BASIC HYPERGEOMETRY OF SUPERSYMMETRIC DUALITIES

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ABSTRACT. We introduce several new identities combining basic hypergeometric sums and integrals. Such identities appear in the context of superconformal index computations for three-dimensional supersymmetric dual theories. We give both analytic proofs and physical interpretations of the presented identities.

1. INTRODUCTION

Recently, there has been renewed interest in basic hypergeometric integrals because of their connection with various branches of mathematical physics, such as supersymmetric field theory, 3-manifold invariants and integrable systems. The purpose of this paper is to state and prove new basic hypergeometric integral identities and give their physical interpretations in terms of superconformal indices.

There is an interesting connection between partition functions of supersymmetric gauge theories on different curved manifolds and certain classes of hypergeometric functions. The first observation of this relation was made by Dolan and Osborn [1]. They found that the superconformal index of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory can be written via elliptic hypergeometric integrals. Similarly, three-dimensional superconformal indices can be expressed in terms of basic hypergeometric integrals (see e.g. [2–5]).

The superconformal index for a three-dimensional $\mathcal{N} = 2$ supersymmetric field theory is defined as

$$\mathrm{Tr}\left[(-1)^{\mathrm{F}} e^{-\beta\{Q,Q^\dagger\}} q^{\frac{1}{2}(\Delta+j_3)} \prod_i t_i^{F_i}\right], \tag{1.1}$$

where the trace is taken over the Hilbert space of the theory, Q and $Q^\dagger$ are supercharges, Δ, j_3 are Cartan elements of the superconformal group and the fugacities t_i are associated with the flavor symmetry group.

Studying the relation between basic hypergeometric integrals and superconformal indices is an important field of research from different points of view (see e.g. [6]). Non-trivial mathematical identities for superconformal indices provide a very powerful tool to check known supersymmetric dualities and to establish new ones. Such identities are also important for better understanding the structure of the moduli of three-dimensional supersymmetric

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theories and supersymmetric dualities. On the other hand, there is an interesting relationship between three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and geometry of triangulated 3-manifolds. The independence of a certain topological invariant of 3-manifolds on the choice of triangulation corresponds to equality of superconformal indices of three-dimensional $\mathcal{N} = 2$ supersymmetric dual theories.

Besides their appearance in supersymmetric field theory, basic hypergeometric integrals discussed in this paper recently appeared in the theory of exactly solvable two-dimensional statistical models [7, 8].

In this paper we extend the results of our previous work [4,5] on superconformal indices to a number of three-dimensional dualities. We provide explicit expressions for the generalized superconformal indices of some three-dimensional $\mathcal{N} = 2$ supersymmetric electrodynamics and quantum chromodynamics in terms of basic hypergeometric integrals.

We will only consider confining theories, which means that the duality leads to a closed form evaluation of a sum of integrals (rather than a transformation between two such expressions). As an example, one of the resulting identities is

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{6} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_\infty}{(q^{N_j+m/2} a_j z, q^{N_j-m/2} a_j/z; q)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{6m}} \frac{dz}{2\pi \mathrm{i} z}$$
$$= \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} a_j^{N_j}} \prod_{1\leq j<k\leq 6} \frac{(q/a_j a_k; q)_\infty}{(a_j a_k q^{N_j+N_k}; q)_\infty} , \quad (1.2)$$

where $|q| < 1$, the parameters a_j are generic and N_j are integers, subject to the balancing conditions $\prod_{i=1}^{6} a_i = q$ and $N_1 + \cdots + N_6 = 0$. Here, we use the standard notation

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j) ,$$

$$(a_1, \ldots, a_m; q)_\infty = (a_1; q)_\infty \cdots (a_m; q)_\infty$$

and the integration is over a positively oriented contour separating sequences of poles going to infinity from sequences going to zero.

The organization of the paper is as follows.

- In Section 2 we outline the superconformal index technique for three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories.
- In Section 3 we discuss supersymmetric dualities and present explicit expressions of superconformal indices for certain supersymmetric dual theories in terms of basic hypergeometric integrals. We present four examples, each leading to an integral evaluation similar to (1.2). Some of these evaluations generalize identities previously obtained in [2–4].

- In Section 4 we give mathematical proofs of the four integral evaluations that were derived using non-rigorous methods in Section 3. This gives a consistency check of the corresponding supersymmetric dualities.
- We review the basic aspects of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with focus on the necessary elements for the superconformal index computations and give some details of index computation in Appendices.

2. 3D SUPERCONFORMAL INDEX

In this section, we recall basic facts related to the superconformal index technique. The presentation closely follows that in [2, 3, 13].

The concept of the superconformal index was first introduced for four-dimensional theories in [14, 15] and later extended to other dimensions. The superconformal index of three-dimensional $\mathcal{N} = 2$ superconformal field theory is a twisted partition function defined on $S^2 \times S^1$ as follows [13, 16, 17]

$$I(q, \{t_i\}) = \mathrm{Tr}\left[(-1)^{\mathrm{F}} e^{-\beta\{Q, Q^\dagger\}} q^{\frac{1}{2}(\Delta + j_3)} \prod_i t_i^{F_i}\right] , \qquad (2.1)$$

where

- the trace is taken over the Hilbert space of the theory on S^2,
- F plays the role of the fermion number which takes value zero on bosons and one on fermions. In presence of monopoles one needs to refine this number by shifting it by $e \times m$, where e and m are electric charge and magnetic monopole charge, respectively. See [18, 19] for a discussion of this issue.
- Δ is the energy (or conformal dimension via radial quantization), j_3 is the third component of the angular momentum on S^2,
- F_i is the charge of global symmetry with fugacity t_i,
- Q is a certain supersymmetric charge in three-dimensional $\mathcal{N} = 2$ superconformal algebra with quantum numbers $\Delta = \frac{1}{2}$ and $j_3 = -\frac{1}{2}$ and R-charge $R = 1$. The supercharges $Q^\dagger = S$ and Q satisfy the anti-commutation relation[1]

$$\frac{1}{2}\{Q, S\} = \Delta - R - j_3 . \qquad (2.2)$$

Only BPS states with $\Delta - R - j_3 = 0$ contribute to the superconformal index. Consequently, the index is β-independent but depends non-trivially on the fugacities t_i and q. The superconformal index counts the number of BPS states weighted by their quantum numbers.

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[1]The full algebra can be found in many places, see e.g., [20].

The superconformal index can be evaluated by a path integral on $S^2 \times S^1$ via the localization technique [21], leading to the matrix integral [13, 17]

$$I(q, \{t_i\}) = \sum_{m \in \mathbb{Z}^{\mathrm{rank}\,G}} \int \frac{1}{|W_m|} e^{-S_{CS}^{(0)}} e^{\mathrm{i}b_0} q^{\frac{1}{2}\epsilon_0} \prod_{j=1}^{\mathrm{rank}\,F} t_j^{q_{0j}}$$

$$\times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{ind}(z_i^n, t^n, q^n; m)\right] \prod_{i=1}^{\mathrm{rank}\,G} \frac{dz_i}{2\pi \mathrm{i} z_i} . \qquad (2.3)$$

The sum in the formula is to be understood as follows. It is a sum over magnetic fluxes $m = (m_1, \ldots, m_{\mathrm{rank}\,G})$ on the two-sphere with

$$m_i = \frac{1}{2\pi} \int_{S^2} F_i , \qquad (2.4)$$

where m_i parametrizes the GNO charge of the monopole configuration[2], in the examples we consider it runs over the integers. The prefactor $|W_m| = \prod_{i=1}^{k} (\mathrm{rank}\,G_i)!$ is the order of the Weyl group of G which is "broken" by the monopoles into the product $G_1 \times G_2 \times \cdots \times G_k$. For instance, in case of $U(N)$ gauge group $|W_m| = \prod N_k!$.

The term

$$S_{CS}^{(0)} = \frac{\mathrm{i}k}{4\pi} \int \mathrm{tr}_{CS}(A^{(0)} dA^{(0)} - \frac{2\mathrm{i}}{3} A^{(0)} A^{(0)} A^{(0)})$$

$$= 2\mathrm{i}\mathrm{tr}_{CS}(gm) , \qquad (2.5)$$

is the contribution of the Chern–Simons term if the action contains such term and

$$b_0 = -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)| \rho(g) \qquad (2.6)$$

is the 1-loop correction to the Chern–Simons term. The tr_{CS} stands for the trace containing the Chern–Simons levels, k is the Chern–Simons level and $\sum_\Phi$ and $\sum_{\rho \in R_\Phi}$ are sums over all chiral multiplets and all weights of the representation R_Φ, respectively. We give the contribution (2.5) for completeness; in all our examples we will consider theories without the Chern–Simons term.

The term q_{0j} in (2.3) is the zero-point contribution to the energy,

$$q_{0j}(m) = -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)| f_j(\Phi) . \qquad (2.7)$$

In addition, there is the contribution from the Casimir energy of the vacuum state on the two-sphere with magnetic flux m,

$$\epsilon_0(m) = \frac{1}{2} \sum_{\Phi} (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(m)| - \frac{1}{2} \sum_{\alpha \in G} |\alpha(m)| , \qquad (2.8)$$

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[2]The operators creating magnetic fluxes are not completely understood yet, for details, see e.g. [17].

where $\sum_{\alpha \in G}$ represents summation over all roots of G, Δ_Φ is the superconformal R-charge of the chiral multiplet Φ and α are the positive roots of the gauge group G.

