Quantum typicality in spin network states of quantum geometry

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In this letter we extend the so-called typicality approach, originally formulated in statistical mechanics contexts, to $SU(2)$ invariant spin network states. Our results do not depend on the physical interpretation of the spin-network, however they are mainly motivated by the fact that spin-network states can describe states of quantum geometry, providing a gauge-invariant basis for the kinematical Hilbert space of several background independent approaches to quantum gravity. The first result is, by itself, the existence of a regime in which we show the emergence of a typical state. We interpret this as the prove that, in that regime there are certain (local) properties of quantum geometry which are "universal". Such set of properties is heralded by the typical state, of which we give the explicit form. This is our second result. In the end, we study some interesting properties of the typical state, proving that the area-law for the entropy of a surface must be satisfied at the local level, up to logarithmic corrections which we are able to bound.

I. INTRODUCTION

In recent years, quantum statistical mechanics and quantum information theory have played an increasingly central role in quantum gravity. Such interplay has proved particularly insightful both in the context of the holographic duality in AdS/CFT [1–5], as well as for the current background independent approaches to quantum gravity, including loop quantum gravity (LQG) [6, 7], the related covariant path-integral spin-foam formulation [8], and group field theory [9].

Interestingly, the different background independent approaches today share a microscopic description of space-time geometry given in terms of discrete, pre-geometric degrees of freedom of combinatorial and algebraic nature, based on spin-network Hilbert spaces [10–13]. In this context then, entanglement has provided a new tool to characterise the quantum texture of space-time in terms of the structure of correlations of the spin networks states.

Along this line, several recent works have considered the possibility to use specific features of the short-range entanglement in quantum spin networks (area law, thermal behaviour) to select quantum geometry states which may eventually lead to smooth spacetime geometry classically [14–18]. This analysis typically concentrated on states with few degrees of freedom, leaving open the question of whether a *statistical* characterisation may reveal new structural properties, independent from the interpretation of the spin network states.

This letter proposes the use of the information theoretical notion of quantum *canonical typicality*, as a statistical tool to characterise universal local features of quantum geometry, going beyond the physics of states with few degrees of freedom.

In quantum statistical mechanics, *canonical typicality* states that almost every pure state of a large quantum mechanical system is such that its reduced density matrix over a sufficiently small subsystem is approximately in the *canonical* state descibed by a thermal distribution à

la Gibbs [19–21].

However, most interestingly, such a statement goes beyond the thermal behaviour. For a generic closed system in a quantum pure state, not described by an eigenstate of some hamiltonian, but still subject to some *global constraint*, the resulting canonical description will not be thermal, but generally defined in relation to the set of constraints considered [22]. Again, in this case, some specific properties of the system emerge at the local level, regardless of the nature of the global state. These properties depend on the physics encoded in the choice of the global constraints.

Within this generalised framework, we exploit the notion of *typicality* to study whether and how "universal" statistical features of the local correlation structure of a spin-network state emerge in relation with the choice of the constraints defining the reduced canonical state of the system.

We focus our analysis on the space of the N-valent $SU(2)$ invariant intertwiners, which are the building blocks of the spin network states. In LQG, such intertwiners can be thought of dually as a region of $3d$ space with an S^2 boundary [11].

We reproduce the typicality statement in the full space of N-valent intertwiners with fixed total area and we investigate the statistical behaviour of the canonical reduced state, dual to a small patch of the S^2 boundary, in the large N limit. Eventually, we study the entropy of such a reduced state and its area scaling behaviour in different thermodynamic regimes.

The content of the letter is organised as follows. Section II introduces the statement of canonical typicality in a formulation particularly suitable for the spin network Hilbert space description. Section III shortly reviews the notion of state of quantum geometry in terms of spin network basis in the kinematic Hilbert space of LQG. In Section IV we reformulate the statement of quantum typicality in this context. We derive the notion of canonical reduced state of the N-valent intertwiner spin network system in Section V and we prove the existence

of a regime of typicality for such system in Section VI. The entropy of the typical state and its thermodynamical interpretation are investigated in Section VII. We conclude in Section VIII with a short discussion of our results. Technical details are given in the Appendices.

II.   THE STATEMENT OF TYPICALITY

In quantum statistical mechanics, canonical typicality states that a typical state of a small subsystem is well approximated by the canonical ensemble. For a random pure state from an energy shell $(\mathcal{E}, \mathcal{E} + \delta\mathcal{E})$,

$$|\phi\rangle = \sum_n c_n |\mathcal{E}_n\rangle \qquad \mathcal{E}_n \in (\mathcal{E}, \mathcal{E} + \delta\mathcal{E}) \qquad (1)$$

the corresponding reduced density matrix $\rho_\phi^S = \mathrm{tr}_E |\phi\rangle\langle\phi|$ for a sufficiently small subsystem S (whose complement is denoted as E) satisfies

$$\rho_\phi^S \simeq \Omega^S \qquad (2)$$

where $\Omega^S = \mathrm{tr}_E[\rho_{micro}]$ is the reduction of the microcanonical density matrix ρ_{micro} defined for the same energy shell. The system is assumed to be large while the shell width $\delta\mathcal{E}$ very small compared with $\mathcal{E}$, but much larger than typical level spacings. In this sense, despite being a purely kinematic statement, canonical typicality can be used as a tool to explain the emergence of statistical ensembles [23].

The thermal form of the canonical state for the system is determined, in the standard statistical setting, by the energy constraints imposed on the state of the full system. However, the statement of canonical typicality is more general as it does not refer to the hamiltonian and thus applies universally to all systems.

For a generic *closed* system (small system S and sufficiently large environment E) in a quantum pure state, subject to a completely arbitrary *global constraint* $\mathcal{R}$, thermalization results from entanglement between the system and the environment [22].

This can be realized quantum mechanically by restricting the allowed states of the system and environment to a subspace $\mathcal{H}_\mathcal{R}$ of the total Hilbert space:

$$\mathcal{H}_\mathcal{R} \subseteq \mathcal{H}_E \otimes \mathcal{H}_S \qquad (3)$$

where $\mathcal{H}_S$ and $\mathcal{H}_E$ are the Hilbert spaces of the system and environment, with dimensions d_S and d_E respectively. In this setting, given an arbitrary pure state from $\phi \in \mathcal{H}_\mathcal{R}$, the reduced state $\rho_S \equiv \mathrm{Tr}_E[|\phi\rangle\langle\phi|]$ will almost always be very close to the canonical state Ω_S.

The canonical state of the system,

$$\Omega_S \equiv \mathrm{Tr}_E[\mathcal{I}_\mathcal{R}]. \qquad (4)$$

is obtained by tracing out the environment from microcanonical (maximally mixed) state $\mathcal{I}_\mathcal{R}$, defined as

$$\mathcal{I}_\mathcal{R} \equiv \frac{\mathbb{1}_\mathcal{R}}{d_\mathcal{R}} \qquad (5)$$

where $\mathbb{1}_\mathcal{R}$ is the projection operator onto $\mathcal{H}_\mathcal{R}$, and $d_\mathcal{R} = \dim \mathcal{H}_\mathcal{R}$. This corresponds to assigning *a priori* equal probabilities to all states of the universe consistent with the constraints $\mathcal{R}$.

