JHEP

Published for SISSA by Springer

Received: November 23, 2016
Accepted: December 25, 2016
Published: January 9, 2017

Non-abelian Z-theory: Berends-Giele recursion for the α'-expansion of disk integrals

Carlos R. Mafra[a,b] **and Oliver Schlotterer**[c]

[a] *STAG Research Centre and Mathematical Sciences, University of Southampton, Southampton, U.K.*

[b] *Institute for Advanced Study, School of Natural Sciences, Einstein Drive, Princeton, NJ 08540, U.S.A.*

[c] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, 14476 Potsdam, Germany*

E-mail: c.r.mafra@soton.ac.uk, olivers@aei.mpg.de

Abstract: We present a recursive method to calculate the α'-expansion of disk integrals arising in tree-level scattering of open strings which resembles the approach of Berends and Giele to gluon amplitudes. Following an earlier interpretation of disk integrals as doubly partial amplitudes of an effective theory of scalars dubbed as Z-theory, we pinpoint the equation of motion of Z-theory from the Berends-Giele recursion for its tree amplitudes. A computer implementation of this method including explicit results for the recursion up to order α'^7 is made available on the website http://repo.or.cz/BGap.git.

Keywords: Effective field theories, Superspaces, Superstrings and Heterotic Strings

ArXiv ePrint: 1609.07078

doi:10.1007/JHEP01(2017)031

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1 Introduction

It is well known that string theory reduces to supersymmetric field theories involving non-abelian gauge bosons and gravitons when the size of the strings approaches zero. Hence, one might obtain a glimpse into the inner workings of the full string theory by studying the corrections that are induced by strings of finite size, set by the length scale $\sqrt{\alpha'}$. One approach to study such α'-corrections to field theory is through the calculation of string scattering amplitudes, see e.g. [1, 2]. Within this framework, higher-derivative corrections are encoded in the α'-expansion of certain integrals defined on the Riemann surface that encodes the string interactions.

In this work, we will mostly study tree-level scattering of open strings, where the Riemann surface has the topology of a disk. As will be reviewed in section 2, the α'-corrections to super-Yang-Mills (SYM) field theory arise from iterated integrals over the disk boundary. These integrals can be characterized by two words P and Q formed from

the n external legs which refer to the integration domain $P = (p_1, p_2, \ldots, p_n)$ and integrand $Q = (q_1, q_2, \ldots, q_n)$ in

$$Z(P|q_1, q_2, \ldots, q_n) \equiv \alpha'^{n-3} \int_{D(P)} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \cdots \, \mathrm{d}z_n}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{R}))} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{q_1 q_2} z_{q_2 q_3} \ldots z_{q_{n-1} q_n} z_{q_n q_1}} \,. \tag{1.1}$$

This paper concerns the calculation of the α'-expansion of these disk integrals in a *recursive* manner for any given domain P and integrand Q. This technical accomplishment is accompanied by conceptual advances concerning the interpretation of disk integrals (1.1) in the light of double-copy structures among field and string theories.

As the technical novelty of this paper, we set up a Berends-Giele (BG) recursion [3] that allows to compute the α'-expansion of the integrals $Z(P|Q)$ and generalizes a recent BG recursion [4] for their field-theory limit to *all orders* of α'. As a result of this setup, once a *finite* number of terms in the BG recursion at the w^{th} order in α' is known, the expansion of disk integrals at *any* multiplicity is obtained up to the same order α'^w. The recursion is driven by simple deconcatenation operations acting on the words P and Q, which are trivially automated on a computer. The resulting ease to probe α'-corrections at large multiplicities is unprecedented in modern all-multiplicity approaches [5, 6] to the α'-expansion of disk integrals.

The conceptual novelty of this article is related to the interpretation of string disk integrals (1.1) as *tree-level amplitudes* in an effective[1] theory of bi-colored scalar fields Φ dubbed as Z-theory [7]. These scalars will be seen to satisfy an equation of motion of schematic structure,

$$\Box\Phi = \Phi^2 + \alpha'^2 \zeta_2 (\partial^2\Phi^3 + \Phi^4) + \alpha'^3 \zeta_3 (\partial^4\Phi^3 + \partial^2\Phi^4 + \Phi^5) + \mathcal{O}(\alpha'^4) \,. \tag{1.2}$$

The above equation of motion is at the heart of the recursive method proposed in this paper; solving it using a perturbiner [8] expansion in terms of recursively defined coefficients $\phi_{A|B}$ is equivalent to a Berends-Giele recursion[2] that computes the α'-expansion of the disk integrals (1.1) as if they were tree amplitudes of an effective field theory,

$$Z(A, n|B, n) = s_A \phi_{A|B} \,. \tag{1.3}$$

Therefore this paper gives a precise meaning to the perspective on disk integrals as Z-theory amplitudes [7] by pinpointing its underlying equation of motion. After this fundamental conceptual shift to extract the α'-expansion of disk integrals from the equation of motion of Φ, its form to all orders in α' is proposed to be

$$\begin{aligned} \frac{1}{2}\Box\Phi = \sum_{p=2}^{\infty} (-\alpha')^{p-2} \int^{\text{eom}} \prod_{i<j}^{p} |z_{ij}|^{\alpha' \partial_{ij}} \\ \times \left(\sum_{l=1}^{p-1} \frac{[\Phi_{12\ldots l}, \Phi_{p,p-1\ldots l+1}]}{(z_{12} z_{23} \ldots z_{l-1,l})(z_{p,p-1} z_{p-1,p-2} \ldots z_{l+2,l+1})} + \mathrm{perm}(2, 3, \ldots, p-1) \right) . \end{aligned} \tag{1.4}$$

[1] The word "effective" deserves particular emphasis since the high-energy properties of Z-theory (and its quantum corrections) are left for future investigations.

[2] For a recent derivation of Berends-Giele recursions for tree amplitudes from a perturbiner solution of the field-theory equations of motion, see [4, 9]. An older account can be found in [8, 10].

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The detailed description of the above result will be explained in section 4, but here we note its remarkable structural similarity with a certain representation of the superstring disk amplitude for massless external states [11]. The $(n-2)!$-term representation which led to the all-order proposal (1.4) has played a fundamental role in the all-multiplicity derivation of local tree-level numerators [4, 12] which obey the duality between color and kinematics [13].

1.1 *Z*-theory and double copies

The relevance of the disk integrals (1.1) is much broader than what the higher-derivative completion of field theory might lead one to suspect. They have triggered deep insights into the anatomy of numerous field theories through the fact that closed-string tree-level integrals (encoding α'-corrections to supergravity theories) boil down to squares of disk integrals through the KLT relations [14]. In a field-theory context, this double-copy connection between open and closed strings became a crucial hint in understanding quantum-gravity interactions as a square of suitably-arranged gauge-theory building blocks [13, 15].

