Continuous point symmetries in Group Field Theories

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We discuss the notion of symmetries in non-local field theories characterized by integro-differential equations of motion, from a geometric perspective. We then focus on Group Field Theory (GFT) models of quantum gravity and provide a general analysis of their continuous point symmetry transformations, including the generalized conservation laws following from them.

Introduction

Symmetry principles are omni-present in modern physics, and especially in the quantum field theory formulation of fundamental interactions. They enter crucially in the very definition of fundamental fields and particles, they dictate the allowed interactions between them, and capture key features of such interactions in terms of conservation laws. They also offer powerful conceptual tools, for example, in the characterization of macroscopic phases of quantum systems, as well as many computational simplifications and powerful mathematical techniques for numerical and analytical calculations.

Group Field Theory (GFT) [1–6] is a promising candidate formalism for a fundamental theory of quantum gravity. It can be seen as generalization of matrix and tensor models for two and higher dimensional gravity in terms of random triangulations [7–10], and as a 2nd quantized, quantum field theoretical reformulation of Loop Quantum Gravity [4, 6]. As such, it is suitable for the adaptation of the standard symmetry analysis techniques from ordinary quantum field theories. This is the goal of the present article. At the same time, the intrinsically non-local structure of GFTs requires non-trivial adaptation of the well-known Lie group-based symmetry analysis and the generally curved nature of the GFT base manifold makes the exact computations and the whole analysis considerably more involved. We will tackle both types of difficulties in the following, but it is the non-local character of the theory that stands out as the truly defining challenge, so we focus mainly on it, and limiting ourselves to the classical aspects of the theory, which are interesting enough. We had laid some of the mathematical foundations of the present analysis in a previous work [11], to which we will often refer in the following.

A possible definition of the non-locality in the case of GFT is the integro-differential structure of the correspondent equations of motion. From this perspective the symmetry analysis of these quantum field theories could be mapped to the symmetry analysis of the integro-differential equations of motion, at least in the classical approximation. The Lie algebra approach to symmetries of integro-differential equations is not new, and there is

a quite extensive literature available on this subject [12–14]. However, in contrast to local field theories, the methods and strategies of analysis strongly vary from case to case, and no universal method is known yet. As a result, we do not know a priori which of the known methods can be suitable adapted for the analysis of group field theories. Particular examples for symmetry calculations for several GFT models were calculated in [15] but no systematic treatment was proposed there.

In this paper we investigate the *variational symmetry* groups of various prominent models in Group Field Theory, that is symmetries of the action and not the (wider) symmetries of the equations of motion. We consider only *point symmetries*, for several interesting models, and investigate the consequences of the derived symmetries in terms of generalized conservation laws, on the basis of the framework performed in [11]. We will also show that, when special types of matter fields are included in the models, a possible definition of "conserved" charges can be derived from the same generalized conservation laws. Also the restriction to point symmetries will be motivated in the following, and we leave the generalization to contact or Lie-Baecklund symmetries to be discussed elsewhere. Similarly, we leave for further work the study of the consequences of the symmetries we identify at the quantum level. While a detailed analysis is of course needed, the main reason for postponing such work is that we do not anticipate any additional difficulty (with respect to the local quantum field theory case) in the derivation of the corresponding Ward identities or in the study of possible anomalies.

The method we use in this paper is, to the large extent, the usual Lie algebra calculations of symmetry groups, suitably modified to adapt to the peculiar features of group field theories.

The presentation is organized as follows. We begin with a brief review on Group Field Theories in section I. Then, after a recap of the definition of various types of symmetries, in the geometric formulation of classical field theories, we proceed with the symmetry analysis in Group Field Theory in section II. In doing this, we clarify how to extend the geometric treatment of field theories to the non-local case. At the end of this section the reader can find the summarized table of symmetry groups for the models under investigation. In section III we present the correspondent Noether currents along with their generalized conservation laws. The case, in which matter fields

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are present, is treated in Section IV, with emphasis on scalar fields that can be used as relational clocks, and the correspondent definition of relational charges.

I.   Group Field Theory

In the first part of this section we give an informal definition of the group field theory formalism and its general features, emphasizing also the connections to other theories of gravity. In the second part we introduce the specific models that we are going to study in the rest of this paper. The notation and conventions introduced in this section will be used throughout the paper.

A Group Field Theory [2] is a quantum field theory on a group manifold defined by a *non-local action* at the classical level, and the corresponding path integral at the quantum level. The fundamental fields of the theory are functions *from a Lie Group* to some vector space (usually, the complex numbers)[1]. In models directly related to lattice gravity and loop quantum gravity, they are often assumed to satisfy a specific *gauge invariance* condition, which indeed provides the perturbative Feynman amplitudes of the model with a lattice gauge theory structure. This possibility is, at its root, allowed by the choice of peculiar non-local pairing of field arguments in the GFT interaction terms. In fact, this has the immediate result that the perturbative expansion of the quantum theory gives Feynman diagrams dual not just to graphs but to cellular complexes, which can also be understood as discretization of some smooth manifold. The Feynman amplitudes, whose explicit expression is of course model-dependent, can be given the form of lattice gravity path integrals [16–18] or, equivalently, spin foam models [19–23]. The latter are a covariant definition of the quantum dynamics of spin networks, the quantum states of Loop Quantum Gravity (LQG) [24, 25]. In fact, GFTs can be understood as a reformulation of the kinematics and dynamics of LQG degrees of freedom in a 2nd quantized framework [4]. The GFT Hilbert space re-organizes the same type of spin networks in a Fock space, whose fundamental quanta are spin network vertices and are created (annihilated) from (into) a Fock vacuum (a state with no geometric nor topological structure) by the action of the GFT field operators, absent any embedding information into any ambient smooth continuum manifold.

In the following we will discuss each point in more detail, in order to convey the general idea and motivation behind each of the above features.

Non-local action.   In local theories the action is an integral functional on the (appropriate) space of fields, whose

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[1] In fact, the GFT formalism includes also the case in which the fields are defined on a finite group, reducing to the *tensor models* formalism. However, for obvious reasons our symmetry analysis would not apply to this case, so we stick to the Lie group setting.

kernel is the Lagrangian. In the geometrical interpretation a Lagrangian is a function on a vector bundle which is usually a jet bundle over a principle G-bundle where G is the fundamental symmetry group of the theory. In usual field theories of fundamental interactions, the base manifold of the vector bundle is interpreted as space time and the fiber is a vector space that carries a representation of G.

In non-local theories we want to maintain this geometrical picture as far as possible, even if in our GFT context the base manifold will not have the interpretation of spacetime (but is rather related to *superspace*, the space of smooth spatial geometries, or minisuperspace, the space of homogeneous spatial geometries; see [26–29]). We assume that a non-local function can be treated as a local function on a higher dimensional space. The drawback of this picture is that different non-local therms are described by different geometrical bundles, and the geometrical structure of the theory strongly depends on the model in question, in contrast to the local case.

In a local theory the action is an integral of a Lagrangian over some domain Ω. In non-local theory it becomes a sum of integrals whose domains are different base manifolds (in particular, of different dimension). In the specific types of non-local field theories we will be concerned with, they correspond to different numbers of copies of a given base manifold. The action can therefore be generally written as

$$S = \sum_i \int_{M_i} L^i \, \mathrm{vol}_i, \qquad (1)$$

where i ranges over the number of different base manifolds M_i with the correspondent Lagrangians L^i. The measure of integration on each manifold defines the volume density, and in the GFT case, will be given by the Haar measure or some other invariant measure, depending on the Lie group chosen as base manifold.

Lie group structure of the base manifold.   Indeed, in Group Field Theory, we require the local base manifold, which matches by definition the domain of each individual GFT field, to be given by some number of copies of a Lie group G. In GFT models of quantum gravity, this is usually chosen to be the local gauge group of gravity, i.e. the Lorentz group or it double cover $SL(2, \mathbb{C})$ (or its Riemannian counterpart $Spin(4)$ for models of gravity in Euclidean signature). For models directly related to Loop Quantum Gravity, the rotation subgroup $SU(2)$ of the Lorentz group becomes the relevant base manifold, via appropriate conditions imposed on the GFT fields (and quantum states) at the dynamical level.

The number of copies of the group G defining the local base manifold is usually the topological dimension of the cellular complexes dual to the Feynman diagrams of the model, chosen to match the dimension of the continuum spacetime one aims to reconstruct from the quantum dynamics of the model. Clearly, to have a physical connection to General Relativity we need to develop and understand models in four dimensions. However, since

these tend to be naturally more involved than their lower dimensional counterparts, several two and three dimensional models have been studied, which allow to investigate important mathematical and conceptual problems of the theory, in the setting with reduced complexity.

Gauge invariance of fields In some GFT models the fields are required to satisfy a so-called gauge invariance condition. Explicitly it means that for any h in the diagonal subgroup $G_D = \{(g, \cdots, g) \in G^{\times n} | g \in G\}$ the fields satisfy

$$\phi \circ R_h = \phi, \tag{2}$$

where $R_h : G^{\times n} \to G^{\times n}$ denotes the right multiplication by $h \in G_D$. Due to this gauge invariance condition, the base manifold of the GFT fields is effectively reduced to a quotient of n copies of the group under the diagonal group action

$$G \times \cdots \times G/G_D.$$

This condition is imposed for many different physical considerations. In particular, in gauge invariant GFT models the perturbative Feynman amplitudes of the theory take the form of lattice gauge theories (on the cellular complex dual to each Feynman diagram) and the quantum states become those of a lattice gauge theory with gauge group G (in particular, for $G = SU(2)$, a complete orthonormal basis is given by spin networks, and the same amplitudes can be equivalently written as spin foam models).

Other defining features of different GFT models

Beside the choice of base manifold, there are various other ingredients that have to be specified, in order to fully define a GFT model, and this even before one chooses a functional form for the interaction kernels. The main differences between various models include:

Presence (absence) of derivatives in the local Lagrangian - dynamical (static) models The GFT models proposed at first, for a study of topological field theories of BF type and, later, for 4d quantum gravity described as a constrained BF theory and in absence of matter fields, only possessed a "mass" term, and no derivatives of the fields in their quadratic, local part of the Lagrangian. In local QFT, this would imply a trivial dynamics, and therefore one could label these models "static". However, in contrast to local field theories, the transition functions, and more generally both the classical and quantum dynamics in "static" group field theories are still highly non-trivial due to their non-local nature. More recently, GFT models which include derivative terms in the quadratic part of the action have been studied extensively. They can be motivated in various ways, the first being that renormalizability seems to require them, at least for some models in which a non-trivial dynamical term will be generated by the RG-flow [30, 31], and therefore needs to be included in the theory space.

Non-local structure of the Lagrangian - combinatorics As said, the main feature distinguishing GFTs from ordinary local field theories is the combinatorial pattern of relations between field arguments in the GFT interactions. In principle many different non-local interaction terms can be included in the action. A preference of one model over another can be given conclusively only by extracting its physical consequences. We will discuss the possible combinatorics and their consequences rather extensively in the following. Here we only stress that a detailed analysis of symmetries of the corresponding models is going to be useful also for choosing one combinatorial structure over another.

Number of different fundamental fields - colored theories One can also consider GFT models involving more than a single fundamental field. If a Group Field Theory model involves more than one field, we call the model *colored* and distinguish different fields by an additional index, calling it the color index. Colored GFTs were introduced for the first time in [32] for models aimed at describing simplicial quantum gravity, and topological BF theories discretized on simplicial complexes. Indeed, this step immediately led to a large number of interesting mathematical results and powerful new techniques, in the GFT context as well as for the simpler tensor models [8]. In particular, Feynman diagrams generated by non-colored simplicial GFT models can be dual to very singular simplicial complexes, while Feynman diagrams of colored models are much more regular, and their topology can be reconstructed to a much greater extent [33].

Quantum statistics An additional assumption on the theory is its quantum statistics, i.e. whether the GFT quanta of a given model are bosonic, fermionic or of other nonstandard statistics. In local, spacetime-based quantum field theories the quantum statistics is highly constrained by powerful spin-statistics theorems, linking the quantum statistics to the spin of the quanta, under the assumption of Lorentz or Poincare invariance of the theory. No similar spin-statistics theorem is available, yet, in the GFT framework. First and foremost, this is due to the fact that the base manifold in group field theories is not directly associated with space time, thus we have no obvious symmetry requirement to impose, like Lorentz or Poincare invariance. Second, as we have already stressed as a motivation for our work, very little is known about GFT symmetries, from a full classification of them in specific models to their general consequences, on the statistics of the same models and on their physics. This paper is meant to partially fill this gap.

Notation

Throughout the paper we will denote an element of the Lie group $G^{\times n}$ as

$$\vec{g} = (g_1, g_2, g_3, \cdots, g_n). \tag{3}$$

Differential operators with an index will refer to operators that act on the correspondent copy of the group. For example,

$$\nabla_1 (\vec{g}_{123}) = (\nabla g_1, g_2, \cdots, g_n).\qquad(4)$$

Differential operators with more than one index refer to a sum of individual operators as

$$\nabla_{123} = \nabla_1 + \nabla_2 + \nabla_3.\qquad(5)$$

If used without further clarification, an integral symbol (without the explicit measure) denotes an integral over all variables that appear under the symbol. Integration over each single group element is performed with the Haar measure on G

$$\int \phi(g_1, g_2, g_3) = \int \mathrm{d}g_1 \mathrm{d}g_2 \mathrm{d}g_3 \, \phi(g_1, g_2, g_3).\qquad(6)$$

This notation will be used a bit differently in section III, where we will point out the differences explicitly. The fields are complex scalar fields. The upper script of the field denotes the color and the subscript denotes the field's dependence on the variables

$$\phi^c_{1,2,\cdots,n} := \phi^c(g_1, g_2, \cdots, g_n).\qquad(7)$$

A. Overview of the models discussed in the paper

We are going to present the major distinctions of combinatorial structures of models discussed in the following. The general structure of the GFT actions has the form of (1) as

$$S[\phi] = S^{loc}[\phi] + S^{nloc}[\phi].\qquad(8)$$

We assume that the local, quadratic part of the action is defined as

$$S^{loc}[\phi] = \int_{G^{\times n}} \kappa \nabla \bar{\phi} \cdot \nabla \phi + m \, \bar{\phi}\phi,\qquad(9)$$

where ∇ is the gradient on the group G and the $\cdot$ denotes the contraction of the two vectors. We will also treat cases in which κ is zero, meaning that the model is a static one. In the following, we distinguish between three different types of models - simplicial, tensorial and geometrical. The corresponding interaction parts S^I are presented below and a concise summery of the interaction terms used is given in the table (I).

