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Exploring Perturbative Conformal Field Theory in Mellin space

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Abstract: We explore the Mellin representation of correlation functions in conformal field theories in the weak coupling regime. We provide a complete proof for a set of Feynman rules to write the Mellin amplitude for a general tree level Feynman diagram involving only scalar operators. We find a factorised form involving beta functions associated to the propagators, similar to tree level Feynman rules in momentum space for ordinary QFTs. We also briefly consider the case where a generic scalar perturbation of the free CFT breaks conformal invariance. Mellin space still has some utility and one can consider non-conformal Mellin representations. In this context, we find that the beta function corresponding to conformal propagator uplifts to a hypergeometric function.

Contents

1 Introduction

It has been realised in the last few years, beginning with the pioneering work of Mack [1, 2] (see also [3]), that Mellin space provides the natural setting for the study of Conformal Field theories (CFTs). The Mellin transform of a CFT correlator is a meromorphic function in the Mellin variables. In particular, for a four point function, the isolated simple poles indicate the conformal dimensions of the operators in the spectrum whereas the residues at these poles contain information about the 3-point couplings. Thus the CFT data (operator dimensions and OPE coefficients) is at once made manifest in the Mellin space representation. Mellin amplitudes are also conformally invariant making conformal symmetry manifest in Mellin space.

Usually in quantum field theory, we Fourier transform the position space correlators to write Feynman rules in momentum space. The important advantage in doing so is that translation invariance leads to momentum conservation and the position space integrals are reduced to simple products in momentum space at tree level. In momentum space, conformal transformations have a non-linear action and as a result the conventional way of doing perturbative QFT in momentum space is not so advantageous for CFTs.

Various important features of QFT such as locality, causality and unitarity can be understood in terms of the analytic properties of momentum space amplitudes. The isolated poles of the momentum space propagator correspond to single-particle states and the branch cuts on the real axis give the multi-particle states (Kählen Lehmann spectral representation) and the amplitudes factorise on residues at the poles to lower point amplitudes. In a CFT, we do not have single particle states characterised by the masses since mass is a dimensionful parameter. Hence the propagators in momentum space have branch cuts extending to the origin. In the radial quantization of CFT, the dilatation operator acts as the Hamiltonian. The eigenvalues of this operator are discrete for $d > 2$. This discrete set of operators appear in the operator product expansion (OPE) as the exchanged primaries and descendants in an interacting CFT (in $d > 2$) . So it is desirable to have a representation for correlation functions in CFTs that makes this discrete spectrum manifest. As shown by Mack, it turns out that Mellin space provides such a representation.

The analogy of the Mellin space CFT correlators with scattering amplitudes is also striking. This has been explored in the context of the AdS/CFT correspondence. Following Mack, the application of the Mellin representation of conformal correlation functions was explored at strong coupling for large N CFTs using tree level Witten diagrams in AdS [4–10]. While at tree level, there seem to be a set of Feynman rules to write the Mellin amplitudes, the loop level seems to be significantly more involved. In the flat space limit of AdS/CFT, a relation between the bulk scattering amplitude and the CFT Mellin amplitudes was also suggested in [4, 7] and later put on a firm footing in [8, 11–13] (see also [14]). To be precise, the flat space S-matrix is expressed as an integral transform of the CFT Mellin amplitude and the Mellin variables, in the flat space limit, turn into flat space kinematic invariants (the Mandelstam variables). This scheme also relates the S-Matrix program in QFTs to the Bootstrap program in CFTs. Our work, however, has a different focus and does not use AdS/CFT. We consider *weakly* coupled CFTs and attempt to formulate Feynman rules in Mellin space for perturbative field theory computations.

The Mellin representation for tensor operators and the factorization of Mellin amplitudes was studied in [13]. The Mellin representation has also been explored in the context of minimal model CFTs in [15] and for open string amplitudes in [16]. It was explored in the weak coupling regime in [17, 18] in the context of SYM and has also been used to calculate corrections beyond the planar limit to the 4-point function of a primary in $\mathcal{N}=4$ SYM in [19]. For some more applications in the context of $\mathcal{N}=4$ SYM, see also [20–24]. Feynman rules for tree level diagrams in the Mellin space were stated in [17, 18] after considering a few examples. However a proof of these rules for a general tree level Feynman diagram was not provided.

The goal of our note is to further explore the suitability of the Mellin representation for studying perturbative CFTs. We consider an exactly marginal perturbation around a free CFT and investigate whether it is possible to obtain a set of Feynman rules that can be used to calculated Mellin amplitudes. For simplicity, we restrict to scalar operators throughout the paper. We present a complete derivation of the Feynman rules associated to tree level amplitudes in complete generality. For this purpose, we also develop a diagrammatic algorithm to write down the Mellin amplitude for any Feynman diagram (upto arbitrary loop order) as an integral over Schwinger parameters corresponding to the internal propagators in the diagram. We further relax the conformality of the integrals, we consider, to study Mellin amplitudes in free CFTs with a generic perturbation. It turns out that when we consider integrals that enjoy a scale covariance only (as opposed to the full conformal covariance) the corresponding "Mellin amplitudes" can be interpreted as "off-shell" quantities that reduce to the "on-shell" conformal Mellin amplitudes under an LSZ like prescription.

For application to many well-known conformal field theories (say, $\mathcal{N}=4$ Super-Yang Mills), we need to extend these rules to tensor and fermionic operators as well. We leave this along with the task of obtaining Feynman rules for loop amplitudes to future work.

The plan of this note is as follows. In section 2, we give a quick review of the Mellin amplitude for conformal field theories. In section 3, we consider some simple tree level Feynman diagrams involving only scalar fields and derive their Mellin amplitude. This is to introduce the general strategy that we follow for deriving the Mellin amplitude of a general tree level Feynman diagram. In section 4, we provide a general derivation for the Feynman rules for tree level diagrams. To this end, we develop an algorithmic method for writing down the Mellin amplitude for an arbitrary Feynman diagram (tree as well as loops) as an integral over the Schwinger parameters for the internal propagators. In section 5, we consider the Mellin amplitudes for loop diagrams involving scalar fields. Even though we can write an integral expression for the Mellin amplitude for such diagrams, we have not been able to perform these integrals so as to obtain a set of Feynman rules. In section 6, we extend the notion of Mellin amplitude to include generic scalar deformations of a free CFT which may break conformal invariance. In particular, we consider a tree level diagram with a single internal line (involving scalar fields) in Mellin space for such theories. The appendices elaborate on our notations and conventions, contain some properties of the Mellin transform and a few useful identites. Many elaborate details of the calculations are also relegated to the appendices.

Throughout the draft, the space-time Lorentz indices will be suppressed. We shall use the indices $\{i, j, \cdots\}$ for external vertices and the indices $\{a, b, \cdots\}$ for internal vertices. For convenience, we shall use the upstair indices for denoting the external vertices and the lower indices for denoting the internal vertices. This turns out to be useful for us mainly because of the fact that our analysis does not depend on how many external legs are attached to a given internal vertex. This will become clear when we consider explicit calculations. More details on the notations and convention can be found in the appendix A.

2 Mellin Amplitude

The Mellin amplitude for an arbitrary n-point function is defined by the Mellin transformation of the position space correlation function [1, 2]

$$A\left(\{x^i\}\right) = \prod_{1\le i<j\le n}\left(\int_{-i\infty}^{i\infty}\frac{ds^{ij}}{2\pi i}\,\Gamma\left(s^{ij}\right)\left(x^i-x^j\right)^{-2s^{ij}}\right)\prod_{i=1}^{n}\delta\left(\Delta^i-\sum_{j=1}^{n}s^{ij}\right)M\left(\{s^{ij}\}\right) \tag{2.1}$$

Here s^{ij} are the Mellin variables and $M\left(\{s^{ij}\}\right)$ is defined to be the Mellin amplitude. The variable Δ^i is the scaling dimension of the operator inserted at x^i. One strips $M\left(\{s^{ij}\}\right)$ of the factors of $\Gamma(s^{ij})$ for convenience. This turns out to be particularly useful for large N gauge theories (in the context of the AdS/CFT correspondence) where these Gamma functions account for the poles corresponding to the multi-particle states whereas $M\left(\{s^{ij}\}\right)$ accounts for poles corresponding to the single particle states.

The following are some important points to be noted:

1. The delta function constraints in the definition of Mellin amplitude (2.1) ensure the covariance of $A(\{x_i\})$ under conformal transformations. More precisely, under inversion

$$\left(x^i-x^j\right)^2 \to \frac{\left(x^i-x^j\right)^2}{(x^i)^2(x^j)^2}$$

The correlation function $A\left(\{x^i\}\right)$ transforms as

$$A\left(\{x^i\}\right) \to \left[\prod_{i=1}^{n}(x^i)^{-2\Delta^i}\right]\,A\left(\{x^i\}\right)$$

The delta function constraints $\sum_{j\neq i} s^{ij} = \Delta^i$ ensure that both sides of (2.1) transform in the same way.

2. The Mellin amplitude $M(\{s^{ij}\})$ is manifestly conformally invariant. The conformal transformations act on the position space variables x^i. The x^i dependence of the expression (2.1) and the delta functions imposing constraints on the Mellin variables ensure that $A(\{x^i\})$ is conformally covariant.

3. The Mellin variables s^{ij} are symmetric in i and j. So the number of Mellin variables s^{ij} is $n(n-1)/2$. However, due to the n delta function constraints, the number of independent Mellin variables is only $\frac{n(n-3)}{2}$. This is also the number of independent cross-ratios for n points and the number of Mandelstam invariants for an $n-$point scattering amplitude.

4. The delta function constraints can be solved in terms of the "dual Mellin momenta" [1] with $s^{ij} = k^i \cdot k^j$ and $(k^i)^2 = -\Delta^i$ and overall Mellin momentum conservation $\sum_i k^i = 0$. These are fictitious momenta associated with each x^i. We refer to $(k^i)^2 = -\Delta^i$ as the "on-shell" condition for Mellin momenta.

5. Using the dual Mellin momenta mentioned above, one can define "dual Mandelstam variables" and express the Mellin amplitude in terms of these. To see an explicit example [25], we consider a 4-point function and define the Mandelstam variables s and t as

$$s \equiv -(p^1+p^2)^2 = \Delta^1+\Delta^2-2s^{12} \qquad , \qquad t \equiv -(p^1+p^3)^2 = \Delta^1+\Delta^3-2s^{13}$$

In terms of these Mandelstam variables, we can express the 4-point amplitude as

$$A(x^i) = \left[\prod_{i<j}(x^{ij})^{-2\Delta^{ij}}\right] \quad A(u,v)$$

where, $\Delta^{ij} \equiv \Delta^i - \Delta^j$, u and v are the usual 4-point cross ratios and $A(u,v)$ is the inverse Mellin transform with respect to the above Mandelstam variables

$$\begin{aligned} A(u,v) = &\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\, M(s,t)u^{\frac{s}{2}}v^{-\frac{s+t}{2}}\Gamma\left[\frac{\Delta^1+\Delta^2-s}{2}\right]\Gamma\left[\frac{\Delta^1+\Delta^3-t}{2}\right] \\ &\Gamma\left[\frac{\Delta^3+\Delta^4-s}{2}\right]\Gamma\left[\frac{\Delta^2+\Delta^4-t}{2}\right]\Gamma\left[\frac{s+t-\Delta^2-\Delta^3}{2}\right]\Gamma\left[\frac{s+t-\Delta^1-\Delta^4}{2}\right] \end{aligned}$$

The above equation illustrates the fact that the position space correlator can be expressed as the inverse Mellin transform of the Mellin amplitude and that the kinematical variables in the Mellin space are analogous to the Mandelstam variables.

3 Some Examples of Tree Diagrams

In this section we consider a few simple tree level examples which will illustrate the general strategy we shall follow for deriving the Mellin amplitude of Feynman diagrams involving only scalar fields. Specifically, we shall be looking at the contact interaction diagram, the diagram with one internal propagator and the diagram with two internal propagators. The

Mellin amplitude for the contact interaction diagram and one propagator diagram were presented in [17]. We begin with these examples for pedagogy and completeness of our presentation.

3.1 Contact Interaction

The position space Feynman diagram for the contact interaction is shown in Figure 1. In this diagram, N external lines are meeting at the vertex u. We denote the scaling dimension of the field correponding to the external vertex x^i by Δ^i. As mentioned earlier, we choose to place the index upstairs to keep the notation compact when we discuss more complicated Feynman diagrams.

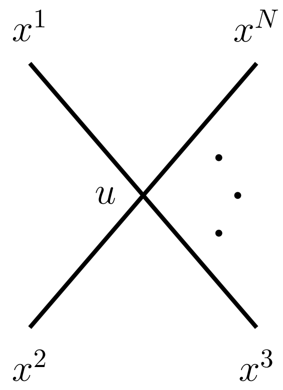

Figure 1. Contact Interaction Diagram

The position space correlation function corresponding to the contact interaction is given by

$$I = \int \frac{d^D u}{2(2\pi)^{D/2}} \left[\prod_{i=1}^{N} (x^i - u)^{-2\Delta^i} \Gamma(\Delta^i) \right] \tag{3.1}$$

The factors of $\Gamma(\Delta^i)$ and π have been included for the sake of convenience[1] later on. We follow these conventions throughout the draft.

The expression is covariant under conformal transformations provided we impose the following 'conformality condition' on the conformal dimensions

$$\sum_{i=1}^{N} \Delta_i = D \tag{3.2}$$

[1] In the rest of the draft, we denote the measure as

$$\frac{d^D u}{2(2\pi)^{D/2}} \equiv \mathcal{D}u$$

We now introduce a Schwinger parameter for each propagator via the identity

$$\frac{1}{(x-y)^{2\Delta}} = \frac{1}{\Gamma(\Delta)} \int_0^\infty d\alpha\ \alpha^{\Delta-1} \exp\left[-\alpha(x-y)^2\right] \tag{3.3}$$

Using this identity in (3.1) gives,

$$I = \prod_{i=1}^N \left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i-1}\right] \int \frac{d^D u}{2(2\pi)^{D/2}} \exp\left[-\left(\sum_{i=1}^N \alpha^i (x^i-u)^2\right)\right]$$

The factors of $\Gamma(\Delta^i)$ present in (3.1) are cancelled by the corresponding factors in (3.3).

Performing the Gaussian integration over u, we obtain

$$I = \frac{1}{2}\prod_{i=1}^N \int_0^\infty d\alpha^i \left(\frac{(\alpha^i)^{\Delta^i-1}}{(\sum_i \alpha^i)^{\frac{D}{2}}}\right) \exp\left[-\frac{1}{\sum_i \alpha^i}\left(\sum_j \sum_{i<j} \alpha^i \alpha^j (x^{ij})^2\right)\right] \tag{3.4}$$

where, $x^{ij} \equiv x^i - x^j$.

We shall now render (3.4) particularly suitable for imposing the conformality conditions (3.2). For this, we insert the following partition of unity in (3.4)

$$1 = \int_0^\infty dv\ \delta\left(v - \sum_i \alpha^i\right)$$

We then rescale the Schwinger parameters, $\alpha^i \to \sqrt{v}\ \alpha^i$ and perform the integration over the auxiliary variable v using delta function. The end result is

$$I = \prod_{i=1}^N \left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i-1}\right] \left(\sum_i \alpha^i\right)^{\sum_i \Delta^i - D} \exp\left(-\sum_j \sum_{i<j} \alpha^i \alpha^j (x^{ij})^2\right) \tag{3.5}$$

We now impose the conformality condition (3.2) on equation (3.5) to obtain

$$I = \prod_{i=1}^N \left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i-1}\right] \exp\left(-\sum_j \sum_{i<j} \alpha^i \alpha^j (x^{ij})^2\right) \tag{3.6}$$

To proceed further, we now use the inverse Mellin transform representation of the exponential function

$$e^{-x} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds\ \Gamma(s) x^{-s} \tag{3.7}$$

The contour of integration is along the imaginary axis with an infinitesimal dent at the origin so as to put the pole at the origin to the left of the contour.