Double-copy structures have recently been identified in the tree-level amplitudes of additional field theories [16]. For instance, classical Born-Infeld theory [17] which governs the low-energy effective action of open superstrings [18] turned out to be a double copy of gauge theories and an effective theory of pions known as the non-linear sigma model (NLSM) [19–23], see [24–27] for its tree-level amplitudes. As a string-theory incarnation of the Born-Infeld double copy, tree-level amplitudes of the NLSM have been identified as the low-energy limit of the disk integrals in the scattering of *abelian* gauge bosons [7]. This unexpected emergence of pion amplitudes exemplifies that disk integrals also capture the interactions of particles that cannot be found in the naive string spectrum.[3]

Moreover, the entire tree-level S-matrix of massless open-superstring states can be presented as a double copy of SYM with α'-dependent disk integrals [5]. Their Z-theory interpretation in [7] was driven by the quest to identify the second double-copy ingredient of the open superstring besides SYM. In view of the biadjoint-scalar and NLSM interactions in the low-energy limit of Z-theory, its full-fledged α'-dependence describes effective higher-derivative deformations of these two scalar field theories [7]. As a double-copy component to complete SYM to the massless open-superstring S-matrix, the collection of effective interactions encompassed by Z-theory deserve further investigations.

In this work, we identify the equation of motion (1.4) of the full non-abelian Z-theory, where the integration domain of the underlying disk integrals endows the putative scalars Φ with a second color degree of freedom. By the results of [5], disk integrals in their interpretation as Z-theory amplitudes obey the duality between color and kinematics due to Bern, Carrasco and Johansson (BCJ) [13] in one of their color orderings. Hence, the effective theories gathered in Z-theory are of particular interest to advance our understanding of the BCJ duality. The abelian limit of Z-theory arises from disk integrals without any notion of color ordering in the integration domain and has been studied in [7] as a factory

[3] See [28] for a string-theory realization of the NLSM through toroidal compactifications in presence of worldsheet boundary condensates.

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for BCJ-satisfying α'-corrections to the NLSM. The present article extends this endeavor such as to efficiently compute the doubly-partial amplitudes of effective bi-colored theories with BCJ duality in one of the gauge groups and explicitly known field equations (1.4).

1.2 Outline

This paper is organized as follows: following a review of disk integrals and the Berends-Giele description of their field-theory limit in section 2, the Berends-Giele recursion for their α'-corrections and the resulting field equations of non-abelian Z-theory are presented in section 3. The mathematical tools to control the equations of motion to all orders in the fields and derivatives by means of suitably regularized polylogarithms are elaborated in section 4. In section 5, the Berends-Giele recursion is extended to closed-string integrals over surfaces with the topology of a sphere before we conclude in section 6. Numerous appendices and ancillary files complement the discussions in the main text.

The BG recursion that generates all terms up to the ${\alpha'}^7$-order in the α'-expansion of disk integrals at arbitrary multiplicity as well as the auxiliary computer programs used in their derivations can be downloaded from [29].

2 Review and preliminaries

In this section, we review the definitions and symmetries of the disk integrals under investigations as well as their appearances in tree amplitudes of massless open-string states. We also review the recent Berends-Giele approach to their field-theory limit in order to set the stage for the generalization to α'-corrections.

2.1 String disk integrals

We define a cyclic chain $C(Q)$ of worldsheet propagators z_{ij}^{-1} with $z_{ij} \equiv z_i - z_j$ on words $Q \equiv q_1 q_2 \ldots q_n$ of length n as

$$C(Q) \equiv \frac{1}{z_{q_1 q_2} z_{q_2 q_3} \cdots z_{q_n q_1}} \ . \tag{2.1}$$

Then, the iterated disk integrals on the real line that appear in the computation of open-superstring tree-level amplitudes are completely specified by two words P and Q,

$$Z(P|Q) \equiv {\alpha'}^{n-3} \int_{D(P)} \frac{\mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_n}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{R}))} \prod_{i<j}^{n} |z_{ij}|^{\alpha' s_{ij}} C(Q) \,, \tag{2.2}$$

where $P \equiv p_1 p_2 \ldots p_n$ encodes the domain of the iterated integrals,

$$D(P) \equiv \{(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n, \ -\infty < z_{p_1} < z_{p_2} < \ldots < z_{p_n} < \infty\} \ . \tag{2.3}$$

Mandelstam variables $s_{ij\ldots p}$ involving legs $i, j, \ldots, p$ are defined via region momenta $k_{ij\ldots p}$,

$$k_{ij\ldots p} \equiv k_i + k_j + \ldots + k_p \,, \qquad s_{ij\ldots p} \equiv \frac{1}{2} k_{ij\ldots p}^2 \,, \tag{2.4}$$

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and the more standard open-string conventions for the normalization of α' (which would cause proliferation of factors of two) can be recovered by globally setting $\alpha' \to 2\alpha'$ everywhere in this work. In the sequel, we refer to the word P as the *integration region* or *domain* and to Q as the *integrand* of (2.2), where P is understood to be a permutation of Q. The inverse volume $\mathrm{vol}(\mathrm{SL}(2,\mathbb{R}))$ of the conformal Killing group of the disk instructs to mod out by the redundancy of Möbius transformation $z \to \frac{az+b}{cz+d}$ (with $ad-bc=1$). This amounts to fixing three positions such as $(z_1, z_{n-1}, z_n) = (0, 1, \infty)$ and to inserting a compensating Jacobian:

$$\int_{D(12\ldots n)} \frac{\mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_n}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{R}))} = z_{1,n-1} z_{1,n} z_{n-1,n} \int_{z_1 \le z_2 \le z_3 \le \ldots \le z_{n-2} \le z_{n-1}} \mathrm{d}z_2\, \mathrm{d}z_3\, \ldots\, \mathrm{d}z_{n-2}\,. \tag{2.5}$$

Given that the words P and Q in the disk integrals (2.2) encode the integration region $D(P)$ in (2.3) and the integrand $C(Q)$ in (2.1), respectively, there is in general no relation between $Z(P|Q)$ and $Z(Q|P)$. This can already be seen from the different symmetries w.r.t. variable P at fixed Q on the one hand and variable Q at fixed P on the other hand.

