wave packets with the same mean energy. Generalizations to more general neutrino spectra is straightforward. Note that a system of wave packets with the energy distribution function  $g_\omega$  is equivalent to a system of neutrinos with well defined energies and spectrum  $g_\omega$  [35], to which our treatment will therefore also apply.

The flavour content of a given  $\omega$ -mode is defined by the projection of  $\vec{P}_\omega$  on the  $z$ -axis in the flavour space; for  $\nu_e$  and  $\bar{\nu}_x$  this projection is positive, whereas for  $\bar{\nu}_e$ ,  $\nu_x$  it is negative. For systems consisting initially of arbitrary numbers of the flavour eigenstate neutrinos  $\nu_e$  and  $\nu_x$  and of their antineutrinos the initial conditions for the individual  $\omega$ -modes are therefore

$$\vec{P}_\omega(0) = P_0 g_\omega \vec{n}_z. \quad (10)$$

The initial condition for the global flavour spin vector is then

$$\vec{P}(0) = P_0 \vec{n}_z. \quad (11)$$

The parameter  $P_0$  (i.e. the initial length of the vector  $\vec{P}$ ) could in principle be set equal to one through a proper redefinition of the neutrino self-interaction strength  $\mu$ . However, in certain situations it may be convenient to define  $\mu$  as being proportional to the number density of just one neutrino species (e.g.,  $\bar{\nu}_e$  [20]). We therefore keep  $P_0$  as a free parameter.

Integrating eq. (2) over  $\omega$ , we obtain the EoM for  $\vec{P}$ :

$$\dot{\vec{P}} = \vec{B} \times \vec{S}, \quad \text{where} \quad \vec{S} \equiv \int d\omega \omega \vec{P}_\omega. \quad (12)$$

From the conservation of  $|\vec{P}_\omega|$  and the initial condition (11) it follows that for  $t > 0$  the length of the vector  $\vec{P}$  satisfies  $P(t) \equiv |\vec{P}(t)| \leq P_0$ . At the same time, eq. (12) implies

$$\vec{P} \cdot \vec{B} = \text{const.} = \vec{P}(0) \cdot \vec{B} = -c_{20} P_0. \quad (13)$$

Therefore,

$$c_{20} P_0 \leq P(t) \leq P_0. \quad (14)$$

From the EoMs (2) and (12) it follows that the following quantity is an integral of motion [10, 12]:

$$\mathcal{E} \equiv \vec{B} \cdot \vec{S} + \frac{\mu P^2}{2} = \text{const.} \quad (15)$$

It can be interpreted as the total energy of a system of ‘spins’  $\vec{P}_\omega$  with magnetic moments characterized by ‘gyromagnetic ratios’  $\omega_i$ . The quantity  $\vec{S}$  is then the total magnetic moment of the system. In this interpretation, the first term in (15) describes the interaction of the spins with the ‘external magnetic field’  $\vec{B}$ , whereas the second term describes the spin-spin interaction [5, 10, 12]. The system of self-interacting neutrinos is thus mathematically equivalent to the system of classical magnetic dipoles with the Hamiltonian given by eq. (15). Many properties of the former can therefore be understood from the properties and symmetries of the latter [36]. The initial conditions (10) imply that

$$\vec{S}(0) = \omega_0 P_0 \vec{n}_z, \quad \text{where} \quad \omega_0 = \langle \omega \rangle \equiv \int g_\omega \omega d\omega. \quad (16)$$

Here  $\omega_0$  is the mean neutrino ‘energy’ corresponding to the spectrum  $g_\omega$ . For symmetric spectra (such as e.g. the Gaussian spectrum (9))  $\omega_0$  coincides with the frequency at which  $g_\omega$  reaches its peak value. Substituting eqs. (11) and (16) into (15), we find

$$\frac{\mu P^2}{2} + \vec{B} \cdot \vec{S} = \frac{\mu P_0^2}{2} - c_{20} \omega_0 P_0. \quad (17)$$

### 3 The formalism of spectral moments

Let us introduce the spectral moments  $\vec{K}_n(t)$  of the flavour spin vector:

$$\vec{K}_n(t) = \int d\omega \omega^n \vec{P}_\omega(t), \quad n \geq 0. \quad (18)$$

The integral on the right hand side of (18) is well defined as far as the neutrino spectrum  $g_\omega$  goes to zero fast enough for  $|\omega| \rightarrow \infty$  (this is the case e.g. for the Gaussian spectrum and for any spectrum which vanishes outside a finite interval of  $\omega$ , such as the box-type spectrum). Note that the quantities  $\vec{P}$  and  $\vec{S}$  discussed in Section 2 are just particular cases of the spectral moments:

$$\vec{P}(t) = \vec{K}_0(t), \quad \vec{S}(t) = \vec{K}_1(t). \quad (19)$$

Rewriting eq. (2) as  $\dot{\vec{P}}_\omega = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_\omega$ , multiplying by  $\omega^n$  and integrating over  $\omega$ , we obtain

$$\dot{\vec{K}}_n = \vec{B} \times \vec{K}_{n+1} + \mu \vec{P} \times \vec{K}_n. \quad (20)$$

From this equation it is straightforward to find a scalar relation between the derivatives of  $\vec{K}_n$  and  $\vec{K}_{n+1}$ :<sup>1</sup>

$$\vec{B} \cdot \dot{\vec{K}}_{n+1} + \mu \vec{P} \cdot \dot{\vec{K}}_n = 0. \quad (21)$$

For  $n = 0$  this gives  $\vec{B} \cdot \dot{\vec{S}} + \mu \vec{P} \cdot \dot{\vec{P}} = 0$ , which can be immediately integrated. The result is the already known conservation law for  $\mathcal{E}$ , eq. (15). For  $n = 1$  we obtain

$$\vec{B} \cdot \dot{\vec{K}}_2 + \mu \vec{P} \cdot \dot{\vec{S}} = 0. \quad (22)$$

Let us show that this relation can also be integrated. Indeed, from eq. (12) it follows that  $\vec{S} \cdot \dot{\vec{P}} = 0$ . Therefore,  $\vec{P} \cdot \dot{\vec{S}} = (d/dt)(\vec{P} \cdot \vec{S})$ . This allows us to integrate eq. (22), which yields another conservation law satisfied by a system of self-interacting neutrinos:<sup>2</sup>

$$\tilde{\mathcal{E}} \equiv \vec{B} \cdot \vec{K}_2 + \mu \vec{P} \cdot \vec{S} = \text{const}. \quad (23)$$

While eq. (15) is well known, the conservation law (23) is new. The constant on its right hand side can be readily found if one observes that the initial conditions (10) imply

$$\vec{K}_n(0) = P_0 \langle \omega^n \rangle \vec{n}_z, \quad \text{where} \quad \langle \omega^n \rangle \equiv \int d\omega \omega^n g_\omega. \quad (24)$$

One can then rewrite eq. (23) as

$$\vec{B} \cdot \vec{K}_2 + \mu \vec{P} \cdot \vec{S} = P_0 [\mu P_0 \omega_0 - c_{20} (\omega_0^2 + \sigma_\omega^2)]. \quad (25)$$

Here we have used the relation  $\langle \omega^2 \rangle = \omega_0^2 + \sigma_\omega^2$ , which is just the definition of the variance  $\sigma_\omega^2$  of the  $\omega$ -spectrum.

## 4 Late-time regime of collective neutrino oscillations

### 4.1 Previous studies

Oscillations in a dense uniform and isotropic neutrino gas have been extensively studied in the literature (see, e.g., [1, 2, 4–8]). However, to the best of our knowledge, the only paper that dealt with the late-time decoherence effects on synchronized neutrino oscillations was Ref. [8].<sup>3</sup> Here we briefly review its main results.

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<sup>1</sup>Alternatively, one can derive eq. (21) by noting that eq. (2) implies  $\vec{H}_\omega \cdot \dot{\vec{P}}_\omega = 0$ . Multiplying this relation by  $\omega^n$  and integrating over  $\omega$  immediately yields eq. (21).

<sup>2</sup>Alternatively, one can derive (23) by multiplying eq. (4.10) of Ref. [37] by  $\omega^2$  and integrating it over  $\omega$ . We thank Baha Balantekin for this comment.

<sup>3</sup>Decoherence effects on flavour transformations of supernova neutrinos was also studied in Ref. [24]. However, the influence of the coherence loss on collective neutrino oscillations has not been considered there.

The authors studied decoherence effects numerically and (under certain assumptions) analytically. As particular examples, two different neutrino spectra were considered: the Gaussian spectrum of unit variance ( $\sigma_\omega = 1$ ) and the box-type spectrum of overall width 2 (in units of some fiducial frequency). As an order parameter characterizing decoherence, the deviation of the transverse (with respect to the vector  $\vec{B}$ ) component of the global flavour spin vector  $\vec{P}$  from its initial value was taken. This choice, first suggested in [4], is well justified: Indeed, decoherence leads to a shrinkage of  $\vec{P}$ .<sup>4</sup> Since the longitudinal (with respect to  $\vec{B}$ ) component of  $\vec{P}$ ,  $\vec{P}_\parallel \equiv (\vec{P} \cdot \vec{B})\vec{B}$ , is conserved, only the transverse component  $\vec{P}_\perp \equiv \vec{P} - (\vec{P} \cdot \vec{B})\vec{B}$  can shrink as a result of decoherence. A convenient choice for the order parameter is therefore  $R_A \equiv P_\perp/P_\perp(0) = P_\perp/(s_{20}P_0)$ ;  $R_A = 1$  would then correspond to perfect coherence, whereas  $R_A = 0$  would imply complete decoherence [4].

The authors of [8] found three regimes of neutrino oscillations in the system, depending on the value of neutrino self-interaction parameter  $\mu$ .

- (1) Large  $\mu$  regime – “perfect synchronization”. Despite differences in  $\omega$ , all  $\vec{P}_\omega$  evolve in a synchronized way, starting practically immediately from time  $t = 0$ . The flavour spin vector  $\vec{P}$  exhibits a simple precession around  $\vec{B}$  with the frequency  $\omega_0$  equal to the mean frequency of the neutrino spectrum:

$$\dot{\vec{P}} = \omega_0 \vec{B} \times \vec{P}. \quad (26)$$

The length of the vector  $\vec{P}$  is conserved and is given by its initial value:  $P = P_0$ .