One can calculate the single letter index

$$\mathrm{ind}(z,t,q;m) = -\sum_{\alpha \in G} e^{\mathrm{i}\alpha(g)} q^{\frac{1}{2}|\alpha(m)|} \tag{2.9}$$

$$+ \sum_\Phi \sum_{\rho \in R_\Phi} \left[e^{\mathrm{i}\rho(g)} \prod_j t_j^{f_j} \frac{q^{\frac{1}{2}|\rho(m)| + \frac{1}{2}\Delta_\Phi}}{1-q} - e^{-\mathrm{i}\rho(g)} \prod_j t_j^{-f_j} \frac{q^{\frac{1}{2}|\rho(m)| + 1 - \frac{1}{2}\Delta_\Phi}}{1-q} \right] \; .$$

Here, the first term is the contribution of the vector multiplets and the second line is the contribution of matter multiplets, labeled by Φ, where j runs over the rank of the flavor symmetry group. Given the single letter index it is a combinatorial problem [22, 23] to compute the full multi-letter index. The result is given by the so-called "plethystic" exponential

$$\exp\left(\sum_{n=1}^\infty \frac{1}{n} \mathrm{ind}(z^n, t^n, q^n; m) \right) . \tag{2.10}$$

For instance, let us consider the $\mathcal{N} = 2$ theory with $U(N)$ gauge group. Then, the chiral multiplet Φ with R-charge r in the fundamental representation of the gauge group contributes to the single-letter index as

$$\sum_{i=1}^N \left[zt^{f(\Phi)} \frac{q^{\frac{r}{2} + \frac{|m_i|}{2}}}{1-q} - z^{-1} t^{-f(\Phi)} \frac{q^{1 - \frac{r}{2} + \frac{|m_i|}{2}}}{1-q} \right] \; . \tag{2.11}$$

After the "plethystic" exponential one obtains the contribution of the chiral multiplet to the index

$$\prod_{i=1}^N \frac{(q^{1 - \frac{r}{2} + \frac{|m_i|}{2}} t^{-f(\Phi)} z_i^{-1}; q)_\infty}{(q^{\frac{r}{2} + \frac{|m_i|}{2}} t^{f(\Phi)} z_i; q)_\infty} \; . \tag{2.12}$$

Similarly the contribution of the vector multiplet to the single-letter index is

$$- \sum_{i,j=1,\ldots,N, \, i \neq j} q^{\frac{1}{4}|m_i - m_j|} \frac{z_i}{z_j} \; , \tag{2.13}$$

and the multi-letter index gets the form

$$q^{-\sum_{1 \leq i < j \leq N} \frac{|m_i - m_j|}{2}} \prod_{i,j=1,\ldots,N, \, i \neq j} \left(1 - \frac{z_i}{z_j} q^{\frac{|m_i - m_j|}{2}} \right) \; . \tag{2.14}$$

Our main interest is the so-called generalized superconformal index which includes integer parameters corresponding to global symmetries. In [3] Kapustin and Willett pointed out that one can generalize the superconformal index of three-dimensional supersymmetric gauge theory by considering the theory in a non-trivial background gauge field coupled to the global symmetries of the theory. As a result the superconformal index includes new discrete parameters for global symmetries; we do not sum over these parameters. In case

of the generalized superconformal index the contribution (2.12) has the form

$$\prod_{i=1}^{N} \frac{(q^{1-\frac{r}{2}+\frac{|m_i+f(\Phi)n_\Phi|}{2}}t^{-f(\Phi)}z_i^{-1};q)_\infty}{(q^{\frac{r}{2}+\frac{|m_i+f(\Phi)n_\Phi|}{2}}t^{f(\Phi)}z_i;q)_\infty} \ , \tag{2.15}$$

where the parameters n_Φ are new discrete variables. It is convenient to express the index as a product of contributions from chiral and vector multiplets

$$I(q,\{t_a\},\{n_a\}) = \sum_{m_1,\ldots,m_{\mathrm{rank}(G)}} \frac{1}{|W_m|} \oint \prod_{j=1}^{\mathrm{rank}\,G} \frac{dz_j}{2\pi\mathrm{i}z_j} Z_{\mathrm{gauge}}(z_j,m_j;q)$$

$$\times \prod_\Phi Z_\Phi(z_j,m_j;t_a,n_a;q) \ , \tag{2.16}$$

where

$$Z_{\mathrm{gauge}}(z_j,m_j;q) = \prod_{\alpha\in\mathrm{ad}(G)} q^{-\frac{1}{2}|\alpha(m)|}\left(1-e^{\alpha(g)}q^{\frac{|\alpha(m)|}{2}}\right) \tag{2.17}$$

and

$$Z_\Phi = \prod_{\rho\in R_\Phi}\left(q^{\frac{1-r_\phi}{2}}\prod_j e^{-\mathrm{i}\rho(g)}t(\Phi)^{-f(\Phi)}\right)^{\frac{1}{2}|\rho(m)+f(\Phi)n(\Phi)|}$$

$$\times \frac{(e^{-\mathrm{i}\rho(g)}t(\Phi)^{-f(\Phi)}q^{\frac{1}{2}|\rho(m)+f(\Phi)n(\Phi)|+\frac{1-r_\Phi}{2}};q)_\infty}{(e^{\mathrm{i}\rho(g)}t(\Phi)^{f(\Phi)}q^{\frac{1}{2}|\rho(m)+f(\Phi)n(\Phi)|+\frac{r_\Phi}{2}};q)_\infty} \ . \tag{2.18}$$

Here $\mathrm{ad}(G)$ stands for the adjoint representation of the gauge group G. Note that we do not write the contribution of the Chern–Simons term in (2.16), since as we mentioned before we consider theories without this term.

It is worth to mention that the three-dimensional superconformal index can be constructed from the so-called holomorphic blocks [24] due to its factorization property [2, 25–29], i.e. the superconformal index can be expressed in terms of two identical $3d$ holomorphic blocks $\mathcal{B}(x;q)$ as[3]

$$\sum_c \mathcal{B}_c(x;q)\mathcal{B}_c(\tilde{x};\tilde{q}) \ . \tag{2.19}$$

It is possible to obtain the factorized superconformal index directly from the localization technique via the so-called Higgs branch localization [30, 31].

3. Integral identities from 3d dualities

In [32] Seiberg found that there exist pairs of different four-dimensional $\mathcal{N}=1$ supersymmetric gauge theories which describe the same physics in the infrared limit. This is called supersymmetric duality. Since its proposal a large number of dualities in various dimensions have been found.

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[3]Geometrically it means that the index can be obtained by gluing two solid tori. In this context $B_c(x;q)$ are partition functions on solid tori.

In this section, we study three-dimensional $\mathcal{N} = 2$ supersymmetric dualities [12, 33–35] and demonstrate the matching of the superconformal index for dual theories. The superconformal index technique is one of the main tools for establishing and checking supersymmetric dualities.

In this work, we consider only confining theories, i.e. theories whose infrared limit can be described in terms of gauge invariant composites (mesons and baryons) and without dual quarks. There are definitely more confining supersymmetric theories in three dimensions (for recent discussions, see [36, 37]). We restrict our attention to samples of theories with $U(1)$ (supersymmetric quantum electrodynamics) and $SU(2)$ (supersymmetric quantum chromodynamics) gauge symmetry. We also limit ourselves to the cases of vanishing Chern-Simons term; however, one can add such a term to the action of the theories considered in the paper.

Note that similar results for $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions were intensively studied in [1, 38, 39]. All $3d$ dualities considered in the next section can be obtained via dimensional reduction from $4d$ dualities. However, obtaining the right duality in three dimensions is more tricky (for details see [19, 40]). The main issue is that the reduction procedure and renormalization group flow from ultraviolet to infrared do not commute with each other. This happens because of an anomalous $U(1)$ symmetry in $4d$, which one needs to break in $3d$ theory. This can be done by adding a monopole operator to the $3d$ Lagrangians. To be more precise we need to add the effective superpotential $W = \eta X$ to the Lagrangian of electric theory and $W = \tilde{\eta}\tilde{X}$ to the magnetic theory (dual theory), where X is a monopole operator and η is the $4d$ instanton factor.

In our examples we give only the necessary input to compute the superconformal index and do not discuss other aspects of dual theories. As for many other dualities in physics, systematic proofs of supersymmetric dualities are absent and the superconformal index computations do not constitute a proof of the duality. There are other important arguments for three-dimensional supersymmetric dualities, i.e. study of superpotentials for interactions among chiral multiplets [19], brane construction (see e.g, [35, 41]), contact terms (see e.g., [42, 43]) and other powerful methods very much in the spirit of the superconformal index such as study of sphere partition functions [44, 45], ellipsoid partition functions [40, 46, 47], lens partition functions [48, 49], etc.

The 't Hooft anomaly matching conditions which played a central role in checking Seiberg dualities for $\mathcal{N} = 1$ supersymmetric gauge theories become useless in three dimensions since, unlike four-dimensional gauge theories, in three dimensions there are no chiral anomalies. In three dimensions it is possible to have a classical Chern-Simons term which breaks parity. One can then use the matching condition for the parity anomaly; however, conditions for discrete anomalies are weaker than those for continuous anomalies.

In what follows, we omit the R-charges for chiral multiplets, since the superconformal indices of dual theories match for arbitrary assignment of the R-charge [13]. The correct R-charges for matter fields in the infrared fixed points can be obtained by the so-called Z-extremization procedure [45].

As a final remark, let us comment that the matching of superconformal indices for dual pairs were studied mainly by expanding in terms of fugacities [13, 50–52] and only in a few works [2–4] authors give rigorous proofs of the index identities.

Below we give explicit expressions of generalized superconformal indices for some theories. Equality of indices for dual theories leads to integral evaluations, which will be proved rigorously in Section 4.

Example 1.

We first consider a **Theory A** and its low-energy description **Theory B** which can be described purely in terms of composite gauge singlets.

- **Theory A**: Supersymmetric Quantum Chromodynamics with $SU(2)$ gauge group and with $SU(6)$ flavor group, chiral multiplets in the fundamental representation of the gauge group and the flavor group, a vector multiplet in the adjoint representation of the gauge group. Note that in case of $SU(2)$ gauge theories the fundamental and antifundamental representations are equivalent, therefore we have $SU(6)$ flavor group rather than $SU(3) \times SU(3) \times U(1)$.