Concretely, such a behaviour can be stated as a theorem [22], showing that for an arbitrary $\varepsilon > 0$, the distance between the reduced density matrix of the system $\rho_S(\phi)$ and the canonical state Ω_S is given probabilistically by

$$\frac{\mathrm{Vol}\left[\phi \in \mathcal{H}_\mathcal{R} \mid D(\rho_S(\phi), \Omega_S) \geq \eta\right]}{\mathrm{Vol}\left[\phi \in \mathcal{H}_\mathcal{R}\right]} \leq \eta' \qquad (6)$$

where the *trace-distance* D is a metric [1] on the space of the density matrices [24], while

$$\eta' = 4\,\mathrm{Exp}\left[-\frac{2}{9\pi^3}d_\mathcal{R}\varepsilon^2\right], \quad \eta = \varepsilon + \frac{1}{2}\sqrt{\frac{d_S}{d_E^{\mathrm{eff}}}}, \quad (7)$$

with the environment effective dimension defined as

$$d_E^{\mathrm{eff}} \equiv \frac{1}{\mathrm{Tr}_E\left[(\mathrm{Tr}_S \mathcal{I}_\mathcal{R})^2\right]} \geq \frac{d_\mathcal{R}}{d_S}. \qquad (8)$$

The bound in Eq. (6) states that the fraction of the volume of the states which are far away from the canonical state Ω_S more than η decreases exponentially with the dimension of the "allowed Hilbert space" $d_\mathcal{R} = \dim\mathcal{H}_\mathcal{R}$ and with $\varepsilon^2 = \left(\eta - \frac{1}{2}\sqrt{\frac{d_S}{d_E^{\mathrm{eff}}}}\right)^2$. This means that, as the dimension of the Hilbert space $d_\mathcal{R}$ grows, there is a huge fraction of states which is concentrated around the canonical state.

The proof of the result relies on the *concentration of measure phenomenon*. The key argument of the above result consists in the *Levy*-lemma, which we report for completeness in Appendix A.

The goal of the paper is to reproduce this argument for a peculiar class of high dimensional quantum systems, the quantum spin network states. These states provide a gauge-invariant basis for the kinematical Hilbert space of several different background independent approaches to quantum gravity, including loop quantum gravity, spinfoam gravity and group field theories.

III.   STATES OF QUANTUM GEOMETRY

In loop quantum gravity, the spin network states provide a *kinematical* description of quantum geometry, in terms of superpositions of graphs Γ labelled by group or Lie algebra elements representing holonomies of the gravitational connection and their conjugate triad [6, 11].

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[1] We use the definition $D(\rho_1, \rho_2) = \frac{1}{2}\sqrt{(\rho_1 - \rho_2)^\dagger(\rho_1 - \rho_2)}$.

These states are constructed as follows. To each edge $e \in \Gamma$, one associates $SU(2)$ irreducible representations (irrep) labelled by a half-integer $j_e \in \mathbb{N}/2$ called spin. The representation (Hilbert) space is denoted V^{j_e} and has a dimension $d_{j_e} = 2j_e + 1$. To each vertex v of the netwwork, one attaches an intertwiner $\mathcal{I}_v$, which is $SU(2)$-invariant map between the representation spaces V^{j_e} associated to all the edges e meeting at the vertex v,

$$\mathcal{I}_v : \bigotimes_{e \text{ ingoing}} V^{j_e} \to \bigotimes_{e \text{ outgoing}} V^{j_e} \tag{9}$$

One can alternatively consider $\mathcal{I}_v$ as a map from $\otimes_{e \in v} V^{j_e} \to \mathbb{C} \simeq V^0$ and call the intertwiner an invariant tensor or a singlet state between the representations attached to all the edges linked to the considered vertex. Once the j_e's are fixed, the intertwiners at the vertex v actually form a Hilbert space, which we will call

$$\mathcal{H}_v \equiv \text{Int}_v[\bigotimes_e V^{j_e}] \tag{10}$$

A spin network state $|\Gamma, \{j_e\}, \{\mathcal{I}_v\}\rangle$ is defined as the assignment of representation labels j_e to each edge and the choice of a vector $|\{\mathcal{I}_v\}\rangle \in \otimes_v \mathcal{H}_v$ for the vertices. The spin network state defines a wave function on the space of discrete connections $SU(2)^E / SU(2)^V$,

$$\phi_{\{j_e\},\{\mathcal{I}_v\}}[g_e] = \langle h_e | \mathcal{I}_v \rangle \equiv \text{tr} \bigotimes_e D^{j_e}(h_e) \otimes \bigotimes_v \mathcal{I}_v \tag{11}$$

where we contract the intertwiners $\mathcal{I}_v$ with the (Wigner) representation matrices of the group elements g_e in the chosen representations j_e.

Therefore, upon choosing a basis of intertwiners for every assignment of representations j_e, the spin networks provide a basis of the space of wave functions associated to the graph Γ,

$$\mathcal{H}_\Gamma = L_2[SU(2)^E / SU(2)^V] = \bigoplus_{\{j_e\}} \bigotimes_v \mathcal{H}_v. \tag{12}$$

Such discrete and algebraic objects provide a description of the fundamental excitations of quantum space-time. From a geometrical point of view, classically, given a cellular decomposition of a three-dimensional manifold, a spin-network graph with a node in each cell and edges connecting nodes in neighbouring cells is said to be dual to this triangulation. Therefore, each edge of the graph is dual to a surface patch intersecting the edge and the half-integer defines the area of such patch. Analogously, vertices of a spin network can be dually thought of as chunks of volume.

In the following, we will focus on a fundamental building block of a spin network graph, the Hilbert space of a single intertwiner with N legs.

IV. THE SPIN NETWORKS SETTING

Along with the approach of Section II, we consider a large quantum system given by a simple example of spin network state. Our system consists in a collection of N edges, represented by N independent edges states. The Hilbert space of the system is the direct sum over $\{j_i\}$'s of the tensor product of N irreducible representations V^{j_i},

$$\mathcal{H} = \bigoplus_{\{j_i\}} \bigotimes_{i=1}^N V^{j_i}. \tag{13}$$

This set of independent edges plays the role of the "universe". Notice that, despite its extreme simplicity, this system has a huge Hilbert space. The single representation space V^j has finite dimension $d_{j_i} = 2j_i + 1$. However, d_j is summed over all $\{j_i\}$'s. Therefore each Wilson line state (edge) lives in an infinite dimensional Hilbert space. In the following, we will always consider a cut-off in the value of the $SU(2)$ representation labelling the edge,

$$\bigoplus_{\{j_i\}} \to \bigoplus_{\{j_i\}}^{j_i \leq J_{max}}, \quad \text{with} \quad J_{max} \gg 1. \tag{14}$$

This will allow us to deal with a very large but finite dimensional space.

Now, we want to divide the universe into system and environment. We do so simply by defining two subsets of edges E and S, with $\{1, \cdots, k\} \in S$ and $\{k + 1, \cdots, N\} \in E$, such that $k \ll N$. Consequently, we can write the universe Hilbert space as the tensor product $\mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_S$.

The next step toward typicality consists in restricting the allowed states of the system and environment to a subspace of the total Hilbert space.

A. Constrained system

We define the constrained subspace by restricting to the space of $SU(2)$ invariant states in $\mathcal{H}$. This choice reduces the universe Hilbert space to the collection of the $SU(2)$-invariant linear spaces

$$\mathcal{H}_N = \bigoplus_{\{j_i\}} \text{Inv}_{SU(2)}[\bigotimes_{i=1}^N V^{j_i}],$$

spanned by N-valent intertwiner states. Invariance under $SU(2)$ is the the first constraining ingredient defining our subsystem.