1. Simplicial

The interaction part of simplicial models is constructed such that the Feynman diagrams have a particular topological interpretation, i.e. they are simplicial complexes. Let us illustrate in one example how this comes about, in

a simple example: a GFT model for BF theory in three dimensions and the GFT fields subjected to the gauge invariance condition. If the group is chosen to be $SU(2)$, this is a model for 3d gravity in euclidean signature. In this case the simplicial interaction is given by

$$S^I[\phi^c] = \lambda \int_\Omega \phi_{1,2,3}\phi_{3,4,5}\phi_{5,2,6}\phi_{6,4,1} + c.c.\,,\qquad(10)$$

where the integral domain is $\Omega = G^{\times 6}$. If we associate to each field in the interaction a triangle with edges labelled by the three arguments of the same field, the interaction will be associated to a tetrahedron (a 3-simplex) formed by the 4 triangles glued pairwise along common edges, as shown in figure (1).

(a) 3 dim field associated with a triangle

(b) Simplicial interaction in three dimensions associated with a tetrahedron
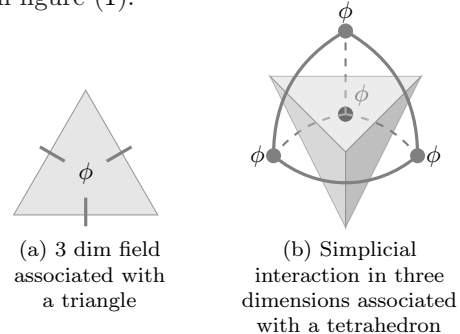
Figure 1: Topological interpretation of the field and the simplicial interaction in three dimensions

The Feynman diagrams will similarly be in correspondence with the simplicial complexes obtained by gluing the different tetrahedra associated to the interaction vertices in the Feynman diagram, along shared triangles, the gluing being identified by a propagator line. Depending on the details of the Feynman diagram, the resulting simplicial complex may or may not be a simplicial manifold and can be quite singular. Moreover, the data present in the GFT diagram are not, in general, sufficient for reconstructing the topology of the same simplicial complex in its entirety. Such technical difficulties are cured by introducing colored fields [33].

The extension to higher dimensions, via extension of the base manifold of the fields, and appropriate pairing of their arguments in the interactions, follows the same criteria and it is straightforward.

Remarks on the combinatorial structure of simplicial models

The original Boulatov model [34] for 3d gravity has a slightly different combinatorial structure from the one we introduced above. It is given by

$$S^I[\phi^c] = \lambda \int_\Omega \phi_{1,2,3}\phi_{1,4,5}\phi_{2,5,6}\phi_{3,6,4} + c.c.,\qquad(11)$$

with an additional invariance of fields under cyclic permutations of the variables. The quantum geometric content of the model is not affected, as it can be seen both in

the group representation, and in the spin representation. In fact, in the Peter-Weyl decomposition both combinatorial structures lead to a 6J symbol, which encodes both the gauge invariance properties and the piece-wise flatness of the simplicial complex generated in the perturbative expansion. However, while the original Boulatov model produces a usual 6J symbol, the interaction pattern we introduced above produces an additional factor (-1) that alternates with the representations involved. In the case of colored models this difference can be absorbed in the redefinition of the fields as $\tilde{\phi}^i = \phi^i \circ P^i$, where P^i being some permutation of the group elements. In this way the exact order of the variables in the field becomes unimportant. We will choose the following combinatorics, since it leads, as we will show, to the largest symmetry group:

$$S^I[\phi^c] = \lambda \int_\Omega \phi^1_{1,2,3} \phi^2_{1,4,5} \phi^3_{6,2,5} \phi^4_{6,4,3} + c.c.. \qquad (12)$$

It is important to mention that if the models are indeed equivalent, their symmetry group should not differ as well. This implies that a particular choice of the combinatorics may simply help to discover symmetries that would still be there for different combinatorics, but would be more difficult to identify. In the next section we will show how these minor combinatorial differences affect the symmetry group.

Notice that, while in the action above we have chosen four GFT fields to appear in one term, with their complex conjugates appearing in the other, our focus here was only the combinatorial structure, and one can devise simplicial interactions involving both the field and its complex conjugate in the same monomial. For example, we can start with the action from equation (10) and color the fields in the way $\phi^1 = \phi^3 = \phi$ and $\phi^2 = \phi^4 = \bar{\phi}$ such that the interaction part coincides with its complex conjugate

$$S^I[\phi^c] = \lambda \int_\Omega \phi_{1,2,3} \bar{\phi}_{3,4,5} \phi_{5,2,6} \bar{\phi}_{6,4,1}. \qquad (13)$$

In this case we refer to the above action as colored, with two colors, even though the model involves only the field ϕ and its complex conjugate. This convention will become handy in the classification of the symmetries in the following.

2. Tensorial models

Tensor models are characterized by an $U(N)$ invariance. Given a rank-n complex tensor $T_{i_1 \ldots i_n}$ with index set of dimension N, it transforms naturally under the group $U(N)^{\times n}$, where $U(N)$ is a unitary $N \times N$ matrix, acting on each of its indices. This is also the natural symmetry of tensor interactions, so that the full theory space is defined to be spanned by all possible monomials in the tensor and its complex conjugate, with their indices

contracted to give unitary invariants [8, 35]. The generalization of the same invariance characterizes the interactions of tensorial GFTs [10, 36]. Hereby a monomial in fields belongs to the theory space if it is invariant under a unitary transformation defined as follows

$$U\phi(\vec{g}) = \int d\vec{h}\, U^1(g_1, h_1)\, U^2(g_2, h_2)\, U^3(g_3, h_3)\, \phi\left(\vec{h}\right),$$

with the requirement on the kernels U^i to satisfy

$$\int_\Omega dh\, U(g, h)\, U^\dagger(h, q) = \delta(g, q), \qquad (14)$$

where $U^\dagger(h, g) := \bar{U}(g, h)$. This conditions requires that two fields which share a group element need to be complex conjugate of each other. It is easy to verify that this excludes the simplicial combinatorics. It is also important to mention that the kernels U^i do not need to be smooth, differentiable or even continuous and for this reason they may include delta distributions. We will come back to this point in the next section, when we discuss the symmetries of the tensorial models.

Note that a dynamical term will in general break the unitary invariance. Therefore, when we refer to tensorial dynamical models in the following, we imply that the unitary invariance characterizes only the interaction part and not of the whole action. Note also that one can have a very similar type of invariance for real GFT fields, with the unitary group replaced by an orthogonal group. The construction proceeds in analogous way.

3. Extended Barrett-Crane model

In four and higher dimensions, gravity can be formulated as a BF theory plus appropriate constraints [37], which are labeled *simplicity constraints*. This goes under the name of Plebanksi formulation of gravity. This formulation provides also the conceptual and technical starting point for the construction of spin foam and group field theory models for 4d quantum gravity. One may call the corresponding GFT models *geometric*, even though one has a direct control only on the discrete (simplicial) geometric interpretation of states and amplitudes, while the reconstruction of continuum geometry requires more work. As an example of these constructions, we deal with the so-called Barrett-Crane model [38], whose detailed treatment in the language of extended Group Field Theory was presented in [18], for the euclidean signature. Here we show just the main features of the model and refer to the cited literature for more details.

The starting point is the GFT model for 4d BF theory based on simplicial interactions, in which the fundamental GFT field is associated to a tetrahedron in 4d, and the interactions involve five GFT fields, paired to represent the gluing of five tetrahedra to form a 4-simplex. The base group manifold of the model is $Spin(4)$.

Simplicity constraints are characterized by a vector in $S^3 \simeq SU(2)$, interpreted as a unit normal vector (in $\mathbb{R}^4$)

	Gauge variant		Gauge invariant	
Dimension	$3D$ and $4D$			
Kinetic part	$\kappa \nabla\phi \cdot \nabla\phi + m\,\phi\phi$			
	$\kappa \neq 0$		$\kappa \neq 0$	$\kappa = 0$
Group	$SU(2)$	$SU(2)$	$SU(2)$	G
Combinatorics	Tensorial	Simplicial	Simplicial	
Colors	-	(un)colored	(un)colored	
Simplicity constrains	-	-	-	Barrett-Crane

Table I: Overview of the models discussed in this paper

of the tetrahedron represented by the field ϕ. In order to keep track of this additional normal vector the local base manifold is extended, so that the field becomes a function on four copies of $Spin(4)$ and one copy of $SU(2)$

$$\phi\left(g^1, g^2, g^3, g^4, k\right) =: \phi_{1,2,3,4,k}, \qquad (15)$$

where $g_i \in Spin(4)$ and $k \in SU(2)$.

The interaction of the model becomes an extended version of the Ooguri interaction given as

$$S^I[\phi] = \int \phi_{1,2,3,4,k_1}\phi_{4,5,6,7,k_2}\phi_{7,3,8,9,k_3}\phi_{9,6,2,10,k_4}\phi_{10,8,5,1,k_5} + c.c.. \qquad (16)$$

The gauge invariance is again written in the usual form as

$$\phi \circ R_h = \phi, \qquad (17)$$

with $h \in Spin(4)_{4D}$. Additionally, the simplicity constraints are imposed by requiring invariance of the fields

$$\phi \circ S = \phi, \qquad (18)$$

under the transformation

$$S : (\vec{g}, k) \mapsto (\vec{g}; k) \cdot \left(\left(k\vec{u}k^{-1}, \vec{u}\right); \mathbb{1}\right), \qquad (19)$$

where $u^j \in SU(2)$. If we write a $Spin(4)$ element in its selfdual and anti-selfdual $SU(2)$ components as $g = (g_-, g_+)$, the above transformation takes the form

$$S(\vec{g}; k) = \left(g_-^j\, ku^j k^{-1}, g_+^j u^j; k\right). \qquad (20)$$

In [18] it has been shown that S and R_h commute as projectors acting on the space of fields, which allows to combine them into a single transformation, which is itself a projector, acting on the GFT fields as

$$\mathcal{S} : (\vec{g}; k) \mapsto \left(\mathbb{1}; h_-^{-1}\right) \cdot (\vec{g}; k) \cdot \left(\left(k\vec{u}k^{-1}, \vec{u},\right); \mathbb{1}\right) \cdot (h_-; h_+),$$

where $h \in Spin(4)$. One can indeed verify that the fields invariant under the above transformation satisfy

$$\phi \circ \mathcal{S} = \phi. \qquad (21)$$

Notice that, since the simplicity and gauge invariance conditions are imposed on the fields via a projector, the imposition of these conditions on all fields appearing in the action is the most natural choice, but any other choice, e.g. imposing them only on the fields appearing in the interactions, would result in the same Feynman amplitudes (but not the same theory, as for example the classical equations of motion would be different). This is not true for other 4d gravity models, where the simplicity constraints take a different form [17].

II. Symmetries

In this section we will present the different notions of symmetry transformations in non-local field theories in general (recalling the geometric construction in the local case, first), and then apply them to Group Field Theory in particular, and derive the symmetry groups for the models introduced above, showing the main steps of the calculations for three-dimensional models.

A. Transformations of local field theory

The geometrical construction of local field theory is very well known, but we will briefly review its main points here because they will be essential in the following discussion of the non-local case.

In the geometrical picture, the Lagrangian is a differentiable (in a sense that needs to be further specified) function on an nth order jet bundle. In order to bring the main idea across without complicating it with technical details, we will assume that the Lagrangian is a function just on a vector bundle. The full construction can be found in usual text books on this subject some of which are [14, 39].

We call the relative vector bundle E, the base manifold of E being M and the fiber being $\mathbb{V}$. Locally, we can think about E as a cross product of $M \times \mathbb{V}$, which we assume for the rest of this discussion. The points on E are then given by $x \in M$ and $u \in \mathbb{V}$, we write $(x, u) \in E$. Hence, the values of the Lagrangian can be denoted as $L(x, u) \in \mathbb{R}$.

We then introduce the physical fields ϕ in the construction. This is done by choosing points of the vector bundle which are given by a smooth section of E. In other words we assume that $u = \phi(x)$.

Assuming that the set of transformations of the theory forms a group G_T, we can write the action of the group on E as

$$g \cdot (x, u) = (\tilde{x}, \tilde{u}) = (C(x, u), Q(x, u)). \qquad (22)$$

The action is thus specified by two functions C and Q. Note, that in general both functions depend on x and $u = \phi(x)$ and are not invertible. However, locally around each point of the bundle these transformations are diffeomorphisms, due the the fact that they represent an action of a Lie group.

We ask for the transformed sections $\tilde{\phi}$ that corresponds to a new point of the bundle, that is $\left(\tilde{x}, \tilde{\phi}(\tilde{x})\right) = (\tilde{x}, \tilde{u})$.

The transformed fields $\tilde{\phi}$ can then be seen as transformed sections under the group action of g. It is a well known result that the transformed fields are given by

$$\tilde{\phi}(\tilde{x}) = Q\left(C^{-1}(\tilde{x}), \phi \circ C^{-1}(\tilde{x})\right), \qquad (23)$$

or, in short,

$$\phi \overset{g}{\mapsto} Q \circ \phi \circ C^{-1}, \qquad (24)$$

at least as long as C is invertible. Hereby, the transformation Q is defined along the fiber and the C^{-1} accounts for the transformation of the base manifold.

We summarize the main properties of the maps Q and C before finishing this part. For a fixed ϕ the base manifold transformation C is a local automorphism

$$C_\phi : M \to M \qquad (25)$$
$$x \mapsto C(x, \phi(x)).$$

And for a given point $x \in M$, the fiber transformation Q is a local automorphism

$$Q_x : \mathbb{V} \to \mathbb{V} \qquad (26)$$
$$\phi(x) \mapsto Q(x, \phi(x)). \qquad (27)$$

B. Transformations of a non-local field theory

We now apply this construction to non-local field theories. As we have pointed out earlier, the action is given by the sum of integrals over Lagrangians

$$S = \int_{M_i} L^i. \qquad (28)$$

Hereby, each of the Lagrangians is a function from a vector bundle E_i to $\mathbb{R}$. For $i \neq j$ the vector bundles E^i and E^j are assumed to be different. If they are not, we can combine the Lagrangians L^i and L^j into a single Lagrangian $L^{ij} = L^i + L^j$.

Following the general construction from the previous section we define a transformation of the theory as transformation of the corresponding vector bundles. Nevertheless, in the non-local case we need to transform different bundles, which is why we say that a group action is given by functions C^i, Q^i such that, for each i, C^i and Q^i are transformations of E^i in the above (local) sense. It is important to realize that these transformations can not be independent from one another, since they represent the same transformation $g \in G_T$. Instead, their mutual relations should be given by the relation between different vector bundles E^i.