Using (3.7) for the exponential factor in (3.6), we obtain,

$$I = \prod_j \prod_{i<j} \left[\int_{-i\infty}^{i\infty} [ds^{ij}] (x^{ij})^{-2s^{ij}} \Gamma(s^{ij})\right] \prod_{i=1}^N \left[\int_0^\infty d\alpha^i (\alpha^i)^{\rho^i-1}\right] \tag{3.8}$$

where s^{ij} (corresponding to x^{ij}) are our Mellin variables and

$$\rho^i \equiv \Delta^i - \sum_{\substack{j=1 \\ j\neq i}}^{N} s^{ij} \quad , \qquad\qquad [ds^{ij}] \equiv \frac{ds^{ij}}{2\pi i}$$

The integral over each of the Schwinger parameters in (3.8) is an integral of an oscillatory function and due to this, each Schwinger parameter integral gives a delta function. More specifically, we can use

$$\int_{-i\infty}^{i\infty} [ds] f(s) \int_0^\infty dt\ t^{s-s_0-1} = \int_{\mathrm{Re}(s_0)-i\infty}^{\mathrm{Re}(s_0)+i\infty} [ds] f(s)\,(2\pi i\delta(s-s_0))$$

This can be taken as an identity in Mellin space (and only in Mellin space). We prove this in the Appendix B.2.

Using the identity above, the expression (3.8) becomes

$$I = (2\pi i)^N \prod_j \prod_{i<j} \left[\int_{-i\infty}^{i\infty} [ds^{ij}](x^{ij})^{-2s^{ij}} \Gamma(s^{ij})\right] \left[\prod_i \delta\left(\Delta^i - \sum_{j\neq i} s^{ij}\right)\right] \tag{3.9}$$

We recognise that the delta function constraints $\rho_i = 0$ are precisely the constraints on the Mellin variables discussed in section 2. These constraints originate from the fact that the position space correlation function is covariant under conformal transformations. We also note that these constraints reduce the number of independent Mellin variables from $N(N-1)/2$ to $N(N-3)/2$ which is also the number of independent cross-ratios for N points[2].

We can now read off the Mellin ampltitude corresponding to the Feynman diagram in Figure 1. Comparing (3.9) with the defining expression of Mellin amplitude (2.1), we find that the Mellin amplitude for contact interaction is just 1, which as promised is conformally invariant.

A careful look at (3.9) tells us that the N delta functions force the $(x^{ij})^{-2s^{ij}}$ terms to combine and form $\frac{N(N-3)}{2}$ cross ratios between the external vertices x^i and some extra factors that give appropriate transformation properties to the position space correlator.

3.2 Tree With One Internal Propagator

The next Feynman diagram that we consider (Figure 2) involves two internal vertices connected by an internal propagator. This example will give us the expression for the scalar propagator in Mellin space. γ denotes the scaling dimension of the internal propagator. The position space expression for this diagram is given by

$$I = \int \mathcal{D}u_1\ \mathcal{D}u_2 \left[\prod_{i\in 1}(x_1^i - u_1)^{-2\Delta_1^i}\Gamma(\Delta_1^i) \prod_{j\in 2}(x_2^j - u_2)^{-2\Delta_2^j}\Gamma(\Delta_2^j)(u_2-u_1)^{-2\gamma}\right]$$

[2]This is true in generic dimensions. In special cases, the number of independent cross ratios may be different

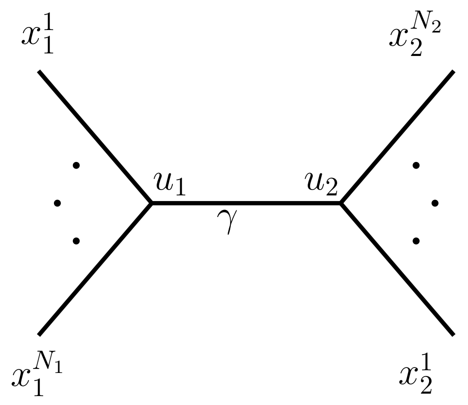

Figure 2. Two vertex

The conformality conditions for the two interaction vertices in this diagram are,

$$\sum_{i\in 1}\Delta_1^i+\gamma=D \quad , \qquad \sum_{i\in 2}\Delta_2^i+\gamma=D \tag{3.10}$$

We again use the identity (3.3) and introduce the Schwinger parameters for each propagator (internal as well as external)

$$I=\left[\prod_{i\in 1}\int_0^\infty d\alpha_1^i(\alpha_1^i)^{\Delta_1^i-1}\prod_{j\in 2}\int_0^\infty d\alpha_2^j(\alpha_2^j)^{\Delta_2^j-1}\frac{1}{\Gamma(\gamma)}\int_0^\infty dt\; t^{\gamma-1}\right]\int \mathcal{D}u_1\;\mathcal{D}u_2$$
$$\exp\left(-\sum_{i\in 1}\alpha_1^i(x_1^i-u_1)^2-\sum_{j\in 2}\alpha_2^j(x_2^j-u_2)^2-t(u_2-u_1)^2\right)$$

where t is the Schwinger parameter for the internal propagator.

Performing the u_1 integration, we obtain

$$I=\frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\prod_{i\in a}\left(\int d\alpha_a^i(\alpha_a^i)^{\Delta_a^i-1}\right)\int_0^\infty dt\; t^{\gamma-1}\int \mathcal{D}u_2\;\exp\left(-\sum_{j\in 2}(x_2^j-u_2)^2\right)$$
$$\left(\sum_{i\in 1}\alpha_1^i+t\right)^{-D/2}\exp\left(-\left(\sum_{i\in 1}\alpha_1^i+t\right)^{-1}\left\{\sum_{(i,j)\in 1}\alpha_1^i\alpha_1^j(x_{11}^{ij})^2+t\sum_{i\in 1}\alpha_1^i(x_1^i-u_2)^2\right\}\right) \tag{3.11}$$

Next, we insert the partition of unity

$$1=\int_0^\infty dy\;\delta\Big(y-\sum_{i\in 1}\alpha_1^i-t\Big)$$

in the integral of (3.11), rescale the Schwinger parameters

$$\alpha_1^i\to\sqrt{y}\;\alpha_1^i \qquad , \qquad t\to\sqrt{y}\;t \tag{3.12}$$

and perform the integration over the variable y using the delta function. The result is

$$I = \frac{1}{\Gamma(\gamma)} \prod_{a=1}^{2} \prod_{i\in a} \left[\int_0^\infty d\alpha_a^i (\alpha_a^i)^{\Delta_a^i - 1} \right] \int_0^\infty dt\, t^{\gamma-1} \Big(\sum_{i\in 1} \alpha_1^i + t \Big)^{\sum_{i\in 1} \Delta_1^i + \gamma - D}$$

$$\exp\Bigg(- \sum_{(i,j)\in 1} \alpha_1^i \alpha_1^j (x_{11}^{ij})^2 \Bigg) \int \mathcal{D}u_2 \exp\Bigg(- \sum_{j\in 2} \alpha_2^j (x_2^j - u_2)^2 - t \sum_{i\in 1} \alpha_1^i (x_i - u_2)^2 \Bigg)$$

Next, we perform the u_2 integration, insert the following partition of unity in the integral

$$1 = \int_0^\infty dy\, \delta\Big(y - \sum_{i\in 2} \alpha_2^i - t \sum_{i\in 1} \alpha_1^i \Big) \quad ,$$

carry out similar rescalings as in (3.12) (but this time, with the variables α_2^i and t) and perform the integration over the auxiliary variable y. This gives,

$$I = \frac{1}{\Gamma(\gamma)} \prod_{i\in 1} \left[\int_0^\infty d\alpha_1^i (\alpha_1^i)^{\Delta_1^i - 1} \right] \prod_{j\in 2} \left[\int_0^\infty d\alpha_2^j (\alpha_2^j)^{\Delta_2^j - 1} \right] \int_0^\infty dt\, t^{\gamma-1}$$

$$\exp\Bigg(-(1+t^2) \sum_{(i,j)\in 1} \alpha_1^i \alpha_1^j (x_{11}^{ij})^2 - \sum_{(i,j)\in 2} \alpha_2^i \alpha_2^j (x_{22}^{ij})^2 + t \sum_{i\in 1} \sum_{j\in 2} \alpha_1^i \alpha_2^j (x_{12}^{ij})^2 \Bigg)$$

$$\Bigg(\Big(1+t^2\Big) \sum_{i\in 1} \alpha_1^i + t \sum_{i\in 2} \alpha_2^i \Bigg)^{\sum_{i\in 1} \Delta_1^i + \gamma - D} \Big(\sum_{i\in 2} \alpha_2^i + t \sum_{i\in 1} \alpha_1^i \Big)^{\sum_{i\in 2} \Delta_2^i + \gamma - D}$$

We impose the conformality conditions (3.10), and then use the identity (3.7) for each exponential factor. After some rearrangement, we obtain,

$$I = \frac{1}{\Gamma(\gamma)} \prod_{(i,j)\in 1+2} \left(\int_{-i\infty}^{i\infty} [ds_{ab}^{ij}] \Gamma(s_{ab}^{ij}) \Big((x_{ab}^{ij})^2 \Big)^{-s_{ab}^{ij}} \right) \prod_{i\in 1} \left(\int_0^\infty d\alpha_1^i (\alpha_1^i)^{\rho_1^i - 1} \right)$$

$$\prod_{j\in 2} \left(\int_0^\infty d\alpha_2^j (\alpha_2^j)^{\rho_2^j - 1} \right) \int_0^\infty dt\, t^{\gamma - s_{12} - 1} (1+t^2)^{-s_{11}} \qquad (3.13)$$

where,

$$\rho_1^i \equiv \Delta_1^i - \sum_{j\in 1} s_{11}^{ij} - \sum_{j\in 2} s_{12}^{ij} \quad , \qquad (i\in 1)$$

$$\rho_2^i \equiv \Delta_2^i - \sum_{j\in 2} s_{22}^{ij} - \sum_{j\in 1} s_{12}^{ji} \quad , \qquad (i\in 2)$$

$$s_{12} \equiv \sum_{i\in 1} \sum_{j\in 2} s_{12}^{ij} \quad ; \qquad s_{aa} \equiv \sum_{1\le i<j\le N_a} s_{aa}^{ij} \quad , \quad a = 1,2$$

Once again, the integrals over the Schwinger parameters α_1^i and α_2^i act as delta functions in the Mellin space (see appendix B.2). This means that the Mellin variables satisfy the constraints

$$\rho_1^i = 0 = \rho_2^i \qquad \forall\ i \qquad (3.14)$$

These reduce the number of independent Mellin variables from $(N_1+N_2)(N_1+N_2-1)/2$ to $(N_1+N_2)(N_1+N_2-3)/2$. Summing over i and using the conformality conditions (3.10) gives useful relations between the Mellin variables

$$\sum_{i\in 1}\Delta_1^i = 2s_{11}+s_{12} = D-\gamma \quad , \qquad \sum_{i\in 2}\Delta_2^i = 2s_{22}+s_{12} = D-\gamma \tag{3.15}$$

By comparing (3.13) with the definition of Mellin amplitude (2.1) (taking into account the constraints (3.14)), we can easily read off the Mellin amplitude to be

$$M(s_{12}) = \frac{1}{\Gamma(\gamma)}\int_0^\infty dt\, t^{\gamma-s_{12}-1}(1+t^2)^{-s_{11}} = \frac{1}{2\Gamma(\gamma)}\beta\Big(\frac{\gamma-s_{12}}{2},\frac{D}{2}-\gamma\Big) \tag{3.16}$$

where we have used (3.15) to simplify the arguments of beta function.

The physical interpretation of the amplitude is clear. We can identify the beta function to be the Mellin space propagator. Moreover, the poles of the beta function have clear physical interpretation. At this stage, it is convenient to introduce dual Mellin momenta. If the Mellin momentum flowing into the internal propagator through the external vertex x_a^i be k_a^i (where we have suppressed the dual spacetime index), then the full momentum propagating through the internal propagator is

$$k = \sum_{i\in 1} k_1^i = -\sum_{i\in 2} k_2^i$$

and the kinematical variable entering into the propagator is

$$s_{12} = \sum_{i\in 1}\sum_{j\in 2} s_{12}^{ij} = \left(\sum_{i\in 1} k_1^i\right)\cdot\left(\sum_{j\in 2} k_2^j\right) = -k^2$$

This means that the poles of the propagator appear at particular values of the k^2, namely

$$s_{12} = -k^2 = \gamma + 2n \qquad ; \quad n = 0,1,2,\cdots$$

These poles correspond to the primary $(n=0)$ and descendant $(n>0)$ propagating states.

3.3 Tree With Two Internal Propagators

In order to check our interpretation of the result (3.16) as the propagator in Mellin space, we consider one more example before generalising to arbitrary tree level diagrams. We consider a Feynman diagram with two internal propagators. The position space expression for this is given by

$$\begin{aligned} I = \int \mathcal{D}u_1\mathcal{D}u_2\mathcal{D}u_3\Bigg[&\prod_{i\in 1}\Big\{(x_1^i-u_1)^{-2\Delta_1^i}\Gamma(\Delta_1^i)\Big\}\prod_{i\in 2}\Big\{(x_2^i-u_2)^{-2\Delta_2^i}\Gamma(\Delta_2^i)\Big\} \\ &\times\prod_{i\in 3}\Big\{(x_3^i-u_3)^{-2\Delta_3^i}\Gamma(\Delta_3^i)\Big\}(u_2-u_1)^{-2\gamma_{12}}(u_2-u_3)^{-2\gamma_{23}}\Bigg] \end{aligned}$$

In this case, the conformality conditions are

$$\sum_{i\in a}\Delta_a^i = D-\gamma_{a,a+1}-\gamma_{a-1,a} \quad , \qquad 1\le a\le 3$$

where, $\gamma_{01}=0=\gamma_{34}$.

For extracting the Mellin amplitude, we follow the same strategy as in the previous examples. However, now we have to choose an ordering of the vertices u_a for conducting the manipulations. All the choices lead to the same result eventually[3]. We follow the order $u_1\to u_2\to u_3$. The final result turns out to be

$$I=\prod_{a=1}^{3}\prod_{b=a}^{3}\left(\prod_{(i,j)\in a+b}\int_{-i\infty}^{i\infty}\left[ds_{ab}^{ij}\right]\Gamma(s_{ab}^{ij})\big(x_{ab}^{ij}\big)^{-2s_{ab}^{ij}}\right)\prod_{a=1}^{3}\left(\prod_{i\in a}\int_0^\infty d\alpha_a^i(\alpha_a^i)^{\rho_a^i-1}\right)M(s_{ab})$$

where,

$$\rho_a^i\equiv\Delta_a^i-\sum_{j\in a}s_{aa}^{ij}-\sum_{\substack{b=1\\ b\neq a}}^{3}\left(\sum_{j\in b}s_{ab}^{ij}\right) \quad , \qquad 1\le a\le 3$$

Again, the integration over the variables α_a^i impose the constraints $\rho_a^i=0$ which can be re-written as (using the conformality conditions)

$$\sum_{i\in a}\Delta_a^i=2s_{aa}+\sum_{\substack{b=1\\ b\neq a}}^{3}s_{ab}=D-\gamma_{a-1,a}-\gamma_{a,a+1} \quad , \qquad 1\le a\le 3$$

Due to these constraints, the number of independent Mellin variables are only $N(N-3)/2$ (where N is total number of external states). The Mellin amplitude is given by

$$\begin{aligned}M(s_{ab}) &= \frac{1}{\Gamma(\gamma_{12})\Gamma(\gamma_{23})}\int_0^\infty dt_{12}(t_{12})^{\gamma_{12}-s_{12}-s_{13}-1}\int_0^\infty dt_{23}(t_{23})^{\gamma_{23}-s_{13}-s_{23}-1}\\ &\quad \big(1+t_{12}^2(1+t_{23}^2)\big)^{-s_{11}}\Big(1+t_{23}^2\Big)^{-s_{22}-s_{12}} \\ &=\left[\frac{1}{2\Gamma(\gamma_{12})}\beta\Big(\frac{\gamma_{12}-s_{12}-s_{13}}{2},\frac{D}{2}-\gamma_{12}\Big)\right]\left[\frac{1}{2\Gamma(\gamma_{23})}\beta\Big(\frac{\gamma_{23}-s_{23}-s_{13}}{2},\frac{D}{2}-\gamma_{23}\Big)\right]\end{aligned} \tag{3.17}$$

This result, being a product of two beta functions with appropriate arguments, is consistent with our interpretation of the Mellin space propagator (3.16). Moreover, the poles of the propagator occur when the negative of the total Mellin momenta squared flowing through it is equal to the conformal dimension of the primary and descendants. This is easily seen by introducing the dual Mellin momenta and writing the arguments of beta functions in terms of these momenta.