2.1.1 Symmetries of disk integrals in the integrand

The manifest cyclic symmetry and reflection (anti-)symmetry of the integrand $C(Q)$ in (2.1) directly propagates to the disk integrals

$$Z(P|q_2 q_3 \ldots q_n q_1) = Z(P|q_1 q_2 \ldots q_n)\,, \qquad Z(P|\tilde{Q}) = (-1)^{|Q|} Z(P|Q)\,, \tag{2.6}$$

where $|Q| = n$ denotes the length of the word $Q = q_1 q_2 \ldots q_n$, and the tilde in $\tilde{Q} = q_n \ldots q_2 q_1$ is a shorthand for its reversal. Moreover, the disk integrals satisfy [5] the Kleiss-Kuijf relations [30],

$$Z(P|A,1,B,n) = (-1)^{|A|} Z(P|1,\tilde{A} \sqcup\!\sqcup B, n)\,, \tag{2.7}$$

or equivalently [31, 32], the vanishing of pure shuffles in $n-1$ legs,

$$Z(P|A \sqcup\!\sqcup B, n) = 0 \quad \forall\ A, B \neq \emptyset\ . \tag{2.8}$$

The shuffle operation in (2.7) and (2.8) is defined recursively via [33]

$$\emptyset \sqcup\!\sqcup A = A \sqcup\!\sqcup \emptyset = A, \qquad A \sqcup\!\sqcup B \equiv a_1(a_2 \ldots a_{|A|} \sqcup\!\sqcup B) + b_1(b_2 \ldots b_{|B|} \sqcup\!\sqcup A)\,, \tag{2.9}$$

and it acts linearly on the parental objects, e.g. $Z(123|1(2 \sqcup\!\sqcup 3)) = Z(123|123) + Z(123|132)$. Finally, integration by parts yields the same BCJ relations among permutations of $Z(P|Q)$ in Q as known from [13] for color-stripped SYM tree amplitudes [5]

$$0 = \sum_{j=2}^{n-1} k_{q_1} \cdot k_{q_2 q_3 \ldots q_j} Z(P|q_2 q_3 \ldots q_j q_1 q_{j+1} \ldots q_n)\ . \tag{2.10}$$

Note that neither (2.7) nor (2.10) depends on the domain P, and they allow to expand any $Z(P|Q)$ in an $(n-3)!$-element basis $\{Z(P|Q_i),\ i = 1, 2, \ldots, (n-3)!\}$ at fixed P [13]. The symmetries (2.6), (2.7) and (2.10) known from SYM interactions crucially support the interpretation of $Z(P|Q)$ as doubly partial amplitudes [7].

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2.1.2 Symmetries of disk integrals in the domain

As a consequence of the form of the integration region $D(P)$ in (2.3), disk integrals obey a cyclicity and parity property in the domain $P = p_1 p_2 \dots p_n$,

$$Z(p_2 p_3 \dots p_n p_1|Q) = Z(p_1 p_2 \dots p_n|Q)\,, \qquad Z(\tilde{P}|Q) = (-1)^{|P|} Z(P|Q)\,, \tag{2.11}$$

which tie in with the simplest symmetries (2.6) of the integrand Q. However, the Kleiss-Kuijf symmetry (2.7) and BCJ relations (2.10) of the integrand do not hold for the integration domain P in presence of α'-corrections. This can be seen from the real and imaginary part of the monodromy relations [34, 35] (see [36] for a recent generalization to loop level)

$$0 = \sum_{j=2}^{n-1} \exp\big[i\pi\alpha' k_{p_1} \cdot k_{p_2 p_3 \dots p_j}\big] Z(p_2 p_3 \dots p_j p_1 p_{j+1} \dots p_n|Q)\ . \tag{2.12}$$

Nevertheless, (2.12) is sufficient to expand any $Z(P|Q)$ in an $(n{-}3)!$-element basis $\{Z(P_i|Q),\ i = 1, 2, \dots, (n{-}3)!\}$ at fixed Q [34, 35].

2.2 Open superstring disk amplitudes

The n-point tree-level amplitude $A^{\rm open}$ of the open superstring takes a particularly simple form once the contributing disk integrals are cast into an $(n{-}3)!$ basis via partial fraction (2.8) and integration by parts (2.10) [11, 37]:

$$A^{\rm open}(1, P, n{-}1, n) = \sum_{Q \in S_{n-3}} F_P{}^Q A^{\rm SYM}(1, Q, n{-}1, n) \tag{2.13}$$

While all the polarization dependence on the right hand side has been expressed through the BCJ basis [13] of SYM trees $A^{\rm SYM}$, the entire reference to α' stems from the integrals

$$\begin{aligned} F_P{}^Q \equiv (-\alpha')^{n-3} \int_{0 \le z_{p_2} \le z_{p_3} \le \dots \le z_{p_{n-2}} \le 1} \mathrm{d}z_2\, \mathrm{d}z_3\, \dots\, \mathrm{d}z_{n-2} \prod_{i<j}^{n-1} |z_{ij}|^{\alpha' s_{ij}}\, \frac{s_{1q_2}}{z_{1q_2}} \left(\frac{s_{1q_3}}{z_{1q_3}} + \frac{s_{q_2q_3}}{z_{q_2q_3}} \right) \\ \times \left(\frac{s_{1q_4}}{z_{1q_4}} + \frac{s_{q_2q_4}}{z_{q_2q_4}} + \frac{s_{q_3q_4}}{z_{q_3q_4}} \right) \dots \left(\frac{s_{1q_{n-2}}}{z_{1q_{n-2}}} + \frac{s_{q_2q_{n-2}}}{z_{q_2q_{n-2}}} + \dots + \frac{s_{q_{n-3}q_{n-2}}}{z_{q_{n-3}q_{n-2}}} \right)\,, \end{aligned} \tag{2.14}$$

where $P = p_2 p_3 \dots p_{n-2}$ and $Q = q_2 q_3 \dots q_{n-2}$ are permutations of $23 \dots n{-}2$. The original derivation [11, 37] of (2.13) and (2.14) has been performed in the manifestly supersymmetric pure spinor formalism [38], where the SYM amplitudes $A^{\rm SYM}$ in (2.13) have been identified from their Berends-Giele representation in pure spinor superspace [39]. Hence, (2.13) applies to the entire ten-dimensional gauge multiplet in the external states.[4]

[4] A bosonic-component check of the formula (2.13) at multiplicity $n \le 7$ within the RNS formalism has been performed in [40].

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2.2.1 *Z*-theory

After undoing the SL(2, $\mathbb{R}$)-fixing in (2.5), the integrals ${F_P}^Q$ can be identified as a linear combination of disk integrals (2.2) [5],

$$ {F_P}^Q = \sum_{R\in S_{n-3}} S[Q|R]_1 Z(P|1,R,n,n{-}1)\,, \tag{2.15} $$

where P, Q and R are understood to be permutations of $2, 3, \ldots, n{-}2$. The symmetric $(n{-}3)!\times(n{-}3)!$ matrix $S[Q|R]_1$ encodes the field-theory KLT relations [41, 42] (see also [43] for the α'-corrections to $S[Q|R]_1$) and admits the following recursive representation [7],

$$ S[A,j|B,j,C]_i = (k_{iB}\cdot k_j) S[A|B,C]_i, \qquad S[\varnothing|\varnothing]_i \equiv 1\,, \tag{2.16} $$

in terms of multiparticle momenta (2.4). Hence, the n-point open-superstring amplitude (2.13) with any domain P can be obtained from the KLT formula,

$$ A^{\rm open}(P) = \sum_{Q,R\in S_{n-3}} Z(P|1,R,n,n{-}1) S[R|Q]_1 A^{\rm SYM}(1,Q,n{-}1,n) \tag{2.17} $$

upon replacing the right-moving SYM trees via $\tilde{A}^{\rm SYM}(1,R,n,n{-}1) \to Z(P|1,R,n,n{-}1)$ [5]. The KLT form of (2.17) reveals the double-copy structure of the open-superstring tree-level S-matrix which in turn motivated the proposal of [7] to interpret disk integrals as doubly partial amplitudes. The specification of disk integrals by two cycles P, Q identifies the underlying particles to be bi-colored scalars, and we collectively refer to their effective interactions that give rise to tree amplitudes $Z(P|Q)$ as Z-theory.