- **Theory B**: no gauge symmetry, fifteen chiral multiplets in the totally antisymmetric tensor representation of the flavor group.

This duality was considered in [53] where the authors presented the sphere partition functions for dual theories. It is analogous to the four-dimensional duality for similar theories [1] and can be obtained by dimensional reduction.

Using the group-theoretical data (see Appendix B) it is straightforward to compute explicitly the generalized superconformal indices, and due to the supersymmetric duality we find the basic hypergeometric integral identity

$$\sum_{m \in \mathbb{Z}} \oint \frac{dz}{4\pi \mathrm{i} z} q^{-|m|} (1 - q^{|m|} z^2)(1 - q^{|m|} z^{-2}) \, (-q^{\frac{1}{2}})^{\sum_{i=1}^{6}(\frac{|n_i+m|}{2} + \frac{|n_i-m|}{2})}$$

$$\times z^{-\sum_{i=1}^{6}(\frac{|n_i+m|}{2} - \frac{|n_i-m|}{2})} \prod_{j=1}^{6} a_j^{-\frac{|n_j+m|}{2} - \frac{|n_j-m|}{2}} \frac{\left(q^{1 + \frac{|n_j+m|}{2}}/a_j z, q^{1 + \frac{|n_j-m|}{2}} z/a_j; q\right)_\infty}{\left(q^{\frac{|n_j+m|}{2}} a_j z, q^{\frac{|n_j-m|}{2}} a_j/z; q\right)_\infty}$$

$$= (-q^{\frac{1}{2}})^{\sum_{1 \leq j < k \leq 6} \frac{|n_j + n_k|}{2}} \prod_{1 \leq j < k \leq 6} (a_j a_k)^{-\frac{|n_j + n_k|}{2}} \frac{\left(q^{1 + \frac{|n_j + n_k|}{2}} a_j^{-1} a_k^{-1}; q\right)_\infty}{\left(q^{\frac{|n_j + n_k|}{2}} a_j a_k; q\right)_\infty} \qquad (3.1)$$

with the balancing conditions

$$\prod_{i=1}^{6} a_i = q \ \text{ and } \ \sum_{i=1}^{6} n_i = 0 \, . \qquad (3.2)$$

This identity describes confinement without breaking of the "chiral symmetry". The left side of the expression (3.1) includes the contributions of twelve chirals and a vector multiplet, while the right hand side contains the contribution of fifteen chirals. From the fact that all physical degrees of freedom of Theory B are gauge invariant there is no integration on the right hand side.

It is worth mentioning that the duality considered in the example is a special case of the duality claimed in [19], where the theory A is the three-dimensional $SP(2N)$ SQCD with $2N_f$ fundamentals and theory B is the $SP(2N_f - 2N - 4)$ theory with $2N_f$ fundamentals. Such duality is qualitatively similar to $SU(N)$ duality with matter in the fundamental representation of the gauge group. In case of $N = 2$ one can consider the theory A as $SU(2)$ gauge theory since $SP(2) \simeq SU(2)$.

Note that the balancing conditions are imposed by the effective superpotential and the theories described above are dual only in the presence of certain superpotentials. We refer the interested reader to [19] for more details related to the study of superpotentials for three-dimensional dualities.

In (3.1) we used the absolute values of monopole charges as in the definition of the superconformal index. It is possible to eliminate all absolute values using the elementary identity [18]

$$\frac{(q^{1 + |m|/2}/z; q)_\infty}{(q^{|m|/2} z; q)_\infty} = (-q^{-\frac{1}{2}} z)^{\frac{|m| - m}{2}} \frac{(q^{1 + m/2}/z; q)_\infty}{(q^{m/2} z; q)_\infty}. \qquad (3.3)$$

After such simplification, (3.1) takes the form

$$\sum_{m=\in\mathbb{Z}}^{\infty} \oint \prod_{j=1}^{6} \frac{(q^{1+(m+n_j)/2}/a_j z, q^{1+(n_j-m)/2}z/a_j; q)_\infty}{(q^{(n_j+m)/2}a_j z, q^{(n_j-m)/2}a_j/z; q)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{6m}} \frac{dz}{4\pi \mathrm{i}z}$$

$$= \frac{1}{\prod_{j=1}^{6} a_j^{n_j}} \prod_{1\le j<k\le 6} \frac{(q^{1+(n_j+n_k)/2}/a_j a_k; q)_\infty}{(q^{(n_j+n_k)/2}a_j a_k; q)_\infty} \ .$$

After replacing $(a_j, n_j) \mapsto (a_j q^{N_j/2}, N_j)$, this is (1.2). We give a rigorous mathematical proof of this identity in Theorem 4.1.

The most intriguing physical interpretation of the formula (3.1) stems from the role it plays as a star-triangle relation [7,8] for a certain two-dimensional statistical model.

The integral identity (3.1) can be obtained by reduction [8,54,55] from the similar identity for four-dimensional lens indices. In [8] such reduction was made in the context of integrable statistical models.

The $q \to 1$ limit of (3.1) was discussed in [8]. This limit also has an interpretation in terms of exactly solvable statistical models [56]. From the viewpoint of supersymmetric dualities such reduction [55,57] gives the equality of the sphere partition functions of dual two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theories.

Example 2.

Our second example is again a supersymmetric quantum chromodynamics with a weakly coupled magnetic dual.

- **Theory A**: Supersymmetric Quantum Chromodynamics with $SU(2)$ gauge group and four flavors, chiral multiplets in the fundamental representation of the gauge group and the flavor group, the vector multiplet in the adjoint representation of the gauge group.

- **Theory B**: no gauge degrees of freedom, with six mesons and a singlet chiral field.

According to the supersymmetric duality we have the following integral identity for the generalized superconformal indices:

$$\sum_{m\in\mathbb{Z}} \oint \frac{dz}{4\pi i z} q^{-|m|}(1-q^{|m|}z^2)(1-q^{|m|}z^{-2}) \; (-q^{\frac{1}{2}})^{\sum_{i=1}^{4}(\frac{|n_i+m|}{2}+\frac{|n_i-m|}{2}-n_i)}$$

$$\times \; z^{-\sum_{i=1}^{4}(\frac{|n_i+m|}{2}-\frac{|n_i-m|}{2})} \prod_{j=1}^{4} a_j^{-\frac{|n_j+m|}{2}-\frac{|n_j-m|}{2}+n_j} \frac{(q^{1+\frac{|n_j+m|}{2}}/a_j z, q^{1+\frac{|n_j-m|}{2}} z/a_j; q)_\infty}{(q^{\frac{|n_j+m|}{2}} a_j z, q^{\frac{|n_j-m|}{2}} a_j/z; q)_\infty}$$

$$= (-q^{\frac{1}{2}})^{\sum_{1\le j<k\le 4}\frac{|n_j+n_k|}{2}-\sum_{i=1}^{4}n_i-\frac{|\sum_{i=1}^{4}n_i|}{2}}(a_1 a_2 a_3 a_4)^{\frac{|\sum_{i=1}^{4}n_i|-\sum_{i=1}^{4}n_i}{2}}$$

$$\times \frac{(q^{\frac{|\sum_{i=1}^{4}n_i|}{2}}a_1 a_2 a_3 a_4)_\infty}{(q^{1+\frac{|\sum_{i=1}^{4}n_i|}{2}}/a_1 a_2 a_3 a_4)_\infty} \prod_{1\le j<k\le 4} (a_j a_k)^{-\frac{|n_j+n_k|+(n_j+n_k)}{2}} \frac{(q^{1+\frac{|n_j+n_k|}{2}}/a_j a_k; q)_\infty}{(q^{\frac{|n_j+n_k|}{2}} a_j a_k; q)_\infty}. \qquad (3.4)$$

The ordinary index of the theory A was considered in [58] in the context of global symmetry enhancement. It was shown that the superconformal index of the theory has an extended $SO(10)$ flavor symmetry when coupled to $4d$ multiplets with specific boundary conditions.

Note that one can deform dual theories from Example 1 by adding mass terms for some of the quarks. After integrating out one flavor (massive modes) the theory with the remaining four flavors confines with "chiral symmetry breaking" [7] if we keep a certain superpotential for the theory giving the balancing conditions similar to (3.2). Here the theory A has no superpotential and therefore we obtain the duality (3.4).

Eliminating the absolute values as before, (3.4) can be expressed as

$$\sum_{m\in\mathbb{Z}} \oint \frac{dz}{4\pi i z} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{4m}} \prod_{j=1}^{4} \frac{(q^{1+\frac{n_j+m}{2}}/a_j z, q^{1+\frac{n_j-m}{2}} z/a_j; q)_\infty}{(q^{\frac{n_j+m}{2}} a_j z, q^{\frac{n_j-m}{2}} a_j/z; q)_\infty}$$

$$= \frac{(q^{\frac{\sum_{i=1}^{4}n_i}{2}}a_1 a_2 a_3 a_4)_\infty}{(q^{1+\frac{\sum_{i=1}^{4}n_i}{2}}/a_1 a_2 a_3 a_4)_\infty} \prod_{1\le j<k\le 4} \frac{(q^{1+\frac{n_j+n_k}{2}}/a_j a_k; q)_\infty}{(q^{\frac{n_j+n_k}{2}} a_j a_k; q)_\infty}. \qquad (3.5)$$

This can be recognized as a special case of Proposition 4.2. More precisely, Proposition 4.2 states that (3.5) holds even with the integers n_j replaced by generic complex parameters.

In contrast to four dimensions, there exist supersymmetric dualities for abelian gauge theories in three dimensions. For details of such dualities see e.g. [59]. Below we consider two examples of such dualities.