Assuming that E^0 denotes the vector bundle, whose sections are identified with physical fields $\phi^0 = \phi$ we can quite generally write each E^i as a pull back of n_i copies of E^0 by some embedding $f_i : M^i \to M^{\times n_i}$. These functions f^i encode the combinatorial structure of non-local Lagrangians and provide a relation between different E^i's. Therefore they also give the relations between the sections ϕ^i as $\phi^i = \left(\phi^0\right)^{\times n_i} \circ f$. Knowing how the field ϕ^0 transforms under g automatically implies the transformations of ϕ^i as

$$Q^i \circ \phi^i \circ \left(C^i\right)^{-1} = \left[Q^0 \circ \phi^0 \circ \left(C^0\right)^{-1}\right]^{\times n_i} \circ f. \qquad (29)$$

This relation implicitly defines C^i as

$$f \circ C^i = \left(C^0\right)^{\times n_i} \circ f, \qquad (30)$$

and Q^i as

$$Q^i \circ \phi^i = \left[Q^0 \circ \phi^0\right]^{\times n_i}. \qquad (31)$$

The above equations provide the missing link between the group actions on different vector bundles. However, equation (30) does not always define a local automorphism C^i on M^i. If C^i were an automorphism, equation (30) would imply that, for any $x^i \in M^i$, there exists an $\tilde{x}^i \in M^i$ such that

$$f\left(\tilde{x}^i\right) = \left(C^0\right)^{\times n_i} f\left(x^i\right). \qquad (32)$$

This however, is not always possible as we will see in the following section.

If equation (32) is not satisfied, the group action can not be chosen consistently as a transformation of the vector bundle. In this case, we can define the action of the group directly on the space of fields, using equation (29) as

$$g \cdot (x, u) = (x, Q(x, u, u_x, \cdots)) = \left(x, \phi \circ C^{-1}(x)\right).$$

Here, u_x denotes the coordinates of the Jet space and refers to derivatives of fields at the point x, i.e. $u_x = D\phi|_x$. That the transformation Q needs to depend on the derivatives of fields ϕ is easily seen from the Taylor expansion, since

$$\phi \circ C^{-1}(x) = \phi(x) - D\phi(X_M) + \mathcal{O}\left(X_M^2\right) \quad (33)$$
$$= u - u_x \cdot X_M + \mathcal{O}\left(X_M^2\right), \quad (34)$$

where X_M is the infinitesimal generator of the transformation C. Such transformations Q generalize the notion of point transformations from equation (22) to the so called Lie-Baecklund transformations, which are transformations from the Jet bundle to the vector bundle. In the next section, we will briefly explain how the Lie-Baecklund transformations represent a more general notion of symmetry. However, for reasons that will also become apparent in the next section, we will restrict our analysis to Lie point symmetries.

C. Notions of symmetry transformations

There are many different notions of a continuous symmetry in local field theories. Almost all of them are formulated as diffeomorphisms of the vector bundle of the theory. In order to distinguish between different notions of symmetries we first point out that an action (in local theories) is defined as an integral, and therefore intrinsically depends on the domain of integration over which the Lagrangian is integrated, i.e. $S_\Omega[\phi] = \int_\Omega L$. In this sense we can talk about a family of actions $\{S_{\Omega'}\}$ for all $\Omega' \subseteq \Omega$. In the discussion of symmetries, the dependence of the action on the domain plays a very important role, which we are going to highlight in the following.

1. Point symmetries

The simplest notion of a symmetry of an action is a diffeomorphism on the vector bundle of the theory [39, 40]. As we discussed above choosing a section of the bundle (physical field ϕ) it is possible to locally project the diffeomorphism on the fiber and the base manifold obtaining the transformation function Q and C, which define a transformation of the fields and of the base manifold respectively. A symmetry is then a transformation which does not change the action functional $S_{\Omega'}$ for any subdomain $\Omega' \subset \Omega$

$$S_{\Omega'}[\phi] \to S_{C(\Omega')}\left[\tilde{\phi}\right] = S_{\Omega'}[\phi] \quad \forall \Omega' \subset \Omega \quad . \quad (35)$$

These transformations are of the type (22) and are called *Lie point symmetries* or "*geometrical*" symmetries, because they admit a geometrical interpretation of a flow, being generated by vector fields on the vector bundle of the theory.

The requirement that the symmetry does not change the action for any sub domain Ω' is essential, in order to be able to make point-wise statements, i.e. to derive truly local statements from the existence of the symmetry itself. In the physical literature, this statement is often referred to as the Noether theorem, which allows the derivation of point-wise equations of the form $\operatorname{div}(J) = E_L \cdot \delta\phi$, i.e. the conservation laws of the corresponding field theory.

2. Generalized symmetries

A generalization of the symmetry concept (already introduced by Noether in her original paper [41]) leads to the so called *generalized symmetries*, which are Lie-Baecklund transformations that can change the action by an arbitrary divergence term. That is, for all $\Omega' \subset \Omega$

$$S_{\Omega'}[\phi] \to S_{C(\Omega')}\left[\tilde{\phi}\right] = S_{\Omega'}[\phi] + \int_{\Omega'} \operatorname{div}(\Gamma) \quad \forall \Omega' \subset \Omega \quad . \quad (36)$$

A restriction of such transformations to those that can depend at most on the first order derivatives of fields defines the so called contact symmetries.

In general, a set of Lie-Baecklund symmetries is infinitely large, but it is often the case that also infinitely many such transformations are equivalent, leading to a finite number of inequivalent transformations. Computational algorithms for finding Lie-Baecklund symmetries to a fixed order of derivative dependencies are known and are implemented in a large variety of computer algebra programs [42]. Nevertheless, already for flat base manifolds (and of course, local theories) the explicit calculations are quite challenging.

The reason for looking for a generalized notion of symmetry is the observation that two actions are physically equivalent *if and only if* they differ by a divergence term [39], because the correspondent equations of motion are the same, and this is all that matters in the classical regime. This implies that the physically relevant object is not the action but rather an equivalence class of actions[2].

As in the previous cases, this class of symmetries gives rise to local, point-wise equations. Even more, only in this case, the correspondence between symmetries and

---

[2] Is important to distinguish between the symmetries of the action and symmetries of the correspondent equations of motion (which correspond to extrema of the same action): generalized variational symmetries form a subgroup of Lie-Baecklund transformations of the equations of motion.

divergence-free quantities like $\mathrm{div}\,(J + \Gamma) = E_L \cdot \delta\phi$ is one to one, which is the actual statement of the original Noether theorem.

3. Integral symmetries

An entirely different notion of symmetry arise if we drop the requirement that a symmetry transformation should leave the family of actions $\{S_{\Omega'}\}$ invariant, and instead require the invariance of S_Ω only for a single, fixed integral domain Ω,

$$S_\Omega\,[\phi] \to S_{C(\Omega)}\left[\tilde{\phi}\right] = S_\Omega\,[\phi]. \tag{37}$$

This kind of transformations does not lead to point-wise statements, like conservation laws. Clearly every symmetry of the previous type is also a symmetry of this type but not the other way around.

It is interesting to observe that tensor models and tensorial GFTs invoke exactly this type of symmetries in order to define the theory space, since we require that the corresponding unitary transformations satisfy

$$\int_G \mathrm{d}b\, U\,(g, h)\, U^\dagger\,(h, s) = \delta\left(gs^{-1}\right), \tag{38}$$

only after the integration over the whole group G, and there is no reason to assume that changing the integral domain to a subspace of G would preserve the above equality.

4. Symmetries in non-local field theories

So far the definition of a symmetry was introduced for an action which is given by an integral over a Lagrangian. In the non-local case, as we have explained, the action is given by a sum of such actions each defined on a different base manifold. Therefore, we need to extend the above notions of symmetries to transformations, which are symmetries (in one of the above senses) of each and all individual functions in the action-sum.

This is the only generalization we need, to start analyzing symmetries of the non-local GFT models introduced above.

However, it is important to stress here, that this is not enough to study generalized symmetries of the Lie-Baecklung type. The same motivation that lead to considering them in the local case would apply as well for non-local models. However, contrary to the local case, for non-local field theories the equivalence class of actions that yields the same equations of motion is not under control (to our knowledge). The only thing that we can say is that it does not coincide with the one defined in the local case, because as we pointed out in [11], a divergence term will, in general, change the equations of motion.

One approach to overcome this difficulty would be to discuss directly the symmetries of the corresponding equations of motion, which are integro-differential equation (see [14] for the standard approach of Lie algebra methods in integro-differential equations). However, this is highly non-trivial, in the GFT case. Gauge invariance condition, the structure of the curved base manifold, as well as its large dimension make the usual Lie algebra approach, even more involved.

Also, there is not much more to say about the integral symmetry transformations defining tensorial group field theories, beside what we remarked already, i.e that they provide a natural characterization of the corresponding theory space. For these reasons, we limit our analysis to the Lie point symmetry analysis and postpone the analysis of Lie-Baecklund symmetries to future work.

In the following we will use the definition of a symmetry for a non-local action as follows:

Definition. A symmetry of a non-local action is a transformation that is a Lie-point symmetries of each functional in the action-sum.

D. Symmetry analysis of gauge variant models

We start by performing the standard Lie group analysis of point symmetries [39] in the case of gauge-variant GFT models.

In [11] we have shown, that a symmetry condition of the action can be equivalently formulated on the level of its Lagrangians, leading to a generalized version of Noether theorem, with respect to the local case. More precisely, the symmetry relation can be formulated in the following way

Theorem 1. G is a symmetry group of the action iff the generators of the symmetry (X_V, X_M) satisfy the relation

$$0 = D_J L \cdot D X_Q + D_V L \cdot X_Q + Div\,(L\, X_M), \tag{39}$$

where $X_Q = X_V - X_M\,(\phi)$ and it is assumed that every term is evaluated at some point z of the correspondent base manifold.

The notation that is used in the above equation needs to be further explained:

- X_M is a vector field on the base manifold which coefficients depend on a point of the base manifold and a point on the fiber. In local coordinates (U, x) of the base manifold the vector field can be written as $X_M = X_M^i\,(x, \phi\,(x))\,\partial_i$.

- X_V is a vector field along the fiber of the bundle that in local coordinates $(U, x) \times (V, u)$ can be denoted as $X_V^i = X_V^i\,(x, \phi\,(x))\,\partial_{u^i}$ where $u^i = \phi^i\,(x)$. We will sometimes use the simpler notation $\partial_{\phi(x)}$ or even ∂_ϕ, always referring to ∂_u.

- The assumption of dealing with a geometrical symmetry translates into the restriction of the coefficients of vector fields depending only on x and $\phi(x)$, but not on $\partial\phi$ and higher order derivatives.

- $X_M(\phi)$ is the Lie derivative of ϕ along X_M.

- X_Q is the characteristic vector field, which corresponds to the effective transformation of the fields from equation (24), given by

$$X_Q = \partial_\epsilon|_0 \, Q_\epsilon \circ \phi \circ C_\epsilon^{-1} = X_V - X_M(\phi) \qquad (40)$$

It is also important to spend few words on the different types of derivatives that are used in this geometrical construction.

- The derivative D_V denotes a derivative of the Lagrangian along the coordinates of the fiber. In the common notation we can write ∂_ϕ or δ_ϕ

- D_J denotes the derivative of the Lagrangian with respect to the jet coordinates. In the above notation we can write $\partial_{\partial_i\phi}$ or $\delta_{\partial_i\phi}$.

- The derivative D refers to the total derivative with respect to the base manifold. This means that the implicit dependence on the base manifold through fields needs to be taken into account.

- The partial derivative ∂_i is instead a derivative purely on the explicit dependence of the coordinates. Using above notation we can write

$$D\,f(x,\phi(x)) = \partial_x f + D_V\,f \cdot \partial_x\phi. \qquad (41)$$

We also use the capital letter in $\mathrm{Div}\,(L \cdot X_M)$ for the total derivatives used in the divergence and $\mathrm{div}\,(X_M)$ to denote the divergence taken only with respect to the explicit coordinates.

Equation (39) holds for local as well as non-local Lagrangians. By partial integration equation (39) becomes

$$E_L[X_Q] + \mathrm{Div}\,(D_J L \cdot X_Q + L \cdot X_M) = 0. \qquad (42)$$

Where E_L is the Euler operator acting on the Lagrangian L^3.

Having clarified the terminology and the notation, we can use (39) to derive the most general geometric symmetries of the various GFT models.

We will use a rather standard procedure, based on the following steps:
i) We assume a most general vector field on the vector bundle and insert it in (39), ii) we rearrange the resulting

equation by different powers in derivatives of fields. Since the coefficients X_M^i and X_V^i do not depend on derivatives of the fields, it is possible to extract all powers explicitly, iii) different powers of derivatives of ϕ are linearly independent since the condition (39) has to be satisfied for all fields. For this reason the coefficients in front of each term have to vanish separately. This results in simple differential equations for the coefficients of the vector field which can then be easily solved.

Since the GFT models of interest, here, are defined on many copies of $SU(2)$ the notation can quickly become unreadable. For this reason we summarize the notation used in the rest of this section in the table (II).

u^c	Field value $\phi^c(\vec{g})$	
	c - color of the field,	
u_{iA}^c	Derivative of the field ϕ^c at the point $\vec{g}$	
	i - chart component of the single copy of $SU(2)$	
	A - number of the copy of $SU(2)$	
	iA direction of the derivative $\partial_{iA}\phi|_{\vec{g}}$	
X_M^{iA}	Vector field that acts on the base manifold M	
	i - chart component of the single copy of $SU(2)$	
	A - number of the copy of $SU(2)$	
X_{u^c}	Component of X_V in ∂_{u^c} direction	
	c - color of the transformed field	

Table II: Usage of indices in this section

We denote the vector fields by X_M and X_V and refer to their coefficients in a specific chart by X_M^{iA} and X_{u^c} respectively, i.e.

$$X_M = X_M^{iA}\,\partial_{iA} \qquad X_V = X_{u^c}\,\partial_{u^c} + X_{\bar{u}^c}\,\partial_{\bar{u}^c}. \qquad (43)$$

For the rest of this section we assume the summation convention over repeated indices.