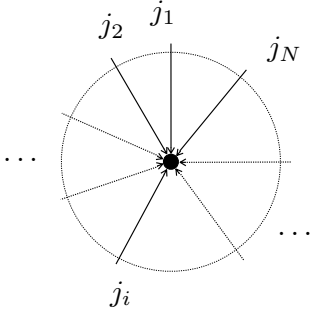The Hilbert space of the N-valent intertwiners natu-

Figure 1. The N-legged intertwiner system describes a closed *2-sphere* pierced by N edges. The N (oriented) edges are dual to the elementary N surfaces comprising the surface of the sphere. The intertwiner contains information on how the elementary surfaces, dual to the links, are combined together to form a surface boundary of the space region dual to the node [6, 7].

rally decomposes into subspaces of constant total area[2]

$$\mathcal{H}_N = \bigoplus_{\{j_i\}} \mathrm{Inv}_{SU(2)}[\bigotimes_{i=1}^{N} V^{j_i}] \tag{15}$$

$$= \bigoplus_{J \in \mathbb{N}} \bigoplus_{\{j_i \mid \sum_l j_i = J\}} \mathrm{Inv}_{SU(2)}[\bigotimes_{i=1}^{N} V^{j_i}] = \bigoplus_J \mathcal{H}_N^{(J)}$$

We further constrain our system by considering only the invariant *tensor product* Hilbert space, with total spin fixed to $J = J_0$ (see Fig. 1).

Eventually, the constrained Hilbert space is given by

$$\mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_N^{(J_0)} = \bigoplus_{\{j_i \mid \sum_l j_i = J_0\}} \mathrm{Inv}_{SU(2)} \left[\bigotimes_{i=1}^{n} V^{j_i} \right] \subseteq \mathcal{H}. \tag{16}$$

It was proven in [26] that each subspace $H_N^{(J)}$ of N-valent intertwiners with fixed total area J carries an irreducible representation of $U(N)$. This remark will prove of fundamental importance in the forthcoming argument. In this context, one can interpret the total area as the total area dual to the set of N legs of the intertwiner. In the semiclassical limit, one can then think of this system dually as a closed surface with area $J_0 \gg l_P^2$, the Planck scale.

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[2] The choice of a linear area spectrum $j \times l_P^2$ is favoured by the forthcoming approach involving the $U(N)$ structure of the intertwiner space.

B.   A basis for the bipartite space

Once the constrained space has been defined, we need a proper basis for the bipartite universe to define the reduced states of the system and environment under study. Following the notation in [27], it is useful to look first at the decomposition of the tensor product space in Eq. (13) in direct sums of irreducible representations, for fixed $\{j_i\}$'s

$$\bigotimes_i V^{j_i} = \bigoplus_q V^q \otimes d_q^{\{j_i\}}, \tag{17}$$

where $d_q^{\{j_i\}}$ is the degeneracy space of states with spin q, depending on the $\{j_i\}$'s. Here one can define a basis vectors of $d_q^{\{j_i\}}$, $|\alpha_q\rangle$, with the α_q label running from 1 to $d_q^S = \dim d_q^{\{j_i\}}$.

Analogously, the product space $\mathcal{H}_E \otimes \mathcal{H}_S$ would decompose as

$$\mathcal{H}_E \otimes \mathcal{H}_S = \bigoplus_{q,r} (V^q \otimes V^r) \otimes (d_q^{\{j_S\}} \otimes d_r^{\{j_E\}}).$$

In this decomposition, due to the $SU(2)$ invariance, the constrained Hilbert space is written as a single direct sum over $q = r$,

$$\mathcal{H}_N^{E,S} \equiv \bigoplus_q (d_q^{\{j_S\}} \otimes d_q^{\{j_E\}}). \tag{18}$$

The expression above corresponds to unfolding the intertwiner state (spin-0 state) on a graph with *two vertices* associated with two *non gauge invariant* tensors intertwining the edges in E and S separately, together with an internal edge linking these two vertices and carrying a representation q.

Then for fixed $\{j_i\}$'s, a suitable basis for such decomposition is given by the vectors [27]

$$|q, \eta_q, \sigma_q\rangle = \tag{19}$$

$$\frac{1}{\sqrt{2q+1}} \sum_{m=-q}^{q} (-1)^{q-m} |q, -m, \eta_q\rangle_E \otimes |q, m, \sigma_q\rangle_S.$$

When we turn on the constrained sum over the $\{j_i\}$'s, the constrained Hilbert space decomposition reads

$$\mathcal{H}_N^{J_0} \equiv \bigoplus_{J_S \leq J_0} \bigoplus_{|\vec{J}_S|} \left(\bigoplus_{\{j_S\}}^{J_S} d_k^{|\vec{J}_S|,\{j_S\}} \otimes \bigoplus_{\{j_E\}}^{J_0 - J_S} d_{N-k}^{|\vec{J}_S|,\{j_E\}} \right). \tag{20}$$

where we use the short notation $\oplus^{(J_0)}$ to indicate the sum over configurations $\{j_E, j_S\}$ satisfying the constraint $\sum_i j_i = J_0$.

Each subspace now has fixed total area $J_S, (J_S - J_0)$ and closure defect $|\vec{J}_S|$. Again we can describe each subspace in terms of a $U(N)$ irreducible representation corresponding to intertwiners states between the $k, (N - k)$

boundary edges and a fictious extra link carrying the closure defect $|\vec{J}_S|$.

Now, we can use a coupling scheme in which we first couple all the irreps within the system and the enviroment, respectively and then we couple the system with the enviroment. We start with vectors

$$\Big|\{j_E, j_S\}; \eta_E, \sigma_S; |\vec{J}_E|, |\vec{J}_S|; |\vec{J}_T| = M_T = 0\Big\rangle \quad (21)$$

where η_E and σ_S are the recoupling quantum numbers necessary to write $SU(2)$ irreps into a coupled basis. However, we want the environment and the system to be decoupled. Using the Clebsh-Gordan coefficients

$$\Big\langle j_1, m_1; j_2, m_2 \Big| \{j_1, j_2\}; |\vec{J}_T| = 0, M_T = 0 \Big\rangle = \quad (22)$$

$$= \delta_{j_1; j_2} \delta_{-m_1; m_2} \frac{(-1)^{j_1 - m_1}}{\sqrt{2j_1 + 1}}$$

generally written here for two $SU(2)$ irreps, we wrote each one of these vectors in a semi-decoupled basis in which all the spins within the system and within the environment, respectively, are coupled, but the environment and the system are not. The *semi-decoupled* basis reads

$$\sum_{M_S, M_E} \frac{(-1)^{|\vec{J}_E| - M_E}}{\sqrt{2|\vec{J}_S| + 1}} \Big|\{j_E\}; \eta_E; |\vec{J}_E|, M_E\Big\rangle_E \quad (23)$$

$$\otimes \Big|\{j_S\}; \sigma_S; |\vec{J}_S|, M_S\Big\rangle_S \, \delta_{-M_E, M_S} \delta_{|\vec{J}_S|, |\vec{J}_E|},$$