Example 3.

- **Theory A**: $d = 3$ $\mathcal{N} = 2$ supersymmetric electrodynamics with $U(1)$ gauge symmetry and six chiral multiplets, half of them transforming in the fundamental representation of the gauge group and another half transforming in the anti-fundamental representation.

- **Theory B**: no gauge degrees of freedom, nine gauge invariant "mesons" transforming in the fundamental representation of the flavor group.

Supersymmetric duality leads to the following identity for the generalized superconformal indices:

$$
\sum_{m \in \mathbb{Z}} \oint \frac{dz}{2\pi i z} (-q^{\frac{1}{2}})^{\sum_{i=1}^{3}(\frac{|m_i+m|}{2}+\frac{|n_i-m|}{2})} z^{-\sum_{i=1}^{3}(\frac{|m_i+m|}{2}-\frac{|n_i-m|}{2})}
$$

$$
\times \prod_{i=1}^{3} a_i^{-\frac{|m_i+m|}{2}} b_i^{-\frac{|n_i-m|}{2}} \frac{(q^{1+\frac{|m_i+m|}{2}}/a_i z, q^{1+\frac{|n_i-m|}{2}} z/b_i; q)_\infty}{(q^{\frac{|m_i+m|}{2}} a_i z, q^{\frac{|n_i-m|}{2}} b_i/z; q)_\infty}
$$

$$
= (-q^{\frac{1}{2}})^{\sum_{i,j=1}^{3} \frac{|m_i+n_j|}{2}} \prod_{i,j=1}^{3} (a_i b_j)^{-\frac{|m_i+n_j|}{2}} \frac{(q^{1+\frac{|m_i+n_j|}{2}}/a_i b_j; q)_\infty}{(q^{\frac{|m_i+n_j|}{2}} a_i b_j; q)_\infty} , \tag{3.6}
$$

where the fugacities a_i and b_i stand for the flavor symmetry $SU(3) \times SU(3)$, z is the fugacity for the $U(1)$ gauge group and the balancing conditions are

$$
\prod_{i=1}^{3} a_i = \prod_{i=1}^{3} b_i = q^{\frac{1}{2}} \quad \text{and} \quad \sum_{i=1}^{3} n_i = \sum_{i=1}^{3} m_i = 0 . \tag{3.7}
$$

Eliminating the absolute values, this identity takes the form

$$
\sum_{m=-\infty}^{\infty} \oint \prod_{i=1}^{3} \frac{(q^{1+(m+m_i)/2}/a_i z, q^{1+(n_i-m)/2} z/b_i; q)_\infty}{(q^{(m+m_i)/2} a_i z, q^{(n_i-m)/2} b_j/z; q)_\infty} \frac{1}{z^{3m}} \frac{dz}{2\pi i z}
$$

$$
= \frac{1}{\prod_{i=1}^{3} a_i^{m_i} b_i^{n_i}} \prod_{i,j=1}^{3} \frac{(q^{1+(m_i+n_j)/2}/a_i b_j; q)_\infty}{(q^{(m_i+n_j)/2} a_i b_j; q)_\infty} .
$$

After the change of variables $z \mapsto -z$, one may check that this is equivalent to Theorem 4.3 below. This identity was first proved in [4] in the special case of ordinary superconformal indices[4], that is, $m_i \equiv n_i \equiv 0$. The general case was presented without proof in [5] .

The expression (3.6) can be written as an integral pentagon identity. Following [4], we introduce the function

$$
\mathcal{B}_m[a, n; b, m] = (-q^{\frac{1}{2}})^{\frac{|n|}{2}+\frac{|m|}{2}-\frac{|n+m|}{2}} a^{-\frac{|n|}{2}} b^{-\frac{|m|}{2}} (ab)^{\frac{|n+m|}{2}}
$$

$$
\times \frac{(q^{1+\frac{|n|}{2}} a^{-1}, q^{1+\frac{|m|}{2}} b^{-1}, q^{\frac{|n+m|}{2}} ab; q)_\infty}{(q^{\frac{|n|}{2}} a, q^{\frac{|m|}{2}} b, q^{1+\frac{|n+m|}{2}} (ab)^{-1}; q)_\infty} , \tag{3.8}
$$

---

[4]Note that the identity of sphere partition functions for this duality was presented in [60, 61].

and rewrite the equality (3.6) in terms of this function. We obtain the following integral pentagon identity in terms of $\mathcal{B}$ functions:

$$\sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi \mathrm{i} z} \prod_{i=1}^{3} \mathcal{B}[a_i z, n_i + m; b_i z^{-1}, m_i - m]$$
$$= \mathcal{B}[a_1 b_2, n_1 + m_2; a_3 b_1; n_3 + m_1] \, \mathcal{B}[a_2 b_1, n_2 + m_1; a_3 b_2, n_3 + m_2] \,, \qquad (3.9)$$

with the balancing conditions (3.7).

The integral identity (3.9) is interesting from the following point of view. There is a recently proposed relation called $3d/3d$ correspondence between $3d$ $\mathcal{N} = 2$ supersymmetric gauge theories and 3-manifolds [62, 63] (see also [18, 64] and earlier works [65, 66]) in similar spirit as the AGT correspondence [67]. This correspondence translates the ideal triangulation of the 3-manifold into mirror symmetry for three-dimensional supersymmetric theories. The independence of the corresponding 3-manifold invariant on the choice of triangulation corresponds to the equality of superconformal indices of mirror dual theories [18]. In this context the identity (3.9) encodes a 3–2 Pachner move for 3-manifolds.

Example 4.

Let us consider another example of abelian duality, namely the well-known XYZ/SQED mirror symmetry [11, 12, 35].

- **Theory A:** $\mathcal{N} = 2$ supersymmetric quantum electrodynamics, with a single $U(1)$ vector multiplet and two chiral multiplets charged oppositely under the gauge group.

- **Theory B:** free Wess–Zumino theory with three chiral multiplets. This theory is often is called the XYZ model in the literature.

In this example we wish to turn on the contribution to the generalized superconformal index of the topological symmetry $U(1)_J$, which is not explicit in the Lagrangian. This hidden symmetry is generated by the current

$$J^\mu \;=\; \varepsilon^{\mu\nu\rho} F_{\nu\rho} \,. \qquad (3.10)$$

The current J^μ is topologically conserved[5] due to the Bianchi identity.

In this case we have a special duality called mirror symmetry which exchanges the Coulomb branch of a theory with the Higgs branch of its mirror dual and vice versa. The duality

---

[5]The corresponding charge is carried by the Abrikosov-Nielsen-Olesen vortices in the Higgs branch of $\mathcal{N} = 2$ theory.

implies the identity

$$
\sum_{s\in\mathbb{Z}} \oint \frac{dz}{2\pi i z}(-1)^{s+m+\frac{|s+m|}{2}+\frac{|s-m|}{2}} z^n w^s (q^{\frac{1}{4}}z\alpha^{-1})^{\frac{|s-m|}{2}}(q^{\frac{1}{4}}z^{-1}\alpha^{-1})^{\frac{|s+m|}{2}}
$$

$$
\times \frac{(z^{\pm}\alpha^{-1}q^{\frac{|s\mp m|}{2}+\frac{3}{4}};q)_\infty}{(z^{\pm}\alpha q^{\frac{|s\pm m|}{2}+\frac{1}{4}};q)_\infty}
$$

$$
=(-1)^{n+m+\frac{|n+m|}{2}+\frac{|n-m|}{2}}(q^{\frac{1}{4}}\alpha w)^{\frac{|m-n|}{2}}(q^{\frac{1}{4}}\alpha w^{-1})^{\frac{|m+n|}{2}}\alpha^{-2|m|}
$$

$$
\times \frac{(\alpha w^{\pm}q^{\frac{|m\pm n|}{2}+\frac{3}{4}},\alpha^{-2}q^{|m|+\frac{1}{2}};q)_\infty}{(\alpha^{-1}w^{\pm}q^{\frac{|m\mp n|}{2}+\frac{1}{4}},\alpha^2 q^{|m|+\frac{1}{2}};q)_\infty}\ , \tag{3.11}
$$

where the fugacity α and the monopole charge m denote the parameters for the axial $U(1)_A$ symmetry, ω and n denote the parameters for the topological $U(1)_J$ symmetry and the discrete parameter s stands for the magnetic charge corresponding to the $U(1)$ gauge group. The factors containing $\pm$ should be interpreted as the product over both choices; for instance,

$$
(z^{\pm}\alpha^{-1}q^{\frac{|s\mp m|}{2}+\frac{3}{4}};q)_\infty = (z\alpha^{-1}q^{\frac{|s-m|}{2}+\frac{3}{4}};q)_\infty (z^{-1}\alpha^{-1}q^{\frac{|s+m|}{2}+\frac{3}{4}};q)_\infty.
$$

Here, we explicitly write the R-charges of chiral multiplets. Due to the permutation symmetry of the superpotential $W = \tilde{q}Sq$ for the theory B, where $q, \tilde{q}, S$ are three chiral multiplets of the theory, one can fix the R-charges[6].

The case $m = n = 0$ of (3.11) was presented in [2, 13] and proven in [2]. The general case was presented, with a slight mistake[7], in [3], where a proof was given for the special case $m = 0$. In Section 4 we give an analytic proof of the general case. More precisely, eliminating the absolute values in (3.11) gives

$$
\sum_{s\in\mathbb{Z}} \oint \frac{dz}{2\pi i z}(-w)^s z^{n-s}\frac{(z^{\pm}\alpha^{-1}q^{\frac{m\mp s}{2}+\frac{3}{4}};q)_\infty}{(z^{\pm}\alpha q^{\frac{m\pm s}{2}+\frac{1}{4}};q)_\infty}
$$

$$
=(-w)^n\frac{(\alpha w^{\pm}q^{\frac{m\pm n}{2}+\frac{3}{4}},\alpha^{-2}q^{m+\frac{1}{2}};q)_\infty}{(\alpha^{-1}w^{\pm}q^{\frac{m\pm n}{2}+\frac{1}{4}},\alpha^2 q^{m+\frac{1}{2}};q)_\infty}\ ,
$$

which can be recognized as the special case $a = b = q^{\frac{1}{4}-\frac{m}{2}}\alpha$, $c = d = q^{\frac{1}{4}+\frac{m}{2}}\alpha$ of Proposition 4.4.