[3] The different choices for this ordering lead to integrals over the Schwinger parameters which are not manifestly equal. For diagrams with higher number of interaction vertices, it can often be difficult to show that these different integrals corresponding to the same Feynman diagram are all equal.

4 General Tree Level Feynman Diagrams

We now consider general tree level Feynman diagrams. We shall show that the Mellin amplitude for an arbitrary tree diagram is given by the product over internal propagators. For each internal propagator, we obtain a factor of beta function with appropriate arguments consistent with the examples considered in the previous section.

In subsection 4.1, we present a diagrammatic algorithm to write down the Mellin amplitude (in terms of integrals over the Schwinger parameters) for any diagram involving only scalar operators. In subsection 4.2, we consider a tree diagram with n internal vertices such that one can go from one end of the diagram to the other end without encountering any branches (figure 9). Finally, in subsection 4.3, we consider a completely general tree diagram.

4.1 Diagrammatic Rules for Writing Mellin Amplitude

In this subsection, we present a diagrammatic technique which will be helpful in directly writing down the Mellin space amplitudes as integrals over the Schwinger parameters. These rules can be used for any tree as well as loop diagrams and will allow us to avoid going through all the algebraic manipulations, as described in the examples of the section 3. Although these rules are quite crucial in our derivation of the Mellin amplitude for a general tree level Feynman diagram, a casual reader may skip this subsection and commence reading from subsection 4.2.

For developing these rules, we shall use a simplified way to represent the Feynman diagrams. In our diagrammatic algorithm, the external lines in a Feynman diagram would not be playing any significant role. Hence, to simplify the diagrammatic representation, we represent the set of external lines attached to an interaction vertex by a small hollow circle at the vertex and the internal propagator by dashed lines. We call this the *skeleton of the Feynman diagram*. The skeleton for the single propagator Feynman diagram we considered earlier, is shown in Figure 3. Note that this way of representing a Feynman diagram is insensitive to the number of external legs attached to any given interaction vertex.

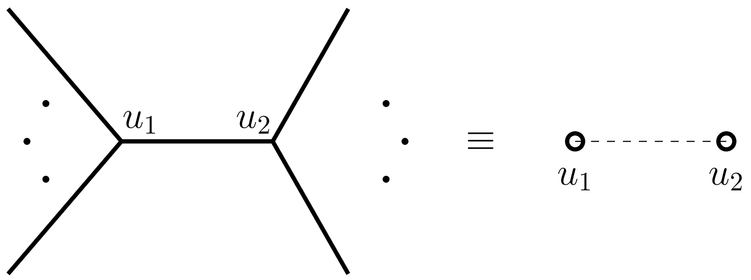

Figure 3. Skeleton of the single propagator diagram

4.1.1 Motivating the rules

To illustrate the diagrammatic rules, it is best to consider an explicit example. We consider the two propagator case of section 3 for this purpose. The skeleton of this Feynman diagram is shown in Figure 4.

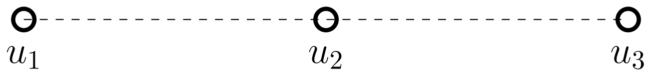

Figure 4. Tree level three vertex

For integrating over the position space vertices, we choose the ordering $u_1 \to u_2 \to u_3$. From the derivation given in the previous section, it is clear that when we integrate over a vertex and perform the associated steps, all the lines connected to that vertex get connected among themselves in the sense

$$\int \mathcal{D}u \prod_i \left(x^i - u\right)^{-2\Delta^i} \longrightarrow \prod \left(x^i - x^j\right)^{-2s^{ij}}$$

We represent this by replacing the small hollow circle associated to that vertex with a bigger circle and the dashed lines connecting the adjacent un-integrated vertices with solid lines. We associate a weight 1 with this bigger circle and a weight t_{12} with the solid line (which is the Schwinger parameter associated with the line joining the vertices 1 and 2). Diagrammatically, we can represent this step as in Figure 5 (from now on, we won't write the vertex indices u_a)

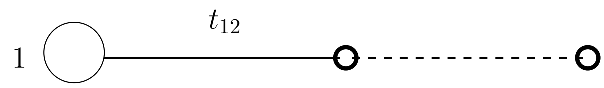

Figure 5. Diagrammatic representation of integration over vertex 1

Next, we perform the integration over the u_2 vertex. The effect of integration over this vertex is represented by replacing the small circle around it with a bigger circle, replacing the dashed line connecting it with u_3 vertex with a solid line and making one more circle around the u_1 vertex. Moreover, we also connect the vertices u_1 and u_3 by a different solid line. Following the derivation, we see that we need to associate a weight 1 with the circle around u_2 vertex, a weight t_{23} with the solid line connecting u_2 and u_3 vertices, a weight $t_{12}t_{23}$ with the solid line connecting u_1 and u_3 and a weight t_{12}^2 with the new circle around u_1. This step, combined with the first step, can be represented as in Figure 6.

Finally, we integrate over the third vertex. The effect of this integration is represented by making the small circle around that vertex bigger. Drawing another circles around the first two vertices each and drawing another solid line connecting the first and second vertices. We associate a weight of 1 for the circle around the third vertex, a weight of t_{23}^2 for the new circle around the second vertex, a weight of $t_{12}^2 t_{23}^2$ for the new circle around the

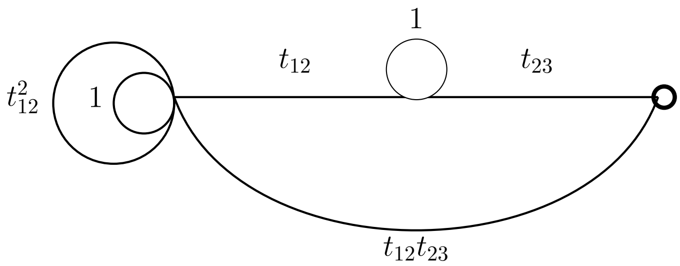

Figure 6. Diagrammatic representation of integration over second vertex

first vertex and a weight of $t_{12}t_{23}^2$ for new solid line connecting the first two vertices. This is shown in Figure 7.

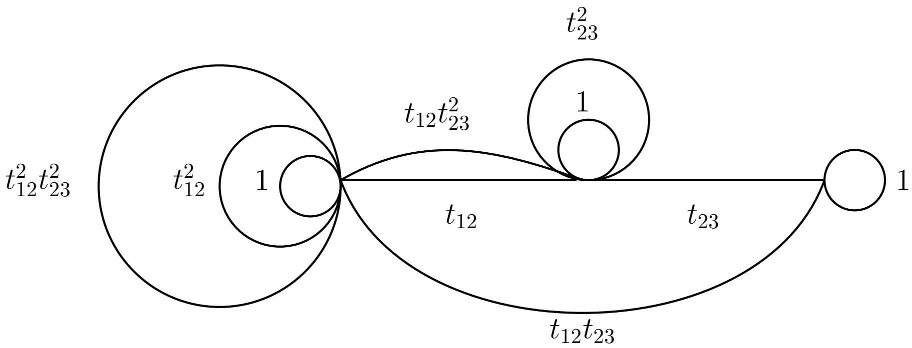

Figure 7. Diagrammatic representation of integration over third vertex

Next, we replace the two lines between the first two vertices by a single line and associate a weight which is sum of the weights of previous two lines. Similarly, we replace the multiple circles at each vertex by a single circle and associate a weight which is sum of the weights of all circles initially present. After combining multiple lines and circles, Figure 7 has been redrawn in Figure 8.

To write the Mellin amplitude,

1. For each initial dashed line between the vertices u_a and u_b, we associate an integral
$$\frac{1}{\Gamma(\gamma_{ab})}\int_0^\infty dt_{ab}\;(t_{ab})^{\gamma_{ab}-1}$$
2. For each solid line and circle in the final diagram, we include in the integrand, the corresponding weight factor raised to the power of s_{ab} where a and b are the two vertices associated with the line or the circle (in which case $a = b$).

Following the steps described above, the Mellin amplitude for the three vertex tree can be

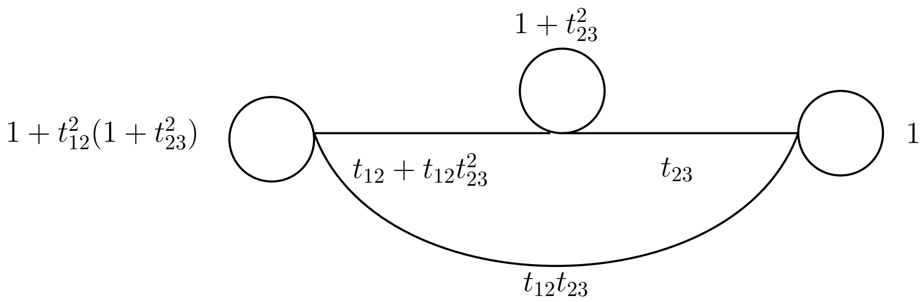

Figure 8. Final step for writing the Mellin amplitude

obtained to be

$$M(s_{ab}) = \frac{1}{\Gamma(\gamma_{12})\Gamma(\gamma_{23})} \int_0^\infty dt_{12}(t_{12})^{\gamma_{12}-1} \int_0^\infty dt_{23}(t_{23})^{\gamma_{23}-1}$$
$$\left[1 + t_{12}^2(1 + t_{23}^2)\right]^{-s_{11}} \left[1 + t_{23}^2\right]^{-s_{22}} \left[t_{12}(1 + t_{23}^2)\right]^{-s_{12}} \left[t_{12}t_{23}\right]^{-s_{13}} \left[t_{23}\right]^{-s_{23}}$$

This is same as the expression (3.17) given in the previous section.

4.1.2 General Rules

We now state the general rules for writing down the Mellin amplitude for any given Feynman diagram involving scalar fields. We start with the skeleton and follow the following steps for each interaction vertex, one at a time. For a general Feynman diagram there is a freedom to choose the order in which the different vertices are integrated over one by one. This procedure works for any chosen ordering.

Diagrammatic representation of integrating over an interaction vertex

At any interaction vertex on the skeleton (which has not been integrated yet), in general, there will be a small hollow circle denoting the external lines, and dashed and solid lines for the internal propagators. To represent the effect of integration over this vertex, we do the following:

1. Replace the small circle with a bigger circle and associate a weight 1.

2. If this vertex is connected by a solid line (with weight t) to another vertex which already has a circle with some weight, draw another circle at that vertex. Associate a weight t^2 to this new circle.

3. If this vertex (that is being integrated over) is connected to another vertex with a dashed line, we replace that dashed line with a solid line and associate a weight equal to the Schwinger parameter for this internal line.

4. After the third step, if this vertex (which is being integrated over) happens to be connected to two or more vertices $\{a\}$ by solid lines with weights $\{t_a\}$, then we join

each pair of those vertices by a solid line as well. To the new line joining vertex a and b, we associate the weight $t_a t_b$.

5. If any two vertices are connected by multiple lines (or any vertex has multiple circles) with each line (or circle) being associated with some weight, we replace them with a single line (or a single circle) with a weight equal to the sum of the weights of the individual lines (or circles). The final diagram should have a single line between any two vertices and a single circle at each vertex.

6. If the vertex (which is being integrated over) is only connected with internal lines but no external lines (in other words, it does not have small circle), then we do not make a bigger circle around it. However, the steps 2-5 are still applicable.

Writing the Mellin amplitude

1. For each initial dashed line between the points a and b, we write an integral

$$\frac{1}{\Gamma(\gamma_{ab})}\int_0^\infty dt_{ab}\,(t_{ab})^{\gamma_{ab}-1}$$

2. For each solid line between the interaction vertices a and b (and a circle at a) in the final diagram, we include in the integrand a factor equal to the corresponding weight raised to the power s_{ab} (s_{aa} for the circle).

Although the output of this procedure is always the Mellin amplitude which is unique, the exact expression for the integrand (function of the Schwinger parameters for the internal lines) depends on the order we choose for integrating over the vertices. Also, it is not easy to show by direct evaluation of the integrals that these different integrals are in fact equal. For our purposes, we shall choose the order for integrating over the vertices that leads to the simplest integral.

4.2 n-Vertex Simple Tree

In this subsection, we consider a tree level diagram with any number of interaction vertices such that all the internal propagators are connected as a single chain. In other words, there are no branches on the skeleton as shown in Figure 9. We refer to this diagram as the *simple n-vertex tree*. The examples considered in section 3 are special cases of this.

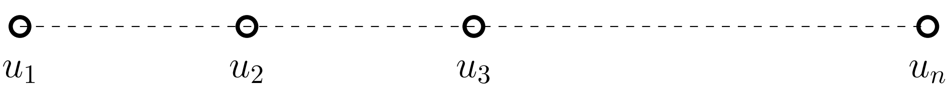

Figure 9. Simple tree with n vertices

The position space amplitude for this diagram is given by the following integral expression

$$I = \prod_{a=1}^{n}\Bigg[\int \mathcal{D}u_a\Bigg\{\prod_{i\in a}(x_a^i-u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\Bigg\}\Bigg\{(u_a-u_{a+1})^{-2\gamma_{a,a+1}}\Bigg\}\Bigg]$$

This expression can be brought to the standard form

$$I = \prod_{a=1}^{n}\prod_{b=a}^{n}\Bigg(\prod_{(i,j)\in a+b}\int_{-i\infty}^{i\infty}\Big[ds_{ab}^{ij}\Big]\,\Gamma(s_{ab}^{ij})\big(x_{ab}^{ij}\big)^{-2s_{ab}^{ij}}\Bigg)\prod_{a=1}^{n}\prod_{i\in a}\Big(2\pi i\delta(\rho_a^i)\Big)M(s_{ab}) \quad (4.1)$$

$$\text{where,} \qquad \rho_a^i \equiv \Delta_a^i - \sum_{j\in a}s_{aa}^{ij} - \sum_{\substack{b=1\\ b\neq a}}^{n}\Bigg(\sum_{j\in b}s_{ab}^{ij}\Bigg) \quad , \qquad 1\le a\le n \quad (4.2)$$

For the standard order of integration over the position space vertices (namely, $u_1 \to u_2 \to \cdots \to u_n$), the Mellin amplitude $M(s_{ab})$ in (4.1) is given by

$$M(s_{ab}) = \frac{1}{\prod_{a=1}^{n-1}\Gamma(\gamma_{a,a+1})}\prod_{a=1}^{n-1}\Bigg\{\int_0^{\infty}dt_{a,a+1}\big(t_{a,a+1}\big)^{R_a-1}\Big(G_a^n\Big)^{-Q_a}\Bigg\} \quad (4.3)$$

where,

$$R_a = \gamma_{a,a+1} - \sum_{b=a+1}^{n}\sum_{c=1}^{a}s_{cb} \quad , \qquad Q_a = \sum_{b=1}^{a}s_{ba} \quad , \qquad 1\le a\le n-1 \quad (4.4)$$

$$G_a^b = 1 + t_{a,a+1}^2(1+t_{a+1,a+2}^2(\cdots\cdots + t_{b-1,b}^2)) \qquad 1\le a\le n-1, \qquad a\le b \quad (4.5)$$

To evaluate this integral, we start with the Schwinger variable $t_{n-1,n}$ and make a coordinate transformation and a rescaling simultaneously

$$1+t_{n-1,n}^2 = y_{n-1} \quad , \qquad y_{n-1}t_{n-2,n-1}^2 \to t_{n-2,n-1}^2$$

By making a further coordinate transformation $y_{n-1}-1 \to y_{n-1}$, the integration over y_{n-1} can be recognised as a beta function and we obtain

$$M(s_{ab}) = \frac{1}{\prod_{a=1}^{n-1}\Gamma(\gamma_{a,a+1})}\prod_{a=1}^{n-2}\Bigg\{\int_0^{\infty}dt_{a,a+1}\big(t_{a,a+1}\big)^{R_a-1}\Big(G_a^{n-1}\Big)^{-Q_a}\Bigg\}$$
$$\frac{1}{2}\beta\Big(\frac{R_{n-1}}{2},\frac{R_{n-2}-R_{n-1}}{2}+Q_{n-1}\Big)$$

We iteratively perform similar steps in order to integrate over the remaining Schwinger parameters and make necessary simplifications to obtain,

$$M(s_{ab}) = \frac{1}{\prod_{a=1}^{n-1}\Gamma(\gamma_{a,a+1})}\prod_{a=1}^{n-1}\frac{1}{2}\beta\Bigg(\frac{\gamma_{a,a+1} - \sum_{b\in L_{a,a+1}}\sum_{c\in R_{a,a+1}}s_{bc}}{2}\ ,\ \frac{D-2\gamma_{a,a+1}}{2}\Bigg) \quad (4.6)$$

where,

$$\sum_{b \in L_{a,a+1}} \sum_{c \in R_{a,a+1}} s_{bc} \equiv \sum_{b=1}^{a} \sum_{c=a+1}^{n} s_{bc}$$

The $L_{a,a+1}$ and $R_{a,a+1}$ appearing in above two equations stand for Left and Right respectively. If we cut the diagram 9 along the propagator $t_{a,a+1}$, the vertices will get divided in two sets. The set $L_{a,a+1}$ includes all the vertices which lie to to left of the cut and the set $R_{a,a+1}$ includes all the vertices which lie to the right of the cut. An example for $n = 4$ is given in Figure 10 in which the sets $L_{3,4}$ and $R_{3,4}$ have been shown.