- (2) Small  $\mu$  regime – complete late-time de-synchronization. At asymptotically large times oscillations completely average out;  $\vec{P}$  aligns with  $\vec{B}$  and remains constant, its length having shrunk to the minimal possible value  $P_{min} = c_{20}P_0$ .
- (3) Intermediate  $\mu$  regime – partial de-synchronization at late times. The evolution of  $\vec{P}$  at asymptotic times is a precession around  $\vec{B}$  with some frequency  $\omega_s$  (in general, different from  $\omega_0$ ), with the length of  $\vec{P}$  satisfying  $P_{min} < P < P_0$ .  $P_\perp$  remains finite.

These results are illustrated by Fig. 1, where the late-time value of  $R_A \equiv P_\perp/(P_0 \sin 2\theta_0)$  is plotted as a function of  $\mu$  for Gaussian and box-type neutrino spectra. A very interesting feature of the  $\mu$ -dependence of  $R_A$  is its threshold behaviour: decoherence is achieved for all values of  $\mu$  below a certain threshold value  $\mu_0$  which depends on the neutrino spectrum and is of order of its effective width  $\sigma_\omega$ . This was not *a priori* expected – it could well be that the complete decoherence would have occurred only in the limit  $\mu \rightarrow 0$ , with the curves on Fig. 1 smoothly approaching the origin.

In addition to numerical studies, in Ref. [8] also an analytical approach was developed, based on the following assumptions and approximations:

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<sup>4</sup>Note that, although the lengths of the vectors  $\vec{P}_\omega$  of the individual modes are conserved by their EoMs, the length of  $\vec{P}$  is in general not conserved.



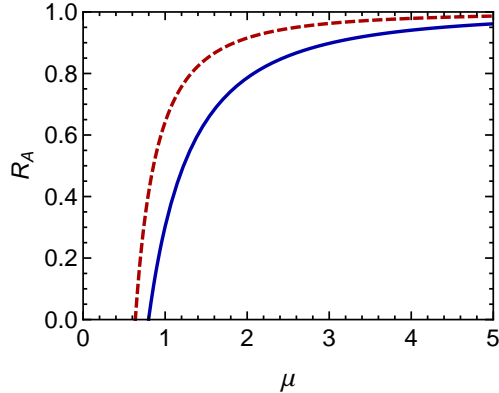


Figure 1: Order parameter  $R_A \equiv P_{\perp}/(P_0 \sin 2\theta_0)$  at late times for Gaussian (solid curve) and box-type (dashed curve) neutrino spectra,  $\theta_0 = \pi/4$ . Reprinted with permission from [8]; copyright 2010 by the American Physical Society.

- (i) It is assumed that the asymptotic (late time) evolution of the global flavour spin vector  $\vec{P}$  is a simple precession around  $\vec{B}$  with a constant frequency  $\omega_s$ .
- (ii) “Sudden approximation”:  $\vec{P}(t)$  is replaced by its asymptotic expression starting immediately at  $t = 0$ .
- (iii) The angles between the individual  $\vec{P}_{\omega}$  and the vectors  $\vec{B}$  and  $\vec{P}$  at the onset of asymptotic regime are taken to be those corresponding to  $t = 0$ .
- (iv) In the corotating frame (the frame rotating around  $\vec{B}$  together with  $\vec{P}$ ) the individual flavour spin vectors  $\vec{P}'_{\omega}$  are replaced by their asymptotic averages assumed to be given by their projections on  $\vec{H}'_{\omega}$ :  $\vec{P}'_{\omega} \rightarrow \langle \vec{P}'_{\omega} \rangle = \frac{\vec{P}'_{\omega} \cdot \vec{H}'_{\omega}}{H'^2_{\omega}} \vec{H}'_{\omega}$ . Here the primed quantities refer to the corotating frame.

The analytical results based on the above approximations reproduced very well the results of the numerical calculations of  $P_{\perp}$  performed in [8]. Yet, the underlying assumptions were of heuristic nature and are certainly rather far from being realistic. Note also that assumptions (ii) and (iii) are not fully consonant: as  $\vec{P}$  is the sum of all  $\vec{P}_{\omega}$ , it does not seem to be consistent to take, at the onset of the asymptotic regime, for  $\vec{P}_{\omega}$  their values specified by the initial conditions and for  $\vec{P}$  its asymptotic value (even if the asymptotic regime sets in immediately after  $t = 0$ ). It is also interesting to note that the analytical results obtained in [8] violate the conservation laws (17) and (25) (except in the limit  $\mu \gg \mu_0$ , where no decoherence occurs). We will discuss this point in more detail in Section 6. We will also explain there the reasons why the analytical approach of [8] works well despite its underlying assumptions being rather unrealistic. In the next subsections we develop two different analytical approaches, based on our spectral moments formalism augmented only by very

simple assumptions about the late-time behaviour of the global flavour spin vector  $\vec{P}$  (such as assumption (i) discussed above), without invoking any additional conjectures.

## 4.2 A simplified analytical approach

Eqs. (20)-(23) and (25) describing flavour evolution of a dense uniform and isotropic neutrino gas are exact and are satisfied for all  $t$ . We shall now assume that at asymptotically large times the evolution of the system is such that the length of the flavour spin vector  $\vec{P}$  is conserved. One example of such an evolution is the simple precession of  $\vec{P}$  around a fixed axis in the flavour space, such as the one described in eq. (26). Thus, our treatment here should be valid if the late-time evolution of the system has the form of synchronized collective oscillations, albeit possibly with the length of the flavour spin vector  $\vec{P}$  decreased by decoherence effects. If not noted otherwise, in what follows we will be considering all the relevant quantities at asymptotically large times, without specifying this explicitly and keeping the same notation for these quantities as before.

From the evolution equation (12) it follows that the conservation of  $P \equiv |\vec{P}|$  at asymptotically large times implies

$$\vec{P} \cdot \dot{\vec{P}} = \vec{P} \cdot (\vec{B} \times \vec{S}) = 0. \quad (27)$$

This condition can be satisfied when one (or more) of the following conditions is satisfied: (a)  $\vec{P} = 0$ ; (b)  $\vec{P} \parallel \vec{B}$ ; (c)  $\vec{S} = 0$ ; (d)  $\vec{S} \parallel \vec{B}$ ; (e)  $\vec{S}_\perp \parallel \vec{P}_\perp$ . Note that in case (e) only transverse components of the vectors  $\vec{S}$  and  $\vec{P}$  enter, because their components along  $\vec{B}$  drop out of eq. (27). We will not consider case (a) (shrinkage of  $\vec{P}$  to zero) which, as follows from (14), can only be realized when  $c_{20} = 0$ . Cases (b), (c) and (d) are of no interest to us either, since they correspond to the situations when at large  $t$  not only  $P \equiv |\vec{P}|$  stays constant, but there is no evolution of  $\vec{P}$  at all. Therefore we concentrate on case (e), in which at asymptotic times  $\vec{S}_\perp$  is parallel or antiparallel to  $\vec{P}_\perp$ , that is  $\vec{S}_\perp = \omega_s \vec{P}_\perp$ . Here we shall make an additional assumption that the longitudinal components of  $\vec{S}$  and  $\vec{P}$  satisfy a similar relation with the same proportionality coefficient (we will lift this extra assumption in the next subsection). In this case we can write

$$\vec{S}(t) = \omega_s \vec{P}(t). \quad (28)$$

In principle, the quantity  $\omega_s$  could be time dependent; however, as we shall show below, eq. (17) implies that it is constant.

Substituting (28) into (12), we find that  $\vec{P}$  satisfies

$$\dot{\vec{P}} = \omega_s \vec{B} \times \vec{P}. \quad (29)$$

Thus, in the considered case the evolution of the global flavour spin vector  $\vec{P}$  at late times is a simple precession around  $\vec{B}$ , and the proportionality coefficient  $\omega_s$  in eq. (28) is just the

frequency of this precession. From eq. (28) and the definitions of  $\vec{P}$  and  $\vec{S}$  it follows that

$$\omega_s = \frac{\int d\omega \omega P_{\omega,i}}{\int d\omega P_{\omega,i}}, \quad (30)$$

where  $P_{\omega,i}$  ( $i = x, y, z$ ) is any of the components of the vector  $\vec{P}_\omega$ . Consistency of our approach requires that the ratio in eq. (30) be independent of  $i$ .

Eq. (29) describes synchronized neutrino oscillations. Substituting (28) into the conservation law for  $\mathcal{E}$  given in eq. (17) and using eq. (13), we obtain

$$\frac{\mu}{2}[P_0^2 - P^2] = c_{20}P_0(\omega_0 - \omega_s). \quad (31)$$

Let us discuss the consequences of this relation. First, we note that all the quantities in it except possibly  $\omega_s$  are constant, so  $\omega_s$  must be constant as well. Second, since the left hand side of (31) is non-negative, so must be its right hand side, i.e.  $\omega_s \leq \omega_0$ . Third, because  $\omega_0$  and  $\omega_s$  are certain averages of  $\omega$  over the same spectrum  $g_\omega$  characterized by an effective width  $\sigma_\omega$ , their difference cannot exceed  $\sigma_\omega$  by much, that is

$$\omega_0 - \omega_s \lesssim \sigma_\omega. \quad (32)$$

From the latter and eq. (31) it immediately follows that in the limit  $\mu \rightarrow \infty$  we must have  $P \rightarrow P_0$  (no shrinkage of  $\vec{P}$  for very large  $\mu$ ). Later on we will also demonstrate that in the limit  $\mu \rightarrow \infty$  the frequency of synchronized oscillations  $\omega_s$  approaches  $\omega_0$ .