---

[6]In the infrared limit the superpotential W must have the R-charge 2, then the R-charge of chiral multiplets of theory B must be $\frac{2}{3}$.

[7] We have an additional phase factor $(-1)^{s+m+\frac{|s+m|}{2}+\frac{|s-m|}{2}}$, which is due to the definition of the fermion number operator F in the definition of the superconformal index [18] (see also [19, 29]). In fact, in general the superconformal indices match for dual theories in presence of this corrected phase factor [19]. Without the phase factor the identity presented by Kapustin and Willett [3] is incorrect. It is actually a good example where the naive choice of the fermion number as $2J_3$ does not work.

The identity (3.11) and related identities can also be written as pentagon identities. In fact, introducing the tetrahedron index [18, 63]

$$\mathcal{I}_q[m, z] = \frac{(q^{1-\frac{m}{2}}/z; q)_\infty}{(q^{-\frac{m}{2}}z; q)_\infty},$$

it follows from Proposition 4.4 that

$$\sum_{s\in\mathbb{Z}} \oint (-w)^s z^{N-s} \mathcal{I}_q[m-s; q^{1/4}\alpha z] \mathcal{I}_q[n+s; q^{1/4}\beta/z] \frac{dz}{2\pi i z}$$

$$= (-w)^N \mathcal{I}_q[m+n; q^{\frac{1}{2}}\alpha\beta] \mathcal{I}_q[n+N; q^{1/4}w/\beta] \mathcal{I}_q[m-N; q^{1/4}/\alpha w].$$

Special cases with $m = n = N = 0$ and $m = n$ (corresponding to (3.11)) were presented earlier in [4], [5], respectively.

One can also consider this duality as a mirror symmetry between $\mathcal{N} = 4$ supersymmetric electrodynamics with a single flavor and its dual theory with a free hypermultiplet. Then we obtain instead of (3.11) the mathematically equivalent identity

$$\alpha^{2|m|} \frac{(\alpha^2 q^{|m|+\frac{1}{2}}; q)_\infty}{(\alpha^{-2}q^{|m|+\frac{1}{2}}; q)_\infty} \sum_{s\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} (-1)^{s+m+\frac{|s+m|}{2}+\frac{|s-m|}{2}}$$

$$\times z^n w^s (q^{\frac{1}{4}}z\alpha^{-1})^{\frac{|s-m|}{2}} (q^{\frac{1}{4}}z^{-1}\alpha^{-1})^{\frac{|s+m|}{2}} \frac{(z^\pm \alpha^{-1} q^{\frac{|s\mp m|}{2}+\frac{3}{4}}; q)_\infty}{(z^\pm \alpha q^{\frac{|s\pm m|}{2}+\frac{1}{4}}; q)_\infty}$$

$$= (-1)^{n+m+\frac{|n+m|}{2}+\frac{|n-m|}{2}} (q^{\frac{1}{4}}\alpha w)^{\frac{|m-n|}{2}} (q^{\frac{1}{4}}\alpha w^{-1})^{\frac{|m+n|}{2}} \frac{(\alpha w^\pm q^{\frac{|m\pm n|}{2}+\frac{3}{4}}; q)_\infty}{(\alpha^{-1}w^\pm q^{\frac{|m\mp n|}{2}+\frac{1}{4}}; q)_\infty} . \qquad (3.12)$$

4. MATHEMATICAL PROOFS OF IDENTITIES

In this section we will use the standard notation of [9]. We will assume that $|q| < 1$. We will also write

$$\theta(z; q) = (z, q/z; q)_\infty .$$

This theta function satisfies the quasi-periodicity

$$\theta(zq^N; q) = \frac{(-1)^N}{q^{\binom{N}{2}}z^N} \theta(z; q) , \qquad N \in \mathbb{Z} . \qquad (4.1)$$

We will formulate four fundamental identities, which evaluate a combination of a basic hypergeometric integral and sum. In each case, we assume that the parameters are generic, so that the poles of the integrand split naturally into geometric sequences converging to 0 and to ∞. The integration is over a positively oriented contour separating these two types of poles.

To prove the first identity, we use the Nasrallah–Rahman integral and the nonterminating Jackson summation, which are top level results for basic hypergeometric integral evaluations and summations, respectively. Consequently, we expect that Theorem 4.1 is a top level result for evaluations of the type considered here, with combined integration and summation.

Theorem 4.1. *Let a_j be generic numbers and N_j integers satisfying $a_1 \cdots a_6 = q$ and $N_1 + \cdots + N_6 = 0$. Then,*

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{6} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_\infty}{(q^{N_j+m/2} a_j z, q^{N_j-m/2} a_j/z; q)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{6m}} \frac{dz}{2\pi i z}$$

$$= \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} a_j^{N_j}} \prod_{1 \leq j < k \leq 6} \frac{(q/a_j a_k; q)_\infty}{(a_j a_k q^{N_j+N_k}; q)_\infty} . \quad (4.2)$$

Proof. Let L denote the left-hand side of (4.2). Note that the poles of the integrand are situated at fixed values of $z q^{m/2}$. Thus, we may replace z by $z q^{-m/2}$ and interchange the sum and the integral. This gives

$$L = \oint \prod_{j=1}^{6} \frac{(q z^\pm/a_j; q)_\infty}{(q^{N_j} a_j z^\pm; q)_\infty} (1-z^2)(1-z^{-2})$$

$$\times {}_8\psi_8 \left(\begin{matrix} q/z, -q/z, a_1/z, \ldots, a_6/z \\ 1/z, -1/z, q/a_1 z, \ldots, q/a_6 z \end{matrix}; q, q \right) \frac{dz}{2\pi i z} . \quad (4.3)$$

By [9, Eq. (III.38)], we may write

$$\prod_{j=1}^{6} (q z^\pm/a_j; q)_\infty (1-z^2)(1-z^{-2}) \, {}_8\psi_8 \left(\begin{matrix} q/z, -q/z, a_1/z, \ldots, a_6/z \\ 1/z, -1/z, q/a_1 z, \ldots, q/a_6 z \end{matrix}; q, q \right)$$

$$= \frac{(q; q)_\infty \prod_{j=1}^{4} (q a_5^\pm/a_j; q)_\infty \theta(a_6 z^\pm; q)(z^{\pm 2}; q)_\infty}{(q a_5^2, a_6 a_5^\pm; q)_\infty}$$

$$\times {}_8W_7(a_5^2; a_5 a_1, a_5 a_2, a_5 a_3, a_5 a_4, a_5 a_6; q, q) + \text{idem}(a_5; a_6) , \quad (4.4)$$

where the second term means that the first term is repeated with a_5 and a_6 interchanged. Using (4.1) to write

$$\theta(a_6 z^\pm; q)_\infty = q^{2\binom{N_6}{2}} a_6^{2N_6} \theta(a_6 q^{N_6} z^\pm; q)_\infty ,$$

this leads to

$$L = q^{2\binom{N_6}{2}} a_6^{2N_6} \frac{(q; q)_\infty \prod_{j=1}^{4} (q a_5^\pm/a_j; q)_\infty}{(q a_5^2, a_6 a_5^\pm; q)_\infty} \, {}_8W_7(a_5^2; a_5 a_1, a_5 a_2, a_5 a_3, a_5 a_4, a_5 a_6; q, q)$$

$$\times \oint \frac{(z^{\pm 2}, q^{1-N_6} a_6^{-1} z^\pm; q)_\infty}{\prod_{j=1}^{5} (q^{N_j} a_j z^\pm; q)_\infty} \frac{dz}{2\pi i z} + \text{idem} \left((a_5, N_5); (a_6, N_6) \right) .$$

Applying the Nasrallah–Rahman identity [9, Eq. (6.4.1)]

$$\oint \frac{(z^{\pm 2}, Bz^{\pm}; q)_\infty}{\prod_{j=1}^{5}(b_j z^{\pm}; q)_\infty} \frac{dz}{2\pi i z} = \frac{2\prod_{j=1}^{5}(B/b_j; q)_\infty}{(q; q)_\infty \prod_{1\le j<k\le 5}(b_j b_k; q)_\infty}, \qquad B = b_1 \cdots b_5,$$

we conclude that

$$L = 2q^{2\binom{N_6}{2}} a_6^{2N_6} \frac{\prod_{j=1}^{4}(qa_5^{\pm}/a_j; q)_\infty \prod_{j=1}^{5}(q^{1-N_6-N_j}/a_j a_6; q)_\infty}{(qa_5^2, a_6 a_5^{\pm}; q)_\infty \prod_{1\le j<k\le 5}(q^{N_j+N_k}a_j a_k; q)_\infty}$$

$$\times\ _8W_7(a_5^2; a_5 a_1, a_5 a_2, a_5 a_3, a_5 a_4, a_5 a_6; q, q) + \text{idem}((a_5, N_5); (a_6, N_6)) .$$

By the non-terminating Jackson summation [9, Eq. (II.25)], this can be simplified to the right-hand side of (4.2). ∎

If one formally replaces 6 by 4 in Theorem 4.1, it is possible to replace the discrete parameters N_j by generic complex numbers. The proof of the corresponding identity is in fact very easy.