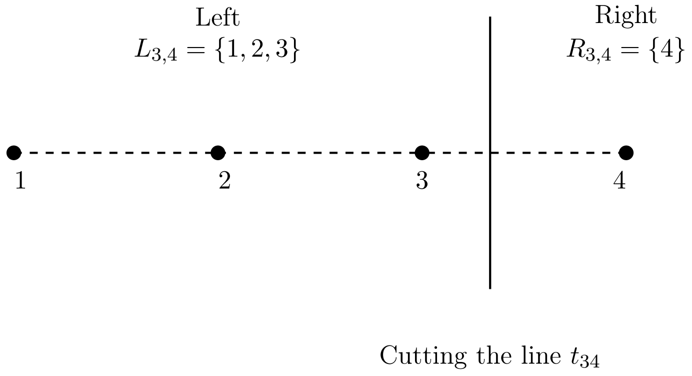

Figure 10. Left and Right of a cut line

The result (4.6) is consistent with the previous examples as the Mellin amplitude is a product over all the propagator factors (each of which is a beta function with appropriate arguments). We can again introduce dual Mellin momenta and replace the Mellin variables in the arguments of beta functions in favour of the total Mellin momenta flowing through the propagator. The propagators develop a pole when the negative of total Mellin momenta squared flowing through it becomes equal to the conformal dimension of a primary or descendant flowing through it.

4.3 General Tree

In this section, we consider a completely general tree Feynman diagram and show that the Mellin amplitude for it can be written in a simple form as product over all the internal propagator factors.

The derivation of Mellin amplitude for such arbitrary tree Feynman diagrams is a graph theoretic exercise and does not shed any light on the physical significance of the result itself. Hence, in this section, we shall only state the final result and discuss it's physical significance. The details of the derivation have been presented in the appendix D.1.

From the diagrammatic rules given in section 4.1, we know that the amplitude can be

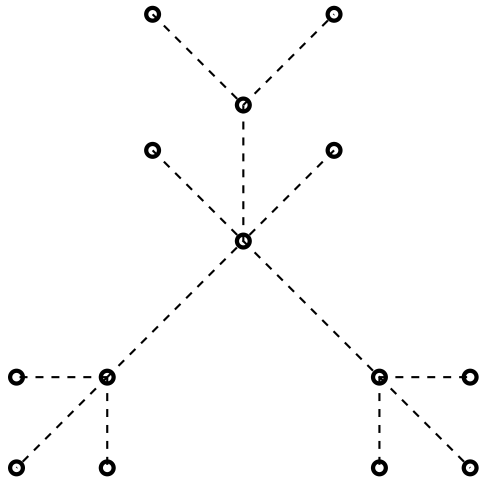

Figure 11. Example of a general tree level Feynman diagram (skeleton)

written in the following form

$$M\left(\{s_{ab}\}\right) = \prod \left[\int_0^\infty dt_{ab} \frac{(t_{ab})^{\gamma_{ab}-1}}{\Gamma(\gamma_{ab})}\right] F\Big(\{t_{ab}\}, \{s_{ab}\}\Big) \tag{4.7}$$

The product runs over all the internal lines[4]. F is function of the Schwinger parameters and the Mellin variables.

The function F depends on the order of integration of the position space vertices and, in general, is a very complicated function of the Schwinger parameters. It turns out that for the tree diagrams, it is possible to make a choice for the order in which the vertices are integrated over such that the integral (4.7) can be performed easily. This has been described in detail in the appendix D.1. With such a choice, the function F can be expressed as

$$F \;=\; \prod_{\text{all propagators}} (t_{ab})^{-P_{ab}} (A_{ab})^{-Q_{ab}}$$

where,

$$P_{ab} = \sum_{c\in L_{ab}} \sum_{d\in R_{ab}} s_{cd} \qquad ; \qquad Q_{ab} = \sum_{\substack{c\in L_{ab}\\ c\neq a}} \sum_{\substack{d\in \tilde{L}_{ab}\\ d\neq a}} s_{cd} \;+ \sum_{d\in L_{ab}} s_{ad}$$

The term involving the double sum in Q_{ab} is absent if there is no branching at the vertex a in the skeleton. The tilde in one of the L in this double sum denotes the fact that we should not include terms of the type s_{cd} where c and d are on the same branch in the set L_{ab}.

To define the function A_{ab}, we shall need a reference vertex which can be chosen freely from any one of the end vertices (a vertex with only one dashed line attached to it) on the

[4]Since we are considering a general tree which may have brances, it is not necessary that neighbouring vertices will always be labelled with consecutive integers.

skeleton. Let that vertex be $\mathcal{P}$ (see figure 15 and the related discussion in appendix D.1), then

$$A_{ab} \equiv 1 + t_{ab}^2 \left(1 + t_{bc}^2 \left(1 + ...(1 + t_{o\mathcal{P}}^2)...\right)\right)$$

b,c,...,o are all on the shortest continuous route from a to the reference vertex $\mathcal{P}$.

The final result, after integrating over all the Schwinger parameters in (4.7), turns out to be a product of beta functions with one beta function for each internal propagator. The arguments of beta functions involve the Left and Right part of the propagator as in the case of simple tree in previous subsection. Since there may be branches in our tree, we need to specify what Left and Right of a cut line mean in this context. As a rule, we refer to the part of the diagram (after the cut) having the reference vertex $\mathcal{P}$ as the Right. With this, we can write the Mellin amplitude for a completely general tree as

$$M\left(\{s_{ab}\}\right) = \prod \frac{1}{2\Gamma(\gamma_{ab})} \beta \left(\frac{\gamma_{ab} - \sum\limits_{c \in L_{ab}} \sum\limits_{d \in R_{ab}} s_{cd}}{2} \ , \ \frac{D}{2} - \gamma_{ab} \right) \tag{4.8}$$

The product is over all the internal propagators of the diagram.

The physical interpretation of the above result becomes clear if we again consider the dual Mellin momenta. As before, we denote the Mellin momentum flowing into the diagram through the external vertex x_a^i by k_a^i. We first note that the constraints on the Mellin variables are automatically satisfied if the total Mellin momentum is conserved. The constraint satisfied by the Mellin variables is

$$\sum_b \left(\sum_{j \in b} s_{ab}^{ij} \right) = \Delta_a^i \qquad \Longrightarrow \qquad k_a^i \cdot \left(\sum_b \sum_{j \in b} k_b^j \right) = 0$$

where, we have used $\Delta_a^i = -(k_a^i)^2$.

The simplest way to satisfy the above equation is by demanding that the total Mellin momenta is conserved, namely $\sum\limits_b \sum\limits_{j \in b} k_b^j = 0$.

Now, the full Mellin momentum propagating through an internal propagator, joining the vertices a and b, is

$$k = \sum_{c \in L_{ab}} \sum_{i \in c} k_c^i = - \sum_{c \in R_{ab}} \sum_{i \in c} k_c^i$$

and the kinematical variable in the propagator is

$$\sum_{c \in L_{ab}} \sum_{d \in R_{ab}} s_{cd} = \sum_{c \in L_{ab}} \sum_{d \in R_{ab}} \sum_{i \in c} \sum_{j \in d} s_{cd}^{ij} = \left(\sum_{c \in L_{ab}} \sum_{i \in c} k_c^i \right) \cdot \left(\sum_{d \in R_{ab}} \sum_{j \in d} k_d^j \right) = -k^2$$

In terms of the Mellin momenta, the expression for the propagator can be written as

$$\frac{1}{2\Gamma(\gamma_{ab})} \beta \left(\frac{1}{2} \left(\gamma_{ab} + k^2 \right), \frac{D}{2} - \gamma_{ab} \right) = \frac{1}{\Gamma(\gamma_{ab})} \sum_{n=0}^{\infty} \frac{\Gamma\left(\gamma_{ab} - \frac{D}{2} + 1 + n\right)}{n! \Gamma\left(\gamma_{ab} - \frac{D}{2} + 1\right)} \frac{1}{k^2 + \gamma_{ab} + 2n}$$

This shows that the total Mellin momentum (squared) flowing through the propagator has poles at $-\gamma_{ab} - 2n$. These correspond to the propagation of a primary field ($n = 0$) and the corresponding descendants ($n > 0$). The above sum representation of the propagator is analogous to the Kählén Lehmann spectral representation in ordinary quantum field theories.

The Feynman rules for tree level Feynman diagrams in perturbative CFT for scalar fields is now obvious. The propagator for any internal line is given by (4.8) and we simply multiply all the propagator factors of the diagram.

We would like to wrap up this discussion with a brief recapitulation of the most important points we have learnt so far. Mellin space provides a manifest conformally invariant representation for correlation functions in a CFT. At tree level, there exist a set of Mellin space Feynman rules that can be associated with Feynman diagrams involving scalar operators. Some linear combinations of the Mellin variables that appear in the propagators can be interpreted as Mandelstam variables constructed out of the (hypothetical) external Mellin momenta flowing into the diagram. The invariance of the amplitude under special conformal transformation allows for a statement of conservation of Mellin momentum. All the Mellin variables (or equivalently all the Mandelstam variables) are not independent and the number of independent Mellin variables is equal to the number of independent cross ratios between the external vertices in the diagram. Mellin space also allows a spectral representation for the correlation functions as any propagator in the diagram has a discrete infinite set of poles corresponding to the exchanged primary field and its descendants.

5 One-Loop Feynman Diagram

After deriving the Mellin space Feynman rules for tree level diagrams, the next step is to consider loop diagrams. We have not yet been able to derive the Feynman rules for loop diagrams. In this section, we shall content ourselves with the expression for the one-loop Mellin amplitude as an integral over the internal Schwinger parameters. It may be possible to derive the loop Feynman rules in Mellin space using an approach similar to the one presented in appendix D.2 which treats the n-vertex simple tree in a way different from what we have already seen in section 4.

The position space amplitude for the loop Feynman diagram in figure (12) is given by

$$I = \prod_{a=1}^{n} \left[\int \mathcal{D}u_a \prod_{i\in a} \left\{ (x_a^i - u_a)^{-2\Delta_a^i} \Gamma(\Delta_a^i) \right\} \right] \prod_{b=1}^{n} (u_b - u_{b+1})^{-2\gamma_{b,b+1}} \tag{5.1}$$

where $n + 1 \equiv 1$.

5.1 One-Loop Mellin Amplitude

The Mellin amplitude for the n-vertex one-loop diagram can be derived using the position space amplitude (5.1) by following the same procedure as in the previous sections for tree

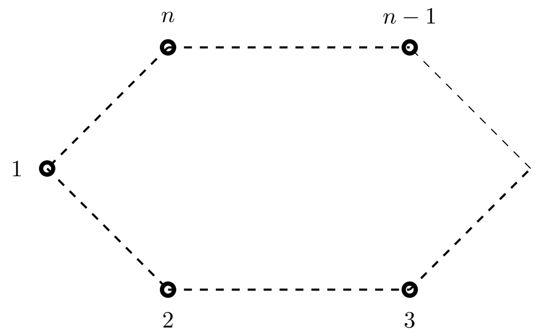

Figure 12. One loop diagram with n internal vertices

diagrams. The Mellin amplitude is written as integral over the Schwinger parameters and the integrand depends upon the order in which we perform the integration over the interaction vertices in position space. For the cyclic order of integration $(u_1 \to u_2 \to \cdots \to u_n)$, the Mellin amplitude turns out to be

$$M(s_{ab}) = \prod_{a=1}^{n} \left[\int_0^\infty dt_{a,a+1} \frac{(t_{a,a+1})^{\gamma_{a,a+1}-1}}{\Gamma(\gamma_{a,a+1})} \right] \prod_{a=1}^{n-1} \prod_{b=a}^{n} \left(\tilde{H}_a^b + \tilde{K}_a \tilde{K}_b \right)^{-s_{ab}} \tag{5.2}$$

where,

$$\begin{aligned}
\tilde{H}_a^b &\equiv t_{a,a+1} \cdots t_{b-1,b} G_b^{n-1} && 1 \leq a < b \leq n-1 \\
\tilde{H}_a^a &\equiv G_a^{n-1} && 1 \leq a \leq n-1 \\
\tilde{H}_a^n &\equiv 0 && 1 \leq a \leq n \\
\tilde{K}_a &\equiv \left(t_{1n} t_{12} \cdots t_{a-1,a} G_a^{n-1} + t_{a,a+1} \cdots t_{n-1,n} \right) && 1 \leq a \leq n-1 \\
\tilde{K}_n &= 1
\end{aligned}$$

We have defined G_a^b in eq. A.1.

5.2 A Consistency Check

For the consistency of the diagrammatic algorithm (which works for any Feynman diagram) given in section 4.1, the one loop amplitude considered above should reduce to the n-vertex simple tree amplitude when we cut a propagator of the loop.[5] This indeed turns out to be the case which is a sanity check for the result (5.2).

It turns out that removing different propagators corresponds to the tree result written in different forms. These different forms correspond to choosing different order of integration for the position space vertices of the tree. For example, setting $t_{a,a+1}$ to zero gives the corresponding result for n vertex simple tree for which the integration has been performed

[5] Operationally, this can be done by "removing" the integration over the corresponding Schwinger parameter.

in the order $u_{a+1} \to u_{a+2} \to \cdots \to u_n \to u_1 \to \cdots \to u_a$. As a special case, the limit $t_{1n} \to 0$ gives the result for the standard order of integration $(u_1 \to \cdots \to u_n)$ over the position space vertices.

5.3 Special Case: Loop With 3 Internal Vertices

The conformal Mellin amplitude of one loop diagram with 3 internal vertices ("delta" diagram) can be exactly evaluated in terms of a tree amplitude ("star" tree diagram). This happens due to the standard "star-delta" relation in an analogy with a similar result in electrical circuits.