Eq. (31) also allows us to clarify the meaning of the formal limit  $\mu \rightarrow \infty$ . In practical terms, this limit means that  $\mu$  becomes large compared to some quantity of dimension of energy characterizing the neutrino system under consideration. In our case this could be e.g.  $\omega_0$ ,  $\omega_s$  or  $\sigma_\omega$ . We shall now show that this characteristic parameter is actually  $\sigma_\omega$ . Indeed, let us rewrite (31) as

$$\frac{P_0^2 - P^2}{P_0^2} = 2c_{20} \frac{\omega_0 - \omega_s}{\mu P_0} \lesssim 2c_{20} \frac{\sigma_\omega}{\mu P_0}, \quad (33)$$

where the approximate inequality follows from (32). From (33) one can immediately see that ‘perfect synchronization’, when the asymptotic value of  $P$  coincides with its initial value  $P_0$  and decoherence effects are negligible, is achieved for  $\mu P_0 \gg \sigma_\omega$ . At the same time it is irrelevant whether or not  $\mu P_0$  is large compared to  $\omega_0$  (or to  $\omega_s$ ). This is in accord with the fact that  $\omega_0$  or  $\omega_s$  can always be eliminated by going into a proper rotating frame. Thus, one can expect noticeable decoherence effects only for  $\mu P_0 \lesssim \sigma_\omega$ .

Eq. (31) relates two unknowns – the late-times value of  $P$  and the frequency of synchronized oscillations  $\omega_s$ . To find these quantities, we need one more relation between them. As such, we will use the new conservation law (23) that was derived in Section 3. Let us first note that eqs. (28) and (29) together with time independence of  $\omega_s$  imply that at asymptotic times the vector  $\vec{S}$  satisfies the EoM similar to (29),

$$\dot{\vec{S}} = \omega_s \vec{B} \times \vec{S}, \quad (34)$$

and so its length is conserved:  $\vec{S} \cdot \dot{\vec{S}} = 0$ . On the other hand, from eq. (20) with  $n = 1$  we have an exact relation

$$\dot{\vec{S}} = \vec{B} \times \vec{K}_2 + \mu \vec{P} \times \vec{S}. \quad (35)$$

The conservation of the length of  $\vec{S}$  then implies

$$\vec{S} \cdot (\vec{B} \times \vec{K}_2) = 0. \quad (36)$$

Following now the arguments similar to those given just below eq. (27), we conclude that eq. (36) is nontrivially realized only if at asymptotically large times  $\vec{K}_2$  is parallel or antiparallel to  $\vec{S}$  (and therefore also to  $\vec{P}$ ), that is  $\vec{K}_2(t) = \omega_1 \vec{S}(t)$ . Comparing then eqs. (35) and (34), we find  $\omega_1 = \omega_s$ , that is, asymptotically,

$$\vec{K}_2(t) = \omega_s \vec{S}(t) = \omega_s^2 \vec{P}(t). \quad (37)$$

Substituting this into (25) yields

$$-\omega_s^2 c_{20} P_0 + \omega_s \mu P^2 = P_0 [\mu P_0 \omega_0 - c_{20} (\omega_0^2 + \sigma_\omega^2)]. \quad (38)$$

Excluding now  $P^2$  from eqs. (31) and (38), we find the following equation for  $\omega_0 - \omega_s$ :

$$(\omega_0 - \omega_s)^2 - \frac{\mu P_0}{c_{20}} (\omega_0 - \omega_s) + \sigma_\omega^2 = 0. \quad (39)$$

Its solution is

$$\omega_0 - \omega_s = \frac{1}{2c_{20}} \left[ \mu P_0 - \sqrt{\mu^2 P_0^2 - 4c_{20}^2 \sigma_\omega^2} \right]. \quad (40)$$

We have discarded the solution with the plus sign in front of the square root because for  $\mu P_0 \gg \sigma_\omega$  it would lead to  $\omega_0 - \omega_s \gg \sigma_\omega$ , in contradiction with (32).

Let us now discuss eq. (40). First, we notice that it exhibits a threshold behaviour: the real solution only exists (and therefore synchronized oscillations can only take place) for  $\mu > \mu_0$ , where  $\mu_0$  is given by

$$\mu_0 P_0 = 2c_{20} \sigma_\omega. \quad (41)$$

Next, we find that for  $\mu P_0 \gg 2c_{20} \sigma_\omega$  (i.e. far above the threshold)

$$\omega_0 - \omega_s \simeq c_{20} \frac{\sigma_\omega^2}{\mu P_0} \ll \sigma_\omega. \quad (42)$$

Thus, for large  $\mu P_0$  the precession frequency  $\omega_s \rightarrow \omega_0$ . The difference  $\omega_0 - \omega_s$  reaches its maximum at the threshold  $\mu = \mu_0$ :

$$\omega_0 - \omega_s(\mu_0) = \sigma_\omega. \quad (43)$$

By using  $\omega_0 - \omega_s$  from eq. (40) in eq. (31) (or equivalently in eq. (38)), one can find the asymptotic value of  $P$  as a function of  $\mu$ :

$$P = P_0 \left(1 - \frac{\mu_0^2}{\mu^2}\right)^{1/4}. \quad (44)$$

The above results for the asymptotic regime have several attractive features: they demonstrate in a very simple way the existence of the threshold  $\mu_0$  below which no synchronized oscillations are possible, give the correct order-of-magnitude estimate for its value ( $\mu_0 P_0 \sim \sigma_\omega$ ), and lead to a reasonable behaviour of  $P$  far above the threshold. However, eq. (44) gives a wrong value of  $P$  at the threshold:  $P = 0$ . This is obviously incorrect because, due to the conservation of  $\vec{P} \cdot \vec{B} = -c_{20} P_0$ , the length of  $\vec{P}$  cannot be smaller than  $c_{20} P_0$ . A possible reason for this is that eq. (28) involves an additional assumption about the longitudinal components of  $\vec{P}$  and  $\vec{S}$  which actually does not directly follow from eq. (27). It can be shown that this additional assumption (and so the relation in eq. (28)) is actually well satisfied far above the threshold  $\mu_0$  but breaks down close to the threshold.

Indeed, we have found that for  $\mu \gg \mu_0$  the asymptotic length of  $\vec{P}$  coincides with  $P_0$ . This means that the time interval between  $t = 0$  and the onset of the asymptotic regime, during which  $P$  may shrink, is essentially zero, and the  $P$ -preserving synchronized oscillations described by eq. (29) start practically immediately at  $t = 0$ . Thus,  $P$  is conserved at all times. From eq. (15) it then follows that  $\vec{B} \cdot \vec{S}$  is also conserved at all times. Together with the relation  $\vec{S}_\perp(t) = \omega_s \vec{P}_\perp(t)$  this means that, starting practically from  $t = 0$ , the vector  $\vec{S}$  precesses around  $\vec{B}$  with the angular velocity  $\omega_s$ , just as  $\vec{P}$  does. Since  $\vec{S}$  and  $\vec{P}$  are collinear at  $t = 0$  (both pointing in the  $z$ -direction in the flavour space), they will then remain collinear at all times. Obviously, this argument fails close to the threshold  $\mu_0$ .

In the next subsection we shall lift the additional assumption about the longitudinal components of  $\vec{S}$  and  $\vec{P}$ . As we shall see, the assumption about the asymptotic behaviour of  $P_\perp$  will also need to be corrected.

### 4.3 An analytical approach valid for all $\mu$

We shall now attempt to develop an approximate analytical approach valid for all  $\mu$ .

#### 4.3.1 Approximation of constant asymptotic value of $P$

Let us first assume, as we did in the Section 4.2, that the evolution of the system at asymptotically large times conserves the length of the vector  $\vec{P}$ , that is, eq. (27) is satisfied. As was discussed above, a nontrivial realization of this condition requires the following relation between the transverse components of  $\vec{P}$  and  $\vec{S}$  at late times:

$$\vec{S}_\perp(t) = \omega_s \vec{P}_\perp(t). \quad (45)$$

Unlike we did in Section 4.2, we will not make here the additional assumption  $\vec{S}_{\parallel} = \omega_s \vec{P}_{\parallel}$ , which is valid only far above the threshold  $\mu_0$ .

Substituting eq. (45) into eq. (12), we again obtain the EoM (29) describing the precession of the flavour spin vector  $\vec{P}$  around  $\vec{B}$  with the angular velocity  $\omega_s$ . In our analysis in Section 4.2 we found that the parameter  $\omega_s$  introduced through eq. (28) was time-independent; this followed from the conservation law (15) (along with  $\vec{B} \cdot \vec{P} = \text{const.}$  and the late-time relation  $P = \text{const.}$ ). Here we cannot use the same argument, as the parameter  $\omega_s$  now relates only the transverse components of the vectors  $\vec{P}$  and  $\vec{S}$ , whereas the conservation law (15) constrains only the longitudinal component of  $\vec{S}$ . Instead, we will *assume* that the parameter  $\omega_s$  can be considered as constant at asymptotically large times. Thus, our assumption here about the late-time behaviour of the flavour spin vector in fact coincides with assumption (i) discussed in Section 4.1. However, we will *not* use listed there assumptions (ii) - (iv), employed in [8]; our spectral moments formalism will allow us to fully determine the late-time behaviour of the system basing solely on assumption (i).

It is easy to demonstrate by induction that at asymptotic times all the spectral moments  $\vec{K}_n$  satisfy the evolution equations similar to (29) with all  $\vec{K}_{n\perp}$  being collinear with  $\vec{P}_{\perp}$ :

$$\dot{\vec{K}}_n = \omega_s \vec{B} \times \vec{K}_n, \quad \vec{K}_{n\perp} = \alpha_n \vec{P}_{\perp}, \quad (46)$$

where  $\alpha_n$  is constant. Similar relations hold also for the flavour spin vectors of the individual modes  $\vec{P}_{\omega}$ . The derivation is especially simple in the corotating frame, see the Appendix. As shown there, for  $P_{\perp} \neq 0$  one can then find the ratio of the longitudinal and transverse components of the asymptotic vectors  $\vec{P}_{\omega}$ . Combining this with the normalization condition  $P_{\omega\parallel}^2 + P_{\omega\perp}^2 = P_{\omega}^2 = P_0^2 g_{\omega}^2$  allows one to determine  $P_{\omega\parallel}$  and  $P_{\omega\perp}$ . With the expressions for these quantities at hand, the longitudinal and transverse components of all the spectral moments  $\vec{K}_n$  at asymptotic times can then be found from eq. (18). However, the direct calculation shows that, except in the limit  $\mu \rightarrow \infty$ , the values of  $P_{\perp}$  obtained in this way do not reproduce correctly the results of numerical integration of the exact EoMs; in particular, no threshold behaviour in  $\mu$  is found. This means that the assumption that  $P \equiv |\vec{P}|$  approaches a constant value at asymptotically large times, on which our consideration here was thus far based, must actually be incorrect.