Note that disk amplitudes of the bosonic string are conjectured in [44] to also admit the form (2.13) or (2.17), with α'-dependent kinematic factors $A^{\rm SYM}(1,Q,n{-}1,n) \to B(1,Q,n{-}1,n;\alpha')$ that also satisfy the KK- and BCJ relations.

2.2.2 α'-expansion of disk amplitudes

The α'-expansion of disk amplitudes (2.13), i.e. their Taylor expansion in the dimensionless Mandelstam invariants $\alpha' s_{ij}$, involves multiple zeta values (MZVs),

$$ \zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2}\ldots k_r^{-n_r}\,, \qquad n_r \geq 2\,. \tag{2.18} $$

The MZV in (2.18) is said to have depth r and weight $w = n_1 + n_2 + \ldots + n_r$ (which is understood to be additive in products of MZVs). While the four-point instance of (2.14),

$$ \begin{aligned} {F_2}^2 &= \exp\left(\sum_{n=2}^{\infty} \frac{\zeta_n}{n}(-\alpha')^n \left[s_{12}^n + s_{23}^n - (s_{12}+s_{23})^n\right]\right) \\ &= 1 - \alpha'^2\zeta_2 s_{12}s_{23} + \alpha'^3\zeta_3 s_{12}s_{23}(s_{12}+s_{23}) \\ &\quad -\alpha'^4\zeta_4 s_{12}s_{23}\left(s_{12}^2 + \frac{s_{12}s_{23}}{4} + s_{23}^2\right) + \mathcal{O}(\alpha'^5)\,, \end{aligned} \tag{2.19} $$

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boils down to a single entry with Riemann zeta values ζ_n of depth $r = 1$ only, disk integrals at multiplicity $n \geq 5$ generally involve MZVs of higher depth $r \geq 2$, see [45] for a recent closed-form solution at five points. It has been discussed in the literature of both physics [37, 46, 47] and mathematics [48–50] that the disk integrals (2.2) at any multiplicity exhibit uniform transcendentality: their α'^w-order is exclusively accompanied by products of MZVs with total weight w.

The basis of functions ${F_P}^Q$ in (2.15) is particularly convenient to directly determine the α'-expansion of the open-string amplitudes (2.13) [6] and to describe their pattern of MZVs[5] [47, 55]. At multiplicities five, six and seven, explicit results for the leading orders in the α'-expansion of ${F_P}^Q$ are available for download on [56].

2.2.3 Basis-expansion of disk integrals

In setting up the Berends-Giele recursion for the fundamental objects $Z(A|B)$ of this work, it is instrumental to efficiently extract their α'-expansion from the basis functions ${F_P}^Q$. However, solving the mediating BCJ and monodromy relations can be very cumbersome, and the explicit basis expansions spelled out in [5] only address an $(n-2)!$ subset of integrands B. These shortcomings are surpassed by the following formula,

$$Z(1, P, n-1, n|R) = \sum_{Q \in S_{n-3}} {F_P}^Q m(1, Q, n-1, n|R)\,, \tag{2.20}$$

where $m(A|B)$ denote the doubly partial amplitudes of biadjoint ϕ^3-theory which arise in the field-theory limit of disk integrals [57]

$$m(A|B) = \lim_{\alpha' \to 0} Z(A|B)\ . \tag{2.21}$$

Note the striking resemblance of the formulas (2.20) and (2.13), which further point out the similar roles played by the amplitudes $A^{\text{open}}(P)$ and $Z(P|Q)$ of string and Z-theory.

2.3 Berends-Giele recursion for the field-theory limit

The task we want to accomplish in this paper concerns the computation of the α'-expansion of the disk integrals (2.2) in a recursive and efficient manner. In the field-theory limit $\alpha' \to 0$, all-multiplicity techniques have been developed in [37], and a relation to the inverse KLT matrix (2.16) has been found in [5]. The equivalent description of the $\alpha' \to 0$ limit in terms of doubly partial amplitudes (2.21) [57] has inspired a recent Berends-Giele description [4] via bi-adjoint scalars $\Phi^{(0)} \equiv \Phi^{(0)}_{a|b} t^a \otimes \tilde{t}^b$. The latter take values in the tensor product of two gauge groups with generators t^a and $\tilde{t}^b$ as well as structure constants f^{acd} and $\tilde{f}^{bgh}$, respectively.

The superscript of the biadjoint scalar $\Phi^{(0)}$ indicates that this is the $\alpha' \to 0$ limit of the Z-theory particles Φ whose interactions give rise to the disk integrals $Z(P|Q)$ as their doubly partial amplitudes. The non-linear field equations in the low-energy limit

$$\Box \Phi^{(0)}_{a|b} = f_{acd} \tilde{f}_{bgh} \Phi^{(0)}_{c|g} \Phi^{(0)}_{d|h} \tag{2.22}$$

[5] After pioneering work in [51–54], the α'-expansion of disk integrals at multiplicity $n \geq 5$ has later been systematically addressed via all-multiplicity techniques based on polylogarithms [5] and the Drinfeld associator [6] (see also [55]).

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with d'Alembertian $\Box \equiv \partial^2$ will later be completed such as to incorporate the α'-corrections in $Z(P|Q)$. One can solve (2.22) through a perturbiner [8] expansion[6],

$$\begin{aligned}\Phi^{(0)} &= \sum_{a_1,b_1} \phi^{(0)}_{a_1|b_1}\, e^{k_{a_1}x}\, t^{a_1} \otimes \tilde{t}^{b_1} + \sum_{a_1,a_2,b_1,b_2} \phi^{(0)}_{a_1a_2|b_1b_2}\, e^{k_{a_1a_2}x}\, t^{a_1}t^{a_2} \otimes \tilde{t}^{b_1}\tilde{t}^{b_2} \\ &\quad + \sum_{a_1,a_2,a_3,b_1,b_2,b_3} \phi^{(0)}_{a_1a_2a_3|b_1b_2b_3}\, e^{k_{a_1a_2a_3}x}\, t^{a_1}t^{a_2}t^{a_3} \otimes \tilde{t}^{b_1}\tilde{t}^{b_2}t^{b_3} + \cdots \\ &= \sum_{A,B} \phi^{(0)}_{A|B}\, e^{k_Ax}\, t^A \otimes \tilde{t}^B\,, \end{aligned} \tag{2.23}$$

which resums tree-level subdiagrams and is compactly written as a sum over all words A, B with length $|A|, |B| \geq 1$ in the last line. We are using the collective notation