Proposition 4.2. *For a_j and b_j generic,*

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{4} \frac{(q^{1+m/2}/a_j z, q^{1-m/2}z/a_j; q)_\infty}{(q^{m/2}b_j z, q^{-m/2}b_j/z; q)_\infty} \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi i z}$$

$$= \frac{2(b_1 b_2 b_3 b_4; q)_\infty}{(q/a_1 a_2 a_3 a_4; q)_\infty} \prod_{1\le j<k\le 4} \frac{(q/a_j a_k; q)_\infty}{(b_j b_k; q)_\infty} . \quad (4.5)$$

Proof. With L the left-hand side of (4.5), the identity (4.3) is replaced by

$$L = \oint \prod_{j=1}^{4} \frac{(qz^{\pm}/a_j; q)_\infty}{(b_j z^{\pm}; q)_\infty}(1-z^2)(1-z^{-2})$$

$$\times\ _6\psi_6 \left(\begin{matrix} q/z, -q/z, a_1/z, \ldots, a_4/z \\ 1/z, -1/z, q/a_1 z, \ldots, q/a_4 z \end{matrix}; q, \frac{q}{a_1 \cdots a_4} \right) \frac{dz}{2\pi i z} .$$

Applying Bailey's summation [9, Eq. (II.33)] gives

$$L = \frac{(q; q)_\infty \prod_{1\le j<k\le 4}(q/a_j a_k; q)_\infty}{(q/a_1 a_2 a_3 a_4; q)_\infty} \oint \frac{(z^{\pm 2}; q)_\infty}{\prod_{j=1}^{4}(b_j z^{\pm}; q)_\infty} \frac{dz}{2\pi i z},$$

which reduces the result to the Askey–Wilson integral [9, Eq. (6.1.4)]. ∎

The next result was obtained in [4] for $M_j \equiv N_j \equiv 0$ and announced in [5] in general.

Theorem 4.3. *Let a_j, b_j be generic numbers and M_j, N_j integers satisfying $a_1 a_2 a_3 = b_1 b_2 b_3 = q^{1/2}$ and $M_1 + M_2 + M_3 = N_1 + N_2 + N_3 = 0$. Then,*

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{3} \frac{(q^{1+m/2}/a_j z, q^{1-m/2}z/b_j; q)_\infty}{(q^{M_j+m/2}a_j z, q^{N_j-m/2}b_j/z; q)_\infty} \frac{(-1)^m}{z^{3m}} \frac{dz}{2\pi i z}$$

$$= \frac{1}{\prod_{j=1}^{3} q^{\binom{M_j}{2}+\binom{N_j}{2}} a_j^{M_j} b_j^{N_j}} \prod_{j,k=1}^{3} \frac{(q/a_j b_k; q)_\infty}{(a_j b_k q^{M_j+N_k}; q)_\infty} . \quad (4.6)$$

Proof. This can be proved similarly as the special case treated in [4], so we will be very brief. Shrinking the contour of integration to zero, we pick up residues at the poles

$$z = q^{k-\frac{m}{2}+N_j} b_j , \qquad j = 1, 2, 3, \quad k \geq \max(0, m - N_j) .$$

Working out the sum of residues explicitly, the left-hand side of (4.6) can be written

$$L = \frac{(-1)^{N_1}}{q^{\frac{3}{2}N_1^2} b_1^{3N_1}} \frac{(qb_1/b_2, qb_1/b_3; q)_\infty}{(q^{N_2-N_1}b_2/b_1, q^{N_3-N_1}b_3/b_1; q)_\infty} \prod_{j=1}^{3} \frac{(q/a_j b_1; q)_\infty}{(q^{N_1+M_j}a_j b_1; q)_\infty}$$

$$\times {}_3\phi_2 \left(\begin{matrix} q^{M_1+N_1}a_1 b_1, q^{M_2+N_1}a_2 b_1, q^{M_3+N_1}a_3 b_1 \\ q^{1+N_1-N_2}b_1/b_2, q^{1+N_1-N_3}b_1/b_3 \end{matrix} ; q, q \right) {}_3\phi_2 \left(\begin{matrix} a_1 b_1, a_2 b_1, a_3 b_1 \\ qb_1/b_2, qb_1/b_3 \end{matrix} ; q, q \right)$$

$$+ \text{idem} \left((b_1, N_1); (b_2, N_2), (b_3, N_3) \right) . \quad (4.7)$$

Let

$$x_1 = b_1 (qb_1/b_2, qb_1/b_3; q)_\infty \prod_{j=1}^{3} (a_j b_2, a_j b_3; q)_\infty \, {}_3\phi_2 \left(\begin{matrix} a_1 b_1, a_2 b_1, a_3 b_1 \\ qb_1/b_2, qb_1/b_3 \end{matrix} ; q, q \right)$$

and let x_2 and x_3 be defined by the same expression with b_1 interchanged by b_2 and b_3, respectively. Then, by the nonterminating q-Saalschütz summation [9, Eq. (II.24)],

$$x_2 - x_1 = b_2 \theta(b_1/b_2; q) \prod_{j=1}^{3} \theta(a_j b_3; q) . \quad (4.8)$$

Let $\tilde{x}_j$ denote the result of replacing a_j by $a_j q^{M_j}$ and b_j by $b_j q^{N_j}$ in x_j. By (4.1), under the same change of variables, the right hand side of (4.8) is divided by $C = \prod_{j=1}^{3} q^{\binom{M_j}{2}+2\binom{N_j}{2}} a_j^{M_j} b_j^{2N_j}$. Thus, if we define $y_j = C\tilde{x}_j$, then $y_2 - y_1 = x_2 - x_1$. By symmetry, $y_j = x_j + D$, where D is independent of j. It follows that

$$(x_3 - x_2)x_1 y_1 + (x_1 - x_3)x_2 y_2 + (x_2 - x_1)x_3 y_3 = (x_2 - x_1)(x_3 - x_2)(x_3 - x_1) .$$

After simplification, this identity reduces to the desired result. $\qquad \square$

We conclude with the following identity. Note that the parameter t can be removed by scaling $z \mapsto tz$, but it seems useful to keep it.

Proposition 4.4. *For a, b, c, d and t generic parameters and integer N, such that $|q^{\frac{N+1}{2}}a^{-1}| < |t| < |q^{\frac{N-1}{2}}b|$,*

$$\sum_{m=-\infty}^{\infty} \oint \frac{(q^{1+m/2}/az, q^{1-m/2}z/b; q)_\infty}{(q^{m/2}cz, q^{-m/2}d/z; q)_\infty} t^m z^{N-m} \frac{dz}{2\pi i z}$$

$$= t^N \frac{(q/ab, -q^{\frac{1+N}{2}}ct, -q^{\frac{1-N}{2}}d/t; q)_\infty}{(cd, -q^{\frac{1+N}{2}}/at, -q^{\frac{1-N}{2}}t/b; q)_\infty} . \quad (4.9)$$

Proof. Replacing z by $zq^{-m/2}$ and changing the order of summation, we find that the left-hand side is given by

$$L = \oint \frac{(q/az, qz/b; q)_\infty}{(cz, d/z; q)_\infty} z^N {}_1\psi_1 \left(\begin{matrix} b/z \\ q/az \end{matrix}; q, -\frac{q^{\frac{1-N}{2}}t}{b} \right) \frac{dz}{2\pi i z} .$$

Applying Ramanujan's summation [9, Eq. (II.29)] gives

$$L = \frac{(q, q/ab; q)_\infty}{(-q^{\frac{1+N}{2}}/at, -q^{\frac{1-N}{2}}t/b; q)_\infty} \oint \frac{\theta(-q^{\frac{N+1}{2}}z/t; q)}{(cz, d/z; q)_\infty} z^N \frac{dz}{2\pi i z}$$

(the restriction on t is needed here for convergence). It remains to prove that

$$\oint \frac{\theta(-q^{\frac{N+1}{2}}z/t; q)}{(cz, d/z; q)_\infty} z^N \frac{dz}{2\pi i z} = t^N \frac{(-q^{\frac{1+N}{2}}ct, -q^{\frac{1-N}{2}}d/t; q)}{(q, cd; q)_\infty} . \quad (4.10)$$

To this end, we expand the integral as the sum of residues at the points $z = q^k d$, $k \geq 0$. By [9, Eq. (4.10.8)], under the additional assumption $|q^{\frac{1+N}{2}}t/d| < 1$, the sum of residues converges and can be computed by the q-binomial theorem [9, Eq. (II.3)]. Since the left-hand side of (4.10) is analytic in t for $t \neq 0$, the result holds also without the restriction $|q^{\frac{1+N}{2}}t/d| < 1$. $\qquad\square$

Using (3.3), it is easy to see that the case $a = b = c = d$, $N = 0$ is equivalent to the identity proved in Appendix A1 of [2]. It may be remarked that our proof of the general case is simpler.

5. CONCLUSIONS

Similarly to four-dimensional dualities [38, 39], equality of the superconformal indices for dual theories in three dimensions leads to new non-trivial integral identities [2–4]. We have presented four new identities for basic hypergeometric integrals. More concretely, we studied the generalized superconformal index of confining theories in three dimensions that has the form of a basic hypergeometric integral. This kind of result is important for better understanding the structure of three-dimensional supersymmetric dualities. Most dualities discussed in the work are known in the literature, but the verification of these dualities using the superconformal index technique is new.

We also presented so-called pentagon identities. They are especially interesting from the geometrical point of view, which interprets the pentagon relation as the $3 - 2$ Pachner move in the context of the $3d - 3d$ correspondence. This relates different decompositions of a polyhedron with five ideal vertices into ideal tetrahedra.

It would be interesting to study more general $SU(N)$ gauge theories, other gauge groups and other confining theories.

APPENDIX A. A SHORT REVIEW OF $3d\ \mathcal{N} = 2$ THEORIES

The subject is very broad, and we only discuss basic facts needed to obtain our results in Section 2. We refer the reader to [10–12] for more details.