This point is partly illustrated by Fig. 2, which shows that for  $\mu$  exceeding the critical value  $\mu_0$  (but not far above it) the quantities  $P_{\perp}$  and  $P$  keep oscillating around their mean values even at very late times. Obviously, this figure cannot serve as a proof that the oscillations will survive for all times (which actually follows from our analytical results), but it shows that these oscillations continue even up to  $t = 10000$  (in the units in which  $\omega_0 = 1$ ), which is several orders of magnitude larger than the naively expected coherence length  $L_{\text{coh}} \simeq 1/(2\sigma_{\omega})$ . The situation is different for  $\mu \gg \mu_0$ , when  $P$  is practically constant at all times.

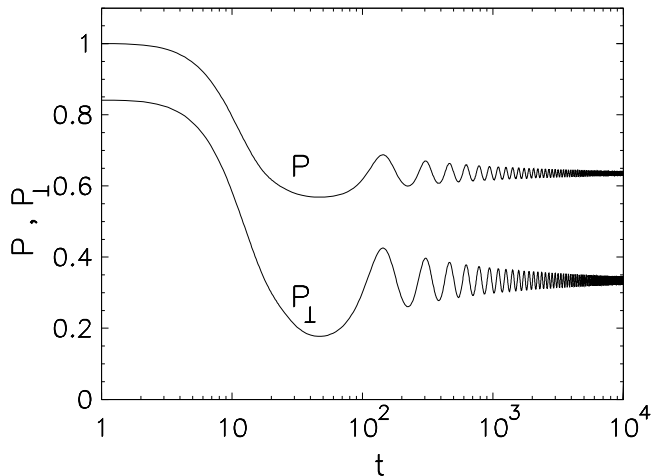


Figure 2: Time dependence of  $P$  (upper curve) and  $P_{\perp}$  (lower curve). Gaussian neutrino spectrum,  $P_0 = 1$ ,  $\omega_0 = 1$ ,  $\sigma_{\omega} = 0.1$ ,  $\theta_0 = 0.5$ ,  $\mu = 0.108$ .

### 4.3.2 Invoking the averaging procedure

As the amplitude of the late-time oscillations of  $P_{\perp}$  is relatively small, it seems to be a reasonable approximation to replace it at asymptotic times by its mean value, found by averaging over these oscillations. However, as our analysis in the previous subsection shows, this should be done through a consistent averaging procedure rather than by simply assuming that at large times  $P_{\perp} = \text{const}$ . To carry out the averaging systematically, we average the EoMs of the flavour spin vectors over a very large (formally infinite) time interval. It is convenient to do this in the corotating frame, where the late-time precession of the flavour spin vector  $\vec{P}$  is ‘rotated away’ from its evolution, and its direction at late times is essentially fixed (see the Appendix). Note that going to the corotating frame does not change the longitudinal (with respect to  $\vec{B}$ ) components of the flavour spin vectors and spectral moments as well as the lengths of their transverse components.

The relevance of the averaging procedure for studying the behaviour of the system at late times follows from the well known property of infinite-interval averages, that for any function  $f(t)$  that is integrable on any finite interval in  $[0, \infty)$ , the average can be found as

$$\langle f(t) \rangle = \lim_{t \rightarrow \infty} f(t) \quad (47)$$

provided that the limit on the right hand side exists. If at  $t \rightarrow \infty$  the limit in (47) does not exist but  $f(t) \rightarrow f_0 + \text{oscillating terms}$ , where  $f_0$  is a finite constant and the oscillating terms are of finite amplitude and average to zero, then  $\langle f(t) \rangle = f_0$ .

As demonstrated in the Appendix, the averaging procedure allows us to find the ratio of the averages of the transverse and longitudinal components of  $\vec{P}_{\omega}$ , which can be cast into the form

$$\langle \vec{P}_{\omega \perp} \rangle = f(\omega, \mu) \langle \mu \vec{P}_{\perp} \rangle,$$

$$\langle P_{\omega\parallel} \rangle = f(\omega, \mu) [\omega - \omega_r(\mu)], \quad (48)$$

with as yet unknown function  $f(\omega, \mu)$ . Note that we cannot combine the ratio  $\langle P_{\omega\perp} \rangle / \langle P_{\omega\parallel} \rangle$  with the normalization condition for  $P_\omega^2$  in order to find  $\langle P_{\omega\perp} \rangle$  and  $\langle P_{\omega\parallel} \rangle$  separately since these averages do not satisfy the same normalization condition as the un-averaged quantities  $P_{\omega\perp}$  and  $P_{\omega\parallel}$ . Therefore, in order to determine  $\langle P_{\omega\parallel} \rangle$  and  $\langle \vec{P}_{\omega\perp} \rangle$  one needs one more relation between them. As shown in the Appendix, such a relation can be found by considering the average  $\langle \vec{P}_\omega \cdot \vec{H}_\omega \rangle$  in the corotating frame. Together with eq. (48), this gives

$$f(\omega, \mu) = P_0 g_\omega \frac{-c_{20}(\omega - \omega_r) + s_{20} \langle \mu P_\perp \rangle}{(\omega - \omega_r)^2 + \langle \mu P_\perp \rangle^2}. \quad (49)$$

Here we have introduced the notation

$$\omega_r \equiv \omega_s(\mu) + c_{20} \mu P_0. \quad (50)$$

Next, we calculate the averages<sup>5</sup>

$$\langle P_\perp \rangle = \int d\omega \langle P_{\omega\perp} \rangle, \quad \langle P_\parallel \rangle = \int d\omega \langle P_{\omega\parallel} \rangle. \quad (51)$$

From eqs. (48), (49) and (13) it follows that these relations can be rewritten as

$$1 = \mu P_0 \int d\omega g_\omega \frac{-c_{20}(\omega - \omega_r) + s_{20} \langle \mu P_\perp \rangle}{(\omega - \omega_r)^2 + \langle \mu P_\perp \rangle^2}, \quad (52)$$

$$-c_{20} = \int d\omega g_\omega \frac{-c_{20}(\omega - \omega_r) + s_{20} \langle \mu P_\perp \rangle}{(\omega - \omega_r)^2 + \langle \mu P_\perp \rangle^2} (\omega - \omega_r). \quad (53)$$

For any neutrino spectrum function  $g_\omega$  and a given  $\mu$ , these two equations can be solved for the two unknowns,  $\omega_s(\mu)$  and  $\langle P_\perp(\mu) \rangle$ . Note that eq. (52) has been obtained by dividing both sides of the first equality in (51) by  $\langle P_\perp \rangle$ , and so it is a necessary condition for the existence of a nontrivial solution  $\langle P_\perp \rangle \neq 0$ .

Eqs. (52) and (53) coincide with eq. (17) of [8] (notice that our definition of the sign of  $c_{20}$  is opposite to theirs). However, we did not use the ‘sudden approximation’ and the assumption that at the onset of the asymptotic regime the flavour spin vectors of the individual modes  $\vec{P}_\omega$  have the same directions as they have at  $t = 0$ , which were employed in [8]. As was pointed out in Section 4.1, these assumptions are in general not well justified. Our consideration was instead based on the averaging procedure performed at the level of EoMs and the assumption that at asymptotically large times  $P_\perp = |\vec{P}_\perp|$  undergoes only relatively small oscillations (though it is not constant). The latter is confirmed by our numerical calculations.

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<sup>5</sup>Note that  $\langle \vec{P}_{\omega\perp} \rangle$  and  $\langle \mu \vec{P}_\perp \rangle$  are parallel, which follows from the first equation in (48).



After a little algebra one can obtain from eqs. (52) and (53) a simpler pair of equations [8]:

$$\frac{s_{20}}{\mu} = P_0 \int d\omega g_\omega \frac{\kappa}{(\omega - \omega_r)^2 + \kappa^2}, \quad (54)$$

$$-\frac{c_{20}}{\mu} = P_0 \int d\omega g_\omega \frac{(\omega - \omega_r)}{(\omega - \omega_r)^2 + \kappa^2}. \quad (55)$$

Here the notation  $\kappa \equiv \langle \mu P_\perp \rangle$  has been introduced. Noting that at the limit  $\mu \rightarrow \mu_0$  one has  $\kappa \rightarrow 0$  and the integrand of (54) goes to  $g_\omega \pi \delta[\omega - \omega_r(\mu_0)]$ , one finds that in this limit eqs. (54) and (55) become

$$\frac{s_{20}}{\mu_0} = P_0 \pi g_{\omega_r(\mu_0)}, \quad -\frac{c_{20}}{\mu_0} = \mathcal{P} \int d\omega g_\omega \frac{P_0}{\omega - \omega_r(\mu_0)}, \quad (56)$$

where  $\mathcal{P}$  stands for the Cauchy principal value. These equations can be solved to find the threshold value  $\mu_0$  and  $\omega_r(\mu_0)$  [8].

