Near integrability of kink lattice with higher order interactions

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Abstract

In the paper, we make use of Manton's analytical method to investigate the force between kink and the anti-kink with large distance in $1 + 1$ dimensional field theory. The related potential has infinite order corrections of exponential pattern, and coefficients for each order are determined. These coefficients can also be obtained by solving the equation of the fluctuation around the vacuum. At the lowest order, the kink lattice represents the Toda lattice. With higher order correction terms, the kink lattice can represent one kind of the generic Toda lattice. With only two sites, the kink lattice is classically integrable. If the number of sites of the lattice is larger than two, the kink lattice is not integrable but a near integrable system. We take use of the Flaschka's variables to study the Lax pair of the kink lattice. These Flaschka's variables have interesting algebraic relations and the non-integrability can be manifested. We also discussed the higher Hamiltonians for the deformed open Toda lattice, which has a similar result as the ordinary deformed Toda.

Keywords: Integrable system, Soliton,, Toda lattice

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1. Introduction

Solitons are non dissipative objects which occurs in many physical models. In many works, people found the solitonic waves from the integrable system by solving the equations directly. In the other direction, a few works on constructing the many body systems even the integrable systems from the soliton objects are given. Since solitons can be approximated by the quasi particles from the dynamical point of view, one can construct the integrable models from solitons in principle. This work is dedicated to construct the Toda lattice from kink lattice.

The interaction between kink and anti-kink is important for the scattering processes. Manton proposed an analytical method to calculate such static force (Manton, 1979, 2004; Vachaspati, 2006), and found that the force is universal for many scalar field models in 1+1 dimensional field theory. In Manton's work, only the leading order interaction between the kink pair are considered. Recently, we studied the interaction up to the second order, and found it to be universal for φ^4, φ^6 and φ^8 models (He, 2016). In this work, we aim to investigate the static force with infinite order corrections. We study the φ^4 theory as an illustration. The coefficients for each order are determined. We found that the dynamics of excitations around the vacuum determine the interaction pattern. The interaction between the kink pair may play an important role in the kink collision phenomenon, especially to explain the bounce resonance (Dorey, 2011; Gani, 2015).

The Toda Lattice is one of the most representative and fundamental system of finite dimension Hamiltonian integrable systems (Toda, 1967). And its integrability was established by Flaschka, Henon and Manakov (Flaschka, 1974; Henon, 1974; Manakov, 1974). In our research, the form of the Hamiltonian of kink Lattice shares a great similarity with the Hamiltonian of Toda Lattice. Reserving the leading order interaction, the kink lattice is exactly the same with the Toda lattice. When the higher order interactions are included, the kink lattice turns out to be one special deformed Toda lattice. Sawada and Kotera has proved that Toda's potential is the unique integrable potential potential of Hénon's type (Sawada, 1976). So the high order terms will break the integrability of the kink lattice. Kink lattice with high order corrections become the near or quasi integrable system (Gignoux, 2009; Ferreira, 2010, 2016). We show such near integrability in terms of the

Flaschka variables. The "Lax pair" has been constructed for the kink lattice and the generalized deformed Toda lattice. This "Lax pair" can represent the integrable Lax pair of Toda lattice, and can show how the high order interactions break the integrability. We find some non-trivial algebra relations for the Flaschka variables. Several cases have been discussed to show the rich physics of the kink lattice. The paper is organized as the following. We construct the kink lattice in the Sec.2. Many aspects of the kink lattice are discussed in Sec. 3. Conclusions and discussions are given in the last section.

2. The kink lattice

For constructing the kink lattice, we use the φ^4 kink for illustration. The $1 + 1$ dimension Lagrangian for the φ^4 theory reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \lambda(\varphi^2 - v^2)^2, \tag{1}$$

where $\mu = 0, 1$ and λ is the coupling constant. The Euler-Lagrange equation is written as

$$\ddot{\varphi} - \varphi'' + \frac{dV}{d\varphi} = 0. \tag{2}$$

The soliton exists in the static condition, the energy of the system is given by

$$E = \int_{-\infty}^{+\infty} (\frac{1}{2}\varphi'^2 \pm V(\phi))dx \geq \mp \int_{-\infty}^{+\infty} \sqrt{2V(\varphi)}d\varphi, \tag{3}$$