$$t^A \equiv t^{a_1}t^{a_2}\ldots t^{a_{|A|}}\,, \qquad \tilde{t}^B \equiv t^{b_1}t^{b_2}\ldots t^{b_{|B|}} \tag{2.24}$$

for products of Lie-algebra generators associated with multiparticle label $A = a_1a_2\ldots a_{|A|}$ and $B = b_1b_2\ldots b_{|B|}$. The coefficients in (2.23) are recursively determined by the non-linear field equations (2.22) [4],

$$s_A\phi^{(0)}_{A|B} = \sum_{\substack{A_1A_2=A\\ B_1B_2=B}} \big(\phi^{(0)}_{A_1|B_1}\phi^{(0)}_{A_2|B_2} - \phi^{(0)}_{A_1|B_2}\phi^{(0)}_{A_2|B_1}\big)\,, \tag{2.25}$$

and referred to as Berends-Giele double currents $\phi^{(0)}_{A|B}$. The notation $\sum_{A_1A_2=A}$ and $\sum_{B_1B_2=B}$ instructs to sum over deconcatenations $A = a_1a_2\ldots a_{|A|}$ into non-empty words $A_1 = a_1a_2\ldots a_j$ and $A_2 = a_{j+1}\ldots a_{|A|}$ with $j = 1, 2, \ldots, |A|-1$ and to independently deconcatenate B in the same manner. The initial conditions for the recursion in (2.25),

$$\phi^{(0)}_{i|j} = \delta_{i,j}\,, \tag{2.26}$$

guarantee that $\phi^{(0)}_{A|B}$ vanishes unless A is a permutation of B and yield expressions such as

$$\phi^{(0)}_{12|12} = -\phi^{(0)}_{12|21} = \frac{1}{s_{12}}\,, \quad \phi^{(0)}_{123|123} = \frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}}\,, \quad \phi^{(0)}_{123|312} = -\frac{1}{s_{12}s_{123}} \tag{2.27}$$

at the two- and three-particle level.

As shown in [4], the field-theory limits of the disk integrals (2.2) and thereby the doubly partial amplitudes (2.21) are given by the Berends-Giele double currents $\phi^{(0)}_{A|B}$,

$$m(A,n|B,n) = s_A\phi^{(0)}_{A|B}\,. \tag{2.28}$$

Given the cyclic symmetry (2.6) of $Z(P|Q)$ in the word Q, one can always choose the last letter of the integrand $Q \equiv (B,n)$ to coincide with the last letter of the integration region $P \equiv (A,n)$ as has been done in (2.28). The recursive definition of $\phi^{(0)}_{A|B}$ in (2.25) gives rise to an efficient algorithm to obtain the field-theory limit of disk integrals $Z(A,n|B,n)$ directly from the two words A, B encoding the integrand and integration domain, respectively.

Furthermore, the BG double currents allow the inverse of the KLT matrix (2.16) to be obtained without any matrix algebra [4],

$$S^{-1}[P|Q]_1 = \phi^{(0)}_{1P|1Q}\,. \tag{2.29}$$

[6] See [8, 58, 59] for perturbiner solutions to self-dual sectors of four-dimensional gauge and gravity theories (see also [10]) and [9] for perturbiners in ten-dimensional SYM.,FTlimit

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2.3.1 Example application of the Berends-Giele recursion

The computation of the field-theory limit of the five-point disk integral

$$m(13524|32451) = \lim_{\alpha'\to 0} \alpha'^2 \int_{D(13524)} \frac{\mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_5}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{R}))} \prod_{i<j}^{5} |z_{ij}|^{\alpha' s_{ij}} \frac{1}{z_{32}z_{24}z_{45}z_{51}z_{13}} \tag{2.30}$$

using the Berends-Giele formula (2.28) proceeds as follows. First, one exploits the cyclic symmetry of the integrand to rotate its labels until the last leg matches the last label of the integration region. After applying (2.28) one obtains,

$$m(13524|32451) = m(13524|51324) = s_{1352}\phi^{(0)}_{1352|5132} = \phi^{(0)}_{135|513}\phi^{(0)}_{2|2}\,. \tag{2.31}$$

Terms such as $\phi^{(0)}_{1|5}\phi^{(0)}_{352|132}$ following from the deconcatenation (2.25) have been dropped from the last equality because the condition (2.26) implies that $\phi^{(0)}_{1|5} = 0$. In addition, the overall factor s_{1352} from (2.28) cancels the propagator $1/s_{1352}$ in the current $\phi^{(0)}_{1352|5132}$. Recursing the above steps until no factor of $\phi^{(0)}_{A|B}$ remains yields,

$$m(13524|32451) = \phi^{(0)}_{135|513} = \frac{1}{s_{135}}\Big(-\phi^{(0)}_{13|13}\phi^{(0)}_{5|5}\Big) = -\frac{1}{s_{135}}\phi^{(0)}_{13|13} = -\frac{1}{s_{135}s_{13}}\,, \tag{2.32}$$

in agreement with the expression for the doubly partial amplitude $m(13524|32451)$ that follows from the methods of [57]. In the next section this method will be extended to compute the α'-corrections of string disk integrals.

3 Berends-Giele recursion for disk integrals

In this section, we develop a Berends-Giele recursion[7] for the full-fledged disk integrals $Z(P|Q)$ defined in (2.2). The idea is to construct α'-dependent Berends-Giele double currents $\phi_{A|B}$ such that the integrals $Z(P|Q)$ including α'-corrections are obtained in the same manner as their field-theory limit in (2.28),

$$Z(A,n|B,n) = s_A\phi_{A|B}\,. \tag{3.1}$$

And similarly, the α'-corrected BG double currents $\phi_{A|B}$ in (3.1) will be given by the coefficients of a perturbiner expansion analogous to (2.23),

$$\Phi = \sum_{A,B} \phi_{A|B} e^{k_A\cdot x}\, t^A \otimes \tilde{t}^B\,, \tag{3.2}$$

that solves non-linear equations of motions which can be viewed as an augmentation of (2.22) by α'-corrections. The field equation obeyed by the perturbiner (3.2) will be interpreted as the equation of motion of Z-theory, the collection of effective theories involving

[7] For a review of the Berends-Giele recursion for gluon amplitudes [3] which is adapted to the current discussion, see section 2 of [4].

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bi-colored scalars encoding all the α'-corrections relevant to the open superstring [7]. In addition, the BG double currents above are subject to the initial and vanishing condition

$$\phi_{i|j} = \delta_{i,j}\,, \qquad \phi_{A|B} = 0\,, \qquad \text{unless } A \text{ is a permutation of } B. \tag{3.3}$$

Given their role in equation (3.1), the words A and B on the BG double current $\phi_{A|B}$ will be referred to as the *integration domain* A and the *integrand* B, respectively.