As was mentioned above, given the neutrino spectrum  $g_\omega$ , eqs. (52) and (53) (or equivalently (54) and (55)) can be solved numerically for  $\omega_s$  and  $\kappa = \langle \mu P_\perp \rangle$ . We will, however, consider now the simple box-type spectrum

$$g_\omega = \frac{1}{2\sigma} \cdot \begin{cases} 1, & |\omega - \omega_0| \leq \sigma, \\ 0, & |\omega - \omega_0| > \sigma \end{cases}, \quad (57)$$

for which the explicit analytical solution of the problem can be found. Note that the parameter  $\sigma$  here is related to the variance  $\sigma_\omega^2 \equiv \langle \omega^2 \rangle - \langle \omega \rangle^2$  as  $\sigma_\omega^2 = \frac{1}{3}\sigma^2$ . As the  $\omega$ -spectrum (57) is flat, the threshold value  $\mu_0$  can be immediately found from the first equation in (56):

$$\mu_0 P_0 = \frac{2}{\pi} s_{20} \sigma. \quad (58)$$

For  $\mu \geq \mu_0$  eqs. (54) and (55) yield

$$s_{20} \frac{2\sigma}{\mu P_0} = \arctan\left(\frac{\omega_0 - \omega_r + \sigma}{\kappa}\right) - \arctan\left(\frac{\omega_0 - \omega_r - \sigma}{\kappa}\right), \quad (59)$$

$$-c_{20} \frac{4\sigma}{\mu P_0} = \ln \frac{(\omega_0 - \omega_r + \sigma)^2 + \kappa^2}{(\omega_0 - \omega_r - \sigma)^2 + \kappa^2}. \quad (60)$$

For the consistency of the last relation it is necessary that  $(\omega_0 - \omega_r)$  be negative. The pair of transcendental equations (59) and (60) admits analytical solution for the two unknowns,  $\kappa$  and  $\omega_0 - \omega_r$ . Noting that  $\langle P_\perp \rangle = \kappa/\mu$ , we obtain

$$\langle P_\perp \rangle = \frac{2\sigma}{\mu} \cdot \frac{\sin(s_{20} \frac{2\sigma}{\mu P_0})}{\exp(c_{20} \frac{2\sigma}{\mu P_0}) + \exp(-c_{20} \frac{2\sigma}{\mu P_0}) - 2 \cos(s_{20} \frac{2\sigma}{\mu P_0})}, \quad (61)$$

$$\omega_0 - \omega_r = -\sigma \cdot \frac{\exp(c_{20} \frac{2\sigma}{\mu P_0}) - \exp(-c_{20} \frac{2\sigma}{\mu P_0})}{\exp(c_{20} \frac{2\sigma}{\mu P_0}) + \exp(-c_{20} \frac{2\sigma}{\mu P_0}) - 2 \cos(s_{20} \frac{2\sigma}{\mu P_0})}. \quad (62)$$

The asymptotic precession frequency  $\omega_s(\mu)$  can then be found from eqs. (62) and (50).

In Fig. 3 we compare the results of direct numerical integration of EoMs of the flavour spin vectors with the results of the approximate analytical approach described here. In the left panel we plot the order parameter  $R_A = \langle P_\perp \rangle / P_\perp(0) = \langle P_\perp \rangle / (s_{20} P_0)$ , which describes decoherence effects, as a function of  $\mu$ . The results are presented for two neutrino spectra: the Gaussian spectrum (9) and the box-type spectrum of eq. (57). The right panel of Fig. 3 presents a similar comparison for the asymptotic precession frequency  $\omega_s$ .

For the Gaussian spectrum the numerical solution of EoMs was obtained considering  $N_\omega = 3000$  modes uniformly distributed in the interval  $(\omega_0 - 4\sigma_\omega, \omega_0 + 4\sigma_\omega)$ . The same number of modes was used for the box-type spectrum. We considered the flavor evolution in the time range  $t \in [0, 10000]$ . The error due to the discretization of the neutrino spectrum was estimated by doubling the number of the modes and is completely negligible ( $\sim 10^{-5}$ ). For the average value of  $P_\perp$  at late times we took the arithmetic mean of the last maximum and last minimum of the oscillating curve before  $t = 10000$  (see Fig. 2). The quantity  $\omega_s$  was found numerically from eq. (45).

For the approach based on the averaging procedure, in the case of the Gaussian spectrum we solve eqs. (54) and (55) numerically, whereas for the box-type spectrum the fully analytical solution in eqs. (61) and (62) is used. Note that near coincidence of the  $R_A$  curves corresponding to the two spectra that we used is a curious accident of our choice of the value of  $\theta_0$ : we have checked that for other choices the curves are clearly distinguishable, though similar in shape. It can be seen from the figure that the analytical approach reproduces the results of numerical integration of the exact EoMs extremely well. This is in accord with the conclusions of Ref. [8], where eqs. (54) and (55) were first obtained.

## 5 Decoherence by wave packet separation: Adiabaticity and adiabaticity violation

Let us try to understand the obtained above results on coherence and decoherence of the oscillations in a dense neutrino gas from the viewpoint of wave packet separation. The split-up of wave packets in configuration space occurs as a consequence of the difference of group velocities of the wave packets of different neutrino states composing the initially produced neutrino flavour eigenstate. Since only the states that diagonalize the Hamiltonian of the neutrino system (propagation eigenstates) have well-defined group velocities, our discussion will be in terms of these eigenstates.<sup>6</sup>

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<sup>6</sup>Collective neutrino oscillations have been previously considered from the viewpoint of the propagation eigenstate basis in [38, 39], but the issue of wave packets and their separation has not been addressed there.

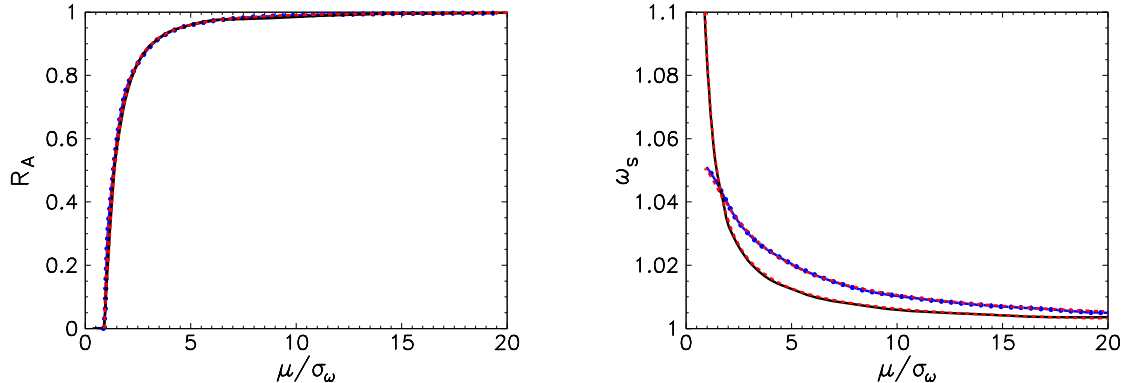


Figure 3: Left panel: Order parameter  $R_A \equiv \langle P_\perp \rangle / (P_0 \sin 2\theta_0)$ . Dash-dotted (blue) curve: numerical integration of EoMs for Gaussian spectrum, solid (black) curve: same for box-type spectrum. Dashed (red) curves: results of the analytical approach for these spectra (see the text). For both spectra  $P_0 = 1$ ,  $\omega_0 = 1$ ,  $\theta_0 = 0.5$ . For Gaussian spectrum we use  $\sigma_\omega = 0.2$ , for box-type spectrum  $\sigma = 0.2$ , which corresponds to  $\sigma_\omega = \sigma/\sqrt{3} \simeq 0.115$ . Note that near coincidence of  $R_A$  curves corresponding to the two considered spectra is an accident of our choice of  $\theta_0$ . Right panel: same as left one, but for  $\omega_s$ .

Why doesn't decoherence by wave packet separation occur in very dense neutrino gases, when the values of the neutrino self-interaction parameter  $\mu$  are far above the threshold  $\mu_0$ ? One might suspect that in this case the dynamics of neutrino evolution is such that the difference of the group velocities of different propagation eigenstates  $\Delta v_g$  vanishes and they all propagate with the same speed. It is, however, easy to make sure that this is not the case, and  $\Delta v_g$  does *not* vanish. As we shall see, it is nevertheless possible to qualitatively understand the absence of decoherence in the large  $\mu$  limit as well as partial decoherence for  $\mu$  only slightly exceeding  $\mu_0$  in terms of the behaviour of the neutrino propagation eigenstates.

For neutrino systems with time-dependent Hamiltonians, such as the one described by the Hamiltonian vector (6), the propagation eigenstates cannot be defined universally, i.e. in a time-independent way. However, at any instant of time  $t$  they can be defined as the states diagonalizing the Hamiltonian at this particular time. These are the so-called instantaneous eigenstates of the Hamiltonian. At a given time  $t$  the neutrino flavour eigenstates can be written as linear superpositions of the instantaneous propagation eigenstates. As in ordinary matter, these superpositions are determined in the 2-flavour case by the mixing angle in matter  $\theta = \theta(t)$ .

The propagation eigenstates evolve independently in the adiabatic limit, i.e. when the neutrino Hamiltonian changes relatively slowly, so that the system has enough time to 'adjust' itself to the changing conditions. In terms of the flavour spin vectors, adiabaticity means that the rate of evolution of  $\vec{H}_\omega$  is small compared to the frequency  $H_\omega$  of precession

of the individual  $\vec{P}_\omega$  around their  $\vec{H}_\omega$ . In this case, in the course of their precession around  $\vec{H}_\omega$ , the vectors  $\vec{P}_\omega$  ‘track’ the movements of  $\vec{H}_\omega$ . If the adiabaticity is violated, the propagation eigenstates do not evolve independently; instead, they can go into each other during the evolution of the neutrino system. As a measure of adiabaticity violation one can choose the ratio  $\gamma$  of the off-diagonal element of the Hamiltonian of the system in the propagation eigenstate basis to the difference of its diagonal elements. Adiabatic regime corresponds to  $\gamma \ll 1$ , whereas  $\gamma \gg 1$  would mean maximal violation of adiabaticity. It should be noted that, while for the oscillations in ordinary matter adiabaticity may only be violated in the case of non-uniform matter, in dense neutrino environments adiabaticity violation may occur even in the case of constant neutrino density. This happens because in the latter case the Hamiltonian of the system depends not only on the overall density of the neutrino gas, but also on its flavour composition, which changes with time.

Consider now several regimes of the evolution of the system, depending on the values of the parameter  $\mu$ .

- I.  $\mu P_0 \gg \omega_0$ . In this case the Hamiltonian vector  $\vec{H}_\omega = \omega \vec{B} + \mu \vec{P}$  nearly coincides with  $\mu \vec{P}$ . Since at  $t = 0$  all the individual flavour spin vectors  $\vec{P}_\omega$  as well as their sum  $\vec{P}$  point in the  $z$ -direction, all  $\vec{P}_\omega$  are practically collinear with their  $\vec{H}_\omega$ . From EoM (2) it then follows that there is essentially no flavour evolution in this case.

As the Hamiltonian of the system remains practically constant, the adiabaticity condition is very well satisfied. The propagation eigenstates are well defined and evolve independently. The condition  $\mu P_0 \gg \omega_0$  means that the mixing is strongly suppressed in the neutrino gas, so that the initially produced flavour state practically coincides with one of the propagation eigenstates rather than being a nontrivial superposition of different eigenstates. Therefore, no wave packet separation occurs (a propagation eigenstate cannot ‘separate with itself’), and hence there is no decoherence.