the equal sign stands for the Bogomol'nyi bound. The BPS equation for the kink is written as

$$\varphi' = \pm\sqrt{2V(\varphi)}. \tag{4}$$

The kink solution must interpolate between v and $-v$. Here, we define $(\varphi_{-\infty}, \varphi_\infty) = (-v, v)$ as the kink, and $(v, -v)$ as the antikink. The kink solution can be written as

$$\varphi(x) = v\tanh\left[\sqrt{2\lambda}v(x - x_0)\right], \tag{5}$$

where x_0 denotes the position of the kink. The antikink solution can be obtained by replacing x to $-x$.

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The momentum of the system is written as [1]

$$P = -T_{01} = -\int_{-\infty}^{b} \dot{\varphi}\varphi' dx. \tag{6}$$

Here we consider the generic case rather than the static case. The classical force is the derived by

$$F = \frac{dP}{dt} = \left[-\frac{1}{2}(\dot{\varphi}^2 + \varphi'^2) + V(\varphi) \right]_{-\infty}^{b}, \tag{7}$$

which is valid also for the motion of the field. We don't consider the static kink solution up to this step.

Now we consider the static interaction between the kink φ_1 and the antikink φ_2, whose positions are x_{01} and x_{02} respectively. φ_1 and φ_2 are written as

$$\varphi_1 = v \tanh[\sqrt{2\lambda}v(x - x_{01})], \qquad \varphi_2 = v \tanh[-\sqrt{2\lambda}v(x - x_{02})]. \tag{8}$$

We set $R = x_{02} - x_{01} > 0$ to be large but not infinite. The multiple kink solutions are represented by $\varphi = \varphi_1 + \varphi_2 - v$, where v is the vacuum at the center of the pair. Now φ is independent of the time. One can omit the first term in (7). At $-\infty$, both φ_1 and φ_2 approach the vacuum, and their derivatives are zero. So, the force is related only to the pressure at b. We also set b to be the center of the pair, i.e., $b = \frac{1}{2}(x_{01} + x_{02})$.

At point b, both φ_1 and φ_2 approach the vacuum v. One can set

$$\varphi_1 \equiv v + \chi_1, \qquad \varphi_2 \equiv v + \chi_2, \tag{9}$$

where $\chi_{1,2}$ denotes the perturbation field around the vacuum. The potential $V(\varphi)$ can be expanded around the vacuum v as a power series of the perturbation χ(Manton, 1979), i.e.,

$$V(v + \chi) = \sum_{n=0}^{\infty} \frac{1}{n!} V_{(n)} \chi^n, \tag{10}$$

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[1]The momentum tensor is defined as $T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\partial\mu\phi}\partial_\nu\phi - g_{\mu\nu}\mathcal{L}$.

where $V_{(n)} = \frac{d^n V}{d^n \varphi}|_{\varphi = v}$. The equation in (2) indicates us that χ satisfies the following equation

$$\frac{d^2 \chi}{dx^2} - \tilde{m}^2 \chi = \sum_{n=2}^{\infty} \frac{1}{n!} V_{(n+1)} \chi^n, \tag{11}$$

where $\tilde{m} = 2\sqrt{2\lambda} v$ is the mass of φ. Near the vacuum, χ can be expanded as

$$\chi(x) = \sum_{k=1}^{\infty} a_k \exp(-k\tilde{m}|x - x_0|). \tag{12}$$

a_1 take arbitrary values, other coefficients can be determined subsequently

$$a_1, \qquad a_2 = a_1^2 \frac{V_{(3)}}{6\tilde{m}^2}, \qquad a_3 = a_1^3 \left(\frac{V_{(3)}^2}{48\tilde{m}^4} + \frac{V_{(4)}}{48\tilde{m}^2} \right), \qquad \dots \tag{13}$$

In the other side, the kink solution enables us to expand φ field by Taylor expansion. One can obtain the expression for $\chi_{1,2}$ from (8), i.e.,

$$\chi_1 = 2v \sum_{k=1}^{\infty} (-1)^k e^{-k\tilde{m}(x - x_{01})}, \qquad \chi_2 = 2v \sum_{k'=1}^{\infty} (-1)^{k'} e^{k'\tilde{m}(x - x_{02})}. \tag{14}$$