V. REDUCED STATE OF THE SYSTEM

The maximally mixed state on $\mathcal{H}_{\mathcal{R}}$ is defined as the only state compatible with the constraint and with the "a priori equal probabilities" principle. This is formally given by

$$\mathcal{E}_{\mathcal{R}} \equiv \frac{1}{d_{\mathcal{R}}} \mathbb{1}_{\mathcal{R}} = \frac{1}{d_{\mathcal{R}}} P_{\mathcal{R}}, \quad (24)$$

where now the $P_{\mathcal{R}}$ operator projects states of $\otimes_l \mathcal{H}^{j_l}$ onto the $SU(2)$ gauge invariant subspace with fixed total spin number. Using the semi-decoupled basis introduced in the previous paragraph, we can write the projector as

$$P_{\mathcal{R}} = \sum_{\{j_E, j_S\}}^{(J_0)} \sum_{\eta_E, \sigma_S} \sum_{|\vec{J}_S|, M_S, M'_S} \frac{(-1)^{M_S + M'_S}}{d_{J_S}} \cdot$$

$$\cdot \Big|\{j_E, j_S\}; \eta_E, \sigma_S; |\vec{J}_S|, -M_S; |\vec{J}_S|, M_S\Big\rangle \quad (25)$$

$$\otimes \Big\langle \{j_E, j_S\}; \eta_E, \sigma_S; |\vec{J}_S|, -M'_S; |\vec{J}_S|, M'_S\Big|$$

The dimension of the constrained Hilbert space $d_{\mathcal{R}} \equiv \dim(\mathcal{H}_{\mathcal{R}})$ counts the degeneracy of the N-valent intertwiners with fixed total spin J_0. Given the equivalence between the space $\mathcal{H}_N^{(J_0)}$ of N-valent intertwiners with

fixed total area $\sum_i j_i = J_0$ (including the possibility of trivial SU(2) irreps) and the irreducible representation of $U(N)$ formalism for $SU(2)$ intertwiners [26], $d_{\mathcal{R}}$ can be calculated as the dimension of the equivalent maximum weight $U(N)$ irrep with Young tableaux given by two horizontal lines with equal number of cases J_0,

$$d_{\mathcal{R}} = \frac{1}{J_0 + 1} \binom{N + J_0 - 1}{J_0} \binom{N + J_0 - 2}{J_0} \quad (26)$$

A. The Canonical state

The canonical reduced state is defined as the partial trace of $\mathcal{E}_{\mathcal{R}}$ over the environment,

$$\Omega_S \equiv \mathrm{Tr}_E \, \mathcal{E}_{\mathcal{R}}$$

The easiest way to perform the partial trace is to use the coupled basis for the environment, i.e. the set of quantum numbers $\Big(\{j_E\}, \eta_E, |\vec{J}_E|, M_E\Big)$. This gives

$$\Omega_S = \frac{1}{d_{\mathcal{R}}} \sum_{\{j_E, j_S\}}^{(J_0)} \sum_{\eta_E, \sigma_S} \sum_{|\vec{J}_S|, |\vec{J}_E|, M_S} \frac{\delta_{|\vec{J}_S|, |\vec{J}_E|}}{d_{J_S}}$$

$$\Big|\{j_S\}, \sigma_S, |\vec{J}_S|, M_S\Big\rangle \Big\langle \{j_S\}, \sigma_S, |\vec{J}_S|, M_S\Big| \quad (27)$$

In order to evaluate the canonical weight, we need to single out the sums over the environment quantum numbers. The constrained sum can be written as[3]

$$\sum_{\{j_E, j_S\}}^{(J_0)} \to \sum_{J_S \leq J_0/2} \sum_{\{j_S\}}^{(J_S)} \sum_{\{j_E\}}^{(J_0 - J_S)} \quad (28)$$

Therefore, by summing over the environment quantum numbers, we get

$$\Omega_S = \sum_{J_S \leq J_0/2} \sum_{\sigma_S, |\vec{J}_S|, M_S} \sum_{\{j_S\}}^{J_S} \frac{D_{(N-k)}(|\vec{J}_S|, J_0 - J_S)}{d_{J_S} \, d_{\mathcal{R}}} \cdot$$

$$(29)$$

$$\cdot \Big|\{j_S\}, \sigma_S, |\vec{J}_S|, M_S\Big\rangle \Big\langle \{j_S\}, \sigma_S, |\vec{J}_S|, M_S\Big|$$

The canonical weight, given by

$$D_{(N-k)}(|\vec{J}_S|, J_0 - J_S) \equiv \sum_{\{j_E\}}^{(J_0 - J_S)} \sum_{\eta_E} \sum_{|\vec{J}_E|} \delta_{|\vec{J}_S|, |\vec{J}_E|},$$

counts the degeneracy of the space $\bigoplus_{\{j_E\}}^{J_0 - J_S} D_{N-k}^{|\vec{J}_S|, \{j_E\}}$, defined by a fixed value of the environment total area

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[3] Given $k \ll N - k$, the bound $J_S \leq J_0/2$ enforces the condition $J_E \geq J_S$.
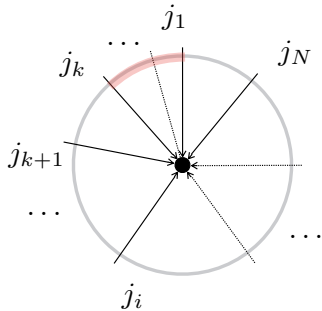
Figure 2. A local patch of the 2d sphere (in red), associated to a subset of intertwined links $\{j_1, \cdots, j_k\}$ defining the "system". The "environment" is identified with the complementary 2d-surface associated to the set of links $\{j_{k+1}, \cdots, j_N\}$, with $N \gg k$.

$J_E = J_0 - J_S$ and a fixed total spin (closure defect) $|\vec{J}_E| = |\vec{J}_S|$, as described in Eq. (20). This dimension can be calculated again by using the $U(N)$ formalism, in terms of the dimension of the corresponding non trivial $U(N)$ irreps [26],

$$\dim R_N^{l_1, l_2} = \frac{l_1 - l_2 + 1}{l_1 + 1} \binom{N + l_1 - 1}{l_1} \binom{N + l_2 - 2}{l_2} \tag{30}$$

In our notation, for the environment we have $l_1 + l_2 = 2J_E$ while $|\vec{J}_E| = \frac{l_1 - l_2}{2}$. Then, for each subspace with given fixed quantum numbers of the system, we have

$$D_{(N-k)}(|\vec{J}_E|, J_E) \equiv \frac{2|\vec{J}_E| + 1}{J_E + |\vec{J}_E| + 1} \cdot$$
$$\cdot \binom{(N - k) + J_E + |\vec{J}_E| - 1}{J_E + |\vec{J}_E|} \binom{(N - k) + J_E - |\vec{J}_E| - 2}{J_E - |\vec{J}_E|} \tag{31}$$

We define $W_{\mathcal{E}} \equiv D_{(N-k)}(|\vec{J}_S|, J_0 - J_S)$, and $W_{\mathcal{S}} \equiv D_{(k)}(|\vec{J}_S|, J_S)$, the dimension for the system degeneracy space with fixed area J_S and closure defect $|\vec{J}_S|$, derived from the equivalent $U(N)$ representation as for the case of the environment in Eq. (31).

In these terms, we can easily check that the reduced state is correctly normalised[4]

$$\mathrm{Tr}[\Omega_S] = \frac{1}{d_{\mathcal{R}}} \sum_{J_S \leq J_0/2} \sum_{|\vec{J}_S|} W_{\mathcal{S}} W_{\mathcal{E}} = 1. \tag{32}$$

The canonical weight $W_{\mathcal{E}}$ encodes all the information about the local structure of correlations of the reduced

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[4] Notice that the sum over M_S canceled the factor d_{J_S} in Eq. (30).

intertwiner state. The specific form of this factor tells us about the physics of the system, defined by the specific choice of constraints: the gauge symmetry and the fixed total area constraints. Given the global constraint, the split in system and environment breaks the gauge symmetry. System and environment can share both total area and total spin. This is why, beside the expected dependence on the total area of the system J_S, the canonical weight carries some interesting extra information on the local closure defect $|\vec{J}_S|$.