- II.  $\sigma_\omega \ll \mu P_0 \lesssim \omega_0$ . This case is most simply considered in the corotating frame, in which one has to replace  $\omega \rightarrow \omega' = (\omega - \omega_s)$ . Since  $|\omega - \omega_s| \lesssim \sigma_\omega$ , the condition  $\sigma_\omega \ll \mu P_0$  implies  $\mu P_0 \gg |\omega'|$ . This case then reduces to the previous one. In the corotating frame one has good adiabaticity and suppressed mixing, which means no wave packet separation and therefore no decoherence. Since decoherence is a physical process, it does not depend on the frame in which the evolution of the system is considered; therefore no decoherence occurs in the original flavour frame either. The individual flavour spin vectors  $\vec{P}_\omega$  as well as the global flavour spin  $\vec{P}$  are practically constant in the corotating frame, which means that in the original frame they all precess around  $\vec{B}$  with the same frequency  $\omega_s$ , that is, synchronized oscillations occur.

It is interesting to interpret this case also directly in the original flavour frame, without any reference to the corotating one. The condition  $\sigma_\omega \ll \mu P_0 \lesssim \omega_0$  means that  $\gamma \gtrsim 1$ , that is adiabaticity is either moderately or strongly violated in the original frame.<sup>7</sup>

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<sup>7</sup>Note that the degree of adiabaticity is frame-dependent because it is not a physically observable quantity.

Strong violation of adiabaticity would mean that the propagation eigenstates are not a physically meaningful notion; even though they are mathematically well defined, they are strongly mixed and go into each other in the course of the evolution of the system. Group velocities are then not well defined either, and no wave packet separation can occur. To put it slightly differently, the initially produced propagation eigenstates with larger and smaller group velocities will fully (or almost fully) interchange on a time scale  $\tau$  that is short compared to the naively expected coherence length  $L_{\text{coh}}$ . The slow state becomes the fast one and vice versa; as a result, a small wave packet separation over the period  $\tau$  is compensated during the next period  $\tau$ . As a consequence of frequent shuffling of the fast and slow propagation eigenstates, no noticeable wave packet separation occurs.

- III.  $\mu$  is slightly above  $\mu_0 \sim \sigma_\omega$ . This case can be understood similarly to the previous one; the difference is that now  $\gamma \lesssim 1$ , i.e. adiabaticity is only slightly violated. The shuffling of the fast and slow propagation eigenstate still occurs, but with an amplitude that is less than one: only a fraction of the fast propagation eigenstate goes into the slow one and vice versa. As a result, at late enough times a split-up of the wave packets occurs, but the strengths of the separated wave packets are uneven: the probability of finding a neutrino in one of them is smaller than in the other. The wave packet separation is then only partial, which means that only partial decoherence has occurred.
- IV.  $\mu < \mu_0$ . In this case  $\gamma = 0$  (perfect adiabaticity), so that the propagation eigenstates are well defined and evolve independently. The mixing angle at production is of order of vacuum mixing angle  $\theta_0$ , which means that the produced neutrino state is a nontrivial superposition of the two propagation eigenstates. At late times the complete wave packet separation occurs, leading to complete decoherence.

More detailed discussion of the impact of adiabaticity and adiabaticity violation on decoherence of synchronized neutrino oscillations will be given in [25].

## 6 Summary and discussion

We have revisited synchronized neutrino oscillations in dense uniform and homogeneous neutrino gases. Our goal was twofold: (i) to give an approximate analytical description of the synchronized neutrino oscillations and of the de-synchronization phenomenon, and (ii) to interpret de-synchronization in terms of late-time decoherence. To this end, we first developed an exact formalism of spectral moments of the flavour spin vectors, and then applied it to find approximate analytical descriptions of decoherence effects in the system. Our spectral moments formalism also allowed us to find a previously unknown conservation law satisfied by the quantities characterizing a homogeneous and isotropic neutrino gas.

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Only the probabilities of flavour transitions have direct physical meaning.

In our first analytical approach, we assumed that at asymptotically large times the global flavour spin vector  $\vec{P}$  undergoes a simple precession around the vector  $\vec{B}$  that describes the vacuum contribution to the neutrino Hamiltonian, with the length of  $\vec{P}$  remaining constant but possibly being smaller than its initial value  $P_0$ . The shrinkage of  $\vec{P}$  signifies a partial or complete decoherence. We have found that this regime can only be nontrivially realized if the transverse (with respect to  $\vec{B}$ ) components of  $\vec{P}$  and of the vector  $\vec{S}$  defined in eq. (12) satisfy  $\vec{S}_\perp = \omega_s \vec{P}_\perp$ , where  $\omega_s$  is the precession frequency. We have additionally assumed that the longitudinal components of  $\vec{P}$  and  $\vec{S}$  satisfy a similar relation, i.e.  $\vec{S}_\parallel = \omega_s \vec{P}_\parallel$ . This allowed us to find (in Section 4.3.1) simple analytical expressions for the asymptotic value of  $P_\perp$  and the precession frequency  $\omega_s$  as functions of the neutrino self-interaction parameter  $\mu$ . In particular, we found a simple and direct mathematical explanation of the existence of the threshold value  $\mu_0$ , below which the complete decoherence occurs: for  $\mu < \mu_0$  the quadratic equation from which  $P = |\vec{P}|$  is determined has no real solutions. We thus confirmed the conclusions of Ref. [8], where the existence of the threshold  $\mu_0$  and the threshold behaviour of the asymptotic value of  $P_\perp$  were previously found.

The described approach, in addition to explaining the existence of the threshold  $\mu_0$  and providing the correct order-of-magnitude estimate of its value ( $\mu_0 \sim \sigma_\omega$ , where  $\sigma_\omega$  is the effective width of the neutrino spectrum in the variable  $\omega = \Delta m^2 / (2p)$ ), predicted the correct behaviour of asymptotic  $P_\perp$  at  $\mu \gg \mu_0$ . It, however, failed to reproduce the correct values of  $P_\perp$  near the threshold  $\mu_0$ . One possible reason for this can be traced back to the fact that our additional assumption  $\vec{S}_\parallel = \omega_s \vec{P}_\parallel$  is actually satisfied with a good accuracy far above the threshold but breaks down close to it. We therefore attempted to find an analytic approach based solely on the condition  $\vec{S}_\perp = \omega_s \vec{P}_\perp$ , which directly follows from the assumption of constant asymptotic  $P$ . By making use of our spectral moments formalism, we then found that this assumption cannot be exact as it leads to controversial results, at least for  $\mu$  only slightly above the threshold  $\mu_0$ . This has been confirmed by our numerical calculations, which showed that for these values of  $\mu$  both  $P$  and  $P_\perp$  do not become constant even at extremely late times, far above the naively expected coherence time  $t_{\text{coh}} \simeq L_{\text{coh}}$ . Instead, they continue oscillating around their mean values, though with relatively small amplitudes (see Fig. 2).

The smallness of the late-time oscillations of  $P$  and  $P_\perp$  suggests that they may be replaced at asymptotic times by their average values. However, as our analysis shows, this cannot be done by simply assuming them to be constant; instead, a consistent averaging procedure must be employed. We performed such a procedure in Section 4.3.2 by averaging the EoMs of the flavour spin vectors  $\vec{P}_\omega$  over a very large time interval. The averaging is most simply done in the corotating frame, i.e. in the frame rotating around  $\vec{B}$  with the angular velocity  $\omega_s$  (see the Appendix). Our analysis in Section 4.3.2 was based on two simple observations:

- Large-interval time averages are dominated by the late-time behaviour of the system, so that by studying such averages one gains information about the asymptotic regime of evolution of the system.

- The smallness of the amplitude of the late-time oscillations of  $P_{\perp}(t)$  means that in certain time integrals related to the averaging procedure  $P_{\perp}(t)$  can be replaced by its asymptotic average value and pulled out of the integral.

No further assumptions or approximations were used.

The developed approach led to very simple analytical expressions for the averaged transverse and longitudinal components of the vector  $\vec{P}_{\omega}$  in terms of the asymptotic value of  $P_{\perp}$  and  $\omega_s$ , which can then be found from consistency conditions. Our numerical calculations demonstrate that the asymptotic  $P_{\perp}$  and  $\omega_s$  thus obtained reproduce very well the results of direct numerical integration of exact EoMs for  $\vec{P}_{\omega}$  (see Fig. 3).

The analytic expressions obtained in Section 4.3.2 coincide with those found in Ref. [8] basing on different approximations and assumptions. Those assumptions are in general not realistic and, as was pointed out by the authors of [8] themselves, may actually be badly violated. How can then one understand the success of the analytical approach of [8] for describing the asymptotic values of  $P_{\perp}$  and  $\omega_s$ ? As follows from our analysis, this is a consequence of cancellation of two large errors. The authors of [8] used the ‘sudden approximation’ and the assumption that the values of the vectors  $\vec{P}_{\omega}$  at the onset of the asymptotic regime coincide with their initial values at  $t = 0$  (see the discussion in our Section 4.1). This allowed them to make the following replacements at asymptotic times (in our notation):

$$P_{\omega\parallel}(t) \rightarrow P_{\omega\parallel}(0), \quad \vec{P}_{\perp}(t) \cdot \vec{P}_{\omega\perp}(t) \rightarrow P_{\perp}(t)P_{\omega\perp}(0), \quad (63)$$

and as a result to replace

$$(\omega - \omega_r)P_{\omega\parallel}(t) + \mu\vec{P}_{\perp}(t) \cdot \vec{P}_{\omega\perp}(t) \rightarrow P_0g_{\omega}[-c_{20}(\omega - \omega_r) + s_{20}\mu P_{\perp}(t)] \quad (64)$$

(cf. the numerators of the integrands in eqs. (16) and (17) of [8]). As follows from the calculations presented in the Appendix of our paper, although each of the two replacements in eq. (63) is unjustified, the errors introduced by these replacements in the expression on the left hand side of eq. (64) nearly cancel each other by virtue of the relation in eq. (A6), which is a direct consequence of the EoM for  $\vec{P}_{\omega}$  in the corotating frame. Thus, our results provide a justification of the analytical approach of Ref. [8].