Here we expand the field near the vacuum v at point b. For $a_1 = -2v$, it can be verified from (13) that $a_2 = 2v$, $a_3 = -2v$, etc. This proves that the coefficients of the Taylor expansion are indeed the solutions for the equation in (11). The agreement between these two theories is not coincident, since the static field equation in (11) is just the spatial derivation of the BPS equation in (4) for $1 + 1$ dimensional scalar theory. Since the static field equation are of second differential, a_1 are arbitrary. However, if we consider the first differential BPS equation, a_1 can be determined uniquely at the boundary. The χ field denotes the perturbation around the vacuum in the theory. The argument here indicates that the static kink configuration can be used to study the dynamics of excitations classically.

Now we consider static force between kink and antikink in (7). The first time differential term can be ignored for the static condition. Then, the force can be calculated by the Taylor expansion above

$$F = -\frac{1}{2}(\chi_1' + \chi_2')^2 + \sum_{n=0}^{\infty} \frac{V_{(n)}}{n!}(\chi_1 + \chi_2)^n. \tag{15}$$

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For the φ^4 potential, V can only be expanded up to the forth derivation $V_{(4)}$. Substituting the expression for χ, one obtains the results

$$F = 8\tilde{m}^2 v^2 \sum_{n=1}^{\infty} \alpha_n e^{-\frac{n+1}{2}\tilde{m}R}, \tag{16}$$

where $\alpha_n = -\frac{(-1)^n}{3}(n+2n^3)$. In (16), the first term disappears since $\chi_1' + \chi_2' = 0$ at point $x = b$. Thus, the force originates completely from the potential term. The sign of each order correction is against the next order, which indicates an alternative attractive or repulsive force. The force in (16) includes all orders of corrections. It was mentioned that the force should be universal for all species of kink and antikink interaction in one dimension(Manton, 1979), which needs to be verified for the φ^6, φ^8 and sine-Gordon theories. It is well-known that for there are no interaction between BPS solitons. The analytical method to obtain the static force between kink and anti-kink here considers the point b, at which the kink and anti-kink are actually non-BPS. The perturbation around the vacuum represents the non-BPS excitation mode in the background of kink and anti-kink. These modes actually affect the soliton scattering. Therefore, the analytical method here can be generalized to study other kinds of soliton interactions.

With the normalization $\lambda = 1$ and $v = \frac{1}{2\sqrt{2}}$, one can write out the interaction energy according to $F = dU/dR$. The potential U is given by

$$U = \sum_{n=1}^{\infty} \beta_n e^{-\frac{n+1}{2}R}, \tag{17}$$

where $\beta_n = (-1)^n \frac{2(n+2n^3)}{3(n+1)}$. One can construct the kink lattice along one line with kink antikink alternating. The $i-$th kink or antikink experiences the following force

$$\frac{dp_i}{dt} = \sum_{n=1}^{\infty} \alpha_n \left[-e^{-\frac{n+1}{2}(q_{i-1}-q_i)} + e^{-\frac{n+1}{2}(q_i-q_{i+1})} \right]. \tag{18}$$

Thus, the kink lattice becomes a kind of deformed Toda lattice[2]. In this way, we have constructed the integrable system from the soliton section directly.

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[2]Here, the deformed Toda lattice has the same Lagrangian with the kink lattice. The generic Toda lattice in this work refers to the deformed Toda lattice with generalized coupling coefficients.

The produced deformed Toda lattice system has not been studied up to our knowledge.