The local part of the action is given by

$$L(u^c, u_J^c) = \sum_{iA} \kappa\,\bar{u}_{iA}^c u_{iA}^c + m\,\bar{u}^c\bar{u}, \qquad (44)$$

where the sum over A ranges in $\{1, \cdots, n\}$ (the $SU(2)$ copies of the local base manifold)

---

[3] For local theories E_L coincides with the equations of motion and the above equation becomes the usual Noether identity. For non-local theories, however, E_L does not coincide with the equations of motion, due to their integro-differential structure. In this case

further work needs to be done to provide a connection between the equations of motion and the divergence terms. The resulting relation is shown in the next section of this paper and is carefully derived in [11].

The above symmetry condition equation (39) implies

$$X_M(L) + L\operatorname{Div}(X_M) + 2\kappa\,\partial_{iA}\phi^c \cdot D^{iA}(X_{cQ}) + 2m\phi^c\,(X_{cQ}) = 0. \tag{45}$$

Explicitly sorting the terms by powers of u_{iA}^c we get

$$0 = \left[m\,|u^c|^2\operatorname{div}(X_M) + m\bar{u}^c X_{u^c} + mu^c X_{\bar{u}^c}\right] \tag{46}$$

$$+\operatorname{Re}\left[u_{nA}^t\right]\left[m\,|u^c|^2\left(D_{\bar{u}^t}X_M^{nA} + D_{u^t}X_M^{nA}\right) + \kappa g_A^{nm}\left(\partial_{mA}X_{u^t} + \partial_{mA}X_{\bar{u}^t}\right)\right] \tag{47}$$

$$+\imath\operatorname{Im}\left[u_{nA}^t\right]\left[m\,|u^c|^2\left(D_{u^t}X_M^{nA} - D_{\bar{u}^t}X_M^{nA}\right) + \kappa g_A^{nm}\left(\partial_{mA}X_{\bar{u}^t} - \partial_{mA}X_{u^t}\right)\right] \tag{48}$$

$$+\kappa\operatorname{Re}\left[\bar{u}_{nA}^c u_{mA}^c\right]\left[\frac{1}{2}X_M^{iB}\partial_{iB}g_A^{nm} - 2g_A^{ni}\,\partial_{iA}X_M^{mA} + g_A^{nm}\left\{(D_{u^c}X_{u^c} + D_{\bar{u}^c}X_{\bar{u}^c}) + \operatorname{div}(X_M)\right\}\right] \tag{49}$$

$$-2\kappa\operatorname{Re}\left[\bar{u}_{nA}^c u_{mB\neq A}^c\right]\left[g_A^{ni}\,\partial_{iA}X_M^{mB\neq A}\right] \tag{50}$$

$$+\kappa g_A^{nm}\operatorname{Re}\left[\bar{u}_{nA}^c u_{mA}^{t\neq c}\right]\left[D_{u^{t\neq c}}X_{cV} + D_{\bar{u}^c}X_{\bar{u}^{t\neq c}V}\right] \tag{51}$$

$$+\imath\kappa g_A^{nm}\operatorname{Im}\left[\bar{u}_{nA}^c u_{mA}^{t\neq c}\right]\left[D_{u^{t\neq c}}X_{u^c} - D_{\bar{u}^c}X_{\bar{u}^{t\neq c}}\right] \tag{52}$$

$$+\kappa g_A^{nm}\operatorname{Re}\left[\bar{u}_{nA}^c \bar{u}_{mA}^t\right]\left[D_{\bar{u}^t}X_c + D_{u^t}X_{\bar{u}^c}\right] \tag{53}$$

$$+\imath\kappa g_A^{nm}\operatorname{Im}\left[\bar{u}_{nA}^c \bar{u}_{mA}^t\right]\left[D_{\bar{u}^t}X_{u^c} - D_{u^t}X_{\bar{u}^c}^\dagger\right] \tag{54}$$

$$+2\kappa\bar{u}_{nA}^c u_{iA}^c\operatorname{Re}\left[u_{mA}^t\right]\left[-2g_A^{nm}\left(D_{u^t}X_M^{iA} + D_{\bar{u}^t}X_M^{iA}\right) + g_A^{ni}\left(D_{u^t}X_M^{mA} + D_{\bar{u}^t}X_M^{mA}\right)\right] \tag{55}$$

$$+\imath 2\kappa\bar{u}_{nA}^c u_{iA}^c\operatorname{Im}\left[\bar{u}_{mA}^t\right]\left[-2g_A^{nm}\left(D_{u^t}X_M^{iA} - D_{\bar{u}^t}X_M^{iA}\right) + g_A^{ni}\left(D_{u^t}X_M^{mA} - D_{\bar{u}^t}X_M^{mA}\right)\right] \tag{56}$$

$$+\bar{u}_{nA}^c u_{iB\neq A}^c\operatorname{Re}\left[u_{mA}^t\right]\left[-2\kappa g_A^{nm}\left(D_{u^t}X_M^{iB\neq A} + D_{\bar{u}^t}X_M^{iB\neq A}\right)\right] \tag{57}$$

$$+\imath\bar{u}_{nA}^c u_{iB\neq A}^c\operatorname{Im}\left[\bar{u}_{mA}^t\right]\left[-2\kappa g_A^{nm}\left(D_{u^t}X_M^{iB\neq A} - D_{\bar{u}^t}^\dagger X_M^{iB\neq A}\right)\right]. \tag{58}$$

This equation has to hold true for arbitrary fields u^c and u_{iA}^c. However, the parts in brackets do not depend on u_{iA}^c, which implies that each line has to vanish individually[4]. The consequences of these equations read:

1. Equations (51) and (52) imply that the vector field component X_{u^c} depend only on the field colors they transform that is (no summation)

$$X_{u^c} = X_{u^c}(\vec{g}, u^c, \bar{u}^c) \qquad X_{\bar{u}^c} = X_{\bar{u}^c}(\vec{g}, u^c, \bar{u}^c).$$

2. Equations (53) and (54) additionally imply that the vector fields X_{u^c} do not depend on the complex conjugate of the field, that is

$$X_{u^c} = X_{u^c}(\vec{g}, u^c) \qquad X_{\bar{u}^c} = X_{\bar{u}^c}(\vec{g}, \bar{u}^c).$$

3. Equations (57) and (58) tell us that the vector fields that transform the base manifold do not depend on

the field values u^c i.e. $X_M^A = X_M^A(\vec{g})$. From this condition, equations (55) and (56) are automatically satisfied.

4. Due to the above, equations (47) and (48) reduce to

$$\partial_{mA}X_{u^t} = 0 = \partial_{mA}X_{\bar{u}^c}. \tag{59}$$

That is, the vector fields do not explicitly depend on the points in the base manifold

$$X_{cV} = X_{u^c}(u^c) \qquad X_{\bar{u}^c} = X_{\bar{u}^c}(\bar{u}^c).$$

5. Equation (46), together with the above conclusion, restricts the vector fields to a specific form

$$X_{u^c} = C\,u^c \qquad X_{\bar{u}^c} = \bar{C}\,\bar{u}^c, \tag{60}$$

where C is an arbitrary constant that satisfies

$$\operatorname{div}(X_M) = -C - \bar{C}. \tag{61}$$

6. The above condition reduces equations (49) and (50) to

$$X_M^{iB}\partial_{iB}g_A^{nm} - 2g_A^{ni}\,\partial_{iA}X_M^{mA} - 2g_A^{mi}\,\partial_{iA}X_M^{nA} = 0$$
$$g_A^{ni}\,\partial_{iA}X_M^{mB\neq A} + g_B^{ni}\,\partial_{iA}X_M^{mA\neq B} = 0.$$

---

[4] Notice that, if we allowed for derivative dependence of the coefficients $\chi = \chi(x, \phi(x), D\phi|_x)$ and similar for the ξ, we could not argue that the terms with different powers of derivatives of ϕ have to vanish independently, since the terms in brackets would also contain derivatives of the fields.

These two equations are the only ones that are not trivial to solve. However, although lengthy, their solution can be found in a straightforward way. The solution in Hopf coordinates (η, ξ, χ)[5] reads as

$$X_M^{\eta A} = C_1 \sin \xi_A \sin \chi_A \tag{62}$$
$$+C_2 \cos \xi_A \sin \chi_A$$
$$+C_3 \sin \xi_A \cos \chi_A$$
$$+C_4 \cos \xi_A \cos \chi_A$$

$$X_M^{\xi A} = \frac{\cos \eta_A}{\sin \eta_A} \partial_{\xi A} X_M^{\eta A} + C_5 \tag{63}$$

$$X_M^{\chi A} = -\frac{\sin \eta_A}{\cos \eta_A} \partial_{\chi A} X_M^{\eta A} + C_6, \tag{64}$$

where C_i's are arbitrary constants.

Setting subsequently C_i to one and the rest of the coefficients to zero we obtain, for each copy of the group A, six linearly independent vector fields given by

$$v_1 = \begin{pmatrix} \sin(\xi)\sin(\chi) \\ \cot(\eta)\sin(\xi)\cos(\chi) \\ -\tan(\eta)\sin(\xi)\cos(\chi) \end{pmatrix} \tag{65}$$

$$v_2 = \begin{pmatrix} \cos(\xi)\sin(\chi) \\ -\cot(\eta)\sin(\xi)\sin(\chi) \\ -\tan(\eta)\cos(\xi)\cos(\chi) \end{pmatrix} \tag{66}$$

$$v_3 = \begin{pmatrix} \sin(\xi)\cos(\chi) \\ \cot(\eta)\cos(\xi)\cos(\chi) \\ \tan(\eta)\sin(\xi)\sin(\chi) \end{pmatrix} \tag{67}$$

$$v_4 = \begin{pmatrix} \cos(\xi)\cos(\chi) \\ -\cot(\eta)\sin(\xi)\cos(\chi) \\ \tan(\eta)\cos(\xi)\sin(\chi) \end{pmatrix}, \tag{68}$$

and

$$v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{69}$$

It is a direct calculation to check that these vector fields are divergence free, $\text{div}(V_i) = 0$. This fact, together with equation (61), implies

$$X_{u^k} = \imath C_k \, u^k \qquad X_{\bar{u}^k} = -\imath C_k \, \bar{u}^k \quad, \tag{70}$$

which generates the usual $U(1)$ symmetry of fields for each color.

In order to find the symmetry group generated by the fields $v_1, \cdots, v_6$, we look at their algebra. The six dimensional Lie algebra of $v_1, \cdots, v_6$ is given in table (III)

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	v_5	v_6	0	$-v_2$	$-v_3$
v_2	$-v_5$	0	0	v_6	v_1	$-v_4$
v_3	$-v_6$	0	0	v_5	$-v_4$	v_1
v_4	0	$-v_6$	$-v_5$	0	v_3	v_2
v_5	v_2	$-v_1$	v_4	$-v_3$	0	0
v_6	v_3	v_4	$-v_1$	$-v_2$	0	0

Table III: Lie algebra of symmetry vector fields

We can split this algebra into $\mathfrak{su}(2) \times \mathfrak{su}(2)$ by taking the following linear combinations

$$
\begin{aligned}
l_1 &= \frac{v_5 - v_6}{2} & r_1 &= \frac{v_5 + v_6}{2} \\
l_2 &= \frac{v_3 - v_2}{\sqrt{2}} & r_2 &= \frac{v_3 + v_2}{\sqrt{2}} \\
l_3 &= \frac{v_4 + v_1}{\sqrt{2}} & r_3 &= \frac{v_4 - v_1}{\sqrt{2}}
\end{aligned}
\tag{71}
$$

The commutators for l_i and r_i become

$$[l_1, l_2] = l_3 \qquad\qquad [r_1, r_2] = r_3 \tag{72}$$
$$[l_1, l_3] = -l_2 \qquad\qquad [r_1, r_3] = -r_2 \tag{73}$$
$$[l_2, l_3] = 2l_1 \qquad\qquad [r_2, r_3] = 2r_1 \tag{74}$$

$$[l_i, r_j] = 0 \quad \forall i, j \in \{1, 2, 3\} \tag{75}$$

A closer inspection shows that l_i and r_i form a set of left and right invariant vector fields on $SU(2)$, respectively [43].

Since the above algebra was derived for each copy of the group A, the whole symmetry group of the local part of the action becomes

$$[SU(2) \times SU(2)]^{\times 3} \times U(1)^{\times Nc}, \tag{76}$$

acting on the base manifold by left and right multiplication as

$$L_{\vec{\eta}} \circ R_{\vec{\mu}}(\vec{g}) = \left(\eta_1 g_1 \mu_1, \, \eta_2 g_2 \mu_2, \, \eta_3 g_3 \mu_3 \right), \tag{77}$$

for any $\vec{\eta}, \vec{\mu} \in SU(2)^{\times 3}$ and on fields by multiplication with a $U(1)$ phase.

It is now trivial to insert this transformations in the interaction part of the action in order to verify which of the transformations remains a symmetry. It is easy to see that the symmetry group is preserved for tensorial interactions.

Indeed, this is a remarkable feature of tensorial GFTs, which can be also stated as follows: by their very definition, the symmetry group of tensorial interactions is

---

[5] In this coordinates the metric on $SU(2)$ is given by $g = \mathrm{d}\eta^2 + \sin^2\eta\,\mathrm{d}\xi^2 + \cos^2\eta\,\mathrm{d}\chi^2$.

the same as that of the local part of the action. In this sense, this is a confirmation of the very motivation for introducing tensorial interactions as encoding the correct new notion of *locality* for tensorial field theories [9]. This is due to the fact that both of the symmetry groups we have found above form particular cases of the unitary transformations characterizing tensorial interactions, as the $U(1)$ transformations are implemented by

$$U(g,h) = \delta\left(gh^{-1}\right)e^{i\theta}, \qquad (78)$$

and the left (right) multiplication by the group is obtained as

$$U(g,h) = \delta\left(L_1 g h^{-1}\right) \quad U(g,h) = \delta\left(g R_1 h^{-1}\right). \qquad (79)$$

In the case of simplicial models the status of both the $SU(2)$ and the $U(1)$ group as symmetries of the full theory depend on the specific interaction in question. We need, then, to check explicitly the condition on the group action (32). We postpone the verification of this condition to the next section, since it will be the main tool in the analysis of gauge invariant models.

E. Gauge invariant models

In this section we study more in detail the symmetry group of simplicial GFT interactions, and we show how the treatment can be significantly simplified in the presence of gauge invariance. Contrary to the previous case, we will use the interaction part to classify the symmetry group, subsequently checking which of the symmetries represent also a symmetry of the local action. The first part of the treatment is independent of the local part of the action and holds for a large number of base group manifolds, which is why we do not specify the group at the beginning of the section. However, in order to verify the symmetry group for the local part we need to know the exact structure of the differential operator involved and so we need to specify the underlying group as well. From this point onwards, we specialize the notation to the $n = 3$ case for simplicity of exposition. The extension to generic n is straightforward.