3.1 Symmetries of the full Berends-Giele double currents

In the representation (3.1) of the disk integrals, their parity symmetries (2.7) and (2.11) can be manifested if the double currents $\phi_{A|B}$ satisfy

$$\phi_{A|B} = (-1)^{|A|-1}\phi_{\tilde{A}|B} = (-1)^{|B|-1}\phi_{A|\tilde{B}}\,, \tag{3.4}$$

upon reversal of either the integration domain A or the integrand B. Similarly, the Kleiss-Kuijf relations (2.8) of the disk integrals follow from the shuffle symmetry[8] of $\phi_{A|B}$ within the integrand B,

$$\phi_{A|P\sqcup Q} = 0 \quad \forall\ P, Q \neq \emptyset\,. \tag{3.5}$$

Note that $\phi_{A|B}$ does *not* exhibit shuffle symmetries in the integration domain A: the α'-correction in the monodromy relations [34, 35], more specifically in the real part of (2.12), yields non-zero expressions[9] $\mathcal{O}((\alpha'\pi)^2)$ for $\phi_{P\sqcup Q|B}$. As a consequence, the perturbiner (3.2) is Lie-algebra valued w.r.t. the $\tilde{t}^b$ generators [31] but not w.r.t. the t^a generators. That is why the Z-theory scalar Φ is referred to as *bi-colored* rather than *biadjoint*.

The symmetries (3.4) and (3.5) will play a fundamental role in the construction of ansaetze for the α'-corrections of the Berends-Giele double currents, see appendices A and B for further details.

3.2 The α'^2-correction to Berends-Giele currents of disk integrals

Assuming that the α'^2-corrections of the integrals (2.2) can be described by Berends-Giele double currents as in (3.1), dimensional analysis admits two types of terms at this order. They have the schematic form $k^2\phi^3$ and ϕ^4 since ϕ has dimension of k^2, and the α'^2-terms contain a factor of k^4 compared to the leading contribution from ϕ^2 in (2.25). Therefore, an ansatz for $s_A\phi_{A|B}$ at this order must be based on a linear combination of

$$\sum_{\substack{A_1A_2A_3=A\\B_1B_2B_3=B}} (k_{A_i}\cdot k_{A_j})\phi_{A_1|B_k}\phi_{A_2|B_l}\phi_{A_3|B_m}\,, \qquad \sum_{\substack{A_1\ldots A_4=A\\B_1\ldots B_4=B}} \phi_{A_1|B_p}\phi_{A_2|B_q}\phi_{A_3|B_r}\phi_{A_4|B_s}\,, \tag{3.6}$$

[8] In the mathematics literature, objects T_B satisfying the symmetry $T_{P\sqcup Q} = 0$ for any $P, Q \neq \emptyset$ are known as "alternal moulds", see e.g. [60].

[9] Since the monodromy relations only differ from the KK relations by rational multiples of π^{2n} or $(\zeta_2)^n$, the sub-sector of $Z(A, n|B, n)$ without any factors of ζ_2 still satisfies shuffle symmetries, e.g. $\phi_{P\sqcup Q|B}\big|_{\zeta_{2n+1}} = 0$, also see [61] for analogous statements for the heterotic string and section 5 for implications for a Berends-Giele approach to closed-string integrals.

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see the explanation below (2.25) for the deconcatenations $A = A_1A_2A_3$ and $A = A_1 \dots A_4$ into non-empty words. By the initial condition (3.3), $\phi_{A|B}$ vanishes unless A is a permutation of B, so there is no need to consider momentum dependence of the form $(k_{A_i} \cdot k_{B_j})$ or $(k_{B_i} \cdot k_{B_j})$.

The most general linear combination of the terms (3.6) contains $36 + 24 = 60$ parameters. Imposing the symmetries (3.4) and (3.5) reduces them to $6 + 4 = 10$ parameters, see appendix A for the implementation of the shuffle symmetry. Then, matching the outcomes of (3.1) with the known α'^2-order of various integrals at four and five points fixes six parameters, leaving a total of four free parameters. The α'^2-order of $(n \geq 6)$-point integrals does not provide any further input: as we have checked with all the known $(n \leq 9)$-point data [56], they are automatically reproduced for any choice of the four free parameters. This is where the predictive power of the Berends-Giele setup kicks in: a finite amount of low-multiplicity data — the coefficients of $k^2\phi^3$- and ϕ^4-terms (3.6) at the α'^2-order — determines the relevant order of disk integrals at *any* multiplicity.

3.2.1 Free parameters versus Z-theory equation of motion

It is not surprising that the ansatz based on (3.6) is not completely fixed (yet) by matching the data. The reason for this can be seen from the interpretation of the Berends-Giele recursion method as the perturbiner solution (2.23) to the Z-theory equation of motion with the schematic form $\Box\Phi = \Phi^2 + \mathcal{O}(\Phi^3)$. Self-contractions $(k_{A_i} \cdot k_{A_i})$ signal the appearance of $\Box\Phi = \Phi^2 + \alpha'^2\zeta_2\Phi^2\Box\Phi + \dots$ on the right hand side, where $\Box\Phi$ along with $\alpha'^2\zeta_2\Phi^2$ can be replaced by the entire right hand side. The result $\Box\Phi = \Phi^2 + \alpha'^2\zeta_2\Phi^2(\Phi^2 + \alpha'^2\zeta_2\Phi^2\Box\Phi) + \dots$ in turn leads to another appearance of $\Box\Phi$ at higher orders in α' and the fields. In order to obstruct an infinite iteration of the field equations, we fix three additional parameters by demanding absence of $(k_{A_i} \cdot k_{A_i})$ with $i = 1, 2, 3$ and thereby leave one free.

The last free parameter reflects the freedom to perform field redefinitions. Terms of the form $\alpha'^2\zeta_2\Box(\Phi^3)$ on the right hand side of $\Box\Phi$ can be absorbed via $\Phi' \equiv \Phi - \alpha'^2\zeta_2\Phi^3$, i.e. the right-hand side of $\Box\Phi'$ will no longer contain the term $\alpha'^2\zeta_2\Box(\Phi^3)$ in question. This leftover freedom can be fixed by requiring the absence of the dot product $(k_{A_1} \cdot k_{A_3})$ among the leftmost and the rightmost slot-momentum[10] in the deconcatenation $A = A_1A_2A_3$ in (3.6). Like this, ambiguities to shift $\Box\Phi$ by a total d'Alembertian $\Box(\dots)$ are systematically avoided while preserving the manifest parity property (3.4) in A.

At the end of the above process, one finds the unique recursion that generates the α'^2 terms in the low-energy expansion of disk integrals at any multiplicity via (3.1):

$$
\begin{aligned}
s_A\phi_{A|B} = & \sum_{\substack{A_1A_2=A\\B_1B_2=B}} (\phi_{A_1|B_1}\phi_{A_2|B_2} - \phi_{A_1|B_2}\phi_{A_2|B_1}) \\
& + \alpha'^2\zeta_2 \sum_{\substack{A_1\dots A_3=A\\B_1\dots B_3=B}} \big[(k_{A_1} \cdot k_{A_2})\big(\phi_{A_1|B_1}\phi_{A_2|B_3}\phi_{A_3|B_2} - \phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3} \\
& \qquad + \phi_{A_1|B_3}\phi_{A_2|B_1}\phi_{A_3|B_2} - \phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_1}\big)
\end{aligned}
\tag{3.7}
$$

[10] In general, in a p-fold deconcatenation $\sum_{A=A_1\dots A_p}\sum_{B=B_1\dots B_p}$, the dot product $(k_{A_1} \cdot k_{A_p})$ among the leftmost and the rightmost momentum will not be included into an ansatz for $s_A\phi_{A|B}$ at given order in α'. This freezes the freedom to perform field redefinitions while preserving the manifest parity property (3.4) in A.