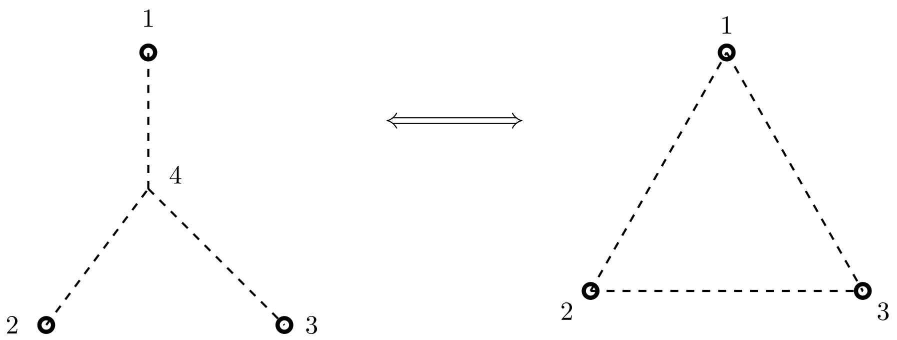

Figure 13. Skeleton of the "star" and the "delta"

The position space expression for the star diagram is

$$I_{star} = \int \mathcal{D}u_4 \prod_{a=1}^{3}\left[\int \mathcal{D}u_a \left\{\prod_{i\in a}(x_a^i - u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\}(u_a - u_4)^{-2\gamma'_{a,4}}\right]$$

To show the equivalence with the 3 vertex loop, we need to perform the integration over the central vertex u_4. For this vertex, we perform the standard algebraic steps as in section 3.1 and obtain

$$\begin{aligned} I_{star} &= \\ &\prod_{a=1}^{3}\left\{\int \mathcal{D}u_a \prod_{i\in a}(x_a^i - u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\}\prod_{a=1}^{3}\left(\int_0^\infty dt_{a,4}\,\frac{(t_{a,4})^{\gamma'_{a,4}-1}}{\Gamma(\gamma'_{a,4})}\right) \\ &\left(\prod_{a=1}^{3}\int [ds_a]\Gamma(s_a)\right)\Big\{t_{14}t_{24}(u_1-u_2)\Big\}^{-2s_3}\Big\{t_{14}t_{34}(u_1-u_3)\Big\}^{-2s_2}\Big\{t_{24}t_{34}(u_2-u_3)\Big\}^{-2s_1} \end{aligned}$$

Integration over the Schwinger parameters $t_{a,4}$ give 3 delta functions. We can thus perform the 3 integrals over s_a. The resulting expression is proportional to the 3 vertex loop amplitude in position space, i.e.

$$\begin{aligned} I_{star} &= \frac{\Gamma(\gamma_{12})\Gamma(\gamma_{23})\Gamma(\gamma_{13})}{\prod_{a=1}^{3}\Gamma(\gamma'_{a,4})}\prod_{a=1}^{3}\left\{\int \mathcal{D}u_a \prod_{i\in a}(x_a^i - u_a)^{-2\Delta_a^i}\Gamma(\Delta_a^i)\right\} \\ &\quad\times (u_1-u_2)^{-2\gamma_{12}}(u_1-u_3)^{-2\gamma_{13}}(u_2-u_3)^{-2\gamma_{23}} \end{aligned}$$

where,

$$\gamma'_{14} = \gamma_{12} + \gamma_{13} \qquad , \quad \gamma'_{24} = \gamma_{12} + \gamma_{23} \qquad , \quad \gamma'_{34} = \gamma_{13} + \gamma_{23}$$

Thus, if we represent the internal propagator of 3 vertex loop by γ_{ab} and those of the star diagram by γ'_{a4}, then the above relation says

$$I_{star}(s_{ab}, \gamma'_{ab}) = \frac{\Gamma(\gamma_{12})\Gamma(\gamma_{23})\Gamma(\gamma_{13})}{\Gamma(\gamma'_{14})\Gamma(\gamma'_{24})\Gamma(\gamma'_{34})} \quad \times \quad I_{delta}(s_{ab}, \gamma_{ab})$$

The star diagram is a tree diagram and its Mellin space amplitude can be easily written down using the Feynman rules given in the previous sections. Thus, we find the 3 vertex loop amplitude in Mellin space to be

$$I_{delta} = \frac{1}{8\Gamma(\gamma_{12})\Gamma(\gamma_{23})\Gamma(\gamma_{13})} \beta\left(\frac{\gamma_{12}+\gamma_{13}-s_{12}-s_{13}}{2}, \frac{D}{2}-\gamma_{12}-\gamma_{13}\right)$$
$$\beta\left(\frac{\gamma_{12}+\gamma_{23}-s_{12}-s_{23}}{2}, \frac{D}{2}-\gamma_{12}-\gamma_{23}\right)\beta\left(\frac{\gamma_{13}+\gamma_{23}-s_{13}-s_{23}}{2}, \frac{D}{2}-\gamma_{13}-\gamma_{23}\right)$$

6 Non-Conformal Mellin Amplitudes

In this section, we revisit some of the tree level Feynman diagrams we have been considering so far. However, this time we relax the conformality conditions imposed on them. The motivation for defining these "non-conformal Mellin amplitudes" comes from noting that exactly marginal deformations of CFTs are rare (generally arising only in some special supersymmteric gauge theories).

We have been considering CFTs whose Lagrangian descriptions are in terms of scalar fields. Since the couplings generically run with the energy scale, the beta function is non-zero and conformal invariance is broken. Thus, even a classically marginal perturbation generally breaks conformal invariance once quantum effects are included[6]. We thus study these non-conformal Mellin amplitudes by considering a generic scalar perturbation around a free CFT that may not preserve any of the conformal or scale symmetry. At an operational level, we relax the conformality conditions that we have been imposing at each interaction vertex.

6.1 Some Examples

As a concrete example, we consider the simple n-vertex tree of figure 9. The conformal Mellin amplitude for this diagram was given in (4.3). If we do not impose the conformality conditions, then instead of (4.1), we obtain

$$I = \prod_{a=1}^{n}\prod_{b=a}^{n}\left(\prod_{(i,j)\in a+b}\int_{-i\infty}^{i\infty}\left[ds_{ab}^{ij}\right]\Gamma(s_{ab}^{ij})\left(x_{ab}^{ij}\right)^{-2s_{ab}^{ij}}\right)\tilde{M}(s_{ab})$$

[6]We would like to thank R. Loganayagam for discussions on this issue.

where,

$$\begin{aligned}
\tilde{M}(s_{ab}) &= \prod_{a=1}^{n}\left(\prod_{i\in a}\int_0^\infty d\alpha_a^i(\alpha_a^i)^{\rho_a^i-1}\right)\prod_{a=1}^{n-1}\left\{\int_0^\infty \frac{dt_{a,a+1}}{\Gamma(\gamma_{a,a+1})}\big(t_{a,a+1}\big)^{R_a-1}\Big(G_a^n\Big)^{-Q_a}\right\}\\
&\quad\times\prod_{a=1}^{n}\left\{\sum_{b=1}^{a-1}\left(H_b{}^a\sum_{i\in b}\alpha_b^i\right)+\sum_{b=a}^{n}\left(H_a{}^b\sum_{i\in b}\alpha_b^i\right)\right\}^{-\lambda_a}\\
&\equiv \delta\left(\sum_{a=1}^{n}(\lambda_a-\rho_a)\right)M(s_{ab})
\end{aligned}\tag{6.1}$$

where ρ_a^i is defined in equation (4.2) and

$$\begin{aligned}
G_a^c &= 1+t_{a,a+1}^2(1+t_{a+1,a+2}^2(\cdots\cdots+t_{c-1,c}^2)) \qquad 1\le a\le n-1\\
H_a{}^b &= t_{a,a+1}t_{a+1,a+2}\cdots t_{b-1,b}G_b{}^n \quad,\qquad H_a{}^a = G_a{}^n\\
\lambda_a &= D-\sum_{i\in a}\Delta_a^i-\gamma_{a-1,a}-\gamma_{a,a+1} \quad,\qquad 1\le a\le n\\
\rho_a &= \sum_{i\in a}\rho_a^i \quad,\qquad 1\le a\le n
\end{aligned}$$

It is a simple exercise to extract the overall delta function from the expression of $\tilde{M}(s_{ab})$ as we have done in equation (6.1). We shall take $M(s_{ab})$ in (6.1) to be the definition of the "non-conformal Mellin amplitude" for n-vertex simple tree[7].

One crucial limitation of this treatment that should be noted here is that the delta function in (6.1) is graph dependent and consequently the definition of $M(s_{ab})$ is also graph dependent. Therefore, although we can calculate $M(s_{ab})$ for individual diagrams, that is not exactly equal to doing perturbation theory in Mellin space. The delta function emerges in this context from the fact that the position space integrals still scale in a given way, although this scaling property depends on the particular graph being considered and may not refer to any symmetry of the theory itself.

The conformality condition amounts to setting all λ_a equal to zero. However, we work with non zero λ_a. We have not been able to obtain any simple set of Feynman rules from the general expression given in (6.1). We shall content ourselves with the special cases of $n=1$ and $n=2$ which give us some interesting results.

6.1.1 Contact Interaction

For a single vertex (i.e. $n=1$), the expression (6.1) gives

$$\tilde{M}(s^{ij}) = \prod_{i=1}^{N}\left(\int_0^\infty d\alpha^i(\alpha^i)^{\rho^i-1}\right)\Big(\sum_{i=1}^{N}\alpha^i\Big)^{\sum_{i=1}^{N}\Delta^i-D}$$

[7] We are using the same symbol $M(s_{ab})$ to denote both conformal as well as non-conformal Mellin amplitudes. However, the distinction should be clear from the context.

All symbols have their usual meaning as used previously. In the conformal case, the expression above just gives delta function constraints on the Mellin variables. We now evaluate this in the non conformal case. For this, we insert the partition of unity

$$1 = \int_0^\infty dq \;\; \delta\left(q - \sum_{i=1}^{N} \alpha^i\right)$$

in the above integral, make the coordinate transformations $\alpha^i = q\, y^i$ and use the identity

$$\prod_{i=1}^{N} \int_0^\infty dx^i \;\, (x^i)^{\rho^i - 1} \delta\left(1 - \sum_{i=1}^{N} x^i\right) = \frac{\prod_{i=1}^{N} \Gamma(\rho^i)}{\Gamma\left(\sum_{i=1}^{N} \rho^i\right)} \tag{6.2}$$

Using the expression for ρ_i, we finally obtain,

$$M(s^{ij}) = \frac{\prod_{i=1}^{N} \Gamma\left(\Delta^i - \sum_{\substack{j=1 \\ j\neq i}}^{N} s^{ij}\right)}{\Gamma\left(D - \sum_{i=1}^{N} \Delta^i\right)} \tag{6.3}$$

We have used the constraint arising from the overall delta function (see the definition (6.1)) to simplify the arguments of the Gamma function in denominator.

6.1.2 Tree With One Internal Propagator

We next consider the tree diagram with one internal propagator (i.e. $n = 2$). For $n = 2$, the expression (6.1) reduces to

$$\begin{aligned} \tilde{M}(s_{ab}) = \frac{1}{\Gamma(\gamma)} \prod_{a=1}^{2} \prod_{i\in a} \left(\int_0^\infty d\alpha_a^i (\alpha_a^i)^{\rho_a^i - 1}\right) \int_0^\infty dt \; (t)^{\gamma - s_{12} - 1} \big(1 + t^2\big)^{-s_{11}} \\ \left(\big(1 + t^2\big) \sum_{i\in 1} \alpha_1^i + t \sum_{i\in 2} \alpha_2^i\right)^{-\lambda_1} \left(\sum_{i\in 2} \alpha_2^i + t \sum_{i\in 1} \alpha_1^i\right)^{-\lambda_2} \end{aligned} \tag{6.4}$$

where, we have relabelled $t_{12} \to t$ and $\gamma_{12} \to \gamma$ to match with our notation in section 3. After some manipulations (see Appendix E), the non-conformal Mellin amplitude can be extracted to be,

$$\begin{aligned} &M(s_{ab}) = \\ &{}_3F_2\left(\gamma - \frac{D}{2} + \lambda_1 + \lambda_2\,,\; \frac{R_1 + \rho_1 - \lambda_1}{2}\,,\; \frac{R_1 + \rho_2 - \lambda_2}{2}\,;\; \frac{R_1 + \rho_1 + \lambda_1}{2}, \frac{R_1 + \rho_2 + \lambda_2}{2}\,;\, 1\right) \\ &\times \frac{1}{2\Gamma(\gamma)} \prod_{a=1}^{2} \left[\prod_{i\in a} \Gamma(\rho_a^i)\right] \left[\frac{\Gamma\left(\frac{R_1 + \rho_1 - \lambda_1}{2}\right) \Gamma\left(\frac{R_1 + \rho_2 - \lambda_2}{2}\right)}{\Gamma\left(\frac{R_1 + \rho_1 + \lambda_1}{2}\right) \Gamma\left(\frac{R_1 + \rho_2 + \lambda_2}{2}\right)}\right] \end{aligned} \tag{6.5}$$

where, $R_1 = \gamma - s_{12}$ and $\lambda_a = D - \gamma - \sum_{i\in a} \Delta_a^i$.

We now look at the pole structure and the conformal limit of the expression (6.5).

Pole Structure

The poles of the amplitude (6.5) occur when the arguments of the gamma functions in the numerator in second line are zero or negative integers (${}_3F_2$ does not give rise to any pole). Thus, as a function of the Mellin variables, the poles of the amplitude lie at

$$\frac{R_1+\rho_1-\lambda_1}{2} = -n \qquad \Longrightarrow \qquad s_{11}+s_{12} = \sum_{i\in 1}\Delta_a^i + \gamma - \frac{D}{2} + 2n$$

and,

$$\frac{R_1+\rho_2-\lambda_2}{2} = -n' \qquad \Longrightarrow \qquad s_{22}+s_{12} = \sum_{i\in 2}\Delta_a^i + \gamma - \frac{D}{2} + 2n'$$

where, n, n' are zero or arbitrary positive integers, i.e. $0, 1, 2, \cdots$.

This shows that in the non-conformal case there are two sets of poles, which in the conformal limit $\lambda_a \to 0$, coalesce to give the one set of poles we had earlier. It would be interesting to find the physical interpretation, if any, of these two sets of poles.

Conformal Limit

In the conformal limit, we have λ_1, $\lambda_2 \to 0$. To impose this limit, we first take $\lambda_1 \to 0$ keeping λ_2 fixed and non-zero. In this limit, two of the arguments of the ${}_3F_2$ hypergeometric function in (6.5) will become identical and it will thus reduce to a ${}_2F_1$ hypergeometric function

$$\tilde{M}(s_{ab}) = \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\prod_{i\in a}\Gamma(\rho_a^i)\right]\left[\frac{\Gamma\left(\frac{R_1+\rho_2-\lambda_2}{2}\right)}{\Gamma\left(\frac{R_1+\rho_2+\lambda_2}{2}\right)}\right]\delta\Big(\rho_1+\rho_2-\lambda_2\Big)$$
$${}_2F_1\left(\gamma-\frac{D}{2}+\lambda_2\ ,\ \frac{R_1+\rho_2-\lambda_2}{2}\ ;\ \frac{R_1+\rho_2+\lambda_2}{2}\ ;\ 1\right)$$

Now using the Gauss identity (C.9), we obtain after some simplification

$$\tilde{M}(s_{ab}) = \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\prod_{i\in a}\Gamma(\rho_a^i)\right]\left[\frac{\delta\Big(\rho_1+\rho_2-\lambda_2\Big)}{\Gamma(\lambda_2)}\right]\beta\left(\frac{R_1+\rho_2-\lambda_2}{2}\ ,\ \frac{D}{2}-\gamma\right)$$

If we now take the limit $\lambda_2 \to 0$, we recover the Mellin amplitude (3.16) for the conformal case along with the delta function constraints (3.14) on the Mellin variables.

6.2 Scale Invariant Amplitudes and Off-Shell Interpretation

In usual QFTs, we often consider correlation functions in which the external legs are off-shell. One puts these external legs on-shell via the LSZ procedure. It turns out that we can define analogous "off-shell" objects for conformal field theories in Mellin space as well.

For this purpose, we consider position space correlation functions with scale covariance (as in a theory with scale symmetry) [8] although they need not have the full conformal

[8] In our notations, this would be equivalent to setting $\sum_a \lambda_a = 0$ although individual λ_a need not be equal to 0.

covariance. We expect any physically interesting scale invariant theory to be conformally invariant as well (see [26] and the references therein). However, it is still interesting to consider this case, since as we shall argue below, the corresponding "Mellin amplitudes" seem to be "off-shell" quantities that reduce to the "on-shell" Mellin amplitudes[9] of conformally invariant theories through an LSZ like procedure.

We can imagine extending the definition of the Mellin amplitude to scale invariant theories in the following manner

$$A\left(x^i\right) = \prod_{i<j}\left(\int_{-i\infty}^{i\infty} \frac{ds^{ij}}{2\pi i}\Gamma(s^{ij})(x^i - x^j)^{-2s^{ij}}\right)\delta\left(\sum_i \Delta^i - \sum_{i\neq j}\sum_j s^{ij}\right) M\left(s^{ij}\right)$$

As opposed to the N (number of external lines) delta function constraints for the conformal amplitude (2.1), in this case we only have one overall constraint on the Mellin variables resulting from the covariance under scale transformations. Therefore, the number of Mellin variables in this case is only $N(N-1)/2-1$ which is also the correct number of independent kinematical variables in a scale invariant theory.

For this case also, we can introduce the dual Mellin momenta in exactly the same manner as before (see section 2). In terms of these dual momenta, the overall delta function constraint translates to the condition $\sum_i (k^i)^2 = -\sum_i \Delta^i$ which is weaker than the conformal case $(k^i)^2 = -\Delta^i$. However, we can still demand the conservation of these dual momenta, i.e. $\sum_i k^i = 0$[10].