VI. TYPICALITY OF THE REDUCED STATE

With the explicit form of the canonical weight at hands, we can now investigate the distance of the canonical state from a randomly chosen pure state in $\mathcal{H}_{\mathcal{R}}$.

Concretely, following the approach described in Section (II), we want to show that for the overwhelming majority of intertwiner states $|\mathcal{I}\rangle \in \mathcal{H}_{\mathcal{R}} \subseteq \mathcal{H}_E \otimes \mathcal{H}_S$, the trace distance $D(\rho_S, \Omega_S)$ between the reduced density matrix of the system $\rho_S = Tr_E(|\mathcal{I}\rangle\langle\mathcal{I}|)$ and the canonical state $\Omega_S = \mathrm{Tr}_E \mathcal{E}_{\mathcal{R}}$ is extremely small. This amounts to prove two statements: the first one is that the Hilbert space average of such trace distance is itself quite small in the regime in which we are interested in

$$\mathbb{E}[D(\rho_S, \Omega_S)] \ll 1 , \tag{33}$$

where $\mathbb{E}$ indicates the Hilbert space average performed using the unique unitarily invariant Haar measure [21, 24]. The second one is that the fraction of states for which such distance is higher than a certain ϵ is exponentially vanishing in the dimension of the Hilbert space.

Now, following [22], one can simply recast the condition in Eq. (6) with the following bound on the averaged distance,

$$0 \leq \mathbb{E}[D(\rho_S, \Omega_S)] \leq \sqrt{\frac{d_S}{d_E^{\mathrm{eff}}}} \leq \frac{d_S}{\sqrt{d_{\mathcal{R}}}} \tag{34}$$

Therefore, the first step toward the statement of typicality in our context amounts to show that the size of the constrained space is much larger than the size of the system [22].

A. Evaluation of the bound

The Hilbert space of the system consists in the tensor product Hilbert space of the set of irreps V^{j_i} with a given cutoff J_{max}. We assume $J_{max} \geq J_0$, in order to be sure that $\mathcal{H}_N^{(J_0)}$ will always carry an irreducible representation of $U(N)$. Each $SU(2)$ representation space has dimension $d_j = 2j + 1$. Therefore, considering the set k edges

comprising the system, we have

$$d_S \equiv \dim(\mathcal{H}_S) = \prod_{i \in \mathrm{S}} \sum_{j_i=0,1/2,\ldots}^{J_{max}} (2j_i + 1) = \qquad (35)$$

$$= \left(\sum_{j_i=0,1/2,\ldots}^{J_{max}} \right)^k = (2J_{max} + 1)^k (J_{max} + 1)^k$$

Analogously, for the environment we will get $d_E = (2J_{max} + 1)^{N-k} (J_{max} + 1)^{N-k}$.

Next, given in Eq. (26), we can focus on the second inequality in Eq. (34) and define the regime where $\mathbb{E}\left[||\rho_S - \Omega_S||_1\right] \ll 1$.

Consider the ratio

$$\frac{d_S^2}{d_\mathcal{R}} = \frac{(2J_{\max} + 1)^{2k} (J_{\max} + 1)^{2k}}{\frac{1}{J_0+1} \binom{N+J_0-1}{J_0} \binom{N+J_0-2}{J_0}}. \qquad (36)$$

Using Stirling's approximation, $\ln n! = n \ln n - n + O(\ln n)$, one can generally prove that in the regime $n, k \gg 1$ (with $n - k \gg 1$)

$$\log \binom{n}{k} \sim nH\left(\frac{k}{n}\right), \qquad (37)$$

where $H(x) \equiv -x \log x - (1-x) \log(1-x)$.

Moreover, using $\binom{n}{k} \leq \binom{n-1}{k}$ in Eq. (36), for $J_{\max}, N, J_0 \gg 1$, the leading order is given by

$$\frac{d_S^2}{d_\mathcal{R}} \lesssim \frac{e^{2k} J_{\max}^{4k} J_0}{e^{2(N+J_0-1)H\left(\frac{J_0}{N+J_0-1}\right)}} \qquad (38)$$

In the standard context of statistical mechanics, in performing the thermodynamic limit, we have the condition that the density of particles must be finite otherwise the energy density would diverge: $N, V \to +\infty$ with $\frac{N}{V} < +\infty$. Analogously here, with the area is playing here the role of the energy, as we will discuss in Section VII, the correct way of performing the thermodynamic limit consists in taking $N, J_0 \to \infty$ with $\frac{J_0}{N} \equiv j_0 < +\infty$, where j_0 is the average spin of the intertwiner.

Eq. (38) is exponentially decreasing both in the total number of links N and in the total area J_0, while only polynomially increasing in $J_{\max}$. The only condition we need to be sure is fulfilled is $J_{\max} \geq J_0$.

Therefore, writing the cut-off as $J_{\max} = e^{\log J_{max}}$ we have

$$\frac{d_S^2}{d_\mathcal{R}} \sim e^{2k} J_0 \, e^{4k \log J_{max} - 2(N+J_0-1)H\left(\frac{J_0}{N+J_0-1}\right)} \qquad (39)$$

In order to understand if our bound goes to zero in the thermodynamic limit $N, J_0 \gg 1$, we need to study when the exponent of the previous equation becomes negative,

$$(N + J_0 - 1)H\left(\frac{J_0}{N + J_0 - 1}\right) > 2k \log J_{max}. \qquad (40)$$

When $N, J_0 \gg 1$ this quantity behaves as

$$(N + J_0)H\left(\frac{J_0}{N + J_0}\right) > 2k \log J_{max} \qquad (41)$$

which can be written in average spin as

$$\frac{N}{k}(1 + j_0)H\left(\frac{1}{1 + j_0}\right) > 2 \log J_{max}. \qquad (42)$$

Studying the function $(1 + x)H(\frac{1}{1+x})$ we see that it is monotonically increasing and always non-negative. [5]

For $x \ll 1$ we have $(1 + x)H(\frac{1}{1+x}) \sim -x \log x$. Hence, in the regime for which $j_0 \ll 1$ we have

$$\frac{N}{k}(1 + j_0)H\left(\frac{1}{1 + j_0}\right) \sim \frac{N}{k} j_0 \log \frac{1}{j_0} =$$

$$= \frac{N}{k} \frac{J_0}{N} \log \frac{N}{J_0} = \frac{J_0}{k}(\log N - \log J_0) \qquad (43)$$

Defining the relative order of magnitude between N and J_0 as $\log \frac{N}{J_0} = \alpha$ and assuming $\alpha \geq 2$, we obtain that in the regime $N \geq e^2 J_0$ the following condition is enough to guarantee an exponential decreasing of the bound

$$\frac{J_0}{k} > \log J_{max}. \qquad (44)$$

We now look at the opposite regime, $j_0 \gg 1$. Considering that in the regime $x \to \infty$ we have $(1+x)H(\frac{1}{1+x}) \sim \log x$, the condition for an exponential decreasing becomes

$$\frac{N}{k} \log j_0 > 2 \log J_{max} \qquad (45)$$

The last condition tells us that the average distance between a random state and the canonical state is extremely small as long as the fraction of links which defines our system is small enough with respect to the rest of the system.