The Hamiltonian for the kink lattice can be given by the kink momentum and potential directly. For the nonperiodic (open) kink lattice, the Hamiltonian is written as

$$H = \sum_{i=1}^{N} \frac{1}{2}p_i^2 + \sum_{i=1}^{N-1}\sum_{n=1}^{\infty} \left(\beta_n e^{-\frac{1+n}{2}(q_i - q_{i+1})} \right), \tag{19}$$

where N is the number of lattice sites. For the periodic (closed) case, the Hamiltonian is given by

$$H = \sum_{i=1}^{N} \frac{1}{2}p_i^2 + \sum_{i=1}^{N}\sum_{n=1}^{\infty} \left(\beta_n e^{-\frac{1+n}{2}(q_i - q_{i+1})} \right), \qquad q_{N+1} = q_1. \tag{20}$$

In the kink lattice, the periodic condition means to glue the right arm of Nth kink with the left arm of the first kink. The vacua must be correctly connected by the kink and antikink. Therefore, N should be an even integer for the periodic kink lattice. However, there are no such constraint for the periodic deformed Toda lattice, N can be any integer. We will consider the integrability of the kink lattice in the following.

3. Near integrability

Although we obtained the Hamiltonian of the kink lattice, its integrability needs to be verified. The Lax pair representation of the system is essential to prove the integrability classically. The Flaschka's transformation enables us to give the Lax pair (Flaschka, 1974). Instead of variables p_i and q_i, we set new variables to describe the system. For the open Toda lattice, one can set

$$a_i \equiv \sqrt{\sum_{n=1}^{\infty} \left(\beta_n e^{-\frac{1+n}{2}(q_i - q_{i+1})} \right)}, \qquad i = 1, 2, \ldots, N-1 \tag{21}$$

$$b_i \equiv p_i. \tag{22}$$

In terms of a_i and b_i, the Hamiltonian of the open Toda is written as

$$H = \frac{1}{2}\sum_{i=1}^{N} b_i^2 + \sum_{i=1}^{N-1} a_i^2. \tag{23}$$

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The Lax pair satisfies $\dot{L} = [M, L]$, they are assumed to have such formula

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & & 0 \\ a_1 & b_2 & a_2 & & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & a_{N-1} & \\ 0 & \cdots & \cdots & a_{N-1} & b_N \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & -c_1 & 0 & \cdots & & 0 \\ c_1 & 0 & -c_2 & & & \vdots \\ 0 & c_2 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & -c_{N-1} \\ 0 & \cdots & \cdots & c_{N-1} & 0 \end{pmatrix}. \tag{24}$$

Here c_i are unknown parameters. The evolution of L leads to the equations for c_i, i.e.,

$$\dot{a}_i = c_i(b_i - b_{i+1}), \qquad i = 1, \cdots, N-1 \tag{25}$$
$$\dot{b}_1 = -2c_1a_1, \tag{26}$$
$$\dot{b}_i = 2(c_{i-1}a_{i-1} - c_ia_i), \qquad i = 2, \cdots, N-1 \tag{27}$$
$$\dot{b}_N = 2a_{N-1}c_{N-1}. \tag{28}$$

The kink lattice system goes back to the Toda system if we only keep the leading $n = 1$ term of the potential. In the Toda theory, the solution for c_i is simply written as

$$c_i = \frac{\partial a_i}{\partial q_i}. \tag{29}$$

The Hamilton equations of motion of a_i and b_i read

$$\dot{a}_i = c_i(b_i - b_{i+1}), \qquad \dot{b}_i = 2c_i(a_{i-1} - a_i). \tag{30}$$

One can observe that the Hamiltonian equations of motion for q_i and p_i agree with (25) to (28). The Poisson bracketss of a_i and b_i are given by

$$\{a_i, b_i\} = c_i, \qquad \{a_i, b_{i+1}\} = -c_i, \qquad i \leq N-1 \tag{31}$$

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We find that the $(i, i+2)$ component of $[M, L]$ is

$$a_i c_{i+1} - a_{i+1} c_i, \tag{32}$$

which is not zero in general. Thus, the validity of the Lax pair needs the constraint that $a_i c_{i+1} = a_{i+1} c_i$. If one finds a solution for c_i, which satisfies equations of motion in (25) and the constraint $a_i c_{i+1} = a_{i+1} c_i$, we can claim that the system is integrable. Otherwise, the system is not integrable.