1. Admissible base manifold transformations

The combinatorics of the interaction part is encoded in the function f from equations (29) and (30). For example the combinatorial structure of a 3d simplicial interaction is given by

$$f : (g_1, \cdots, g_6)$$
$$\mapsto (g_1, g_2, g_3)(g_3, g_4, g_5)(g_5, g_2, g_6)(g_6, g_4, g_1).$$

Admissible transformations of the base manifold are given by those functions $C : G^{\times 3} \to G^{\times 3}$ that satisfy

the relation (30). Therefore, for any $(g_1, \cdots, g_6) \in G^{\times 6}$, there should exist a point $(\tilde{g}_1, \cdots \tilde{g}_6) \in G^{\times 6}$ such that

$$C^{\times 4} \circ f(g_1, \cdots, g_6) = f(\tilde{g}_1, \cdots, \tilde{g}_6). \qquad (80)$$

Writing C in components as

$$C(\vec{g}) = \left(C^1(\vec{g}), C^2(\vec{g}), C^1(\vec{g})\right), \qquad (81)$$

condition (80) implies

$$C^1(g_1, g_2, g_3) = C^3(g_6, g_4, g_1) \qquad (82)$$
$$C^2(g_1, g_2, g_3) = C^2(g_5, g_2, g_6), \qquad (83)$$

which suggests the following decomposition of C,

$$C(g_1, g_2, g_3) = C^1(g_1) C^2(g_2) C^3(g_3). \qquad (84)$$

Notice that in this case the diffeomorphism properties of C carry over to the components C^i.

According to equation (29), the fields transform under C as

$$\phi \mapsto \tilde{\phi} = \phi \circ C^{-1}. \qquad (85)$$

The field $\tilde{\phi}$ needs to be gauge invariant as well, otherwise the transformation C would leave the allowed space of fields. The gauge invariance of $\tilde{\phi}$ reads

$$\tilde{\phi} \circ R_h = \phi \circ C \circ R_h \overset{!}{=} \phi \circ C = \tilde{\phi}. \qquad (86)$$

Since this has to be true for all gauge invariant fields ϕ, the point $C \circ R_h(\vec{g})$ needs to be in the same orbit (under the multiplication from the right by the diagonal group) as the point $C(\vec{g})$. This means that, for any $h \in G_D$, there should exist an $\tilde{h} \in G_D$ such that

$$C \circ R_h = R_{\tilde{h}} \circ C, \qquad (87)$$

or point-wise

$$C(\vec{g}h) = C(\vec{g})\tilde{h}. \qquad (88)$$

As we show in the appendix (A), this restricts the C, up to discrete transformations, to be of the form

$$C(\vec{g}) = \vec{L} \cdot h^{-1} \vec{g} h, \qquad (89)$$

for some $L \in G^{\times 2}$ and $h \in G_D$.

In the end, the symmetry group of the interaction part becomes

$$G^{\times 2} \times G_D. \qquad (90)$$

It is evident that this group already forms a symmetry group, due to the left invariance of the Haar measure. We can summarize the role of combinatorial structure and the gauge invariance on the transformation group of the base manifolds as follows

$$\text{Diff}\left(\left[G^{\times 3}\right]\right) \underset{\text{combinatorics}}{\longrightarrow} \text{Diff}(G)^{\times 2} \underset{\text{gauge invariance}}{\longrightarrow} G^{\times 2} \times G_{3D}. \tag{91}$$

For higher dimensional models such as the Ooguri model with the interaction given by

$$f : (g_1, \cdots, g_{10}) \mapsto$$
$$(g_1, g_2, g_3, g_4)(g_4, g_5, g_6, g_7)(g_7, g_3, g_8, g_9) \times$$
$$\times (g_9, g_6, g_2, g_{10})(g_{10}, g_8, g_5, g_1),$$

we observe that the above treatment still results in the transformation group

$$G^2 \times G_{4D}, \tag{92}$$

where G^2 acts by left multiplication as

$$(G_1, G_2, G_2, G_1)\,\vec{g} = \left(G_1 g^1, G_2 g^2, G_2 g^3, G_1 g^4\right). \tag{93}$$

Note on differences between simplicial combinatorics

As we mentioned above, the combinatorial structure for simplicial models can vary. This variation is captured by different functions f, as used in the beginning of this section.

For the original Boulatov interaction we get

$$f : (g_1, \cdots, g_6)$$
$$\mapsto (g_1, g_2, g_3)(g_1, g_4, g_5)(g_2, g_5, g_6)(g_3, g_6, g_4).$$

The resulting transformations become

$$C(\vec{g}) = C^1(g_1) C^1(g_2) C^1(g_3), \tag{94}$$

where all the components are the same. It is easy to check that this transformation also respect the cyclic permutation condition and therefore we get the symmetry group of the Boulatov model as

$$G \times G. \tag{95}$$

In the colored case, instead, we get

$$f : (g_1, \cdots, g_6)$$
$$\mapsto (g_1, g_2, g_3)(g_1, g_4, g_5)(g_6, g_2, g_5)(g_6, g_4, g_3),$$

and the resulting admissible transformations are

$$C(\vec{g}) = C^1(g_1) C^2(g_2) C^3(g_3). \tag{96}$$

The group of admissible transformations is therefore

$$G^{\times 3} \times G. \tag{97}$$

2. Symmetries of gauge invariant models

Following the procedure of the previous section, the infinitesimal symmetry condition for the simplicial interaction in three dimensions takes the form

$$0 = D_V L \cdot X_Q + \text{Div}(L\, X_M), \tag{98}$$

Writing the same condition in terms of u^i, and using the fact that the only admissible base manifold transformations are generated by divergence free vector fields, we obtain

$$u^2 u^3 u^4 X_V^1 + u^1 u^3 u^4 X_V^2$$
$$+ u^1 u^2 u^4 X_V^3 + u^1 u^2 u^3 X_V^4 \tag{99}$$
$$+ c.c. = 0.$$

Hereby X^i is evaluated at the point $(\vec{g}_i, \vec{u})$ with $\vec{g}_1 = (g_1, g_2, g_3)$, $\vec{g}_2 = (g_3, g_4, g_5)$, $\vec{g}_3 = (g_5, g_2, g_6)$ and $\vec{g}_4 = (g_6, g_4, g_1)$ and $\vec{u} = \left\{\phi^i(\vec{g})\right\}_{i \in \{1, \cdots, 4\}}$. Notice that $\vec{u}$ is not $\left(u^1, u^2, u^3, u^4\right)$ since the latter tuple is given by $\left\{\phi^i(\vec{g}_i)\right\}_{i \in \{1, \cdots, 4\}}$. Equation (99) needs to hold true for any ϕ^i and any point of the base manifold.

Inserting the formal power series expansion of X_V^i

$$X_V^i(\vec{g}, \vec{u}) = \sum_{\vec{m}} \Theta_{\vec{m}}^i(\vec{g})\, \phi^1(\vec{g})^{m_1} \cdots \phi^4(\vec{g})^{m_4}\, \bar{\phi}^1(\vec{g}) \cdots \bar{\phi}^4(\vec{g}),$$

in equation (99) we observe that all the coefficient functions Θ^i vanish except for one, such that

$$X_V^i(\vec{g}, \vec{u}) = \Theta^i(\vec{g})\, u^i. \tag{100}$$

Equation (99) becomes

$$\Theta^1(\vec{g}_1) + \Theta^2(\vec{g}_2) + \Theta^3(\vec{g}_3) + \Theta^4(\vec{g}_4) = 0. \tag{101}$$

As we show in the appendix (C), the only functions that are gauge invariant and satisfy the above equation are constants,

$$\theta^i = const. \qquad \sum_i \theta^i = 0 \quad . \tag{102}$$

Therefore X_V^i generate the symmetry group $U(1)^{\#c-1}$, where $\#c$ is the number of colors in the interaction part of the model. Notice, that if the model is not colored, that is the number of colors is one, the $U(1)$ symmetry is not present. The overall symmetry group for simplicial models becomes

$$G^{\times n} \times G \times U(1)^{\#c-1}, \tag{103}$$

where n depends on the actual combinatorial pattern, as we have shown. This classification of symmetries also fits

Model	Symmetry Group	Action
$\phi^1_{1,2,3}\phi^2_{1,4,5}\phi^3_{6,2,5}\phi^4_{6,4,3}$	$G^{\times 3} \times U(1)^{\times 3}$	$\vec{g} \ \mapsto L_C(\vec{g})$ $\phi^c \mapsto e^{\imath\theta_c}\phi^c \quad \sum_c \theta^c = 0$ with $C = \left(c^1, c^2, c^3\right)$
$\phi_{1,2,3}\ \phi_{3,4,5}\ \phi_{5,2,6}\ \phi_{6,4,1}$	$G^{\times 2}$	$\vec{g} \ \mapsto L_C(\vec{g})$ with $C = \left(c^1, c^2, c^1\right)$
$\phi^P_{1,2,3}\ \phi^P_{1,4,5}\ \phi^P_{2,5,6}\ \phi^P_{3,6,4}$	G	$\vec{g} \ \mapsto L_C(\vec{g})$ with $C = (c, c, c)$
$\phi_{1,2,3}\ \bar{\phi}_{3,4,5}\ \phi_{5,2,6}\ \bar{\phi}_{6,4,1}$	$G^{\times 2} \times U(1)$	$\vec{g} \ \mapsto L_C(\vec{g})$ $\phi \ \mapsto e^{\imath\theta}\phi$ with $C = \left(c^1, c^2, c^1\right)$
$\phi_{1,2,3,4}\ \phi_{4,5,6,7}\ \phi_{7,3,8,9}\ \phi_{9,6,2,10}\ \phi_{10,8,5,1}$	$SU(2)^{\times 2}$	$\vec{g} \ \mapsto L_C(\vec{g})$ with $C = \left(c^1, c^2, c^2, c^1\right)$
Barrett-Crane	$Spin(4)^{\times 2} \times SU(2)$	$\vec{g} \ \mapsto L_S(\vec{g})$ $\vec{g} \ \mapsto c \cdot (\vec{g}; k) \cdot c^{-1}$ with $S = \left(s^1, s^2, s^2, s^1\right)$

Table IV: Models and their symmetry groups excluding gauge symmetries of the fields. Hereby, $c^i \in SU(2)$ and $s^i \in Spin(4)$ and L_C denotes the left multiplication by C.

the model with the interaction part of the type $\int \phi\bar{\phi}\phi\bar{\phi}$, since we defined it as a model with two different colors. The symmetry group of colored models, which is independent from the precise combinatorial pattern of field arguments, is the largest compatible with the one of the local part of the action (and with gauge invariance), and coincides with the one of the corresponding tensorial model. This gives a different perspective, and confirms, the close relation between colored simplicial models and tensorial ones, highlighted first in [35] in terms of properties of the corresponding functional integrals.

3. Barrett-Crane model

We now briefly discuss the implication of the simplicity constraints, in the Barrett-Crane formulation, on the symmetry group.

Applying the above analysis to the BC model from equation (16) defined by the following function f

$$f : (g_1, \cdots g_{10}; k_1, \cdots, k_4) \mapsto \qquad (104)$$
$$(g_{1,2,3,4}; k_1)(g_{4,5,6,7}; k_2)(g_{7,3,8,9}; k_3)$$
$$\times (g_{9,6,2,10}; k_4)(g_{10,8,5,1}; k_5), \qquad (105)$$

we realize that the symmetry group for the gauge invariant BC model without simplicity constrains would

be that of an extended Ooguri model from equation (92) where the group G is now specified to $Spin(4)$

$$Spin(4)^{\times 2} \times Spin(4) \times G(k). \qquad (106)$$

The group $G(k)$ denotes a group of transformations of the $SU(2)$ element k_i. However, remember that the extension of the GFT field the $SU(2)$ variable k was needed for consistent implication of simplicity constrains and therefore the actual meaning of $G(k)$ is relevant only after the imposition of simplicity constrains.

Equation (106) provides the symmetry group of extended Ooguri model with gauge invariance, in order to obtain the symmetry group of the BC model simplicity constrains need to be further imposed. We refer to the appendix (B) for explicit calculations and state here just the result of imposing the simplicity constrains on the field ϕ, by imposing invariance under the projector $\mathcal{S}$ from equation (20). As we show in the appendix (B), the simplicity constrains

$$\phi \circ \mathcal{S} = \phi, \qquad (107)$$

reduce the symmetry group of the Ooguri model (for the chosen combinatorics) down to

$$Spin(4)^{\times 2} \times SU(2), \qquad (108)$$

where the $SU(2)$ group replaces the $Spin(4) \times G(k)$ part from equation (106) and acts on the elements of the local base manifold of the BC model $Spin(4)^{\times 4} \times SU(2)$ by conjugation as

$$c \circ (\vec{g}_-, \vec{g}_+; k) := (c, c; c) \cdot (\vec{g}_-, \vec{g}_+; k) \cdot (c^{-1}, c^{-1}; c^{-1}).$$

And $Spin(4)^{\times 2}$ acts by the left multiplication

$$(G_1, G_2, G_2, G_1)(\vec{g}; k) = (G_1 g^1, G_2 g^2, G_2 g^3, G_1 g^4; k).$$

The same considerations we have made regarding the dependance of the symmetry group on the combinatorics and on the use of colors apply also to the Barrett-Crane case.

In table (IV) we summarize the symmetries of different interaction terms.

III. Classical currents

We will now derive the (generalized) conservation laws for the symmetries we identified in the last section. Once more we limit ourselves to the classical regime of the GFTs, postponing the analysis of the full quantum theory. Also, we stress again that the conservation laws and corresponding currents, just like the whole kinematics and dynamics of such quantum field theories, should not be interpreted in spatiotemporal or geometric terms, at least in general. Even for GFT models with a direct quantum gravity interpretation, the spatiotemporal and geometric meaning of the various aspects and regimes of each model should be extracted and analyzed with care. On this note, we point out that the classical GFT equations of motion of 4d quantum gravity models, which capture the hydrodynamics of special condensate states of the theory, have been given a cosmological interpretation and have been studied in some detail and with remarkable results in a series of recent works [26–29, 44–48].

A. Conservation laws in non-local field theories

In local field theories there is a conserved current associated to every continuous symmetry of the action given by the famous Noether theorem. However, for non-local theories this result does not hold as such, and must be generalized, due to the fact that the equations of motion become integro-differential equations.

In [11] we derived an equivalent expression for Noether currents for the case of non-local field theories, and for the associated generalized conservation laws. In order to keep the notation simple we present here a simplified version of the theorem, referring to the original work for the full statement.