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$$
\begin{aligned}
&+(k_{A_2}\cdot k_{A_3})\big(\phi_{A_1|B_2}\phi_{A_2|B_1}\phi_{A_3|B_3}-\phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3}\\
&+\phi_{A_1|B_2}\phi_{A_2|B_3}\phi_{A_3|B_1}-\phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_1}\big)\big]\\
+\alpha'^2\zeta_2\sum_{\substack{A_1\ldots A_4=A\\ B_1\ldots B_4=B}}&\big[\phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_4}\phi_{A_4|B_3}-\phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3}\phi_{A_4|B_4}\\
&+\phi_{A_1|B_1}\phi_{A_2|B_3}\phi_{A_3|B_2}\phi_{A_4|B_4}-\phi_{A_1|B_1}\phi_{A_2|B_4}\phi_{A_3|B_2}\phi_{A_4|B_3}\\
&+\phi_{A_1|B_2}\phi_{A_2|B_1}\phi_{A_3|B_3}\phi_{A_4|B_4}-\phi_{A_1|B_2}\phi_{A_2|B_1}\phi_{A_3|B_4}\phi_{A_4|B_3}\\
&-\phi_{A_1|B_2}\phi_{A_2|B_3}\phi_{A_3|B_1}\phi_{A_4|B_4}+\phi_{A_1|B_2}\phi_{A_2|B_4}\phi_{A_3|B_1}\phi_{A_4|B_3}\\
&-\phi_{A_1|B_3}\phi_{A_2|B_1}\phi_{A_3|B_4}\phi_{A_4|B_2}+\phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_4}\phi_{A_4|B_1}\\
&+\phi_{A_1|B_3}\phi_{A_2|B_4}\phi_{A_3|B_1}\phi_{A_4|B_2}-\phi_{A_1|B_3}\phi_{A_2|B_4}\phi_{A_3|B_2}\phi_{A_4|B_1}\\
&+\phi_{A_1|B_4}\phi_{A_2|B_1}\phi_{A_3|B_3}\phi_{A_4|B_2}-\phi_{A_1|B_4}\phi_{A_2|B_2}\phi_{A_3|B_3}\phi_{A_4|B_1}\\
&-\phi_{A_1|B_4}\phi_{A_2|B_3}\phi_{A_3|B_1}\phi_{A_4|B_2}+\phi_{A_1|B_4}\phi_{A_2|B_3}\phi_{A_3|B_2}\phi_{A_4|B_1}\big]\\
+\mathcal{O}(\alpha'^3)\ .&
\end{aligned}
$$

For example, applying the above recursion to the disk integral $Z(13524|32451)$ whose field-theory limit was computed in (2.32) leads to the following result up to α'^2:

$$
Z(13524|32451)=-\frac{1}{s_{13}s_{135}}+\alpha'^2\zeta_2\left(\frac{s_{35}}{s_{135}}+\frac{s_{25}}{s_{13}}-1\right)+\mathcal{O}(\alpha'^3)\,. \tag{3.8}
$$

It is important to emphasize that, while only four- and five-point data entered in the derivation of (3.7), this recursion allows the computation of α'^2 terms of disk integrals at arbitrary multiplicity. The eleven-point example

$$
Z(134582679ba|123456789ab)=\frac{-\alpha'^2\zeta_2}{s_{19ab}s_{ab}s_{345}s_{67}}\left(\frac{1}{s_{34}}+\frac{1}{s_{45}}\right)\left(\frac{1}{s_{1ab}}+\frac{1}{s_{9ab}}\right)+\mathcal{O}(\alpha'^3) \tag{3.9}
$$

with the shorthands $a=10$ and $b=11$ was computed within two seconds on a regular laptop with the program available in [29].

3.2.2 Manifesting the shuffle symmetries of BG currents

The length of the recursion in (3.7) at the $\alpha'^2\zeta_2$ order calls for a more efficient representation. In this subsection, we identify the sums of products of $\phi_{A_i|B_j}$ which satisfy the shuffle symmetries (3.5) in the B_j-slots. This allows to rewrite the recursion (3.7) in a compact form which inspires the generalization to higher orders and clarifies the commutator structure in the Z-theory equation of motion upon rewriting the results in the language of perturbiners (3.2).

In order to do this, recall from the theory of free Lie algebras that all shuffle products are annihilated by a linear map ρ acting on words $(B_1,B_2,\ldots,B_n)$ of n letters B_i which is defined by $\rho(B_i)\equiv B_i$ and [31]

$$
\rho(B_1,B_2,\ldots,B_n)\equiv\rho(B_1,B_2,\ldots,B_{n-1}),B_n-\rho(B_2,B_3,\ldots,B_n),B_1\,. \tag{3.10}
$$

For example, it is easy to see that $\rho(B_1,B_2)=(B_1,B_2)-(B_2,B_1)$ and

$$
\rho(B_1,B_2,B_3)=(B_1,B_2,B_3)-(B_2,B_1,B_3)-(B_2,B_3,B_1)+(B_3,B_2,B_1) \tag{3.11}
$$

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imply the vanishing of $\rho(B_1 \sqcup B_2)$ and $\rho((B_1, B_2) \sqcup B_3)$. Therefore, after defining

$$T^{\text{dom}}_{A_1,A_2,\ldots,A_n} \otimes T^{\text{int}}_{B_1,B_2,\ldots,B_n} \equiv \phi_{A_1|B_1}\phi_{A_2|B_2}\cdots\phi_{A_n|B_n} \ . \tag{3.12}$$

it is straightforward to check that the following linear combinations

$$\begin{aligned} T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n} &\equiv T^{\text{dom}}_{A_1,A_2,\ldots,A_n} \otimes T^{\text{int}}_{\rho(B_1,B_2,\ldots,B_n)} \\ &= T^{B_1,B_2,\ldots,B_{n-1}}_{A_1,A_2,\ldots,A_{n-1}}\,\phi_{A_n|B_n} - T^{B_2,B_3,\ldots,B_n}_{A_1,A_2,\ldots A_{n-1}}\,\phi_{A_n|B_1}\,, \end{aligned} \tag{3.13}$$

with $T^B_A \equiv \phi_{A|B}$ satisfy the shuffle symmetries on the B_j-slots [31],[11]

$$T^{(B_1,B_2,\ldots,B_i)\sqcup(B_{i+1},\ldots,B_n)}_{A_1,A_2,\ldots,A_n} = 0\,, \qquad i = 1,2,\ldots,n-1\,. \tag{3.14}$$