The Flaschka's form indicates that the kink lattice has a generic algebra structure in the dynamics. In the kink lattice, the β_n is determined by α_n. The coefficients can be generalized without hindering the equations of motion (in terms of a_i, b_j and c_k). The integer and half integer index in the exponential term can be generalized to an arbitrary real number. However, the physical system should have finite energy, which puts strong constrains for the coefficients. The Hamiltonian of the proposed most generic open Toda lattice is given by

$$H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N-1} \sum_{n=1}^{\tilde{N}} \beta_n e^{-k_n(q_i - q_{i+1})}. \tag{33}$$

where k_n is positive. $\tilde{N}$ can be infinite when the summation of all terms is convergent. Then, the algorithm of the Flashcka transformation repeats the Lax pair representation as above, with setting

$$a_i \equiv \sqrt{\sum_{n=1}^{\tilde{N}} \beta_n e^{-k_n(q_i - q_{i+1})}}, \qquad b_i \equiv p_i. \tag{34}$$

The Lax representation does not change for this new deformed Toda lattice. The Flaschka's variables enable us to discuss the integrability for several special cases.

3.1. Cases study

First, if we keep only the leading order interaction in the kink lattice, the theory goes back to the Toda theory exactly. The constraint in (32) is satisfied automatically, since

$$c_i = -\frac{1}{2} a_i. \tag{35}$$

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This shows that the Toda lattice is integrable, which is a well known result. The integrals of the motion can be obtained by

$$H_k = \frac{1}{k}\mathrm{Tr}L^k, k = 1, 2, \cdots, N \tag{36}$$

The first invariant H_1 presents the conservation of the momentum. The second invariant $H_2 = \frac{1}{2}\mathrm{Tr}L^2$ is the Hamiltonian. The invariants also satisfy the relation $\{H_i, H_j\} = 0$.

Secondly, if we keep only the leading interaction in the generic Toda lattice in (33), the Hamiltonian in (33) becomes

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N-1} \beta_i e^{-k_i(q_i - q_{i+1})}. \tag{37}$$

which is is the Hamiltonian of the inhomogeneous Toda theory. One can check that

$$a_i c_{i+1} - c_i a_{i+1} = \frac{1}{2}a_i a_{i+1}(k_i - k_{i+1}) \neq 0, \tag{38}$$

which is not zero. Thus, the inhomogeneous Toda lattice is not integrable. Physically, the different couplings between different sites breaks the integrability. We give the $N = 3$ case for illustration. According the the definition in (36), the three variables H_1, H_2 and H_3 are given by

$$H_1 = b_1 + b_2 + b_3, \tag{39}$$

$$H_2 = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2) + a_1^2 + a_2^2, \tag{40}$$

$$H_3 = \frac{1}{3}(b_1^3 + b_2^3 + b_3^3) + (b_1 + b_2)a_1^2 + (b_2 + b_3)a_2^3. \tag{41}$$

The Poisson brackets between them are given by

$$\{H_1, H_2\} = 0, \qquad \{H_1, H_3\} = 0, \qquad \{H_2, H_3\} = (k_2 - k_1)a_1^2 a_2^2. \tag{42}$$

These relations show that the system has momentum and energy conservation, but we don't have the third integrals of motion. H_3 is not a conserved integral of motion. The evolution of H_3 is corrected by the difference of the couplings. The inhomogeneous Toda is not an integral system.

Thirdly, for the only two sites case, i.e. $N = 2$, the deformed Toda theory with high order exponential terms are exact integrable, since there are no constraint conditions anymore. For the same reason, the two sites kink lattice is integrable. For two sites system, we only need two integrals of motion to manifest the integrability. The dynamics of the Lax pair agrees with the Hamiltonian equations of motion, which are given by

$$\dot{a}_1 = c_1(b_2 - b_1), \quad \dot{b}_1 = 2a_1c_1, \quad \dot{b}_2 = -2a_1c_1. \tag{43}$$

One can construct two integrals of motion, which are the momentum and the energy of the system evidently.