Theorem 2. *If a non-local action $S = \int_M S^L + \int_{\tilde{M}} S^I$ is symmetric under a group action generated by the vector*

fields (X_M, X_V) *then the following identity holds for all* i

$$
\begin{aligned}
EL[X_Q] = &\sum_c \int D_F L^I (X_{cQ}) \left[\delta^c - \delta^i\right] \\
&- Div_M \left(D_J L^L [X_Q] + L^L \cdot X_M \right) \\
&- Div_M \left(\int_\Omega D_J L^I (X_Q^c) \delta^c \right) \\
&- \int_{\tilde{\Omega}} Div_{\tilde{M}} \left(L^I \cdot X_{\tilde{M}} \right) \delta^i.
\end{aligned}
\tag{109}
$$

Here $X_{\tilde{M}}$ denotes the vector field of base manifold transformations of $\tilde{M}$ generated by X_M as we discussed in the previous section, $D_F L(X_{cQ})$ denotes the Fréchet derivatives of the Lagrangian in the direction of X_{cQ}, δ^c denotes the delta distribution on the domain of the field of color c and the non-local Lagrangian $L^I = L^I(x, \phi(x), \partial\phi|_x)$ is assumed to be a function on the base manifold, fields at the point, and first derivatives of the fields at the same point. The left hand side denotes the equations of motion contracted with the vector field X_Q.

In the case when the non-local Lagrangian is independent of derivatives of the fields, and the generators of the symmetry group of base manifold transformations are divergence-free, $\mathrm{div}(X_M) = 0$, and when the transformations of the field values is proportional to the field value itself, $Q(\phi_{1,2,3}) \propto \phi_{1,2,3}$, the above identity simplifies significantly to

$$EL[X_Q] = \Delta - \mathrm{Div}_M \left(D_J L^L [X_Q] + L^L \cdot X_M \right), \tag{110}$$

where

$$\Delta = \sum_c \int D_{cV} L^L (X_Q^c) \delta^c, \tag{111}$$

is referred to as correction term. This result explicitly shows that the currents associated to symmetries of the non-local action are no longer conserved. Instead their divergence are dictated by the non-local part of the action.

After imposing equations of motion on the fields, we get the identity that replaces the usual Noether theorem

$$\mathrm{Div}_M \left(D_J L^L [X_Q] + L^L \cdot X_M \right) =: \mathrm{Div}(J) = \Delta. \tag{112}$$

The quantity in brackets on the left hand side is the Noether current of the local part of the action and the right hand side of the equation is the non-vanishing divergence of the current due to non-local structure of the theory.

It becomes now a straightforward calculation to apply the equation (112) to models and symmetries introduced in the previous section. In the rest of this section we summarize the resulting identities.

Model	Symmetry Group	Action
$\phi^1_{1,2,3}\phi^2_{1,4,5}\phi^3_{6,2,5}\phi^4_{6,4,3}$	$SU(2)^{\times 3}$	$\Delta = -2\lambda \sum_c \int \delta^c \ \mathrm{Re}\left[X^{\oplus 3}_{cM}\left(\phi^1\phi^2\phi^3\phi^4\right)\right]$
	$U(1)^{\times 3}$	$\Delta = -\imath 2\lambda \sum_c \theta_c \int \delta^c \ \mathrm{Im}\left(\phi^1\phi^2\phi^3\phi^4\right)$
$\phi_{1,2,3}\,\phi_{3,4,5}\,\phi_{5,2,6}\,\phi_{6,4,1}$	$G^{\times 2}$	$\Delta = -8\lambda \ \mathrm{Re}\left(X^{\oplus 2}_M\left(\phi\right)\int \phi\phi\phi\right)$
$\phi^P_{1,2,3}\,\phi^P_{4,3,5}\,\phi^P_{5,2,6}\,\phi^P_{6,4,1}$	G	$\Delta = -8\lambda \ \mathrm{Re}\left(X_M\left(\phi\right)\int \phi\phi\phi\right)$
$\phi_{1,2,3}\,\bar{\phi}_{3,4,5}\,\phi_{5,2,6}\,\bar{\phi}_{6,4,1}$	$G^{\times 2}$	$\Delta = -4\lambda \ \mathrm{Re}\left(X_M\left(\phi\right)\int \bar{\phi}\phi\bar{\phi}\right)$
	$U(1)$	$\Delta = -4\lambda\theta \ \mathrm{Im}\left(\phi\int \bar{\phi}\phi\bar{\phi}\right)$
$\phi_{1,2,3,4}\,\phi_{4,5,6,7}\,\phi_{7,3,8,9}\,\phi_{9,6,2,10}\,\phi_{10,8,5,1}$	$SU(2)^{\times 2}$	$\Delta = -10\lambda \ \mathrm{Re}\left(X^{\oplus 2}_M\left(\phi\right)\int \phi\phi\phi\phi\right)$
Barrett-Crane	$Spin(4)^{\times 2}$	$\Delta = -10\lambda \ \mathrm{Re}\left(X^{\oplus 2}_M\left(\phi\right)\int \phi\phi\phi\phi\right)$
	$SU(2)$	$\Delta = -10\lambda \ \mathrm{Re}\left(X_M\left(\phi\right)\int \phi\phi\phi\phi\right)$

Table V: Models and their correspondent correction terms. The vector fields X_M are the left invariant vector fields given in equation (71)

Since the local part of all our models is given by equation (9), the Noether current does not change and can be written as

$$J = \kappa \sum_c \left(\nabla\phi^c \cdot \bar{X}_Q + \nabla\bar{\phi}^c \cdot X_Q\right) + L^L \cdot X_M. \quad (113)$$

Note that for the $U(1)$ symmetry $X_M = 0$, and the Noether current becomes proportional to κ. This automatically implies that for all static models the Noether current associated to the $U(1)$ symmetry is zero. The correction term, however, may not trivially vanish. Apart from the values for κ, the models introduced earlier will differ only by the correction term in equation (112). In table (V) we will present the correction terms for discussed models.

The notation in table (V) is as follows. For brevity we do not indicate the base points and write $\int \phi^1\phi^2\phi^3\phi^4$ in order to refer to the non-local part of the Boulatov action. We also write $\phi^1 \int \phi^2\phi^3\phi^4$ for

$$\phi^1_{1,2,3}\int \mathrm{d}g_{4,5,6}\,\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1}. \quad (114)$$

The integral $\int \phi^2\phi^3\phi^4$ can be seen as a function evaluated at the point (g_1, g_2, g_3). We denote the Lie derivative of this function with respect to the vector field X_M as $X_M\left(\int \phi\phi\phi\right)$. For brevity we denote the expression in (114) also by the formal delta distribution

$$\int \phi^1\phi^2\phi^3\phi^4\,\delta^1 = \phi^1_{1,2,3}\int \mathrm{d}g_{4,5,6}\,\phi^2_{3,4,5}\phi^3_{5,2,6}\phi^4_{6,4,1},$$

meaning that the integral over the domain of the field ϕ^1 is to be excluded.

IV. Conserved charges in presence of matter

In this section we will discuss the consequences of the matter coupling introduced in [29] and show that such coupling implies the existence of quantities which are constant in the matter field variable, and can be interpreted as *conserved charges*.

While this could be taken as a fact of purely mathematical interest, it may also indicate some underlying interesting physics, for quantum gravity models. The reason is the following. The type of matter field introduced in [29] was a free, massless, minimally coupled real scalar field, entering as an additional variable in the domain of the GFT fields, for 4d gravity models, whose classical dynamics was then studied. The same classical dynamics was given an interpretation as cosmological dynamics for continuum homogeneous universes, emerging from the GFT system as quantum condensates. As customary in quantum cosmology, and to some extent compulsory in background independent, diffeomorphism invariant theories, the dynamics was expressed in terms of relational observables [49–51]. In particular, the added scalar field was chosen to play the role of relational clock, i.e. the physical time variable in terms of which describing the evolution of all the geometric observables, e.g. the volume of the universe. We refer to [29] for more details. Remarkably, the same variable enters the GFT action just as a standard, local time coordinate would enter an ordinary field theory. This suggests a deeper physical meaning for the charges that, following some symmetry of the corresponding GFT model, are in fact conserved with respect to the same relational time variable/clock. We do not discuss further the possible physical interpre-

tation and confine ourselves to the mathematical analysis of such extended models.

The domain of the GFT field is extended to become

$$\phi : M \times \mathbb{R} \to \mathbb{C}, \tag{115}$$

where M is the base manifold of the correspondent GFT model without matter and $\mathbb{R}$ describes the degree of freedom of a real scalar field. We call the new base manifold $M_{\mathrm{mat}} = M \times \mathbb{R}$, and denote a point on M_{mat} as $(g_1, \cdots, g_n, \varphi)$. The field value at this point is then denoted $\phi(g_1, \cdots, g_n, \varphi) = \phi_{1, \cdots, n, \varphi}$. Intuitively we can think of GFT field ϕ as describing a "chunk" of space in which the scalar field takes the value φ.

The dynamics is then described by an action which is non-local in the group variables, but local in the additional matter field variable. This means that every Lagrangian in the non-local action sum is evaluated at the same value φ:

$$S = \int_{M \times \mathbb{R}} \phi(\vec{g}, \varphi) K(\vec{g}, \varphi) \phi(\vec{g}, \varphi)$$
$$+ \int_{M \times \mathbb{R}} \phi(\vec{g}, \varphi) \cdots \phi(\vec{h}, \varphi) V(\vec{g}, \cdots, \vec{h}, \varphi). \tag{116}$$

with the dependence of the various terms in the action on the additional scalar field being motivated by an analysis of the GFT Feynman amplitudes and their relation with simplicial gravity path integrals, and by the required symmetries of the scalar field dynamics. The further requirements that the scalar field is free, massless and minimally coupled, plus some further approximation motivated by the hydrodynamics setting [29], lead to $K(\vec{g}, \varphi) = \mathcal{K}(\vec{g}) + \Delta_\varphi$, and to a vertex function V that is independent of φ.

A. Conserved charges and symmetries

Locality in the matter field allows to define a local slicing with respect to which we can construct conserved quantities $Q(\varphi)$ for any symmetry of the action, such that $\partial_\varphi Q(\varphi) = 0$. This is easily seen from the equation (109), where the integral domain is now replaced by $M_{\mathrm{mat}} = M \times \mathbb{R}$ and the delta function δ^c that acts on the domain of the field with color c can be written as $\delta_M^c \delta_\mathbb{R}^\varphi$, where δ_M^c acts on the group part of the domain and $\delta_\mathbb{R}^\varphi$ fixes the value of the matter field. Integrating the above equation over M and taking into account that the action is local in the parameter φ, as well as the fact that the base manifold M has no boundary[6], the above equation

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[6] If the underling group of the model has a boundary, then boundary terms need to taken into account.

simplifies to

$$\int_M EL|_\varphi [X_Q] \simeq \partial_\varphi \left(\partial_{\partial_\varphi \phi^c} L^{\mathrm{loc}} [X_Q^c] + \partial_{\partial_t \bar{\phi}^c} L^{\mathrm{loc}} [\bar{X}_Q^c] \right)$$
$$+ \partial_\varphi \left(L^{\mathrm{loc}} \cdot X_\varphi \right)$$
$$+ \partial_\varphi \int_M \left(D_{\partial_\varphi \phi^c} L^{\mathrm{int}} (X_Q^c) + D_{\partial_\varphi \bar{\phi}^c} L^{\mathrm{int}} (\bar{X}_Q^c) \right)$$
$$+ \partial_\varphi \int_M \left(L^{\mathrm{int}} \cdot X_\varphi \right), \tag{117}$$

where the equality is true up to a minus sign. Taking the φ component of the current we get

$$Q(\varphi) := \partial_{\partial_\varphi \phi^c} L^{\mathrm{loc}} [X_Q^c] + \partial_{\partial_\varphi \bar{\phi}^c} L^{\mathrm{loc}} [\bar{X}_Q^c] + L^{\mathrm{loc}} \cdot X_\varphi$$
$$+ \int_M \left(D_{\partial_\varphi \phi^c} L^{\mathrm{int}} (X_Q^c) + D_{\partial_\varphi \bar{\phi}^c} L^{\mathrm{int}} (\bar{X}_Q^c) \right)$$
$$+ \int_M \left(L^{\mathrm{int}} \cdot X_\varphi \right). \tag{118}$$

Due to equation (117), this satisfies on shell

$$\partial_\varphi Q(\varphi) = 0. \tag{119}$$

Since the interaction Lagrangian does not depend on derivatives of ϕ, the conserved charge becomes

$$Q(\varphi) := S|_\varphi \cdot X_\mathbb{R}^\varphi \tag{120}$$
$$+ \int_M \left(\partial_{\partial_\varphi \phi^c} L^{\mathrm{loc}} [X_Q^c] + \partial_{\partial_\varphi \bar{\phi}^c} L^{\mathrm{loc}} [\bar{X}_Q^c] \right),$$

where $S|_\varphi = \int_M L^{\mathrm{loc}}|_\varphi + \int_M L^{\mathrm{int}}|_\varphi$ is the action in equation (116) at a fixed value of the parameter φ.

For example in the case of a $U(1)$ symmetry which is generated by $X_Q = \imath \theta_c \phi^c$, with $\sum_c \theta^c = 0$, we get the conserved charge

$$Q(\varphi) = \imath \sum_c \theta^c \int_M \left(\phi^c \partial_{\partial_t \phi^c} L^{\mathrm{loc}} - \bar{\phi}^c \partial_{\partial_t \bar{\phi}^c} L^{\mathrm{loc}} \right). \tag{121}$$

For the $SU(2)$ symmetry, with $X_Q = -X_M(\phi)$ and X_M being left invariant generators of $SU(2)$ as in equation (71), Q takes instead the form

$$Q(\varphi) = -\int_M \partial_{\partial_\varphi \phi^c} L^{\mathrm{loc}} [X_{cM}(\phi^c)]$$
$$- \int_M \partial_{\partial_\varphi \bar{\phi}^c} L^{\mathrm{loc}} [X_{cM}(\bar{\phi}^c)].$$

This shows that we can easily calculate "conserved" quantities for the symmetries we found earlier in the paper. However, in addition to the symmetries on the group space we may also have symmetries on $\mathbb{R}$ which correspond to symmetries of the matter field, so the symmetry group of the models will be larger.