Contrarily to the case $j_0 \ll 1$, where N and J_0 play a similar role in the evaluation of the dimension of the constrained Hilbert space $d_\mathcal{R}$, the emergence of typicality in the $j_0 \gg 1$ regime is a remarkable new feature. Indeed, the limit $j_0 \gg 1$ heralds in some way a semiclassical limit, in the sense that, on average, the quantum numbers of the area are really high. Such emergence of a canonical behaviour, within a semiclassical regime, is something we may expect to be related to what happens in a black hole.

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[5] In the limit $x \to 0$ we have $(1 + x)H(\frac{1}{1+x}) \to 0$

Eventually, we should consider the regime $j_0 \sim 1$. In this case we have $(1 + j_0)H(\frac{1}{1+j_0}) \sim 2H(1/2) = 2 \log 2$. Therefore, the condition

$$\frac{N}{k} > 2 \log J_{max} \qquad (46)$$

is sufficient for an exponential decreasing of the bound under study.

We conclude that there are large regions in the space of the three parameters $(N, J_{\max}, J_0)$, of physical interest, in which the bound is much smaller than one.

It is important to mention that the results can also be checked numerically, since we have an exact expression for both d_S and $d_{\mathcal{R}}$.

In this sense, we can use a physically motivated argument to provide a meaningful value for the cut-off J_{max} to check the plausibility of the given bounds. As an example, we can consider the scale of the radius of the observed universe L_U as the limit value on the maximal representation of $SU(2)$. Using this argument we obtain the following cut-off

$$J_{max} \sim \frac{c^2}{l_P^2 L_U} \sim 6 \times 10^{122} = e^{122 \times \log 10} \qquad (47)$$

where c is the speed of light, l_P is the Planck length. Putting the numbers in the inequality to study the sign of the exponent we get that the following two conditions are enough to guarantee an exponential decay of the bound on the trace distance:

$$\frac{J_0}{k} \gtrsim 3 \times 10^2 \qquad (N \gg J_0 \gg 1) \qquad (48)$$

$$\frac{N}{k} \gtrsim 6 \times 10^2 \qquad (J_0 \geq N \gg 1) \qquad (49)$$

We see that the bounds can be easily achieved even for such a large value of J_{max}.

B. Levy's lemma

Again, following [22] we can apply Levy's lemma to bound the fraction of the volume of states which are ε more distant than $\frac{d_S}{\sqrt{d_{\mathcal{R}}}}$ from Ω_S as

$$\frac{\text{Vol}\left[|\mathcal{I}\rangle \in \mathcal{H}_{\mathcal{R}} \mid D(\rho_S, \Omega_S) - \frac{d_S}{\sqrt{d_{\mathcal{R}}}} \geq \varepsilon\right]}{\text{Vol}\left[|\mathcal{I}\rangle \in \mathcal{H}_{\mathcal{R}}\right]} \leq B_\epsilon(d_{\mathcal{R}}) \quad (50)$$

$$B_\epsilon(d_{\mathcal{R}}) \equiv 4 \, \text{Exp}\left[-\frac{2}{9\pi^3} d_{\mathcal{R}} \varepsilon^2\right].$$

The dimension $d_{\mathcal{R}}$ can be evaluated numerically because we have an exact expression. We give a numeric example to show that it is not necessary to have huge areas or number of links for the typicality to emerge. Suppose we have a huge sensitivity on the trace distance:

$\epsilon = 10^{-10}$. Moreover, $\frac{2}{9\pi^3} \sim 7 \times 10^{-3}$. With these numbers the left-hand side of the previous equation becomes

$$B_{10^{-10}}(d_{\mathcal{R}}) = 4\text{Exp}\left[-7 \cdot 10^{-23} d_{\mathcal{R}}\right]. \qquad (51)$$

Suppose we look at the most elementary patch, just a few links ($k = 1, 2$). The set of numbers $J_0 = N = 10^4$ gives the following bounds, using a cut-off given by the cosmological horizon

$$\frac{N}{k} \sim 10^4 \gg 6 \times 10^2 \qquad (52\text{a})$$

$$B_{10^{-10}}(d_{\mathcal{R}}) = 4\text{Exp}\left[-5.6 \times 10^{5992}\right] \ll 1 \qquad (52\text{b})$$

As we can see, the typicality emerges quite easily, due to the exponential-like growth of the constrained Hilbert space on the number of links N and on the total area J_0. The bound on the average value of the distinguishability between a random density matrix and the canonical reduced state is satisfied but not even remotely as good as the bound on the volume of states.

The proven typical behaviour indicates the existence of a regime where the properties of the reduced state of the N-valent intertwiner state are *universal*. The structure of correlations carried by the reduced state is independent from the specific shape of the pure intertwiner state and it is *locally* the same everywhere. Due to the global symmetry constraint though, the canonical weight presents a very involved analytic form, despite the extreme simplicity of the system under study. In order to extract some physical information from this coefficient we are going to study its behaviour in the thermodynamic limit.

VII. THERMODYNAMIC LIMIT & AREA LAW

The entropy of the system is given by the von Neumann entropy,

$$S(\Omega_S) = -\text{Tr}[\Omega_S \log \Omega_S] \qquad (53)$$

Given the diagonal form of the canonical reduced density matrix Ω_S in Eq. (30), this can be written as

$$S(\Omega_S) = -\frac{1}{d_{\mathcal{R}}} \sum_{J_S \leq J_0, |\vec{J}_S|} W_S \, W_{\mathcal{E}} \log\left(\frac{W_{\mathcal{E}}}{d_{J_S} \, d_R}\right). \qquad (54)$$

Within the typicality regime, for $N, J_0 \gg 1$, we use again the Stirling approximation for the factorials to simplify the form of the binomial coefficients in $W_{\mathcal{E}}, W_{\mathcal{S}}$ and $d_{\mathcal{R}}$. We will consider two different regimes for this function, associated to the two different thermodynamic limits $N \gg J_0$ and $J_0 \gg N$.

A. regime: $N \gg J_0$

Let us start by studying the case $N \gg J_0$. We single out two different sectors for the entropy function, corresponding to the extrema of the sum over $|\vec{J}_S|, J_S$, namely $|\vec{J}_S|, J_S \ll J_0$ and $(|\vec{J}_S| + J_S \simeq J_0)$. The value of the argument of the logarithm in Eq. (54), for the two sectors, is explicitly calculated in Appendix B.

For $|\vec{J}_S|, J_S \ll J_0$, i.e. when the closure defect is small and J_S is small, we find

$$- \log \left(\frac{W_{\mathcal{E}}}{d_{J_S} d_R} \right) = \tag{55}$$
$$2 J_S \left(1 + \log \frac{N - k}{J_0} \right) - \log \alpha_1(J_S, |\vec{J}_S|),$$

where, up to $O(1/J_0)$ terms,

$$\alpha_1(J_S, |\vec{J}_S|) = \left(1 + \frac{J_S - |\vec{J}_S|}{J_0} \right) \left(1 - \frac{k}{J_0 + N} \right)^{2J_0}.$$
$$\tag{56}$$
$$\cdot \left(1 - \frac{2(N-k)|\vec{J}_S|}{J_0(J_0 + N - k)} \right)^{|\vec{J}_S|}$$

For large J_0 the α_1 term is clearly sub-leading with respect to the $2J_S$ term.