A system with N degrees of freedom is superintegrable if it has $2N - 1$ independent constants of motion. One can further ask whether such system is superintegrable (Agrotis, 2006). Moser proposed to take use of new variables (λ_i, r_i) to replace the variables (a_i, b_i). Defining the function $f(\lambda)$ as

$$f(\lambda) = \frac{1}{\lambda - b_2 - \frac{a_1^2}{\lambda - b_1}}. \tag{44}$$

$f(\lambda)$ can be expanded in a series of powers of $1/\lambda$, i.e.,

$$f(\lambda) = \sum_{j=0}^{\infty} \frac{c_j}{\lambda^{j+1}}, \tag{45}$$

where $c_j = \sum_{i=1}^{N} r_i^2 \lambda_i^j / \sum_{i=1}^{N} r_i^2$. The corresponding relations between a_1, b_i and λ_1, r_i are given by

$$a_1^2 = \frac{r_1^2 r_2^2 (\lambda_2 - \lambda_1)^2}{(r_1^2 + r_2^2)^2}, \quad b_1 = \frac{r_1^2 \lambda_2 + r_2^2 \lambda_1}{r_1^2 + r_2^2}, \quad b_2 = \frac{r_1^2 \lambda_1 + r_2^2 \lambda_2}{r_1^2 + r_2^2}. \tag{46}$$

The equations of motion for λ_i and r_i are

$$\dot{\lambda}_i = 0, \tag{47}$$
$$\dot{r}_i = -\lambda_i r_i. \tag{48}$$

This agrees with the Lax pair representation, i.e.,

$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}. \tag{49}$$

However, such Lax pair representation does not agree with the Hamiltonian equations of motion of the deformed Toda, see Eq. (43). One has $a_1 \neq c_1$. This probably indicates that the deformed Toda is integrable but not super-integrable. In order to prove the superintegrability, one needs to represent c_1 with λ_i and r_i, which is not solved currently.

Forth, for $N \geq 3$ case, there is no solution of c_i which satisfies the equations of motion and the constraint simultaneously. For instance, the $N = 3$ case, there are c_1 and c_2 unknown. There are no self-consistent solutions, or equally, the equation of motion conflicts with the constraint. The component $(c_i a_{i+1} - a_i c_{i+1})$ are high order interaction correction. One can decompose the Hamiltonian in Eq.(19) as

$$H(q, p) = H_0(q, p) + V(q, p), \tag{50}$$

where H_0 represents the Hamiltonian for Toda lattice, and

$$V(q, p) = \sum_{i=1}^{N-1} \sum_{n=2}^{\infty} \beta_n e^{-\frac{1+n}{2}(q_i - q_{i+1})}$$

denotes the high order interactions, which can be considered as the perturbation part. This indicates that the kink lattice is not exact but "near" integrable. One can take use of the canonical perturbation method or to study its dynamics (Gignoux, 2009). The "near" integrable system interpolates between the integrable system and the chaotic system. The kink lattice offers a nice toy model for such near integrable system. Recently, Ferreira et al. take use the parameter expansion method to study the sine-Gordon kink model for the breathers and wobbles phenomenon (Ferreira, 2010, 2016). Our exact analytical results of the kink interaction here can also be used to study such phenomenon.

3.2. Higher Hamiltionians

For the ordinary Toda lattice, one can construct the higher Hamiltonians by taking trace of many powers of the Lax matrix, which are all integrable. The Hamiltonian of the Toda lattice system is given by

$$H = \frac{1}{2} \text{Tr}(L^2). \tag{51}$$

One can construct the systems with higher Hamiltonian, i.e.,

$$H_k = \frac{1}{k} \text{Tr}(L^k), \tag{52}$$

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where k is an integer larger than 2. If L represents the ordinary Toda, then all the systems of higher Hamiltonian are integrable. For each k, one can have a Lax representation, i.e.,

$$\dot{L} = [M_k, L], \tag{53}$$

where M_k is a skew-symmetric matrix.