In general, the symmetry of the matter field will also be strongly model dependent, and have to be investigated on a case by case basis. However, in the case of free, massless, minimally coupled scalar matter, the action is

(and should be) invariant under matter field translations of the form $\varphi \mapsto \varphi + \mu$. The charge for this symmetry will take the following form

$$Q\left(\varphi\right) = - \int_M \left(\partial_{\partial_\varphi \phi^c} L^{\mathrm{loc}} \partial_\varphi \phi^c + \partial_{\partial_\varphi \bar{\phi}^c} L^{\mathrm{loc}} \partial_\varphi \bar{\phi}^c \right) + S|_\varphi.$$

Defining $\Pi^c := \partial_{\partial_\varphi \phi^c} L^{\mathrm{loc}}$, this takes the form of the Legendre transform of the Lagrangian defined by $S|_t$

$$Q\left(\varphi\right) = - \int_M \left(\Pi^c \partial_\varphi \phi^c + \bar{\Pi}^c \partial_\varphi \bar{\phi}^c \right) + S|_\varphi. \qquad (122)$$

This is of course extremely suggestive of a GFT Hamiltonian with respect to the evolution defined by the relational "time" φ, and this is certainly an important point to be investigated further, in both its mathematical and physical consequences.

It is important to note that there are very special conditions that the matter field φ has to satisfy to represent a good relational clock. It is interesting to investigate further also how these conditions, and their relaxation, reflect on the dependence of the GFT action on the same matter field variable, and what field-theoretic consequences they have, in particular concerning the existence and form of the conserved charges we have found. Moreover, it is easy to realize that, if the model has more than one matter field that enters the action locally, the above treatment can be performed for any of the matter fields. In this case, however, above equations will contain additional boundary terms. We leave further analysis of these points to future work.

V. Conclusion and outlook

In this paper we provided an extensive symmetry analysis for various models in Group Field Theory.

We have elucidated the symmetry group of various GFT models, and how it is affected by the various ingredients entering their definition: rank, base group, color, combinatorial structure.

Our main result shows that, apart form the expected symmetry groups of left multiplication and $U(1)$, the discussed models do not posses any other continuous Lie point symmetries. This holds even in the case of static, gauge invariant, models, in which the Lagrangian does not depend on derivatives of fields. This is not obvious since an ordinary local field theory without dynamical terms would possess a fairly large gauge group of diffeomorphisms of the base manifold. However, the presence of the interaction term with a particular combinatorial structure as well as the requirement of gauge invariance insures that the symmetry group becomes very small. In this sense our treatment provides a complete set of point symmetries of discussed models.

Using our previous result on conservation laws for non-local theories we were then able to calculate generalized

"conservation" laws that correspond to continuous symmetries. And were able to show that in particular cases of matter coupling to GFT fields our construction provides a notion of conserved charges, the same way Noether theorem does in local field theories. An existence of conserved quantities shows, that once a matter field satisfies a notion of a "good" clock it also obtains the usual "time" properties in the field theoretical frame-work. As we already mentioned, a lot more should be understood about such conserved charges in GFT models.

It is an exciting and important task to understand the consequences of the GFT symmetry groups on the physics of these models. This is what needs primarily to be addressed in the future.

On the one hand, an understanding of conservation laws in terms of geometrical objects could be a very important step in the development of the theory. Conservation laws and conserved charge equations could provide a field theoretical explanation of cosmological features stemming from the underlying quantum gravity models, in the context of GFT condensate cosmology [29, 47, 52]. The very existence of a condensate phase in GFT models, and more generally their macroscopic phase diagram, currently being explored mainly by FRG methods [36, 53, 54], can now be studied also on the basis of GFT symmetries and corresponding symmetry breaking.

On the other hand a classification of symmetry groups in GFT could be used as a better characterization of the theory space, a crucial ingredient for systematic renormalization group studies [10, 30, 55–57].

In particular this could help clarifying the connection between simplicial and tensorial GFT models. As we noted, a further indication of their close connection has been found already in our analysis, showing that only colored GFT models of simplicial type appear to have an $U(1)$ symmetry as well as the unrestricted translation invariance that is found in tensorial GFTs.

From a more mathematical point of view, it appears to be very interesting to understand the extension of the symmetry groups we considered to Lie-Baecklund or generalized symmetries, which requires a better characterization of the equivalence class of GFT actions leading to the same classical equations of motion.

Finally, we need to go beyond the purely classical analysis performed in this paper, and move to the analysis of the same symmetries we have identified at the quantum level, deriving and studying in detail the corresponding Ward identities, and the issue of possible anomalies.

A. Reduction of transformations due to gauge invariance

From equation (88) the requirement on the transformation reads

$$C(\vec{g}h) = C(\vec{g})\,\tilde{h}. \tag{A1}$$

Writing out this equation in components we get

$$\begin{aligned}
C^1(g_1 h) &= C^1(g_1)\,\tilde{h} \\
C^2(g_2 h) &= C^2(g_2)\,\tilde{h} \\
C^1(g_3 h) &= C^1(g_3)\,\tilde{h},
\end{aligned} \tag{A2}$$

with C^i being a diffeomorphism on the group G. At this point we employ the known relation

$$\mathrm{Diff}(G) \simeq G \times \mathrm{Diff}_{\mathbb{1}}(G), \tag{A3}$$

that states that the group of diffeomorphisms on G is diffeomorphic (as a manifold) to the group G itself (that acts by left multiplication) times a group of diffeomorphisms that stabilizes the identity of G, denoted $\mathrm{Diff}_{\mathbb{1}}(G)$. This implies that every C^i can be written by some $c^i \in G$ and $\mathcal{D}^i \in \mathrm{Diff}_{\mathbb{1}}(G)$ such that $C^i(g) = c^i\,\mathcal{D}^i(g)$ with $\mathcal{D}^i(\mathbb{1}) = \mathbb{1}$. Inserting this relation in the equations (A2) and evaluating it a the point $g_1 = g_2 = g_3 = \mathbb{1}$ we observe

$$c^1 \cdot \mathcal{D}^1(h) = c^1 \cdot \tilde{h} \tag{A4}$$
$$c^2 \cdot \mathcal{D}^2(h) = c^2 \cdot \tilde{h} \tag{A5}$$
$$c^1 \cdot \mathcal{D}^3(h) = c^1 \cdot \tilde{h}, \tag{A6}$$

which, in tern, implies

$$\mathcal{D}^1(h) = \mathcal{D}^2(h) = \mathcal{D}^3(h) = \tilde{h} =: \mathcal{D}(h). \tag{A7}$$

Inserting this relation again in (A2) at an arbitrary point $\vec{g}$ we get for $\mathcal{D}$

$$\mathcal{D}(g_i h) = \mathcal{D}(g_i)\,\mathcal{D}(h). \tag{A8}$$

In other words $\mathcal{D}$ is an homomorphism and therefore an automorphism. On G however, the group of automorphisms splits in the inner automorphisms which are given by a conjugation with a fixed group element and outer automorphisms which are given by automorphisms of the Dynkin diagram of the group and relate to the discreet symmetries. Focusing on continuos transformations we get

$$\mathcal{D}(g) = d \cdot g \cdot d^{-1} \tag{A9}$$

for some fixed $d \in SU(2)$.

B. Barrett-Crane model

In this section we are going to show what are the admissible transformation in the Barrett-Crane model.

In the following we will denote a group element of $Spin(4)$ by its two copies of $SU(2)$ a

$$Spin(4) \ni g = (g_-, g_+),$$

a tuple of four elements is referred to by the vector notation

$$\vec{g} = (\vec{g}_-, \vec{g}_+).$$

We will also sometimes write $g_{1,2,3,4}$ for the tuple of elements (g_1, g_2, g_3, g_4).

A base manifold transformation of the model is denoted by $C : Spin(4)^{\times 4} \times SU(2) \to Spin(4)^{\times 4} \times SU(2)$. We denote the components of this transformation as

$$C(\vec{g}, k) = \left((C_1^-, C_1^+), \cdots, (C_4^-, C_4^+), C_k \right).$$

Here all the component functions $C_i^{\pm}$ are functions on the base manifold and therefore depend on points of the form $(\vec{g}, k)$. However, the combinatorial structure of the BC model dictates the following conditions on the components

$$\begin{aligned}
C_1(g_{1,2,3,4}, k_1) &= C_4(g_{10,8,5,1}, k_5) \\
C_2(g_{1,2,3,4}, k_1) &= C_3(g_{9,6,2,10}, k_4).
\end{aligned}$$

From the above relations we see that the components of the transformation have the following dependences

$$C(g_{1,2,3,4}, k) = (C_1(g_1), C_2(g_2), C_3(g_3), C_4(g_4), C_k(k)).$$

A priori we do not have any additional constrains on the component C_k. However, since C is a diffeomorphism and C_i are diffeomorphisms, the transformation of the normal has to be a diffeomorphism as well [7]. Again invoking the diffeomorphism of manifolds $\mathrm{Diff}(Spin(4)) \simeq Spin(4) \times \mathrm{Diff}_{\mathbb{1}}$ we denote the components of C that belong to $\mathrm{Diff}_{\mathbb{1}}$ by the lower case c.

At this point we remind the reader that in the Barrett-Crane model the gauge invariance of the fields was extended to incorporates simplicity constrains

$$\mathcal{S} : (\vec{g}; k) \mapsto (\mathbb{1}; h_-^{-1}) \cdot (\vec{g}; k) \cdot \left((k\vec{u}k^{-1}, \vec{u},); \mathbb{1} \right) \cdot (h; h_+).$$

Where $\cdot$ stands for the group multiplication and $;$ separates the $Spin(4)$ elements from $SU(2)$. This means that the fields of the model are invariant under $\mathcal{S}$,

$$\phi \circ \mathcal{S} = \phi.$$

Since the fields are transformed under C as $\phi \mapsto \phi \circ C^{-1}$ we again get the following relations for the transformation C

$$\phi \circ C \circ \mathcal{S} = \phi \circ C,$$

---

[7] Notice, that it would not be true if we didn't have restriction on C_i, since then C_i would not be a diffeomorphism and hence neither needs to be C_k.

or equivalently for each $h \in Spin\,(4)$, $u \in SU\,(2)^{\times 4}$ and $g_i \in Spin\,(4)$ there exist $\tilde{u} \in SU\,(2)^{\times 4}$ and $\tilde{h} \in Spin\,(4)$ and $\tilde{k} \in SU\,(2)$ such that

$$C_i\,(g \cdot u_k \cdot h) = C_i\,(g) \cdot \tilde{u}_{C_k} \cdot \tilde{h} \qquad \text{(B1)}$$

$$C_k\,\left(h_-^{-1}kh_+\right) = \tilde{h}_-^{-1}C_k\,(k)\,\tilde{h}_+, \qquad \text{(B2)}$$

where we write $u_k = \left(kuk^{-1}, u\right)$. It is again obvious that the left multiplication by $Spin\,(4)$ is untouched by this transformation, however this is not true for normal component C_k. We first focus on the transformations C_i and treat the normal component C_k afterwards.

From the from of u_k we notice that for $u = \mathbb{1}$ the left hand side does not depend on k and so should't the right hand side. It follows that for $u = \mathbb{1}$ we have $\tilde{u} = \mathbb{1}$. Equation (B1) then reads for the $\text{Diff}_\mathbb{1}$ part,

$$c_i\,(g \cdot h) = c_i\,(g) \cdot \tilde{h}.$$

It follows that c_i is a homomorphism on $Spin\,(4)$ and therefore is either conjugation by a fixed element of $Spin\,(4)$ or a flip of the $SU\,(2)$ parts, which is a discrete transformation. Hence, if c_i is continuous it can be written as

$$c_i\,(g) = s \cdot g \cdot s^{-1},$$

where $g, s \in Spin\,(4)$. This implies

$$\tilde{h} = s \cdot h \cdot s^{-1}.$$

Inserting this relation now in equation (B2) we obtain

$$C_k\,\left(h_-^{-1}kh_+\right) = \left(s_-h_-^{-1}s_-^{-1}\right)\,C_k\,(k)\,\left(s_+h_+s_+^{-1}\right).$$

Splitting C_k in the left multiplication by $SU\,(2)$ and $\text{Diff}_\mathbb{1}$ we get for some fixed $w \in SU\,(2)$

$$w\,c_k\,\left(h_-^{-1}kh_+\right) = \left(s_-h_-^{-1}s_-^{-1}\right)\,w\,c_k\,(k)\,\left(s_+h_+s_+^{-1}\right). \tag{B3}$$

Choosing $h_- = h_+$ and setting $k = \mathbb{1}$ we get

$$w = \left(s_-h_-^{-1}s_-^{-1}\right)\,w\,\left(s_+h_-s_+^{-1}\right),$$

which can only be satisfied if $w = \mathbb{1}$.

Inserting equation (B3) in (B1) and using the fact that c_i is a homomorphism yields

$$c_i\,(u_k) = c_i\,(k, \mathbb{1}) \cdot c_i\,(u, u) \cdot c_i\,\left(k^{-1}, \mathbb{1}\right)$$
$$= c_k\,(k)\,c_i\,(u, u)\,c_k\,\left(k^{-1}\right).$$

Hence, $c_i\,(a, b) = \left(c_k\,(a)\,, c_k\,(b)\right)$ and c_k is a homomorphisms itself. Therefor

$$c_i\,(g) = (s, s) \cdot g \cdot (s, s)^{-1},$$

and $c_k\,(k) = sks^{-1}$.

These are the only admissible transformations that preserve the combinatorial structure of the theory and respect the simplicity constraints together with gauge invariance. Notice that $\mathcal{S}$ itself would fail the requirement (32) and therefor can not be seen as a base manifold transformations, which is why we dot not obtain the symmetry under $\mathcal{S}$ in this approach.

C. Constancy of the phase

In this section we are going to prove the following statement

Theorem. *If for any point $g_1, \cdots, g_6 \in G$ where G is a simple Lie group the following equation holds*

$$\sum_i^4 \theta^i\,(\vec{g}_i) = 0.$$

And for any $h \in G_D\,(2)$ the functions θ^i satisfy

$$\theta^i \circ R_h = \theta^i, \qquad \text{(C1)}$$

then the functions θ^i are constants that add up to zero, $\theta^1 + \theta^2 + \theta^3 + \theta^4 = 0$.

We first prove the following lemma

Lemma 3. *Let θ be a function from a Lie group G to $\mathbb{R}$ such that for all $g \in G$ the difference*

$$\theta\,(gh) - \theta\,(g) = f\,(h)$$

is a function only on the "distance" of the points h. Then f is a homomorphism from the group G to $(\mathbb{R}, +)$.