A.1. **Conventions.** The Clifford algebra in $2 + 1$ dimensions with metric $g_{\mu\nu}$ is

$$\{\gamma_\mu, \gamma_\nu\} \ = \ 2g_{\mu\nu}, \tag{A.1}$$

$$[\gamma_\mu, \gamma_\nu] \ = \ -2\mathrm{i}\epsilon^{\mu\nu\lambda}\gamma_\lambda. \tag{A.2}$$

As a convenient representation we choose γ^μ as

$$(\gamma^1)^\alpha_\beta = \mathrm{i}\sigma_2 \ , \quad (\gamma^2)^\alpha_\beta = \sigma_3 \ , \quad (\gamma^3)^\alpha_\beta = \sigma_1 \ , \tag{A.3}$$

where α, β are spinor indices in the defining representation of $SL(2, \mathbb{R})$. Spinor indices are contracted, raised and lowered with the anti-symmetric matrix

$$C_{\alpha\beta} \ = \ -C_{\beta\alpha} \ = \ C^{\beta\alpha} \ = \ \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{A.4}$$

A.2. $\mathcal{N} = 2$ **SUSY algebra.** Besides the ordinary generators of the Poincaré algebra, the three-dimensional $\mathcal{N} = 2$ SUSY algebra (as for $\mathcal{N} = 1$ SUSY in four dimensions) has four real supercharges. They can be combined into a complex supercharge and its Hermitian conjugate

$$Q_\alpha \ \text{and} \ \bar{Q}_\alpha \ , \tag{A.5}$$

where α is a spinor index which goes from 1 to $2(=\mathcal{N})$. The part of the $\mathcal{N}=2$ SUSY algebra involving the supercharges can be written [11]

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0, \tag{A.6}$$

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^i_{\alpha\beta}P_i + 2i\epsilon_{\alpha\beta}Z, \tag{A.7}$$

where the bosonic generator P_μ is the momentum generator and Z is a central charge which can be thought of as the reduced component of four-dimensional momentum. The automorphism group of the algebra is $U(1)$ R-symmetry which rotates the supercharges

$$[R, Q_\alpha] = -Q_\alpha . \tag{A.8}$$

Here we are interested in superconformal theories. In this case, we have two additional bosonic generators, special conformal transformations K_μ and dilatations D and two fermionic generators, S_α and $\bar{S}_\alpha$. The $\mathcal{N}=2$ superconformal algebra in three dimensions takes the form of the supergroup [45]

$$SO(3,2)_{\text{conf}} \times SO(2)_R \subseteq OSp(2|4) . \tag{A.9}$$

In Euclidean signature it is

$$SO(4,1)_{\text{conf}} \times SO(2)_R \subseteq OSp(2|2,2) . \tag{A.10}$$

The first factor is the conformal group and the second one is the R-symmetry. Note that in the superconformal case the algebra has a distinguished R-symmetry. The important relation of the superconformal algebra for our purposes is

$$\{\bar{Q}_\alpha, \bar{S}_\beta\} = M_{\mu\nu}[\gamma^\mu, \gamma^\nu]_{\alpha\beta} + 2\varepsilon_{\alpha\beta}D - 2\varepsilon_{\alpha\beta}R . \tag{A.11}$$

In particular, we will use the commutation relation

$$\{\bar{Q}_1, \bar{S}_1\} = 2\Delta - 2R - 2j_3 . \tag{A.12}$$

A.3. **Multiplets.** The supersymmetry representations of $3d$ $\mathcal{N}=2$ theories are closely related to the representations of $4d$ $\mathcal{N}=1$ theories and can be directly obtained from these by dimensional reduction.

To obtain irreducible representations one must impose constraints. In order to do so it is useful to define supercovariant derivatives:

$$D_\alpha = \frac{\partial}{\partial\theta_\alpha} - i(\gamma^\mu\bar{\theta})_\alpha\partial_\mu , \tag{A.13}$$

$$\bar{D}_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu . \tag{A.14}$$

The simplest type of superfield is a chiral multiplet Φ. It satisfies the constraint

$$\bar{D}_\alpha\Phi = 0 . \tag{A.15}$$

It can be expanded as

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y) , \tag{A.16}$$

where ϕ is a complex scalar field, ψ is a complex Dirac fermion, F is an auxiliary complex scalar, θ is a Grassmann coordinate and $y^\mu = x^\mu + \mathrm{i}\theta\sigma^\mu\bar{\theta}$.

The vector multiplet consists of a real scalar field σ, a vector field A_μ, a complex Dirac fermion λ and a real auxiliary scalar field D. Its expansion in Wess-Zumino gauge is given by

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) - \theta\bar{\theta}\sigma + \mathrm{i}\theta\theta\bar{\theta}\bar{\lambda}(x) - \mathrm{i}\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) , \tag{A.17}$$

The appearance of a real scalar field σ is due to the component of the four-dimensional vector field in the reduced direction.

APPENDIX B. DETAILS OF EXAMPLE 1

All contributions to the superconformal indices in Example 1 are as follows:

- Contribution of the chiral multiplets

$$\mathrm{ind}_\Phi = \begin{cases} \left[za_i \frac{q^{\frac{|n_i+m|}{2}}}{1-q} - z^{-1}a_i^{-1}\frac{q^{1+\frac{|n_i+m|}{2}}}{1-q} \right] + \left[z^{-1}a_i\frac{q^{\frac{|n_i-m|}{2}}}{1-q} - za_i^{-1}\frac{q^{1+\frac{|n_j-m|}{2}}}{1-q} \right] : \text{Theory A,} \\ \left[a_i a_j \frac{q^{\frac{|n_i+n_j|}{2}}}{1-q} - a_i^{-1}a_j^{-1}\frac{q^{1+\frac{|n_i+n_j|}{2}}}{1-q} \right] \qquad\qquad\qquad\qquad\quad : \text{Theory B.} \end{cases} \tag{B.1}$$

- Contribution of the vector multiplet

$$\mathrm{ind}_{gauge} = \begin{cases} -q^{\frac{1}{2}|m_i|}z^2 - q^{\frac{1}{2}|m_i|}z^{-2} : \text{Theory A,} \\ \text{no vector multiplet} \quad : \text{Theory B.} \end{cases} \tag{B.2}$$

- Other contributions

$$q_0(m) = \begin{cases} -\frac{|n_j+m|}{2} - \frac{|n_j-m|}{2} : \text{Theory A,} \\ -\frac{|n_i+n_j|}{2} \qquad\quad : \text{Theory B.} \end{cases} \tag{B.3}$$

$$e^{\mathrm{i}b_0} = \begin{cases} z^{-\sum_{i=1}^{6}(\frac{|n_i+m|}{2} - \frac{|n_i-m|}{2})} : \text{Theory A,} \\ 0 \qquad\qquad\qquad\qquad\quad : \text{Theory B.} \end{cases} \tag{B.4}$$

REFERENCES

[1] F. Dolan and H. Osborn, "Applications of the superconformal index for protected operators and q-hypergeometric identities to N=1 dual theories," *Nucl. Phys.* **B818** (2009) 137–178, `arXiv:0801.4947 [hep-th]`.

[2] C. Krattenthaler, V. Spiridonov, and G. Vartanov, "Superconformal indices of three-dimensional theories related by mirror symmetry," *JHEP* **1106** (2011) 008, `arXiv:1103.4075 [hep-th]`.

[3] A. Kapustin and B. Willett, "Generalized superconformal index for three dimensional field theories," `arXiv:1106.2484 [hep-th]`.

[4] I. Gahramanov and H. Rosengren, "A new pentagon identity for the tetrahedron index," *JHEP* **1311** (2013) 128, `arXiv:1309.2195 [hep-th]`.

[5] I. Gahramanov and H. Rosengren, "Integral pentagon relations for 3d superconformal indices," *Proc. Symp. Pure Math.* **93** (2016) 165 `arXiv:1412.2926 [hep-th]`.

[6] I. Gahramanov, "Mathematical structures behind supersymmetric dualities," *Arch. Math.* **51**, 5 (2015) `arXiv:1505.05656 [math-ph]`.

[7] I. Gahramanov and V. Spiridonov, "The star-triangle relation and 3d superconformal indices," *JHEP* **1508**, 040 (2015) `arXiv:1505.00765 [hep-th]`.

[8] A. P. Kels, "New solutions of the star-triangle relation with discrete and continuous spin variables," *J. Phys. A* **48**, no. 43, 435201 (2015) `arXiv:1504.07074 [math-ph]`.

[9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Cambridge University Press, second ed., 2004.

[10] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three-dimensions," *in The Mathematical Beauty of Physics* (1996) 333–366, `arXiv:hep-th/9607163 [hep-th]`.

[11] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. Strassler, "Aspects of N=2 supersymmetric gauge theories in three-dimensions," *Nucl. Phys.* **B499** (1997) 67–99, `arXiv:hep-th/9703110 [hep-th]`.

[12] K. A. Intriligator and N. Seiberg, "Mirror symmetry in three-dimensional gauge theories," *Phys. Lett.* **B387** (1996) 513–519, `arXiv:hep-th/9607207 [hep-th]`.

[13] Y. Imamura and S. Yokoyama, "Index for three dimensional superconformal field theories with general R-charge assignments," *JHEP* **1104** (2011) 007, `arXiv:1101.0557 [hep-th]`.

[14] C. Romelsberger, "Counting chiral primaries in N = 1, d=4 superconformal field theories," *Nucl. Phys.* **B747** (2006) 329–353, `arXiv:hep-th/0510060 [hep-th]`.

[15] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, "An index for 4 dimensional super conformal theories," *Comm. Math. Phys.* **275** (2007) 209–254, `arXiv:hep-th/0510251 [hep-th]`.