In the following, we will present the higher Hamiltonian for the deformed Toda lattice, which are near integrable systems. If L represents the deformed Toda lattice in (24), one can also construct the higher Hamiltonian for the generalized Toda lattice. Similar to the generic Toda, systems with higher Hamiltonian are not integrable, but they are near integrable. We find a "Lax representation" for these near integrable system. For instance, we consider the $H_3 = \frac{1}{3}\mathrm{Tr}L^3$ system. The M_3 matrix can be given by

$$M_3 = \begin{pmatrix} 0 & \zeta_1 & \eta_1 & 0 & \cdots & \cdots & 0 \\ -\zeta_1 & 0 & \zeta_2 & \eta_2 & \cdots & \cdots & 0 \\ -\eta_1 & -\zeta_2 & 0 & \zeta_3 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \zeta_{n-2} & \eta_{n-2} \\ \vdots & & & \ddots & -\zeta_{n-2} & 0 & \zeta_{n-1} \\ 0 & 0 & 0 & \cdots & -\eta_{n-2} & -\zeta_{n-1} & 0 \end{pmatrix}. \tag{54}$$

The components are given by

$$\zeta_i = c_i(b_i + b_{i+1}), \qquad\qquad k = 1, \ldots, n-1 \tag{55}$$
$$\eta_i = c_i a_{i+1}. \qquad\qquad k = 1, \ldots, n-2 \tag{56}$$

The Lax representation leads to the equations of motion, which are given by

$$\dot{a}_i = -a_{i-1}c_{i-1}a_i + c_i(a_{i+1}^2 - b_i^2 + b_{i+1}^2), \qquad k = 1, \ldots, n-1 \tag{57}$$
$$\dot{b}_i = -2a_{i-1}c_{i-1}(b_{i-1} + b_i) + 2a_ic_i(b_i + b_{i+1}), \qquad k = 1, \ldots, n \tag{58}$$

where one also sets $a_0 = a_n = c_n = b_0 = b_{n+1} = 0$. Besides that, other non-zero components in the $[M_k, L]$ matrix are proportional to $(a_ic_{i+1} - a_{i+1}c_i)$. We take the $n = 4$ case for illustration. One obtains that

$$[M_3, L]_{13} = (b_2 + b_3)(a_2c_1 - a_1c_2), \tag{59}$$
$$[M_3, L]_{14} = a_3(a_2c_1 - a_1c_2), \tag{60}$$
$$[M_3, L]_{24} = (b_3 + b_4)(a_3c_2 - a_2c_3). \tag{61}$$

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The subscripts 13, 14 and 24 denotes the matrix components. One concludes that these new higher Hamiltonian systems are still near integrable systems. For the ordinary Toda case, $a_i \propto c_i$, all these terms disappear. Thus, the Hamiltonian of the kink lattice is near integrable, the higher Hamiltonian of the kink lattice is still near integrable systems.

3.3. The closed case

For the representation of the closed Toda lattice, one can repeat the technique with the generalized Hamiltonian in (20). Assume that the system is integrable. The Lax formula can be constructed as (Babelon, 2003; He, 2016)

$$
L = \begin{pmatrix}
b_1 & a_1 & 0 & \ldots & \lambda^{-1}a_N \\
a_1 & b_2 & a_2 & & \vdots \\
0 & a_2 & b_3 & \ddots & \vdots \\
\vdots & & \ddots & \ddots & a_{N-1} \\
\lambda a_N & \ldots & \ldots & a_{N-1} & b_N
\end{pmatrix},
\tag{62}
$$

where $a_N = \sqrt{\sum_{n=1}^{\tilde{N}} \left(\beta_n e^{-k_n(q_N - q_1)} \right)}$, and λ is the spectral parameter. The M matrix is given by (Babelon, 2003)

$$
M = \begin{pmatrix}
0 & -c_1 & 0 & \ldots & \lambda^{-1}c_N \\
c_1 & 0 & -c_2 & & \vdots \\
0 & c_2 & 0 & \ddots & \vdots \\
\vdots & & \ddots & 0 & -c_{N-1} \\
-\lambda c_N & \ldots & \ldots & c_{N-1} & 0
\end{pmatrix}.
\tag{63}
$$

The periodic condition is given by $q_{N+i} = q_i$. We list the equations of motion for several cases in the following.

For the $N = 2$ case, the system is integrable. The Hamiltonian is given by

$$
H = \frac{1}{2}(b_1^2 + b_2^2) + a_1^2.
\tag{64}
$$

The equation of motion is given by

$$
\dot{a}_1 = \frac{c_1}{\lambda}(b_2 - b_1) = \lambda c_1(b_2 - b_1), \qquad \dot{b}_1 = a_1 c_1(\frac{1}{\lambda} + \lambda) = -\dot{b}_2.
\tag{65}
$$

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λ is equal to ± 1 for consistence. Thus, the spectral curve reduces to two points trivial case. The two integrals of motion denote the total momentum and energy conservation, respectively.