Proof. From the definition it follows that $f\,(\mathbb{1}) = 0$. Choosing $g = \tilde{g}h^{-1}$ we get

$$f\,(h) = \theta\,(gh) - \theta\,(g) = \theta\,(\tilde{g}) - \theta\,\left(\tilde{g}h^{-1}\right) = -f\,\left(h^{-1}\right).$$

Choosing $g = gh\tilde{h}$ we also get

$$f\,\left(h\tilde{h}\right) = \theta\,\left(gh\tilde{h}\right) - \theta\,(g)$$
$$= \theta\,\left(gh\tilde{h}\right) \pm \theta\,(gh) - \theta\,(g)$$
$$= f\,\left(\tilde{h}\right) + f\,(h).$$

Which concludes the proof. $\qquad \square$

We now prove the above theorem.

Proof. The above equation then reads

$$\theta^1\,(\vec{g}_1) + \theta^2\,(\vec{g}_2) + \theta^3\,(\vec{g}_3) + \theta^4\,(\vec{g}_4) = 0, \qquad \text{(C2)}$$

where $\vec{g}_1 = (g_1, g_2, g_3)$, $\vec{g}_2 = (g_3, g_4, g_5)$, $\vec{g}_3 = (g_5, g_2, g_6)$ and $\vec{g}_4 = (g_6, g_4, g_1)$. Than for any differentiable curve $c : \mathbb{R} \supset I \to SU\,(2)$ with $c\,(0) = \mathbb{1}$ the above equation is true if we replace g_1 by the curve $c\,(t)$. Deriving the resulting equation with respect to the parameter t we get

$$\partial_t \theta^1\,(c\,(t)\,, g_2, g_3) + \partial_t \theta^4\,(g_6, g_4, c\,(t)) = 0.$$

By integration we obtain

$$\theta^1\,(c\,(t)\,, g_2, g_3) - \theta^1\,(\mathbb{1}, g_2, g_3) = \theta_1^1\,(c\,(t))\,,$$

for some function $\theta^1_1 (c(t))$. Applying the same argument to θ^4 we gain the following relations,

$$\theta^1 (g_1, g_2, g_3) = \theta^1_1 (g_1) + \theta^1 (\mathbb{1}, g_2, g_3)$$
$$\theta^4 (g_6, g_4, g_1) = -\theta^1_1 (gd_1) + \theta^4 (\mathbb{1}, g_2, g_3).$$

Inserting these relations into equation (C2) yields

$$\theta^1 (\mathbb{1}, g_2, g_3) + \theta^2 (\vec{g}_2) + \theta^3 (\vec{g}_3) + \theta^4 (g_6, g_4, \mathbb{1}) = 0.$$

Successively performing the same step for all other group elements g_i eventually leads to the separation of the functions θ^i as follows,

$$\theta^i (g_1, g_2, g_3) = \theta^i_1 (g_1) + \theta^i_2 (g_2) + \theta^i_3 (g_3) + \text{const.}^i, \quad (C3)$$

where θ^i_j's satisfy

$$\theta^1_1 = -\theta^4_3 \qquad \theta^1_2 = -\theta^3_2 \qquad \theta^1_3 = -\theta^2_1$$
$$\theta^2_2 = -\theta^4_2 \qquad \theta^2_3 = -\theta^3_1$$
$$\theta^3_3 = -\theta^4_1.$$

Using the requirement on gauge invariance (equation (C1)) yields for any $h \in G_{3D}$

$$\theta^i_1 (g_1 h) + \theta^i_2 (g_2 h) + \theta^i_3 (g_3 h) = \theta^i (\vec{g}).$$

Since this equation needs to hold for any $\vec{g} \in G^{\times 3}$ we get for each θ^i_j the following condition

$$\theta^i_j (gh) - \theta^i_j (g) = f^i_j (h), \qquad (C4)$$

with some functions f^i_j. From the above lemma it follows that f^i_j is a homomorphism from G to $(\mathbb{R}, +)$. Since $(\mathbb{R}, +)$ is abelian and G is simple f is a constant zero function, $f = 0$.

Evaluating equation (C4) at $g = \mathbb{1}$ proves that

$$\theta^i_j = \text{const}^i,$$

which together with equation (C3) proves

$$\theta^i (\vec{g}, \phi^c) = \theta^i,$$

for some constants θ^i. The conditions on the constants follows. $\qquad\square$

[1] D. Oriti, "The Group field theory approach to quantum gravity", arXiv:gr-qc/0607032.

[2] D. Oriti, "The Group field theory approach to quantum gravity: Some recent results", *AIP Conf. Proc.* **1196** (2009) 209–218, arXiv:0912.2441.

[3] D. Oriti, "The microscopic dynamics of quantum space as a group field theory", in "Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity: Cape Town, South Africa", pp. 257–320. 2011. arXiv:1110.5606.

[4] D. Oriti, "Group field theory as the 2nd quantization of Loop Quantum Gravity", *Class. Quant. Grav.* **33** (2016), no. 8, 085005, arXiv:1310.7786.

[5] T. Krajewski, "Group field theories", *PoS* **QGQGS2011** (2011) 005, arXiv:1210.6257.

[6] D. Oriti, "Group Field Theory and Loop Quantum Gravity", 2014. arXiv:1408.7112.

[7] M. Gross, "Tensor models and simplicial quantum gravity in > 2-D", *Nucl. Phys. Proc. Suppl.* **25A** (1992) 144–149.

[8] R. Gurau and J. P. Ryan, "Colored Tensor Models - a review", *SIGMA* **8** (2012) 020, arXiv:1109.4812.

[9] V. Rivasseau, "Quantum Gravity and Renormalization: The Tensor Track", *AIP Conf. Proc.* **1444** (2011) 18–29, arXiv:1112.5104.

[10] S. Carrozza, "Tensorial methods and renormalization in Group Field Theories", PhD thesis, Orsay, LPT, 2013. arXiv:1310.3736.

[11] A. Kegeles and D. Oriti, "Generalized conservation laws in non-local field theories", *J. Phys.* **A49** (2016), no. 13, 135401, arXiv:1506.03320.

[12] Z. J. Zawistowski, "Symmetries of Integro-Differential Equations", *eConf* **C0107094** (2001) 263–270.

[13] N. H. Ibragimov, V. Kovalev, and V. Pustovalov, "Symmetries of integro-differential equations: A survey of methods illustrated by the benny equations", *Nonlinear Dynamics* **28** (2002), no. 2, 135–153.

[14] S. V. Meleshko, Y. N. Grigoriev, N. K. Ibragimov, and V. F. Kovalev, "Symmetries of integro-differential equations: with applications in mechanics and plasma physics", Springer Science & Business Media, 2010.

[15] J. Ben Geloun, "Classical Group Field Theory", *J. Math. Phys.* **53** (2012) 022901, arXiv:1107.3122.

[16] A. Baratin and D. Oriti, "Group field theory with non-commutative metric variables", *Phys. Rev. Lett.* **105** (2010) 221302, arXiv:1002.4723.

[17] A. Baratin and D. Oriti, "Group field theory and simplicial gravity path integrals: A model for Holst-Plebanski gravity", *Phys. Rev.* **D85** (2012) 044003, arXiv:1111.5842.

[18] A. Baratin and D. Oriti, "Quantum simplicial geometry in the group field theory formalism: reconsidering the Barrett-Crane model", *New J. Phys.* **13** (2011) 125011, arXiv:1108.1178.

[19] J. C. Baez, "Spin foam models", *Class. Quant. Grav.* **15** (1998) 1827–1858, arXiv:gr-qc/9709052.

[20] S. Alexandrov and P. Roche, "Critical Overview of Loops and Foams", *Phys. Rept.* **506** (2011) 41–86, arXiv:1009.4475.

[21] J. Ben Geloun, R. Gurau, and V. Rivasseau, "EPRL/FK Group Field Theory", *Europhys. Lett.* **92** (2010) 60008, arXiv:1008.0354.

[22] S. Alexandrov, M. Geiller, and K. Noui, "Spin Foams and Canonical Quantization", *SIGMA* **8** (2012) 055,

arXiv:1112.1961.

[23] A. Perez, "The Spin Foam Approach to Quantum Gravity", *Living Rev. Rel.* **16** (2013) 3, arXiv:1205.2019.

[24] C. Rovelli, "Loop quantum gravity", *Living Reviews in Relativity* **11** (2008), no. 5,.

[25] D.-W. Chiou, "Loop Quantum Gravity", *Int. J. Mod. Phys.* **D24** (2014), no. 01, 1530005, arXiv:1412.4362.

[26] S. Gielen, D. Oriti, and L. Sindoni, "Homogeneous cosmologies as group field theory condensates", *JHEP* **06** (2014) 013, arXiv:1311.1238.

[27] S. Gielen and D. Oriti, "Quantum cosmology from quantum gravity condensates: cosmological variables and lattice-refined dynamics", *New J. Phys.* **16** (2014), no. 12, 123004, arXiv:1407.8167.

[28] S. Gielen, "Quantum cosmology of (loop) quantum gravity condensates: An example", *Class. Quant. Grav.* **31** (2014) 155009, arXiv:1404.2944.

[29] D. Oriti, L. Sindoni, and E. Wilson-Ewing, "Emergent Friedmann dynamics with a quantum bounce from quantum gravity condensates", arXiv:1602.05881.

[30] J. Ben Geloun and V. Bonzom, "Radiative corrections in the Boulatov-Ooguri tensor model: The 2-point function", *Int. J. Theor. Phys.* **50** (2011) 2819–2841, arXiv:1101.4294.

[31] J. Ben Geloun, "On the finite amplitudes for open graphs in Abelian dynamical colored Boulatov–Ooguri models", *J. Phys.* **A46** (2013) 402002, arXiv:1307.8299.

[32] R. Gurau, "Colored Group Field Theory", *Commun. Math. Phys.* **304** (2011) 69–93, arXiv:0907.2582.

[33] R. Gurau, "Lost in Translation: Topological Singularities in Group Field Theory", *Class. Quant. Grav.* **27** (2010) 235023, arXiv:1006.0714.

[34] D. V. Boulatov, "A Model of three-dimensional lattice gravity", *Mod. Phys. Lett.* **A7** (1992) 1629–1646, arXiv:hep-th/9202074.

[35] V. Bonzom, R. Gurau, and V. Rivasseau, "Random tensor models in the large N limit: Uncoloring the colored tensor models", *Phys. Rev.* **D85** (2012) 084037, arXiv:1202.3637.

[36] D. Benedetti, J. Ben Geloun, and D. Oriti, "Functional Renormalisation Group Approach for Tensorial Group Field Theory: a Rank-3 Model", *JHEP* **03** (2015) 084, arXiv:1411.3180.

[37] L. Freidel, K. Krasnov, and R. Puzio, "BF description of higher dimensional gravity theories", *Adv. Theor. Math. Phys.* **3** (1999) 1289–1324, arXiv:hep-th/9901069.

[38] J. W. Barrett and L. Crane, "Relativistic spin networks and quantum gravity", *J. Math. Phys.* **39** (1998) 3296–3302, arXiv:gr-qc/9709028.

[39] P. J. Olver, "Applications of lie groups to differential equations", Springer Science & Business Media, 2000.

[40] S. Lie, "Theorie der transformationsgruppen i", *Mathematische Annalen* **16** (1880), no. 4, 441–528.

[41] E. Noether, "Invariant variation problems", *Transport Theory and Statistical Physics* **1** (1971) 186–207, physics/0503066.

[42] W. Fushchich and V. V. Kornyak, "Computer algebra application for determining lie and lie-bäcklund symmetries of differential equations", *Journal of Symbolic Computation* **7** (1989), no. 6, 611–619.

[43] S. J. Akhtarshenas, "Differential geometry on SU(N): Left and right invariant vector fields and one-forms", *ArXiv e-prints*, 2010 arXiv:1003.2708.

[44] S. Gielen, D. Oriti, and L. Sindoni, "Cosmology from Group Field Theory Formalism for Quantum Gravity", *Phys. Rev. Lett.* **111** (2013), no. 3, 031301, arXiv:1303.3576.

[45] S. Gielen, "Perturbing a quantum gravity condensate", *Phys. Rev.* **D91** (2015), no. 4, 043526, arXiv:1411.1077.

[46] S. Gielen, "Identifying cosmological perturbations in group field theory condensates", *JHEP* **08** (2015) 010, arXiv:1505.07479.

[47] D. Oriti, L. Sindoni, and E. Wilson-Ewing, "Bouncing cosmologies from quantum gravity condensates", arXiv:1602.08271.

[48] M. de Cesare, A. G. A. Pithis, and M. Sakellariadou, "Cosmological implications of interacting Group Field Theory models: cyclic Universe and accelerated expansion", arXiv:1606.00352.

[49] B. Dittrich, "Partial and complete observables for Hamiltonian constrained systems", *Gen. Rel. Grav.* **39** (2007) 1891–1927, arXiv:gr-qc/0411013.

[50] B. Dittrich, "Partial and complete observables for canonical general relativity", *Class. Quant. Grav.* **23** (2006) 6155–6184, arXiv:gr-qc/0507106.

[51] A. Baratin, B. Dittrich, D. Oriti, and J. Tambornino, "Non-commutative flux representation for loop quantum gravity", *Class. Quant. Grav.* **28** (2011) 175011, arXiv:1004.3450.

[52] S. Gielen and L. Sindoni, "Quantum cosmology from group field theory condensates: a review", arXiv:1602.08104.

[53] J. Ben Geloun, R. Martini, and D. Oriti, "Functional Renormalization Group analysis of a Tensorial Group Field Theory on $\mathbb{R}^3$", *Europhys. Lett.* **112** (2015), no. 3, 31001, arXiv:1508.01855.

[54] J. Ben Geloun, R. Martini, and D. Oriti, "Functional Renormalisation Group analysis of Tensorial Group Field Theories on $\mathbb{R}^d$", *Phys. Rev.* **D94** (2016), no. 2, 024017, arXiv:1601.08211.

[55] J. Ben Geloun and V. Rivasseau, "A Renormalizable 4-Dimensional Tensor Field Theory", *Commun. Math. Phys.* **318** (2013) 69–109, arXiv:1111.4997.

[56] S. Carrozza, D. Oriti, and V. Rivasseau, "Renormalization of Tensorial Group Field Theories: Abelian U(1) Models in Four Dimensions", *Commun. Math. Phys.* **327** (2014) 603–641, arXiv:1207.6734.

[57] J. Ben Geloun, "Renormalizable Tensor Field Theories", in "18th International Congress on Mathematical Physics (ICMP2015) Santiago de Chile, Chile, July 27-August 1, 2015". 2016. arXiv:1601.08213.