[16] J. Bhattacharya and S. Minwalla, "Superconformal indices for N = 6 Chern Simons theories," *JHEP* **0901** (2009) 014, `arXiv:0806.3251 [hep-th]`.

[17] S. Kim, "The complete superconformal index for N=6 Chern-Simons theory," *Nucl. Phys.* **B821** (2009) 241–284, `arXiv:0903.4172 [hep-th]`.

[18] T. Dimofte, D. Gaiotto, and S. Gukov, "3-Manifolds and 3d indices," `arXiv:1112.5179 [hep-th]`.

[19] O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, "3d dualities from 4d dualities," `arXiv:1305.3924 [hep-th]`.

[20] F. Dolan, "On superconformal characters and partition functions in three dimensions," *J. Math. Phys.* **51** (2010) 022301, `arXiv:0811.2740 [hep-th]`.

[21] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Comm. Math. Phys.* **313** (2012) 71–129, `arXiv:0712.2824 [hep-th]`.

[22] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, "Counting BPS operators in gauge theories: quivers, syzygies and plethystics," *JHEP* **0711** (2007) 050, `arXiv:hep-th/0608050 [hep-th]`.

[23] B. Feng, A. Hanany, and Y.-H. He, "Counting gauge invariants: The plethystic program," *JHEP* **0703** (2007) 090, `arXiv:hep-th/0701063 [hep-th]`.

[24] C. Beem, T. Dimofte, and S. Pasquetti, "Holomorphic blocks in three dimensions," *JHEP* **1412** (2014) 177, `arXiv:1211.1986 [hep-th]`.

[25] S. Pasquetti, "Factorisation of N = 2 theories on the squashed 3-sphere," *JHEP* **1204** (2012) 120, `arXiv:1111.6905 [hep-th]`.

[26] M. Taki, "Holomorphic blocks for 3d non-abelian partition functions," `arXiv:1303.5915 [hep-th]`.

[27] F. Nieri, S. Pasquetti, and F. Passerini, "3d and 5d gauge theory partition functions as q-deformed CFT correlators," *Lett. Math. Phys.* **105**, no. 1, 109 (2015) `arXiv:1303.2626 [hep-th]`.

[28] C. Hwang, H. C. Kim and J. Park, "Factorization of the 3d superconformal index," JHEP **1408** (2014) 018 `arXiv:1211.6023 [hep-th]`.

[29] C. Hwang and J. Park, "Factorization of the 3d superconformal index with an adjoint matter," JHEP **1511**, 028 (2015) `arXiv:1506.03951 [hep-th]`.

[30] M. Fujitsuka, M. Honda, and Y. Yoshida, "Higgs branch localization of 3d N = 2 theories," *PTEP* **2014** no. 12, (2014) 123B02, `arXiv:1312.3627 [hep-th]`.

[31] F. Benini and W. Peelaers, "Higgs branch localization in three dimensions," *JHEP* **1405** (2014) 030, `arXiv:1312.6078 [hep-th]`.

[32] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys.* **B435** (1995) 129–146, `arXiv:hep-th/9411149 [hep-th]`.

[33] O. Aharony, "IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c)) gauge theories," *Phys. Lett.* **B404** (1997) 71–76, `arXiv:hep-th/9703215 [hep-th]`.

[34] A. Karch, "Seiberg duality in three-dimensions," *Phys. Lett.* **B405** (1997) 79–84, `arXiv:hep-th/9703172 [hep-th]`.

[35] J. de Boer, K. Hori, Y. Oz, and Z. Yin, "Branes and mirror symmetry in N=2 supersymmetric gauge theories in three-dimensions," *Nucl. Phys.* **B502** (1997) 107–124, `arXiv:hep-th/9702154 [hep-th]`.

[36] C. Csaki, M. Martone, Y. Shirman, P. Tanedo, and J. Terning, "Dynamics of 3D SUSY gauge theories with antisymmetric matter," *JHEP* **1408** (2014) 141, `arXiv:1406.6684 [hep-th]`.

[37] A. Amariti, C. Csaki, M. Martone, and N. R.-L. Lorier, "From S-confinement to 3D chiral theories: Dressing the monopoles," `arXiv:1506.01017 [hep-th]`.

[38] V. Spiridonov and G. Vartanov, "Elliptic hypergeometry of supersymmetric dualities," *Comm. Math. Phys.* **304** (2011) 797–874, `arXiv:0910.5944 [hep-th]`.

[39] V. Spiridonov and G. Vartanov, "Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices," `arXiv:1107.5788 [hep-th]`.

[40] V. Niarchos, "Seiberg dualities and the 3d/4d connection," *JHEP* **1207** (2012) 075, `arXiv:1205.2086 [hep-th]`.

[41] A. Amariti, D. Forcella, C. Klare, D. Orlando, and S. Reffert, "The braneology of 3D dualities," `arXiv:1501.06571 [hep-th]`.

[42] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, "Comments on Chern-Simons contact terms in three dimensions," *JHEP* **1209** (2012) 091, `arXiv:1206.5218 [hep-th]`.

[43] A. Amariti and C. Klare, "Chern-Simons and RG flows: Contact with dualities," *JHEP* **1408** (2014) 144, `arXiv:1405.2312 [hep-th]`.

[44] A. Kapustin, B. Willett, and I. Yaakov, "Exact results for Wilson loops in superconformal Chern-Simons theories with matter," *JHEP* **03** (2010) 089, `arXiv:0909.4559 [hep-th]`.

[45] D. L. Jafferis, "The exact superconformal R-Symmetry extremizes Z," *JHEP* **1205** (2012) 159, `arXiv:1012.3210 [hep-th]`.

[46] F. Dolan, V. Spiridonov, and G. Vartanov, "From 4d superconformal indices to 3d partition functions," *Phys. Lett.* **B704** (2011) 234–241, `arXiv:1104.1787 [hep-th]`.

[47] I. B. Gahramanov and G. S. Vartanov, "Superconformal indices and partition functions for supersymmetric field theories," *XVIIth International Congress on Mathematical Physics* (2013) 695–703, `arXiv:1310.8507 [hep-th]`.

[48] Y. Imamura and D. Yokoyama, "S^3/Z_n partition function and dualities," *JHEP* **11** (2012) 122, `arXiv:1208.1404 [hep-th]`.

[49] I. Gahramanov and A. Kels, to appear.

[50] A. Kapustin, H. Kim, and J. Park, "Dualities for 3d Theories with tensor matter," *JHEP* **1112** (2011) 087, `arXiv:1110.2547 [hep-th]`.

[51] H. Kim and J. Park, "Aharony dualities for 3d theories with adjoint matter," *JHEP* **1306** (2013) 106, `arXiv:1302.3645 [hep-th]`.

[52] J. Park and K.-J. Park, "Seiberg-like dualities for 3d N=2 theories with SU(N) gauge group," `arXiv:1305.6280 [hep-th]`.

[53] J. Teschner and G. Vartanov, "6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories," `arXiv:1202.4698 [hep-th]`.

[54] M. Yamazaki, "Four-dimensional superconformal index reloaded," *Theor. Math. Phys.* **174** (2013) 154–166.

[55] F. Benini, T. Nishioka, and M. Yamazaki, "4d index to 3d index and 2d TQFT," *Phys. Rev.* **D86** (2012) 065015, `arXiv:1109.0283 [hep-th]`.

[56] A. P. Kels, "A new solution of the star-triangle relation," *J. Phys.* **A47** no. 5, (2014) 055203, `arXiv:1302.3025 [math-ph]`.

[57] F. Benini and S. Cremonesi, "Partition functions of $\mathcal{N} = (2,2)$ gauge theories on S^2 and vortices," *Comm. Math. Phys.* **334** no. 3, (2015) 1483–1527, `arXiv:1206.2356 [hep-th]`.

[58] I. Gahramanov and G. Vartanov, "Extended global symmetries for 4D $N = 1$ SQCD theories," *J. Phys.* **A46** (2013) 285403, `arXiv:1303.1443 [hep-th]`.

[59] M. J. Strassler, "Confining phase of three-dimensional supersymmetric quantum electrodynamics," *in Shifman, M.A. (ed.): The many faces of the superworld* (1999) , `arXiv:hep-th/9912142 [hep-th]`.

[60] R. Kashaev, F. Luo and G. Vartanov, "A TQFT of TuraevViro type on shaped triangulations," *Annales Henri Poincare* **17** (2016) no.5, 1109 `arXiv:1210.8393 [math.QA]`.

[61] S. Benvenuti and S. Pasquetti, "3d N=2 mirror symmetry, pq-webs and monopole superpotentials," `arXiv:1605.02675 [hep-th]`.

[62] Y. Terashima and M. Yamazaki, "SL(2,R) Chern-Simons, Liouville, and gauge theory on duality walls," *JHEP* **1108** (2011) 135, `arXiv:1103.5748 [hep-th]`.

[63] T. Dimofte, D. Gaiotto, and S. Gukov, "Gauge theories labelled by three-manifolds," *Comm. Math. Phys.* **325** (2014) 367–419, `arXiv:1108.4389 [hep-th]`.

[64] T. Dimofte, "3d superconformal theories from three-manifolds," `arXiv:1412.7129 [hep-th]`.

[65] Y. Terashima and M. Yamazaki, "Semiclassical analysis of the 3d/3d relation," *Phys. Rev.* **D88** no. 2, (2013) 026011, `arXiv:1106.3066 [hep-th]`.

[66] Y. Terashima and M. Yamazaki, "Emergent 3-manifolds from 4d superconformal indices," *Phys. Rev. Lett.* **109** (2012) 091602, `arXiv:1203.5792 [hep-th]`.

[67] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville correlation functions from four-dimensional gauge theories," *Lett. Math. Phys.* **91** (2010) 167–197, `arXiv:0906.3219 [hep-th]`.

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