The contact interaction diagram has only one interaction vertex and ensuring scale invariance automatically ensures conformal invariance as well. However, for more than one interaction vertices, there is a difference between the scale and conformally invariant Mellin amplitudes and the expressions for the former can be obtained by imposing $\sum_a \lambda_a = 0$ on the corresponding non-conformal Mellin amplitudes. For example, for the single propagator case, to obtain the scale invariant Mellin amplitude, we would need to impose $\lambda_1 + \lambda_2 = 0$ on (6.5).

From the examples given in subsection 6.1, we can see that each amplitude has a factor involving the product over Gamma functions, namely, $\prod_i \Gamma(\rho^i)$ (we have suppressed the label for the internal vertices). We also know that the conformal Mellin amplitudes involve product over delta functions with the same arguments, namely, $\prod_i \delta(\rho^i)$. In terms of dual Mellin momenta, we can write

$$\rho^i = \Delta^i - \sum_{j\neq i} s^{ij} = (k^i)^2 + \Delta^i$$

Thus, in the conformal case, the delta functions impose a set of constraints $(k^i)^2 + \Delta^i = 0$ which is the "on-shell" condition for the Mellin momenta. In contrast, for the scale invariant amplitudes, we have Gamma functions with the same arguments for each external leg. In the space of Mellin momenta, the "on-shell point" (or equivalently the conformal theory)

[9] $(k_a^i)^2 = -\Delta_a^i$ being the on-shell condition.

[10] This is not possible when the scale covariance is also absent.

lies at the pole of the Gamma function (where its argument vanishes). This motivates us to interpret these Gamma functions as external leg factors. It is in this sense that the scale invariant amplitudes are "off-shell" objects and imposing the conformality conditions is akin to an LSZ prescription in which the external leg factors are replaced by the corresponding delta functions. It would be very nice to take this observation further and put it on a more rigorous footing.

7 Discussion

In this note, we have taken some steps in formulating Feynman rules in Mellin space for weakly coupled CFTs. For simplicity, we have restricted to scalar operators. We considered first the case when the weakly coupled CFT is defined as an exactly marginal perturbation of a free CFT. In this context, we were able to prove in complete generality that the Mellin amplitude for any tree level Feynman diagram involving only the scalar operators factorises into a product of beta functions (with appropriate arguments involving Mellin variables) each of which is associated with a propagator. The meromorphy of the Mellin amplitudes and the identification of its poles with the exchanged primary and its descendants is manifest from this result. We also gave a diagrammatic algorithm to write down the Mellin amplitude of any diagram (tree as well as loops) in terms of integrals over the internal Schwinger parameters. These are the main results of this note.

Thereafter, we undertook the study of one loop conformal Mellin amplitudes. However, we have not been able, so far, to generalize the tree level Feynman rules to loops. We discussed the scenario where a generic scalar perturbation about a free CFT breaks the conformality of the theory. In particular, even if we add a classically marginal perturbation, the loop corrections will, in general, break the marginality of the interaction rendering the perturbative interacting theory non-conformal. We extended the definition of the Mellin amplitudes as provided by Mack [1] to such a setting where we have only one constraint restricting the number of independent Mellin variables. We calculated some simple examples of these non-conformal Mellin amplitudes. The Beta function conformal propagator uplifts to a ${}_3F_2$ hypergeometric function in the non-conformal case. We also considered position space correlators in theories with scaling symmetry but not the full conformal symmetry. The corresponding Mellin amplitudes seem to be like "off-shell" objects which are related to the "on-shell" conformal Mellin amplitudes through an LSZ like prescription.

One can compare these results at weak coupling to those at strong coupling obtained using Witten diagrams in the dual bulk theory in AdS in [4, 5, 7]. As an example, we look at the result obtained in [5] for the 4-point function exchange Witten diagram involving scalars as shown in Figure 14.

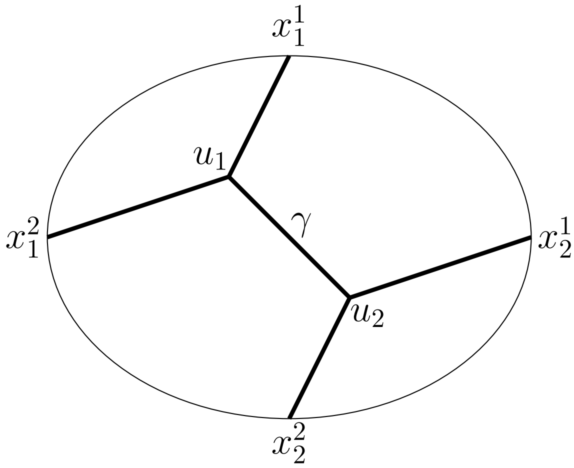

Figure 14. 4-point exchange Witten diagram

The Mellin amplitude for this diagram is given by (for a coupling constant g),

$$M(s_{12}) = \frac{1}{2}\frac{g^2}{(s_{12}-\gamma)}\frac{\Gamma\left(\frac{\Delta_1^1+\Delta_1^2+\gamma-\frac{D}{2}}{2}\right)\Gamma\left(\frac{\Delta_2^1+\Delta_2^2+\gamma-\frac{D}{2}}{2}\right)}{\Gamma\left(1+\gamma-\frac{D}{2}\right)}$$
$$\quad {}_3F_2\left(\frac{2-\Delta_1^1-\Delta_1^2+\gamma}{2},\frac{2-\Delta_2^1-\Delta_2^2+\gamma}{2},\frac{\gamma-s_{12}}{2};\frac{2+\gamma-s_{12}}{2},1+\gamma-\frac{D}{2};1\right)$$

This Mellin amplitude has the same analytic structure as our corresponding result (3.16) for the weak coupling case which is proportional to $\beta\left(\frac{\gamma-s_{12}}{2},\frac{D}{2}-\gamma\right)$.

It would be interesting to understand the extrapolation of the weak coupling results to the strong coupling results (in the particular example we have considered, how the beta function of the weakly coupled regime extrapolates to the ${}_3F_2$ hypergeometric function in the strong coupling regime). In the maximally supersymmetric case, it may be possible to use the integrability of the boundary field theory as well as the string theory in the bulk to understand this interpolation between the results at strong coupling and at weak coupling.

Our analysis throughout was somewhat limited as it dealt only with scalar operators. It is important to extend this perturbative formalism to include tensor and spinor operators as well. There are examples of 2 dimensional CFTs which only involve scalar fields[11]. For such theories, one can try to apply this formalism. However, for application to higher dimensional CFTs, the formalism needs to be extended as mentioned above. It will also be nice if the tree level Feynman rules can be generalized to loop diagrams. One may expect, in analogy with standard momentum space Feynman rules, that the loop amplitudes can be expressed as an integral over undetermined loop variables, with a product of beta functions (with appropriate arguments) in the integrand. If this is the case, it should be possible

[11] We would like to thank R. Loganayagam for drawing our attention to this.

to systematically develop conformal perturbation theory in Mellin space to arbitrary loop order.

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A Notations and Conventions

Notations for Most Used Variables

1. Mellin variable $\equiv s$
2. Internal Schwinger parameters $\equiv t$
3. External Schwinger parameters $\equiv \alpha$
4. Conformal dimension of internal lines $\equiv \gamma$
5. Conformal dimension of external lines $\equiv \Delta$
6. Coordinates of external vertices $\equiv x$
7. Coordinates of internal vertices $\equiv u$
8. Number of space-time dimensions $\equiv D$

Convention for Indices

1. The vertices are labeled by indices $a, b, \cdots$
2. The co-ordinate of the vertices are $u_a, u_b, \cdots$
3. The co-ordinate of all external points attached to the a^{th} vertex are x_a^i (we suppress the spacetime Lorentz index)
4. The conformal dimension of the operator inserted at x_a^i is Δ_a^i

5. The squared distance between two points x_a^i and x_b^j is
$$(x_a^i - x_b^j)^2 \equiv (x_{ab}^{ij})^2 = (x_{ba}^{ji})^2$$
6. The Mellin variable dual to x_{ab}^{ij} is denoted by s_{ab}^{ij} which satisfies
$$s_{ab}^{ij} = s_{ba}^{ji} \qquad \text{and} \qquad s_{aa}^{ii} \equiv 0$$

Convention for Summations and Products

1. If there are N_a external lines meeting at the a^{th} vertex, then we denote
$$\sum_{i\in a} \equiv \sum_{i=1}^{N_a} = \text{ sum over all } \textit{external} \text{ lines connected to the vertex } u_a$$
$$\prod_{i\in a} \equiv \prod_{i=1}^{N_a} = \text{ product over all } \textit{external} \text{ lines connected to the vertex } u_a$$
2. For the double summation and products (which avoid over counting), we use the notations
$$\sum_{1\leq i<j\leq N_a} \sum \equiv \sum_{(i,j)\in a} \qquad ; \qquad \prod_{1\leq i<j\leq N_a} \prod \equiv \prod_{(i,j)\in a}$$
3. If the upper index is *not* mentioned then it implies that upper indices have been summed over all possible values. e.g. for the Mellin variables, we shall use
$$s_{aa} \equiv \sum_{(i,j)\in a} s_{aa}^{ij} \quad ; \quad s_{ab} = s_{ba} \equiv \sum_{i\in a}\sum_{j\in b} s_{ab}^{ij} \quad , \quad (a \neq b)$$

Some Other Conventions

1. Mellin measure $\qquad [ds_{ij}] \equiv \frac{ds_{ij}}{2\pi i}$
2. Position space measure $\qquad \mathcal{D}u \equiv \frac{d^D u}{2(2\pi)^{-D/2}}$
3. Mellin space delta function $\qquad \delta_M(s - s_0) \equiv 2\pi i\ \delta(s - s_0)$

Shorthand Notations

The Schwinger parameters in the integral expression of Mellin amplitude, for n-vertex simple tree and one loop diagrams we consider in this draft, appear in a nice structure. It is useful to introduce a short hand notations for these functions of Schwinger parameters. These notations turn out to be especially convenient for various manipulations. Along with these, we shall also introduce some functions of the Mellin variables.

Set 1 : G_a^c

$$\begin{aligned} &G_a^c \equiv 1 + t_{a,a+1}^2(1 + t_{a+1,a+2}^2(\cdots\cdots + t_{c-1,c}^2)) \qquad && 1 \leq a \leq n-1 \\ &G_a^a \equiv 1 && 1 \leq a \leq n-1 \end{aligned} \tag{A.1}$$

The value of the upstair index c will be greater than or equal to the lower index.

Set 2 : $\tilde{H}_a^b$

$$\begin{aligned}
\tilde{H}_a^b &\equiv t_{a,a+1}\cdots t_{b-1,b}G_b^{n-1} \qquad\qquad 1\leq a<b\leq n-1\\
\tilde{H}_a^a &\equiv G_a^{n-1} \qquad\qquad\qquad\qquad 1\leq a\leq n-1\\
\tilde{H}_a^n &\equiv 0 \qquad\qquad\qquad\qquad 1\leq a\leq n
\end{aligned}$$

$\tilde{H}_a^b$ is not symmetric in its indices. A good mnemonic worth remembering is that upstairs index is always larger than or equal to the downstairs index.

Set 3 : K_a **and** $\tilde{K}_a$

$$\begin{aligned}
K_a &\equiv t_{a,a+1}\cdots t_{n-1,n} \qquad 1\leq a\leq n-1\\
K_n &\equiv 1\\
\tilde{K}_a &\equiv \left(t_{1n}t_{12}\cdots t_{a-1,a}G_a^{n-1}+K_a\right) \qquad 1\leq a\leq n-1\\
\tilde{K}_n &= 1
\end{aligned}$$

Set 4

$$R_a \equiv \gamma_{a,a+1}-\sum_{c=a+1}^{n}\sum_{b=1}^{a}(s_{bc}) \qquad ; \qquad 1\leq a\leq n-1$$

B Mellin Transformation

B.1 Definition and Example

The Mellin transformation of a complex valued function $f(x)$ of real variable x, is defined as

$$\mathcal{M}\{f(x)\}\equiv F(s)=\int_0^\infty x^{s-1}f(x)\,dx \tag{B.1}$$

The complex Mellin variable s is restricted to those values for which the above integral is convergent. In general, the Mellin transform of $f(x)$ exists in a vertical strip in the complex s plane (analytic extension is usually possible).

The inverse Mellin transformation is given by

$$f(x)=\int_{c-i\infty}^{c+i\infty}[ds]\;F(s)\,x^{-s}$$

where the constant c lies within the vertical strip in which the integral in (B.1) converges.

One well known example of the Mellin transform is the Gamma function which can be represented as the Mellin transformation of e^{-x}

$$\Gamma(s)=\int_0^\infty dx\;x^{s-1}e^{-x}$$

with the inverse transformation given by Cahen-Mellin integral

$$\exp(-x)=\int_{c-i\infty}^{c+i\infty}[ds]\;\Gamma(s)\;x^{-s},\quad c>0$$

B.2 Mellin Space Delta Function

In this section we want to show that under certain conditions (specified below), we have

$$I = \int_{-i\infty}^{i\infty} [ds] f(s) \int_0^\infty dt\ t^{s-s_o-1} \equiv \int_{\mathrm{Re}(s_0)-i\infty}^{\mathrm{Re}(s_0)+i\infty} [ds] f(s) \Big(2\pi i\ \delta(s-s_0)\Big) \tag{B.2}$$

i.e. inside the contour integral, the real integral $\int_0^\infty dt\ t^{s-s_0-1}$ behaves as the delta function as long we can shift the contour (as done in the equation above) without crossing any poles. In order to prove our claim, we perform a change of variable $t = e^x$ to get

$$I = \int_{-i\infty}^{i\infty} [ds] f(s) \int_{-\infty}^{\infty} dx\ e^{(s-s_0)x}$$

Now, we note that the contour of integration for the complex variable s is along the imaginary axis from $-i\infty$ to $i\infty$. If we shift the contour from $\mathrm{Re}(s) = 0$ to $\mathrm{Re}(s) = \mathrm{Re}(s_o)$ then, the x integral becomes an oscillatory integral. For this to be valid, we need to ensure that we don't encounter any poles of $f(s)$ while shifting the contour. For our purpose, this turns out to be valid. Now, using the delta function representation

$$\delta(y-a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp\ e^{i(y-a)p}$$

we obtain the desired result

$$I = \int_{-i\infty}^{i\infty} [ds] f(s)\ \ \big[2\pi\delta\big(-i(s-s_0)\big)\big] = \int_{\mathrm{Re}(s_0)-i\infty}^{\mathrm{Re}(s_0)+i\infty} [ds] f(s) \Big(2\pi i \delta\big(s-s_0\big)\Big)$$

C Some Useful Identities

1. The following Mellin-Barnes representation turns out to be very useful

$$\frac{1}{(1+z)^a} = \frac{1}{\Gamma(a)} \int_{-i\infty}^{i\infty} [ds]\ z^{-s} \Gamma(a-s)\Gamma(s) \tag{C.1}$$

2. The first Barnes lemma is

$$\int_{c-i\infty}^{c+i\infty} [ds]\ \beta(a+s, b-s)\beta(c+s, d-s) = \beta(a+d, b+c) \tag{C.2}$$

3. An useful rearrangement identity involving the product of beta functions is

$$\beta(a-u, u)\beta(d-u, k) = \beta(d-u, u)\beta(d, k) \qquad ; \quad \text{provided} \quad a = d+k \tag{C.3}$$

4. The recursive integral form of product of beta functions is

$$\prod_{a=1}^{m-1} \int_0^\infty dt_{a,a+1} \big(t_{a,a+1}\big)^{N_a-1} \Big(G_a^m\Big)^{-L_a} = \prod_{a=1}^{m-1} \frac{1}{2} \beta\Big(\frac{N_a}{2}, L_a + \frac{N_{a-1}-N_a}{2}\Big) \tag{C.4}$$

where $N_0 \equiv 0$ and G_a^m is defined in appendix A.

5. The definition of the Hypergeometric function is

$$ {}_pF_q\Big(a_1,\cdots,a_p\ ;\ b_1,\cdots,b_q\ ;\ x\Big) \;=\; \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\cdots\Gamma(a_p+n)}{\Gamma(b_1+n)\cdots\Gamma(b_q+n)}x^n \quad \text{(C.5)} $$

with $|x|<1$.