On the other hand, the case $(|\vec{J}_S| + J_S \simeq J_0)$ corresponds to a single value of the entropy sum, namely $|\vec{J}_S| = J_S = J_0/2$. In this case, we find

$$- \log \left(\frac{W_{\mathcal{E}}}{d_{J_S} d_R} \right) \simeq J_0 \left(1 + \log \frac{(N-k)}{J_0} + \frac{2k}{N} \right) + \tag{57}$$
$$+ \epsilon \left(1 + \log \frac{(N-k)}{J_0} \right)$$

where $\epsilon = J_0 - A$.

We see that, for both cases in the $N \gg J_0 \gg 1$ regime, the leading term in the logarithm is proportional to twice the value of the total area of the system, $2J_S$.

This allows us to write the entropy of the reduced state as

$$S(\Omega_S) \simeq \beta \langle 2J_S \rangle + \text{small corrections} \tag{58}$$

where $\langle \cdot \rangle$ indicate the weighted average obtained by summing over $J_S, |\vec{J}_S|$, while

$$\beta \equiv \left(1 + \log \frac{N-k}{J_0} \right) \tag{59}$$

formally identified with the "temperature" of the environment, turns out to be a function of the averaged spin of the environment.

In statistical mechanics, the thermal behaviour of the canonical state relies on the constraint of energy conservation. The emergence of the canonical state from the micro-canonical occurs as the degeneracy of the the environment grows exponentially with the energy, hence decreasing exponentially with the system energy.

Despite being quite far from the standard setting, a hint toward a thermodynamical interpretation of the result comes from the $U(N)$ description of the $SU(2)$ intertwiner space. Using the Schwinger representation of the $\mathfrak{su}(2)$ Lie algebra [26, 28], one can describe the N-valent intertwiner state as a set of $2N$ oscillators, a_i, b_j. The quadratic operators $E_{ij} \equiv (a_i^\dagger a_j - b_i^\dagger b_j), with E_{ij}^\dagger = E_{ji}$ acts on couples of punctures (i, j) and form a closed $\mathfrak{u}(N)$ Lie algebra. The $\mathfrak{u}(1)$ Casimir operator is given by the oscillators' energy operator $E \equiv \sum_i E_i$, with $E_i \equiv E_{ii}$, and its value on a state gives twice the sum of the spins on all legs. Therefore, one can interpret E as measuring (twice) the total area of the boundary surface around the intertwiner.

In these terms, constraining the total area is equivalent to fix a shell of eigenvalues (in fact a single eigenvalue) of the energy operator acting on the full system. In the limit $N \gg J_0 \gg 1$, the degeneracy of the single energy level grows exponentially.

For such a reason the area scaling described by Eq. (58) is consistent with a thermal interpretation for our reduced surface state. On the other hand, interestingly, we can interpret the departure from the thermal behaviour à la Gibbs as a signature of the breaking of the global $SU(2)$ symmetry (closure defect), indicated by the explicit dependence of the reduced state on $|\vec{J}_S|$.

B. regime: $J_0 \gg N$

Finally, we study the behaviour of the entropy in the regime $J_0 \gg N$. We focus on the case $A, B \gg N \gg 1$ (See Appendix B). Up to $O(1/J_0)$ the logarithm of the normalised canonical is given by

$$- \log \left(\frac{W_{\mathcal{E}}}{d_{J_S} d_R} \right) \simeq - \log \left(\frac{J_0 e}{N - k} \right)^{-2k} + \frac{3k}{N} - \frac{2k J_S}{J_0} \tag{60}$$

$$- \frac{2J_S + 2|\vec{J}_E|}{J_0} \simeq k \log \left(\frac{J_0 e}{N - k} \right)^2 + \text{small corrs}$$

Interestingly, the leading term does not depend on the quantum numbers of the system. Therefore the entropy is counting the number of orthogonal states on which the canonical state has non-zero support. In this sense, the term $\left[\left(\frac{J_0 e}{N-k} \right)^2 \right]^k$ defines the *effective* dimension of the system, suggesting that in such regime the canonical state has approximately a tensor product structure. This makes the entropy extensive in the number of edges comprising the dual surface of the system.

VIII.   DISCUSSION

In this letter, we extend the so-called typicality approach, originally formulated in statistical mechanics contexts, to a specific class of tensor network states given by $SU(2)$ invariant spin networks. In particular, following the approach given in [22], we investigate the notion of canonical typicality for a simple class of spin network states given by N-valent intertwiner graphs with total fixed area. Our results do not depend on the physical interpretation of the spin-network, however they are mainly motivated by the fact that spin networks provide a gauge-invariant basis for the kinematical Hilbert space of several background independent quantum gravity approaches, including loop quantum gravity, spin-foam gravity and group field theories.

The first result is the very existence of a regime in which we show the emergence of a canonical typical state, of which we give the explicit form. Geometrically, such a reduced state describes a patch of the surface comprising the volume dual to the intertwiner. The structure of correlations described by the state should tell us how local patches glue together to form a closed connected surface in the quantum regime.

We find that, within the typicality regime, the canonical state tends to an exponential to the power of the total spin of the subsystem with an interesting departure from the Gibbs state. The exponential decay à la Gibbs of the reduced state is perturbed by a parametric dependence on the norm of the total angular momentum vector of the subsystem (closure defect). Such a feature provides a signature of the non local correlations enforced by the global gauge symmetry constraint. This is our second result.

We study some interesting properties of the typical state within two complementary regimes, $N \gg J_0 \gg 1$ and $J_0 \gg N \gg 1$. In both cases, we find that the area-law for the entropy of a surface patch must be satisfied at the local level, up to sub-leading logarithmic corrections due to the unavoidable dependence of the state from the closure defect. However, the area scaling interpretation of the entropy in the two regimes is quite different. In the $N \gg J_0 \gg 1$ regime, the result is related to the definition of a generalised Gibbs equilibrium state. The area is playing the role of the energy, as imposed by the specific choice of the global constraint, requiring total area conservation.

On the other hand, in the $J_0 \gg N \gg 1$ regime, the area scaling is given by the extensivity of the entropy in the number of links comprising the reduced state, as for the case of the generalised (non $SU(2)$-gauge invariant) spin networks [29]. In this regimen, each link contributes independently to the result, indicating that the global constraints are very little affecting the local structure of correlations of the spin network state. Still, interestingly, the remainder of the presence of the constraints can be read in the definition of what looks like an effective dimension for the single link Hilbert space.

We interpret these results as the proof that, within the typicality regime, there are certain (local) properties of quantum geometry which are "universal", namely independent of the specific form of the global pure spin network state and descending directly from the physical definition of the system encoded in the choice of the global constraints.

Our interest in testing the notion of typicality in quantum gravity is twofold. It first resides in the kinematic nature of the statement, a fundamental feature to study the possibility of a thermal characterisation of reduced states of quantum geometry regardless of any hamiltonian evolution in time. In this sense, typicality seems to be useful to characterise a notion of equilibrium compatible with the fully constrained gauge dynamics of the theory.

Secondly, we stress the importance of the statistical character of typicality. The statement of typicality generically relies on the high dimensionality of the Hilbert space and on the principle of concentration of measure [30]. For the generic case of a simple intertwiner state, the statistical analysis necessarily requires to consider a system with a *large number* of edges, beyond the very large dimensionality of the Hilbert space of the single constituents (the single edges in this case). In this sense, the presented statistical analysis and thermal interpretation is very different from what recently done in [15–17], considering quantum geometry states characterised by few constituents with a high dimensional Hilbert space. We expect a large number quantum statistical analysis to play a role in facing the problem of the continuum in quantum gravity.