As was pointed out in Section 4.1, the analytical expressions first obtained in [8] and rederived in a different approach in our Section 4.3.2 reproduce very well the results of numerical calculations of the asymptotic values of  $P_{\perp}$ , but they fail to satisfy the conservation laws (17) and (25) (except in the limit  $\mu \gg \mu_0$ ). It is actually easy to understand why this happens. The derivations of these expression involved replacing the vectors  $\vec{P}_{\omega}$  by their averages  $\langle \vec{P}_{\omega} \rangle$ . This allows an accurate determination of the averaged values of the longitudinal and transverse components of the global flavour spin vector  $\vec{P}$ , which are connected to the corresponding components of  $\langle \vec{P}_{\omega} \rangle$  by linear relationships, see eq. (51). At the same time, the conservation laws (17) and (25) contain, along with linear members, terms that are quadratic or bilinear in the components of  $\vec{P}$  and  $\vec{S}$ ; their averages are not connected to

the components of  $\langle \vec{P}_\omega \rangle$  by linear relationships and therefore cannot be reliably found from the latter. The reason for this is essentially that the average of a square is not in general equal to the square of an average. On the other hand, in the limit  $\mu \rightarrow \infty$  the condition  $P_\perp = \text{const.}$  is very well satisfied at all times; in this case it is not necessary to invoke any averaging procedure, and the formulas derived for the averaged individual-mode and global flavour spin vectors are actually valid for the un-averaged quantities as well. The conservation laws (17) and (25) are then satisfied. Indeed, using eqs. (48) and (49) one can readily make sure that for general  $\mu \geq \mu_0$  the differences of the left-hand and right-hand sides of eqs. (17) and (25) are proportional to  $P_\perp - s_{20}P_0 = P_\perp - P_\perp(0)$ . In the limit  $\mu \rightarrow \infty$  this quantity vanishes, and the conservation laws (17) and (25) are fulfilled.

In our analysis we had in mind a system of identical neutrino wave packets, with the function  $g_\omega$  characterizing the energy distribution within each individual wave packet. Such a system is known to be equivalent to a system consisting of neutrinos with well-defined energy and  $g_\omega$  characterizing the energy spectrum of the neutrino ensemble [35]. Our treatment therefore applies to such a system as well.

Decoherence in a system of wave packets can be considered both in the momentum space and in the configuration space. In the latter case it is a consequence of separation of wave packets moving with different group velocities. In Section 5 we presented a qualitative interpretation of our results in terms of the possible wave packet separation. We have shown that all the regimes that we studied (perfect coherence and partial or full decoherence) can be understood from this standpoint. Our qualitative analysis there, however, did not provide a simple explanation of the existence of the threshold  $\mu_0$ , though it explained the complete decoherence for  $\mu < \mu_0$  as being due to perfect adiabaticity.

A dense uniform and isotropic neutrino gas that we considered in the present paper is the simplest possible system in which collective neutrino oscillations can occur. It can probably only very approximately represent the phenomena occurring in dense neutrino gases in the early Universe and to some extent in supernovae. Despite its simplicity, neutrino flavour evolution in this system exhibits a rich variety of possible patterns, the reason being that the equations of motion governing its evolution are highly nonlinear. We have addressed a number of issues pertaining to decoherence effects in this system. Some topics were, however, left out of our discussion. Those include e.g. the questions of what determines the relaxation time (i.e. the time necessary for the asymptotic regime to set in) and the amplitude of the residual oscillations of  $P$  at late times. Hopefully, future studies will address these issues as well as will shed light on decoherence effects in collective neutrino oscillations in more realistic settings.

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## Appendix A: Formalism in the corotating frame and the evolution at asymptotically large times

Consider the evolution of the flavour spin vectors in the corotating frame (i.e. in the frame rotating around  $\vec{B}$  with the angular velocity  $\omega_s$ ). We will mark the flavour spins and related quantities in this frame with a prime. Note that going to the corotating frame does not change the longitudinal (with respect to  $\vec{B}$ ) components of the flavour spin vectors and spectral moments as well as the lengths of their transverse components. We shall be assuming that at  $t = 0$  the corotating frame coincides with the original one, so that the initial conditions for all the primed quantities are the same as for the corresponding unprimed ones.

The EoM of the flavour spin vectors of the individual modes in the corotating frame is

$$\dot{\vec{P}}'_\omega = \vec{H}'_\omega \times \vec{P}'_\omega, \quad (\text{A1})$$

where

$$\vec{H}'_\omega = (\omega - \omega_s)\vec{B} + \mu\vec{P}' = (\omega - \omega_r)\vec{B} + \mu\vec{P}'_\perp. \quad (\text{A2})$$

Here the quantity  $\omega_r$  was defined in eq. (50), and in the last equality we have taken into account that the longitudinal component of  $\vec{P}'$  is conserved and coincides with  $\vec{P}'_\parallel = -c_{20}P_0\vec{B}$ . The evolution equations for the longitudinal and transverse components of  $\vec{P}'_\omega$  read

$$\dot{P}'_{\omega\parallel} = \vec{B} \cdot (\mu\vec{P}'_\perp \times \vec{P}'_{\omega\perp}), \quad (\text{A3})$$

$$\dot{\vec{P}}'_{\omega\perp} = \vec{B} \times [(\omega - \omega_r)\vec{P}'_{\omega\perp} - \mu P'_{\omega\parallel}\vec{P}'_\perp]. \quad (\text{A4})$$

In eq. (A4) the first term in the square brackets describes the precession of  $\vec{P}'_{\omega\perp}$  around  $\vec{B}$ , whereas the second term is responsible for the variations of the length of  $\vec{P}'_{\omega\perp}$ . Integrating eq. (A1) over  $\omega$ , we obtain the EoM of  $\vec{P}'$ :

$$\dot{\vec{P}}' = \vec{B} \times (\vec{S}'_\perp - \omega_s\vec{P}'_\perp). \quad (\text{A5})$$

The EoMs of the spectral moments  $\vec{K}'_n$  can be found by multiplying (A1) by  $\omega^n$  and integrating over  $\omega$ .

A useful relation is obtained by noting that eq. (A1) implies  $\vec{H}'_\omega \cdot \dot{\vec{P}}'_\omega = 0$ , that is

$$(\omega - \omega_r)\dot{P}'_{\omega\parallel} + \mu\vec{P}'_\perp \cdot \dot{\vec{P}}'_{\omega\perp} = 0. \quad (\text{A6})$$

Multiplying this by  $\omega^n$  and integrating over  $\omega$ , one can find a scalar relation between the derivatives of  $\vec{K}'_n$  and  $\vec{K}'_{n+1}$ , analogous to eq. (21).

Consider now the asymptotic regime. If not indicated otherwise, in what follows we will be considering all the relevant quantities at asymptotically large times, without specifying this explicitly and keeping the same notation for these quantities as before.

Let us first assume that the evolution of the flavour spin vector  $\vec{P}$  at late times is the simple precession around  $\vec{B}$  with a constant frequency  $\omega_s$ . This means that in the corotating frame the flavour spin vector  $\vec{P}'$  remains constant at asymptotic times. Eq. (A5) then yields

$$\vec{S}'_{\perp} = \omega_s \vec{P}'_{\perp}, \quad (\text{A7})$$

which, in particular, means that  $\vec{S}'_{\perp}$  is also constant at asymptotic times. At the same time, eq. (15) together with the asymptotic constancy of  $P = P'$  implies  $S'_{\parallel} = \text{const}$ . Thus, we conclude that the vector  $\vec{S}'$  is constant at late times.

Acting by induction, it is then easy to show that all the spectral moments  $\vec{K}'_n$  satisfy similar relations, that is, at asymptotically large times they all remain constant, with their transverse components collinear with  $\vec{P}'_{\perp}$ . (Note that such a behaviour in the corotating frame means that in the original flavour frame they all precess around  $\vec{B}$  with the same constant frequency  $\omega_s$ , remaining in the same plane). From the definition of the spectral moments  $\vec{K}'_n$ , it then follows that the flavour spin vectors  $\vec{P}'_{\omega}$  of the individual  $\omega$ -modes also satisfy similar relations, that is

$$\vec{P}'_{\omega} = \text{const.}, \quad \vec{P}'_{\omega\perp} = a_{\omega} \vec{P}'_{\perp}, \quad (\text{A8})$$

with constant  $a_{\omega}$ . By making use of eqs. (A8) and (A4), one can find the ratio of the longitudinal and transverse components of the asymptotic vector  $\vec{P}'_{\omega}$ . The same relation will also hold for the corresponding unprimed quantities in the original flavour frame. Once the ratio of  $P_{\omega\parallel}$  and  $P_{\omega\perp}$  is known, one can then find these quantities from the normalization condition  $P_{\omega\parallel}^2 + P_{\omega\perp}^2 = P_{\omega}^2 = P_0^2 g_{\omega}^2$ .

With the expressions for  $P_{\omega\parallel}$  and  $P_{\omega\perp}$  at hand, the longitudinal and transverse components of  $\vec{P}$  can be found by integrating over the  $\omega$ -modes. However, the direct calculation shows that, except in the limit  $\mu \rightarrow \infty$ , the values of  $P_{\perp}$  obtained in this way do not reproduce correctly the results of numerical integration of the exact EoMs. This means that the assumption that  $P \equiv |\vec{P}|$  becomes constant at asymptotically large times, on which our consideration here was thus far based, is actually incorrect.

Indeed, it was demonstrated in Section 4.3 that for  $\mu$  exceeding the critical value  $\mu_0$  (but not far above it), the quantities  $P_{\perp}$  and  $P$  do not become constant even at very late times; instead, they keep oscillating around their mean values. The amplitude of the late-time oscillations of  $P_{\perp} = P'_{\perp}$  is relatively small, and therefore it seems to be a reasonable approximation to replace it at asymptotic times by its mean value, found by averaging over these oscillations. However, this should be done through a consistent averaging procedure rather than by simply assuming that at large times  $P'_{\perp} = \text{const}$ .