6. The recursive integration formula for hypergeometric functions is

$$ \begin{aligned} &\beta\left(a_{p+1},b_{q+1}-a_{p+1}\right){}_{p+1}F_{q+1}\Big(a_1,\cdots,a_{p+1}\ ;\ b_1,\cdots,b_{q+1}\ ;\ x\Big) \\ &\qquad = \int_0^1 dt\ (t)^{a_{p+1}-1}(1-t)^{b_{q+1}-a_{p+1}-1}{}_pF_q\Big(a_1,\cdots,a_p\ ;\ b_1,\cdots,b_q\ ;\ tx\Big) \end{aligned} \quad \text{(C.6)} $$

7. Following identities relate two ${}_3F_2$ hypergeometric functions with different arguments

$$ \begin{aligned} {}_3F_2(a_1,a_2,a_3;b_1,b_2;1) \;=\;& {}_3F_2(a_1,b_2-a_2,b_2-a_3;b_2,b_1+b_2-a_1-a_3;1) \\ &\times\frac{\Gamma(b_1)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_1-a_1)\Gamma(b_1+b_2-a_2-a_3)} \end{aligned} \quad \text{(C.7)} $$

and,

$$ \begin{aligned} &{}_3F_2(a_1,a_2,a_3;b_1,b_2;1) \\ &\quad = {}_3F_2(b_1-a_1,b_2-a_1,b_1+b_2-a_1-a_2-a_3;b_1+b_2-a_1-a_2,b_1+b_2-a_1-a_3;1) \\ &\quad\times\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(a_1)\Gamma(b_1+b_2-a_1-a_2)\Gamma(b_1+b_2-a_1-a_3)} \end{aligned} \quad \text{(C.8)} $$

8. The Gauss identity is

$$ {}_2F_1(a_1,a_2;b_1;1) = \frac{\Gamma(b_1)\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)} \quad \text{(C.9)} $$

9. An integral representation of ${}_2F_1$ is

$$ {}_2F_1(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \quad \text{(C.10)} $$

10. Following identity relates two ${}_2F_1$ Hypergeometric function with different arguments

$$ {}_2F_1(a,b;c;z) \;=\; (1-z)^{-a}{}_2F_1\Big(a,c-b;c;\frac{z}{z-1}\Big) \quad \text{(C.11)} $$

D Details of Some Tree Level Calculations

D.1 Mellin Amplitude of a Completely General Tree

In this appendix, we carry out the derivation of Mellin amplitude for an arbitrary tree Feynman diagram and show that the amplitude is given by product of beta functions with one beta function for each internal propagator.

From the diagrammatic rules described in section 4.1, we know that a general Mellin amplitude takes the form

$$M\left(\{s_{ab}\}\right) = \prod \left[\int_0^\infty dt_{ab} \frac{(t_{ab})^{\gamma_{ab}-1}}{\Gamma(\gamma_{ab})}\right] F\Big(\{t_{ab}\}, \{s_{ab}\}\Big) \tag{D.1}$$

The t_{ab} is the Schwinger parameter for the propagator joining the internal vertices a and b and the product is over all the internal propagators of the Feynman diagram. The function F is, in general, an arbitrary function of the Schwinger parameters t_{ab} and the Mellin variables s_{ab}.

As mentioned earlier, the function F, for any given graph, depends on the order of integration over the position space vertices. For a straight chain of propagators (as in section 4.2), the natural choice is to go from one end to the other without any jumps. This results in the simplest expression for F. For a tree with branches, this option is absent. We shall, however, prescribe an order that gives an expression for F such that the integration over the Schwinger parameters can be performed easily.

To specify the ordering, we first choose any one of the vertices, with only one edge attached, to be the reference vertex $\mathcal{P}$ (see figure 15) on the skeleton diagram. In our prescription, the integration over this vertex will be carried out after performing integration over all the other vertices. For all the other vertices, the order is indicated by some arrows on the lines. For drawing these arrows, one needs to follow two rules. The first rule is that, among all the lines meeting at a vertex, there should be only one line with an outgoing arrow. All the other lines attached to that vertex should have ingoing arrows. The second rule is that any given vertex is integrated only after all the other vertices connected to it by (lines with) ingoing arrows have been integrated over. Thus, $\mathcal{P}$ is the only vertex with a single line which has an ingoing arrow and according to our prescription, it is integrated in the end. Figure 15 gives an example of a compatible ordering using arrows. In this example, an order allowed by the above rules is $1 \to 3 \to 2 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10$. This, however, is not the only ordering consistent with our rules and all such allowed orderings are equally good for our purpose.

We can now write down the function F for this prescribed order of integration for any Feynman diagram using the diagrammatic rules described in section 4.1. We state the result here in a graph independent manner. From the diagramatic rules, we know that each pair of vertices a, b (which may be the same point also) of the Feynman diagram contributes a multiplicative factor raised to $-s_{ab}$ to F. In other words, if we denote this factor by $\mathcal{F}_{ab}$,

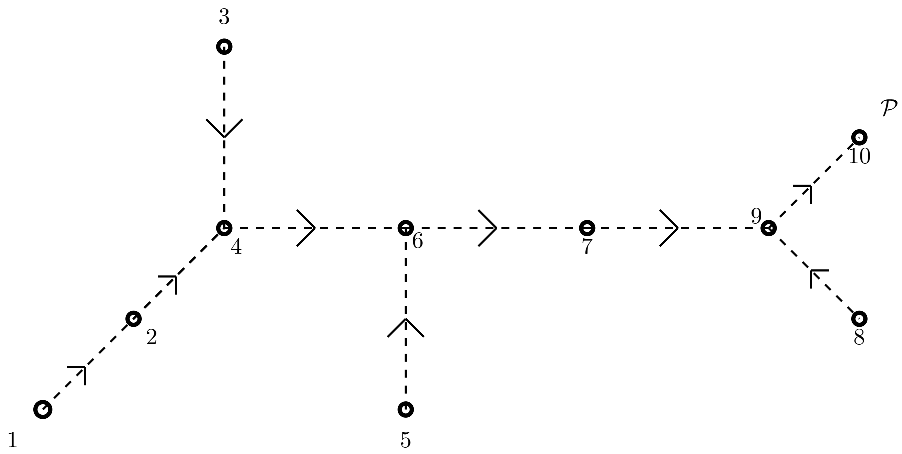

Figure 15. Order of integration over the vertices depicted by the numbers (in increasing order)

then the function F is given by,

$$F = \prod_{a=1}^{N}\prod_{b=1}^{N}(\mathcal{F}_{ab})^{-s_{ab}} \tag{D.2}$$

where N is the total number of internal vertices.

We now describe the functional dependence of $\mathcal{F}$ on Schwinger parameters. For this, we draw the shortest continuous line (without raising the pen) between the two vertices a and b on the skeleton diagram (via the vertices that come on the way). We refer to the set of vertices that we cross as $\mathcal{A}_{ab}$. Since we are considering a tree, there will exist a vertex in this set that is *nearest* to the reference vertex $\mathcal{P}$. The set of vertices on the continuous route from this vertex to $\mathcal{P}$ is denoted by $\mathcal{B}_{ab}$.

To see an example, let us consider the pair of points (3,5) in figure 15. Here, $\mathcal{A}_{35} = \{3, 4, 6, 5\}$. The vertex in the set $\mathcal{A}_{35}$ nearest to $\mathcal{P}$ is 6 and $\mathcal{B}_{35} = \{6, 7, 9, 10\}$.

Since the vertices in the sets $\mathcal{A}_{ab}$ and $\mathcal{B}_{ab}$ form individual chains, there is always a line on the skeleton diagram connecting the consecutive vertices in each of these sets. Below, we shall make use of the Schwinger parameters corresponding to the propagators in these lines.

The functional dependence of $\mathcal{F}_{ab}$ on the Schwinger parameters can now be easily written down by following the diagrammatic rules of section 4.1 and is given by

$$\mathcal{F}_{ab} = \left[\prod_{i=1}^{|\mathcal{A}_{ab}|-1} t_{\mathcal{A}_{ab}(i)\mathcal{A}_{ab}(i+1)}\right] K_{ab} \tag{D.3}$$

where $|\mathcal{A}_{ab}|$ denotes the number of vertices in the set $\mathcal{A}_{ab}$ and the product in the bracket is over those propagators which lie along the shortest continuous line joining the two vertices a and b. The function K_{ab} is given by

$$K_{ab} \equiv 1 + t^2_{\mathcal{B}_{ab}(i)\mathcal{B}_{ab}(i+1)} K_{\mathcal{B}_{ab}(i+1)\mathcal{B}_{ab}(i+2)} \qquad ; \qquad K_{\mathcal{B}_{ab}(|\mathcal{B}_{ab}|-1)\mathcal{B}_{ab}(|\mathcal{B}_{ab}|)} \equiv 1$$

It should be emphasized that the above form of $\mathcal{F}_{ab}$ is true only for the chosen order of integration and will be different if we change the order.

As an example, for the Feynman diagram in figure 15, the factor $\mathcal{F}_{3,5}$ is given by

$$\mathcal{F}_{3,5} = t_{34}t_{46}t_{65}\left(1 + t_{67}^2\left(1 + t_{79}^2\left(1 + t_{9,10}^2\right)\right)\right)$$

By using (D.3) in (D.2) and rearranging the terms, we can write a simplified expression for the function F as

$$F = \prod_{\text{all propagators}} (t_{ab})^{-P_{ab}}(A_{ab})^{-Q_{ab}} \tag{D.4}$$

where,

$$A_{ab} \equiv 1 + t_{ab}^2\left(1 + t_{bc}^2\left(1 + ...(1 + t_{op}^2)...\right)\right)$$

b,c,...,o are all on the shortest continuous route from a to the reference vertex $\mathcal{P}$. In other words, for each propagator on the skeleton graph, we draw the shortest continuous line connecting it with the propagator containing the reference vertex $\mathcal{P}$. $t_{ab}, t_{bc}, \cdots, t_{op}$ are the Schwinger parameters of the successive propagators on this line.

The functions P_{ab} and Q_{ab} in (D.4) are given by

$$P_{ab} = \sum_{c\in L_{ab}}\sum_{d\in R_{ab}} s_{cd} \qquad ; \qquad Q_{ab} = \sum_{\substack{c\in L_{ab}\\ c\neq a}}\sum_{\substack{d\in \tilde{L}_{ab}\\ d\neq a}} s_{cd} \;+ \sum_{d\in L_{ab}} s_{ad}$$

The L_{ab} (R_{ab}) in the above definition refer to the set of vertices which lie to the left (right) of the propagator joining the vertex a and b. By convention, we call the set of vertices which include the reference vertex $\mathcal{P}$ as R_{ab}. The term involving the double sum in Q_{ab} is absent if not more than two lines meet at the vertex a in the skeleton graph. The tilde in one of the L in this double sum denotes the fact that we should not include terms of the type s_{cd} where c and d are on the same branch in the set L_{ab}.

Finally, we now need to integrate over the Schwinger parameters in (D.1). Since the integrals are not factorised, we shall have to carry out one integral at a time and we shall do that in an order compatible with the arrows on the skeleton (integrating over the line involving the reference vertex $\mathcal{P}$ in the end). For example, for the figure 15, a compatible order is $t_{12} \to t_{24} \to t_{34} \to t_{46} \to t_{56} \to t_{67} \to t_{89} \to t_{79} \to t_{9,10}$. Carrying out these integrals in exactly the same way as in the simple tree case in section 4.2 and using the conformality conditions, we obtain the desired result (4.8).

D.2 Mellin-Barnes Approach to n-Vertex Tree

In this appendix, we present an alternative derivation of the factorization of n vertex tree diagram. This derivation makes use of the Barnes' first identity and shows the usefulness of the Barnes integrals for Mellin space.

We start by noting that the Mellin amplitude of n vertex simple tree of section (4.2) can be written in a form which closely resembles the loop amplitude (5.2)

$$M_n(s_{ab}) = \prod_{a=1}^{n-1}\left[\int_0^\infty dt_{a,a+1}(t_{a,a+1})^{\gamma_{a,a+1}-1}\right]\prod_{b=c}^{n}\prod_{c=1}^{n-1}\Big(\tilde{H}_c^b + K_c K_b\Big)^{-s_{cb}}$$

where, the functions $\tilde{H}_a^b$ and K_a are defined in the appendix (A).

The basic idea is to use the identity (C.1) to convert the terms involving sum as products of Mellin Barnes integrals. We then perform the integration over Schwinger parameters. The Mellin Barnes integrations are performed in the end by making repeated use of Barnes first lemma.

Using the identity (C.1) and noting that $\tilde{H}_a^n \equiv 0$ (which means that we only need to introduce $n(n-1)/2$ Barnes variables w_{ab}), we obtain after making use of the definitions of $\tilde{H}_a^b$ and K_a

$$\begin{aligned}M(s_{ab}) = &\prod_{b=1}^{n-1}\prod_{c=b}^{n-1}\left(\int_{-i\infty}^{i\infty}[dw_{bc}]\beta(s_{bc}-w_{bc},w_{bc})\right)\int_0^\infty dt_{n-1,n}\big(t_{n-1,n}\big)^{\mathcal{R}_{n-1}-1}\\ &\prod_{a=1}^{n-2}\left(\int_0^\infty dt_{a,a+1}(t_{a,a+1})^{\mathcal{R}_a-1}(G_a^{n-1})^{-\mathcal{Q}_a}\right)\end{aligned}$$

where,

$$\begin{aligned}\mathcal{R}_a &\equiv R_a - 2\sum_{b=1}^{a} w_{bb} - 2\sum_{b=1}^{a}\sum_{c=1}^{b-1} w_{bc} \qquad ; \qquad 1 \le a \le n-1\\ \mathcal{Q}_a &\equiv \sum_{b=1}^{a}\Big(s_{ab} - w_{ab}\Big) \qquad ; \qquad 1 \le a \le n-1\end{aligned}$$

$t_{n-1,n}$ integral is straightforward and it gives a Mellin space delta function. For integration over other Schwinger parameters, we use the identity (C.4). After simplifying the expressions by making use of the conformal conditions, we obtain

$$M(s_{ab}) = \prod_{b=1}^{n-1}\left\{\prod_{c=b}^{n-1}\int_{-i\infty}^{i\infty}[dw_{bc}]\ \beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\delta_M\Big(\mathcal{R}_{n-1}\Big) \tag{D.5}$$

where, $\gamma_{01} \equiv 0 \equiv \mathcal{R}_0$.

We note that we have obtained beta functions for only $n-2$ propagators. The missing propagator is hidden in the Mellin-Barnes integrals and the Mellin space delta function as we shall show below.

To perform the integration over the w_{ab} variables, we rewrite (D.5) as

$$\begin{aligned}M(s_{ab}) = &\prod_{b=1}^{n-2}\left\{\prod_{c=b}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\ \beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\\ &\prod_{b=1}^{n-1}\left\{\int_{-i\infty}^{i\infty}[dw_{b,n-1}]\ \beta(s_{b,n-1}-w_{b,n-1},w_{b,n-1})\right\}\delta_M\Big(\mathcal{R}_{n-1}\Big)\end{aligned} \tag{D.6}$$

The first line does not involve the $w_{a,n-1}$ variables (note that for n vertex case, only $\mathcal{R}_{n-1}$ involves the $w_{a,n-1}$ variables). This helps in performing the integration over $w_{a,n-1}$ variables. We first use the delta function to get rid of integration over $w_{n-1,n-1}$ and then use the identity (C.2) to perform integration over other $w_{a,n-1}$ variables. After carefully keeping track of various terms, we obtain

$$\begin{aligned}M(s_{ab}) = &\frac{1}{2}\prod_{b=1}^{n-2}\left\{\prod_{c=b}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\ \beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\\ &\beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\ ,\ \frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\right)\end{aligned} \tag{D.7}$$

The factor of $\frac{1}{2}$ in front arises due to the delta function in (D.6).