Finally, it is interesting to consider the proposed "generalised" thermal characterisation of a local surface patch within the standard LQG description of the horizon as a closed surface made of patches of quantized area. Differently from the *isolated horizon* analysis (see e.g. [18, 32]), in our description the thermal character of the local patch is not (semi)classically induced by the thermal properties of a black hole horizon geometry, but emerges from a purely quantuum description. In this sense, our picture goes along with the informational theoretic characterisation of the horizon proposed in [33].

In fact, we expect that typicality could be used to define an information theoretic notion of quantum horizon, as the boundary of a generic region of the quantum space with an emergent thermal behaviour [34].

Appendix A: The Levy-lemma

In order to better understand the result it is useful to look at its most important step, which is the so-called Levy-lemma. Take an hypersphere in d dimensions S^d, with surface area V. Any function f of the point which does not vary too much

$$f : S^d \ni \phi \to f(\phi) \in \mathbb{R} \qquad |\nabla f| \leq 1$$

will have the property that its value on a randomly chosen point ϕ will approximately be close to the mean value.

$$\frac{\text{Vol}\left[\phi \in S^d \, : \, f(\phi) - \langle f \rangle \geq \epsilon \right]}{\text{Vol}\left[\phi \in S^d\right]} \leq 4 \, \text{Exp}\left[-\frac{d+1}{9\pi^3}\epsilon^2\right]$$

The Levy lemma is essentially needed to conclude that all but an exponentially small fraction of all states are quite close to the canonical state.

The effect of such result is that we can re-think about the "a priori equal probability" principle as an "apparently equal probability" stating that: as far as a small system is concerned almost every state of the universe seems similar to its average state, which is the maximally mixed state $\mathcal{E}_{\mathcal{R}} = \frac{1}{d_{\mathcal{R}}}\mathbb{I}_{\mathcal{R}}$.

Appendix B: Approximations

Let us define $A = J_0 - J_S + |\vec{J}_S|$ and $B = J_0 - J_S - |\vec{J}_S|$ and rewrite the formula in Eq. (30) in a more synthetic form

$$W_{\mathcal{E}} = \frac{A - B + 1}{A + 1}\binom{N-k-1+A}{A}\binom{N-k-2+B}{B}$$

In the expression above, we can simplify the factorials

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k! \, (n-1)!}, \qquad \text{(B1)}$$

whenever $n, k \gg 1$, by means of the Stirling approximation $\ln n! = n \ln n - n + O(\ln n)$. In the following, we consider two relevant regimes, $N \gg J_0 \gg 1$ and $J_0 \gg N \gg 1$.

For $N \gg J_0$, we evaluate two cases:

(1)$N \gg A, B \gg 1$ $(|\vec{J}_S|, J_S \ll J_0)$:

$$W_{\mathcal{E}} \simeq \frac{d_{J_S}}{A+1}\left(1 + \frac{N-k-1}{A}\right)^A\left(1 + \frac{A}{N-k-1}\right)^{N-k-1}.$$
$$\text{(B2)}$$
$$\cdot \left(1 + \frac{N-k-2}{B}\right)^B\left(1 + \frac{B}{N-k-2}\right)^{N-k-2} =$$
$$= \frac{d_{J_S}}{A}e^{A+B}\left(1 + \frac{N-k-1}{A}\right)^A\left(1 + \frac{N-k-2}{B}\right)^B$$

where $d_{J_S} = (2|\vec{J}_S| + 1)$.

(2)$N \gg A, B$ with $A \gg 1, B \sim 0$ $(|\vec{J}_S| \simeq J_S \simeq J_0/2)$:

$$W_{\mathcal{E}} \simeq d_{J_S}\left(1 + \frac{N-k-1}{A}\right)^A\left(1 + \frac{A}{N-k-1}\right)^{N-k-1}$$
$$\text{(B3)}$$

Within the regime of approximation $N \gg J_0$, we then study the behaviour of the normalization factor of the canonical state $d_{\mathcal{R}}$. The Stirling approximation of Eq. (26) reads

$$d_{\mathcal{R}} \simeq \frac{1}{J_0+1}\left(1 + \frac{N-1}{J_0}\right)^{J_0}\left(1 + \frac{J_0}{N-1}\right)^{N-1} \quad \text{(B4)}$$
$$\left(1 + \frac{N-2}{J_0}\right)^{J_0}\left(1 + \frac{J_0}{N-2}\right)^{N-2} \simeq \frac{1}{J_0}e^{2J_0}\left(1 + \frac{N}{J_0}\right)^{2J_0}$$

where we used $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$.

In particular, we are interested in the *normalized* canonical weight, given by $W_{\mathcal{E}}/d_{J_S} d_{\mathcal{R}}$. Again, for $N \gg J_0 \gg 1$, we write the two cases:

(1) $N \gg A, B \gg 1$ $(|\vec{J}_S|, J_S \ll J_0)$:

$$\frac{W_{\mathcal{E}}}{d_{J_S} d_{\mathcal{R}}} \simeq e^{-2J_S}\left(1 + \frac{J_S - |\vec{J}_S|}{J_0}\right)\left(1 + \frac{k}{J_0 + N}\right) \quad \text{(B5)}$$
$$\left(1 - \frac{2(N-k)|\vec{J}_S|}{J_0(J_0 + N - k)}\right)^{|\vec{J}_S|}\left(1 + \frac{N-k}{J_0}\right)^{-2J_S}.$$

(2) $N \gg A, B$ with $A \gg 1, B \sim 0$ $(|\vec{J}_S| \simeq J_S \simeq J_0/2)$:

$$\frac{W_{\mathcal{E}}}{d_{J_S} d_{\mathcal{R}}} \simeq \frac{J_0}{A}e^{A - 2J_0}\left(1 + \frac{N-k-1}{A}\right)^A\left(1 + \frac{N}{J_0}\right)^{-2J_0}.$$
$$\text{(B6)}$$

We therefore consider the regime $J_0 \gg N \gg 1$. In this case, for $A, B \gg N \gg 1$ we have

$$W_{\mathcal{E}} \simeq \frac{d_{J_S}}{A+1}\left(1 + \frac{A}{N-k-1}\right)^{N-k-1}. \quad \text{(B7)}$$
$$\cdot \left(1 + \frac{B}{N-k-2}\right)^{N-k-2}e^{2N-2k-3},$$

while

$$d_{\mathcal{R}} \simeq \frac{1}{J_0}\left(1 + \frac{J_0}{N}\right)^{2N-3}e^{2N-3}. \quad \text{(B8)}$$

Thereby, for the normalised canonical weight we find

$$\frac{W_{\mathcal{E}}}{d_{J_S}\,d_{\mathcal{R}}} \simeq \left(\frac{J_0 e}{N-k}\right)^{-2k}\left(1+\frac{J_S-|\vec{J}_S|}{J_0}\right)\cdot \qquad \text{(B9)}$$

$$\cdot\left(1+\frac{2k}{N}\right)^{N}\left(1-\frac{2J_S}{J_0^2}-\frac{|\vec{J}_S|^2}{J_0^2}\right)^{N}\left(1+\frac{3k}{N}\right)\cdot$$

$$\cdot\left(1-\frac{3J_S}{J_0}-\frac{|\vec{J}_S|}{J_0}\right)\left(1-\frac{2J_S}{J_0}\right)^{-k}.$$

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