For the $N = 3$ case, one obtains the equations of motion

$$\dot{b}_i = 2(a_{i-1}c_{i-1} - a_i c_i), \qquad i = 1, 2, 3 \tag{66}$$

$$\dot{a}_1 = c_i(b_i - b_{i+1}) - \frac{1}{\lambda}(a_{i-1}c_{i+1} - a_{i+1}c_{i-1})$$

$$= c_i(b_i - b_{i+1}) - \lambda(a_{i-1}c_{i+1} - a_{i+1}c_{i-1}). \qquad i = 1, 2, 3 \tag{67}$$

Here we have considered the periodic conditions. One can observe that, the constraint terms $a_i c_{i+1} - c_i a_{i+1}$ appears. For ordinary periodic Toda, these constrains disappear. The non-zero constraint will involve λ into the equations of motion. Also, the consistence requires $\lambda = \pm 1$. Thus, the high order perturbation term will trivialize the spectral curve. From these evidence, the higher Hamiltonian systems of the deformed Toda lattice are near or quasi integrable systems (Gignoux, 2009; Ferreira, 2010).

4. Conclusions and Discussions

In this paper, we have calculated the effective potential of the kink and anti-kink pair at large separation. The assumption of the large distance is essential for the Manton's method. The resulted potential contains infinite order corrections of the exponential type. All coefficients for these orders are obtained exactly. Such effective potential may play an important role in the kink antikink collision test. Many work have been done to investigate the kink scattering by the numerical method (Dorey, 2011; Gani, 2015). We observed that the fluctuation of the φ^4 theory around the vacuum satisfies a second differential equation, and its solution agrees with the BPS equation. This phenomenon helps us to calculate the quantum theory from the topological section. If we consider the theory in one time and zero space dimensions, the Lagrangian in Eq.(1) will become the anharmonic oscillator with a double well potential theory (Gildener & Patrascioiu , 1977). The kink solution here will become the instanton solution. Our effective potential of the kink and antikink can help to study the instanton and anti-instanton contribution to the kernel. It can be expected that discrete energy levels for the anharmonic oscillator are related to the instanton configurations. The Manton's method here can also be used to study the kink dynamics in other

theories, for instance the kink solutions (which are confined monopoles on the vortex string) in the SQCD theories (Shifman, 2004; Eto et al., 2011; AlonsoIzquierdo et al., 2008). Several work have studied the dynamics of such kinks (Arai et al., 2014; Harland, 2009; Tong, 2003). It was found that the potential for kink interactions has a similar exponential pattern. The procedure here to calculate the high order potentials and construct the kink lattice can also be applied to these massive sigma model. One can expect that these confined monopoles form an near integrable system. All these expectations are deserved to be discovered in the future.

The kink can be considered as the pseudoparticle, whose dynamics shows rich structures. We present the effective total Hamiltonian for the kink lattice, and generalized the kink lattice to the generic deformed Toda lattice. Keeping only the leading order potential, the kink lattice is exact the Toda lattice, which is integrable. If higher order terms are included, the kink lattice and the deformed Toda lattice are near integrable except for the two sites case. In terms of the Flaschka's variables, we studied the Lax representation of the kink lattice. Although the potential has infinite correction terms, the Lax equation shows a simple algebra structure. The integrability is broken by the high order corrections. This shows that the kink lattice is a near integrable system. It has been pointed out that the Toda potential is the unique potential for the integrability (Sawada, 1976), this theorem indicates that the inclusion of high order corrections will break the integrability. We found that the two sites kink lattice is integrable, which are most related to the breather phenomenon of the kink and antikink. The two sites kink lattice is most probably not superintegrable, i.e., there are no $2N - 1$ integrals for the kink lattice, the and further study is needed to verify this point. The higher Hamiltonians for the kink lattice are constructed, and they are all near integrable system. The evolution of the near integrable system is also interesting, since they will evolve from the integrable to the chaotic systems. The kink lattice is a nice model for the near integrable study.

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