To proceed further, we note that the conformal condition at vertex 1 can be expressed in following way

$$s_{11} = \frac{R_1}{2}+\frac{D}{2}-\gamma_{12}$$

This allows us to use the rearrangement identity(C.3) as

$$\beta\left(s_{11}-w_{11},w_{11}\right)\beta\left(\frac{\mathcal{R}_1}{2},\frac{D-2\gamma_{12}}{2}\right) = \beta\left(\frac{R_1}{2},\frac{D-2\gamma_{12}}{2}\right)\beta\left(\frac{R_1}{2}-w_{11},w_{11}\right)$$

Using this, we can rewrite (D.7) as

$$\begin{aligned}M(s_{ab}) = &\frac{1}{2^2}\beta\Big(\frac{R_1}{2},\frac{D-2\gamma_{12}}{2}\Big)\int_{-i\infty}^{i\infty}[dw_{11}]\beta\left(\frac{R_1}{2}-w_{11},w_{11}\right)\\ &\int_{-i\infty}^{i\infty}[dw_{12}]\beta\left(s_{12}-w_{12},w_{12}\right)\int_{-i\infty}^{i\infty}[dw_{22}]\beta\left(s_{22}-w_{22},w_{22}\right)\\ &\prod_{b}^{n-2}\left\{\prod_{c}^{n-2}\int_{-i\infty}^{i\infty}[dw_{bc}]\ \beta(s_{bc}-w_{bc},w_{bc})\right\}\prod_{a=2}^{n-2}\frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\\ &\beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\ ,\ \frac{R_{n-1}-R_{n-2}+\mathcal{R}_{n-2}}{2}\right)\end{aligned} \tag{D.8}$$

The integrals in the third line do not include the integrations over the (w_{11},w_{12},w_{22}) variables. Moreover, these variables appear only in the form of sum (i.e. $w_{11}+w_{12}+w_{22}$)

in the beta functions of last two lines. Hence, it is useful to use a new set of coordinates as follows

$$\{w_{11}, w_{12}, w_{22}\} \to \{w_{11}, w_{22}, u_1\} \qquad ; \quad u_1 \equiv w_{11} + w_{12} + w_{22}$$

After this coordinate change, the last two lines of (D.8) do not include w_{12} or w_{22} variables anywhere. They just appear in the first two lines. Performing the integration over these variables using Barnes first lemma gives

$$\begin{aligned} M = &\frac{1}{2^3}\beta\Big(\frac{R_1}{2}, \frac{D-2\gamma_{12}}{2}\Big) \int_{-i\infty}^{i\infty} [du_1] \beta\left(\frac{R_1}{2} + s_{12} + s_{22} - u_1, u_1\right) \beta\Big(\frac{R_2}{2} - u_1, \frac{D-2\gamma_{23}}{2}\Big) \\ &\prod_b^{n-2} \left\{ \prod_c^{n-2} \int_{-i\infty}^{i\infty} [dw_{bc}]\ \beta(s_{bc} - w_{bc}, w_{bc}) \right\} \prod_{a=3}^{n-2} \frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2}, \frac{D-2\gamma_{a,a+1}}{2}\Big) \\ &\beta\left(\sum_{a=1}^{n-1} s_{a,n-1} - \frac{R_{n-1} - R_{n-2} + \mathcal{R}_{n-2}}{2}\ ,\ \frac{R_{n-1} - R_{n-2} + \mathcal{R}_{n-2}}{2}\right) \end{aligned}$$

In the next step we use the conformal condition at the vertex 2 to use the rearrangement identity (C.3) for the two beta functions inside the integration in the first line of the above expression. After this, we note that the u_1 variable appears in the combination $u_1 + w_{13} + w_{23} + w_{33}$ in all but one of the beta function. We exploit this by trading the u_1 variable for a new variable u_2 defined as $u_2 = u_1 + w_{13} + w_{23} + w_{33}$. This allows us to perform the integrations over w_{13}, w_{23} and w_{33} variables using the Barnes' first lemma. The end result after this step is

$$\begin{aligned} M = &\frac{1}{2^3}\beta\Big(\frac{R_1}{2}, \frac{D-2\gamma_{12}}{2}\Big)\beta\Big(\frac{R_2}{2}, \frac{D-2\gamma_{12}}{2}\Big) \int_{-i\infty}^{i\infty} [du_2] \beta\left(\frac{R_2}{2} + s_{13} + s_{23} + s_{33} - u_2, u_2\right) \\ &\prod_b^{n-2} \left\{ \prod_c^{n-2} \int_{-i\infty}^{i\infty} [dw_{bc}]\ \beta(s_{bc} - w_{bc}, w_{bc}) \right\} \prod_{a=3}^{n-2} \frac{1}{2}\beta\Big(\frac{\mathcal{R}_a}{2}, \frac{D-2\gamma_{a,a+1}}{2}\Big) \\ &\beta\left(\sum_{a=1}^{n-1} s_{a,n-1} - \frac{R_{n-1} - R_{n-2} + \mathcal{R}_{n-2}}{2}\ ,\ \frac{R_{n-1} - R_{n-2} + \mathcal{R}_{n-2}}{2}\right) \end{aligned}$$

Continuing this process iteratively, i.e. combining the beta functions and then making a change of coordinate (such that only two beta functions involve the appropriate w_{ab} variables), we obtain the desired result. We need to make repeated use of the identity (conformal condition at a^{th} vertex)

$$\frac{R_a}{2} + \sum_{b=1}^{a+1} s_{b,a+1} = \frac{R_{a+1}}{2} + \frac{D}{2} - \gamma_{a+1,a+2} \tag{D.9}$$

In the end step, we obtain the desired result

$$
\begin{aligned}
M(s_{ab}) &= \frac{1}{2}\left[\prod_{a=1}^{n-2}\frac{1}{2}\beta\Big(\frac{R_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)\right]\int_{-i\infty}^{i\infty}[du_{n-3}]\beta\Big(\frac{R_{n-2}}{2}-u_{n-3}\ ,\ u_{n-3}\Big)\\
&\quad\times\beta\left(\sum_{a=1}^{n-1}s_{a,n-1}-\frac{R_{n-1}}{2}+u_{n-3}\ ,\ \frac{R_{n-1}}{2}-u_{n-3}\right)\\
&= \prod_{a=1}^{n-1}\frac{1}{2}\beta\Big(\frac{R_a}{2},\frac{D-2\gamma_{a,a+1}}{2}\Big)
\end{aligned}
$$

where, we have used the identity (D.9) for $a=n-1$ after performing the u_{n-3} integration.

E Details of Calculation in Section 6.1.2

In this appendix, we present the details of the calculation leading to the equation (6.5). Our starting expression is (6.4). We insert a partition of unity in this expression in the form

$$
1=\int dq_a\ \delta\left(q_a-\sum_{i\in a}\Delta_a^i\right)\qquad ;\qquad a=1,2
$$

and make the coordinate transformations $\alpha_a^i=q_a\ y_a^i$ $(a=1,2)$ and then use the identity (6.2) to obtain

$$
\begin{aligned}
\tilde{M}(s_{ab}) &= \frac{1}{\Gamma(\gamma)}\prod_{a=1}^{2}\left[\frac{\prod_{i\in a}\Gamma(\rho_a^i)}{\Gamma(\rho_a)}\right]\int_0^\infty dt\ (t)^{R_1-1}\big(1+t^2\big)^{-s_{11}}\int_0^\infty dq_1\ (q_1)^{\rho_1-1}\\
&\quad\int_0^\infty dq_2\ (q_2)^{\rho_2-1}\big(q_1(1+t^2)+tq_2\big)^{-\lambda_1}\Big(t\ q_1+q_2\Big)^{-\lambda_2}
\end{aligned}
$$

where,

$$
R_1\equiv\gamma-s_{12}\qquad\text{and}\qquad \rho_a\equiv\sum_{i\in a}\rho_a^i\qquad ;\quad a=1,2
$$

To proceed further, we rescale $q_2\to tq_1q_2$ and obtain,

$$
\begin{aligned}
\tilde{M}(s_{ab}) &= \frac{1}{\Gamma(\gamma)}\prod_{a=1}^{2}\left[\frac{\prod_{i\in a}\Gamma(\rho_a^i)}{\Gamma(\rho_a)}\right]\int_0^\infty dt\ (t)^{R_1+\rho_2-\lambda_2-1}\big(1+t^2\big)^{-s_{11}}\int_0^\infty dq_1dq_2\\
&\quad(q_1)^{\rho_1+\rho_2-\lambda_1-\lambda_2-1}\ (q_2)^{\rho_2-1}\Big(1+t^2+t^2q_2\Big)^{-\lambda_1}\Big(1+q_2\Big)^{-\lambda_2}
\end{aligned}
$$

Integration over q_1 gives a delta function using the identity (B.2). We now take out the factor of $1+t^2$ from the term containing the single power of λ_1, rescale $t^2\to t$ and make a further change of coordinates

$$
q_2=\frac{u}{1-u}\qquad ,\qquad t=\frac{v}{1-v}
$$

This gives,

$$\tilde{M}(s_{ab}) = \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\frac{\prod_{i\in a}\Gamma(\rho_a^i)}{\Gamma(\rho_a)}\right]\delta\Big(\rho_1+\rho_2-\lambda_1-\lambda_2\Big)\int_0^1 du\int_0^1 dv\ (v)^{\frac{R_1+\rho_2-\lambda_2}{2}-1}$$
$$(u)^{\rho_2-1}\big(1-u\big)^{\lambda_2-\rho_2-1}\big(1-v\big)^{s_{11}+\lambda_1-\frac{1}{2}(R_1+\rho_2-\lambda_2)-1}\Big(1+\frac{uv}{1-u}\Big)^{-\lambda_1}$$

Now, using the identities (C.10) and (C.11), we obtain

$$\tilde{M}(s_{ab}) = \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\frac{\prod_{i\in a}\Gamma(\rho_a^i)}{\Gamma(\rho_a)}\right]\delta\Big(\rho_1+\rho_2-\lambda_1-\lambda_2\Big)\beta\left(\frac{R_1}{2}+\frac{\rho_2-\lambda_2}{2}\ ,\ \frac{D}{2}-\gamma\right)$$
$$\int_0^1 du\ (u)^{\rho_2-1}\big(1-u\big)^{\rho_1-1}{}_2F_1\left(\lambda_1\ ,\ \frac{D}{2}-\gamma\ ;\ s_{11}+\lambda_1\ ;\ u\right)$$

We have also used

$$s_{11}+\lambda_1-\frac{1}{2}(R_1+\rho_2-\lambda_2) = \frac{D}{2}-\gamma$$

Next, we use the identity (C.6) for $a=2, b=1$ and the identity (C.7) to obtain

$$\tilde{M}(s_{ab}) = \frac{1}{2\Gamma(\gamma)}\prod_{a=1}^{2}\left[\frac{\prod_{i\in a}\Gamma(\rho_a^i)}{\Gamma(\rho_a)}\right]\beta\left(\frac{R_1}{2}+\frac{\rho_1-\lambda_1}{2}\ ,\ \frac{D}{2}-\gamma\right)\beta\Big(\rho_1,\rho_2\Big)$$
$${}_3F_2\left(\frac{D}{2}-\gamma\ ,\ \lambda_2\ ,\ \rho_1\ ;\ s_{11}+\rho_1,\lambda_1+\lambda_2\ ;\ 1\right)\delta\Big(\rho_1+\rho_2-\lambda_1-\lambda_2\Big)$$

In writing the above expression, we have used the fact that ${}_3F_2$ is symmetric in first set of three indices and the second set of two indices.

The above expression is not symmetric in the two vertices. We can put it in a symmetric form by using the identity (C.8). After some simplification, we obtain the desired result (6.5).

References

[1] G. Mack, "D-independent representations of Conformal Field Theories in D dimensions via transformations of auxiliary Dual Resonance Models. Scalar amplitudes", arxiv: 0907.2407 [hep-th]

[2] G. Mack, "D-dimensional Conformal Field Theories with anomalous dimensions as dual resonance models", Bulg. J. Phys.36, 214 (2009); arXiv:0909.1024 [hep-th]

[3] K. Symanzik, "On calculations in Conformal Invariant Field Theories", Lett. Nuovo Cim. 3 (1972) 734-738

[4] J. Penedones, "Writing CFT correlation functions as AdS scattering amplitudes", JHEP 1103, 025 (2011); arXiv:1011.1485 [hep-th]

[5] M. F. Paulos, “Towards Feynman rules for Mellin amplitudes in AdS/CFT”, JHEP10(2011)074; arXiv:1107.1504 [hep-th]

[6] D. Nandan, A. Volovich, C. Wen, “On Feynman rules for Mellin amplitudes in AdS/CFT”, JHEP 1205 (2012) 129 ; arXiv:1112.0305 [hep-th]

[7] A. Liam Fitzpatrick, Jared Kaplan, João Penedones, Suvrat Raju, Balt C. van Rees, “A Natural Language for AdS/CFT Correlators”, JHEP Vol. 2011, No. 11, 95; arXiv:1107.1499 [hep-th]

[8] A. Liam Fitzpatrick, Jared Kaplan, “AdS Field Theory from Conformal Field Theory”, JHEP 1302, 054 (2013); arXiv:1208.0337 [hep-th]

[9] Miguel S. Costa, Vasco Gonçalves, João Penedones, “Spinning AdS Propagators”, JHEP 1409 (2014) 064; arXiv:1404.5625 [hep-th]

[10] Eliot Hijano, Per Kraus, Eric Perlmutter, River Snively, “Witten diagrams revisited: The AdS geometry of conformal blocks”, JHEP 01 (2016) 146; arxiv:1508.00501 [hep-th]

[11] A. Liam Fitzpatrick, Jared Kaplan, “Analyticity and the Holographic S-Matrix”, JHEP10(2012)127 arXiv:1111.6972 [hep-th]

[12] A. Liam Fitzpatrick, Jared Kaplan, “Unitarity and the Holographic S-Matrix”, JHEP10(2012)127 arXiv:1112.4845 [hep-th]

[13] Vasco Gonçalves, João Penedones, Emilo Trevisani, “Factorization of Mellin amplitudes”, JHEP 1510 (2015) 040 ; arXiv:1410.4815 [hep-th]

[14] Miguel F. Paulos, Joao Penedones, Jonathan Toledo, Balt C. van Rees, Pedro Vieira “The S matrix Bootstrap I : QFT in AdS”, arXiv:1607.06109v1 [hep-th]

[15] David A. Lowe, “Mellin transforming the minimal model CFTs: AdS/CFT at strong curvature”, arXiv:1602.0561 [hep-th]

[16] S. Stieberger, T. R. Taylor, “Superstring Amplitudes as a Mellin transform of Supergravity”, Nucl. Phys. B873, 65(2013) arXiv:1303.1532 [hep-th]

[17] M. F. Paulos, M. Spradlin and A. Volovich , “Mellin Amplitudes for Dual Conformal Integrals”, JHEP 1208 (2012) 072; arxiv: 1203.6362

[18] Dhritiman Nandan, Miguel F. Paulos, Marcus Spradlin, Anastasia Volovich, “Star integrals, convolutions and simplices”, JHEP 1305 (2013) 105; arXiv:1301.2500 [hep-th]

[19] Vasco Gonçalves, “Four-point function of $\mathcal{N} = 4$ stress-tensor multiplet at strong coupling”, JHEP 1504 (2015) 150 ; arXiv:1411.1675 [hep-th]

[20] Luis F. Alday, Agnesse Bissi, Tomasz Łukowski, “Lessons from crossing symmetry at large N”, JHEP 1506 (2015) 074 ; arXiv:1410.4717 [hep-th]

[21] A. Belitsky, S. Hohenegger, G. Korchemsky, E. Sokatchev, A. Zhiboedov, “From correlation functions to event shapes”, Nucl. Phys. B 884, 305 (2014) arXiv:1309.0769 [hep-th]

[22] A. Belitsky, S. Hohenegger, G. Korchemsky, E. Sokatchev, A. Zhiboedov, “Event shapes in $\mathcal{N} = 4$ super-Yang-Mills theory”, Nucl.Phys. B884 (2014) 206-256; arXiv:1309.1424 [hep-th].

[23] A. Belitsky, S. Hohenegger, G. Korchemsky, E. Sokatchev, A. Zhiboedov, “Energy-Energy Correlations in $\mathcal{N} = 4$ SYM”, Phys.Rev.Lett. 112 no. 7, (2014) 071601, arXiv:1311.6800v1 [hep-th]

[24] Luis F. Alday and Agnese Bissi "Unitarity and positivity constraints for CFT at large central charge ", arXiv:1606.09593v1 [hep-th]

[25] Miguel F. Costa, Vasco Gonçalves, João Penedones, "Conformal Regge Theory", JHEP 1212, 091 (2012); arXiv:1209.4355 [hep-th]

[26] Yu Nakayama, "Scale invariance vs. conformal invariance"; Phys.Rept. 569 (2015) 1-93; arXiv: 1302.0884v4 [hep-th]