To carry out the averaging procedure systematically, let us average the EoMs of the flavour spin vectors over a very large (formally infinite) time interval. Since the infinite-interval average of the derivative of any bounded function vanishes,<sup>8</sup> eqs. (A3)-(A5) yield

$$\langle \vec{P}'_{\perp} \times \vec{P}'_{\omega\perp} \rangle = 0, \quad (\text{A9})$$

$$(\omega - \omega_r) \langle \vec{P}'_{\omega\perp} \rangle = \mu \langle P'_{\omega\parallel} \vec{P}'_{\perp} \rangle, \quad (\text{A10})$$

$$\langle \vec{S}'_{\perp} \rangle = \omega_s \langle \vec{P}'_{\perp} \rangle. \quad (\text{A11})$$

Note that eq. (A11) is simply the averaged version of eq. (A7).

To draw useful information from eq. (A10), let us note that, while at late times the transverse component of the *global* flavour spin in the corotating frame  $\vec{P}'_{\perp}$  is essentially a fixed-direction vector with its length exhibiting only small oscillations, this is in general not the case for the components of the flavour spin vectors  $\vec{P}'_{\omega}$  of the *individual*  $\omega$ -modes, which undergo large-scale variations.<sup>9</sup> Because infinite-time averages are dominated by the late-time contributions to the averaging integral, it is then a good approximation to replace the nearly constant vector  $\vec{P}'_{\perp}(t)$  by its mean value and pull it out of the integral when calculating the average of  $P'_{\omega\parallel} \vec{P}'_{\perp}$ . This gives

$$\langle P'_{\omega\parallel} \vec{P}'_{\perp} \rangle \simeq \langle P'_{\omega\parallel} \rangle \langle \vec{P}'_{\perp} \rangle. \quad (\text{A12})$$

We will be using the same factorization approximation whenever calculating the averages of the products of  $\vec{P}'_{\perp}$  and any components of  $\vec{P}'_{\omega}$ .

Substituting eq. (A12) into (A10) yields

$$\frac{\langle \vec{P}'_{\omega\perp} \rangle}{\langle P'_{\omega\parallel} \rangle} = \frac{\mu \langle \vec{P}'_{\perp} \rangle}{(\omega - \omega_r)}. \quad (\text{A13})$$

Note that in the factorization approximation eq. (A9) becomes  $\langle \vec{P}'_{\perp} \rangle \times \langle \vec{P}'_{\omega\perp} \rangle = 0$ . This relation does not bring in any new information, as it follows also from (A13).

Eq. (A13) equips us with the ratio of  $\langle \vec{P}'_{\omega\perp} \rangle$  and  $\langle P'_{\omega\parallel} \rangle$ . However, one cannot combine it with the normalization condition for  $P'^2_{\omega}$  in order to find  $\langle \vec{P}'_{\omega\perp} \rangle$  and  $\langle P'_{\omega\parallel} \rangle$  separately, since these averages do not satisfy the same normalization condition as the un-averaged quantities  $\vec{P}'_{\omega\perp}$  and  $P'_{\omega\parallel}$ . Therefore, in order to determine  $\langle P'_{\omega\parallel} \rangle$  and  $\langle \vec{P}'_{\omega\perp} \rangle$ , one needs one more relation between them. To find it, we first rewrite eq. (A13) as

$$\langle \vec{P}'_{\omega\perp} \rangle = f(\omega, \mu) \langle \mu \vec{P}'_{\perp} \rangle,$$

---

<sup>8</sup> Indeed, for a bounded function  $f(t)$  one has  $\langle \dot{f}(t) \rangle \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \dot{f}(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} [f(T) - f(0)] = 0$ .

<sup>9</sup> This can be seen from eq. (A4). For instance, in the limit  $\mu \rightarrow \mu_0$  we have  $P'_{\perp} \rightarrow 0$ , and the second term in the square brackets in this equation vanishes, whereas the first term describes the undamped precession of  $\vec{P}'_{\perp}$  around  $\vec{B}$  in the plane perpendicular to  $\vec{B}$  with the frequency  $\omega - \omega_r$ . For  $\mu > \mu_0$  also the length of  $\vec{P}'_{\omega\perp}$  varies, and therefore so does  $P'_{\omega\parallel}$ .

$$\langle P'_{\omega\parallel} \rangle = f(\omega, \mu) [\omega - \omega_r(\mu)], \quad (\text{A14})$$

with as yet unknown function  $f(\omega, \mu)$ . Next, consider the time average of the quantity  $\vec{P}'_{\omega} \cdot \vec{H}'_{\omega}$ . Using eq. (A14) and the definition of  $\vec{H}'_{\omega}$  given in eq. (A2), one can express this average through  $f(\omega, \mu)$ :

$$\langle \vec{P}'_{\omega} \cdot \vec{H}'_{\omega} \rangle = f(\omega, \mu) [(\omega - \omega_r)^2 + \langle \mu P_{\perp} \rangle^2]. \quad (\text{A15})$$

In obtaining this relation we have used the factorization approximation for  $\langle \vec{P}'_{\omega\perp} \cdot \vec{P}'_{\perp} \rangle$ . On the other hand, at a given time  $t_1$ , for the un-averaged quantity  $\vec{P}'_{\omega}(t_1) \cdot \vec{H}'_{\omega}(t_1)$  one can write

$$\begin{aligned} \vec{P}'_{\omega}(t_1) \cdot \vec{H}'_{\omega}(t_1) &= (\omega - \omega_r) P'_{\omega\parallel}(t_1) + \mu \vec{P}'_{\perp}(t_1) \cdot \vec{P}'_{\omega\perp}(t_1) \\ &= (\omega - \omega_r) P'_{\omega\parallel}(0) + \mu \vec{P}'_{\perp}(t_1) \cdot \vec{P}'_{\omega\perp}(0) \\ &\quad + \{(\omega - \omega_r)[P'_{\omega\parallel}(t_1) - P'_{\omega\parallel}(0)] + \mu \vec{P}'_{\perp}(t_1) \cdot [\vec{P}'_{\omega\perp}(t_1) - \vec{P}'_{\omega\perp}(0)]\}. \end{aligned} \quad (\text{A16})$$

Let us now consider the expression in the curly brackets in (A16). We have

$$\begin{aligned} &\{(\omega - \omega_r)[P'_{\omega\parallel}(t_1) - P'_{\omega\parallel}(0)] + \mu \vec{P}'_{\perp}(t_1) \cdot [\vec{P}'_{\omega\perp}(t_1) - \vec{P}'_{\omega\perp}(0)]\} \\ &= \int_0^{t_1} dt [(\omega - \omega_r) \dot{P}'_{\omega\parallel}(t) + \mu \vec{P}'_{\perp}(t) \cdot \dot{\vec{P}}'_{\omega\perp}(t) - \mu \vec{P}'_{\perp}(t) \cdot \dot{\vec{P}}'_{\omega\perp}(t)] \\ &\quad + \mu \vec{P}'_{\perp}(t_1) \cdot [\vec{P}'_{\omega\perp}(t_1) - \vec{P}'_{\omega\perp}(0)] \\ &= -\mu \int_0^{t_1} dt \vec{P}'_{\perp}(t) \cdot \dot{\vec{P}}'_{\omega\perp}(t) + \mu \vec{P}'_{\perp}(t_1) \cdot [\vec{P}'_{\omega\perp}(t_1) - \vec{P}'_{\omega\perp}(0)]. \end{aligned} \quad (\text{A17})$$

Here in going from lines 2, 3 to line 4 we have used relation (A6). We will eventually be interested in the time average of eq. (A17). Since the integrals related to infinite-interval averages are dominated by the contributions of late times in the integration intervals, we can concentrate on the large  $t_1$  limit of (A17). The integral in the last line of this equation is also dominated by the late-time contributions, as its integrand is not suppressed at large  $t$ . Since  $\vec{P}'_{\perp}(t)$  is nearly constant at asymptotic times, one can approximately replace it by its value at  $t = t_1$  and pull it out of the integral. The two terms in last line in eq. (A17) then cancel each other, which means that the average of the expression in the curly brackets in (A16) can be neglected. Thus, the averaging of eq. (A16) yields

$$\langle \vec{P}'_{\omega} \cdot \vec{H}'_{\omega} \rangle \simeq (\omega - \omega_r) P'_{\omega\parallel}(0) + \vec{P}'_{\omega\perp}(0) \cdot \langle \mu \vec{P}'_{\perp} \rangle. \quad (\text{A18})$$

The quantities  $\vec{P}'_{\omega\perp}(0)$  are constant vectors in the corotating frame which lie in the  $x'z'$  plane and all point in the same direction, irrespectively of the value of  $\omega$ . This follows from the fact that at  $t = 0$  they coincide with the corresponding  $\vec{P}_{\omega\perp}$ , and the initial conditions for  $\vec{P}_{\omega}$  in eq. (10) mean that all  $\vec{P}_{\omega\perp}(0)$  point in the same direction. The averaged global flavour spin  $\langle \vec{P}'_{\perp} \rangle$  is also a constant vector in the corotating frame. Its direction can only

depend on the vectors characterizing the neutrino system under consideration and should be given by their linear superposition. Those are the vector  $\vec{n}'_z(0) = \vec{n}_z$ , which specifies the initial conditions for the flavour spin vectors, and  $\vec{B}$ . The latter, as well as the longitudinal component of the former, cannot enter in the definition of a transverse vector, whereas the transverse component of  $\vec{n}'_z(0)$  defines the direction of  $\vec{P}'_{\omega\perp}(0)$ . We therefore conclude that  $\langle\vec{P}'_{\perp}\rangle$  and  $\vec{P}'_{\omega\perp}(0)$  must be collinear. Thus, one can rewrite eq. (A18) as

$$\begin{aligned}\langle\vec{P}'_{\omega}\cdot\vec{H}'_{\omega}\rangle &\simeq (\omega - \omega_r)P'_{\omega\parallel}(0) + \mu\langle P'_{\perp}\rangle P'_{\omega\perp}(0) \\ &= [-c_{20}(\omega - \omega_r) + s_{20}\mu\langle P_{\perp}\rangle]P_0g_{\omega}.\end{aligned}\tag{A19}$$

Here we have taken into account that the initial conditions for the primed and the corresponding unprimed components of the flavour spin vectors are the same and that  $P'_{\perp} = P_{\perp}$ . Combining eqs. (A19) and (A15), one arrives at the expression for  $f(\omega, \mu)$  given in eq. (49).

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