The first few examples of (3.13) read as follows,

$$\begin{aligned} T^{B_1,B_2}_{A_1,A_2} &\equiv \phi_{A_1|B_1}\phi_{A_2|B_2} - \phi_{A_1|B_2}\phi_{A_2|B_1}\,, \\ T^{B_1,B_2,B_3}_{A_1,A_2,A_3} &\equiv \phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3} - \phi_{A_1|B_2}\phi_{A_2|B_3}\phi_{A_3|B_1} \\ &\quad -\phi_{A_1|B_2}\phi_{A_2|B_1}\phi_{A_3|B_3} + \phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_1}\,, \\ T^{B_1,B_2,B_3,B_4}_{A_1,A_2,A_3,A_4} &\equiv \phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3}\phi_{A_4|B_4} - \phi_{A_1|B_2}\phi_{A_2|B_1}\phi_{A_3|B_3}\phi_{A_4|B_4} \\ &\quad -\phi_{A_1|B_2}\phi_{A_2|B_3}\phi_{A_3|B_1}\phi_{A_4|B_4} + \phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_1}\phi_{A_4|B_4} \\ &\quad -\phi_{A_1|B_2}\phi_{A_2|B_3}\phi_{A_3|B_4}\phi_{A_4|B_1} + \phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_4}\phi_{A_4|B_1} \\ &\quad +\phi_{A_1|B_3}\phi_{A_2|B_4}\phi_{A_3|B_2}\phi_{A_4|B_1} - \phi_{A_1|B_4}\phi_{A_2|B_3}\phi_{A_3|B_2}\phi_{A_4|B_1}\,, \end{aligned} \tag{3.15}$$

and their shuffle symmetries (3.14) are easy to verify, starting with

$$T^{B_1,B_2}_{A_1,A_2} = -T^{B_2,B_1}_{A_1,A_2}\,, \quad T^{B_1,B_2,B_3}_{A_1,A_2,A_3} + T^{B_1,B_3,B_2}_{A_1,A_2,A_3} + T^{B_3,B_1,B_2}_{A_1,A_2,A_3} = 0\,. \tag{3.16}$$

Moreover, the ρ-map in (3.10) exhausts all tensors of the type (3.12) subject to shuffle symmetry in the B_j-slots it acts on [31, 62]. Hence, a BG recursion which manifests the shuffle symmetry in the B_j-slots is necessarily expressible in terms of $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ in (3.13).

Rather surprisingly, it turns out that the definition (3.13) not only manifests the shuffle symmetries on the B_j-slots but also implies *generalized Jacobi identities* with respect to the A_j-slots. In other words, the above $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ satisfy the same symmetries as the nested commutator $[[\ldots[[A_1, A_2], A_3]\ldots], A_n]$, see appendix A.2 for a proof.

3.2.3 Simplifying the α'^2-correction to BG currents

As discussed in the previous subsection, the BG double current can always be written in terms of $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ from the definition (3.13). For example, the expression (3.7) becomes

$$\begin{aligned} s_A\phi_{A|B} = &\sum_{\substack{A=A_1A_2\\B=B_1B_2}} T^{B_1,B_2}_{A_1,A_2} - \alpha'^2\zeta_2 \sum_{\substack{A=A_1\ldots A_3\\B=B_1\ldots B_3}} \Big[(k_{A_2}\cdot k_{A_3})T^{B_1,B_2,B_3}_{A_1,A_2,A_3} + (k_{A_1}\cdot k_{A_2})T^{B_1,B_2,B_3}_{A_3,A_2,A_1}\Big] \\ &+\alpha'^2\zeta_2 \sum_{\substack{A=A_1\ldots A_4\\B=B_1\ldots B_4}} \Big(T^{B_1,B_2,B_3,B_4}_{A_1,A_2,A_4,A_3} - T^{B_1,B_2,B_3,B_4}_{A_1,A_2,A_3,A_4} - T^{B_1,B_2,B_3,B_4}_{A_1,A_3,A_4,A_2} + T^{B_1,B_2,B_3,B_4}_{A_1,A_3,A_2,A_4}\Big) \\ &+\mathcal{O}(\alpha'^3)\,. \end{aligned} \tag{3.17}$$

[11] The parenthesis around the B labels signifies that the shuffle product treats the (multiparticle) labels B_j as single entries, e.g. $(B_1, B_2)\sqcup(B_3) = (B_1, B_2, B_3) + (B_1, B_3, B_2) + (B_3, B_1, B_2)$.

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From a practical perspective, it could be a daunting task to convert a huge expression in terms of $\phi_{A_i|B_j}$ such as (3.7) into linear combinations of $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ on the right-hand side of (3.17). Fortunately, since both the BG double current and $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ satisfy generalized Jacobi identities in the A_j-slots, an efficient algorithm due to Dynkin, Specht and Wever [63] can be used to accomplish this at higher orders in α'. See the appendix A.3 for more details.

3.3 The perturbiner description of α'-corrections

The recursion (3.17) for the coefficients $\phi_{A|B}$ of the perturbiner (3.2) can be rewritten in a more compact form by defining the shorthand

$$[[\ldots[[\Phi_{i_1},\Phi_{i_2}],\Phi_{i_3}],\ldots,\Phi_{i_{p-1}}],\Phi_{i_p}] \equiv \sum_{\substack{A_1,A_2,\ldots,A_p\\ B_1,B_2,\ldots,B_p}} e^{k_{A_1\ldots A_p}\cdot x}\, T^{B_1,B_2,\ldots,B_p}_{A_{i_1},A_{i_2},\ldots,A_{i_p}}\, t^{A_1A_2\ldots A_p}\otimes \tilde{t}^{B_1B_2\ldots B_p}\,, \tag{3.18}$$

which exploits the generalized Jacobi symmetry of the A_j-slots in $T^{B_1,B_2,\ldots,B_p}_{A_{i_1},A_{i_2},\ldots,A_{i_p}}$. That is, the numeric indices $i_1,i_2,\ldots,i_p$ of the various *formal* perturbiners Φ_i in the commutator match the ordering of the labels within the A-slots in $T^{B_1,B_2,\ldots,B_p}_{A_{i_1},A_{i_2},\ldots,A_{i_p}}$, while the ordering of the B-slots is always the same. Finally, the color degrees of freedom enter in a global multiplication order; $t^{A_1A_2\ldots A_p}\otimes\tilde{t}^{B_1B_2\ldots B_p}$.

The above definition implies that the Berends-Giele recursion (3.17) condenses to,

$$\begin{aligned}\frac{1}{2}\Box\Phi \,=\, &[\Phi_1,\Phi_2]-\alpha'^2\zeta_2\big(\partial_{23}[[\Phi_1,\Phi_2],\Phi_3]-\partial_{12}[\Phi_1,[\Phi_3,\Phi_2]]\big)\\ &+\alpha'^2\zeta_2\big([[\Phi_1,\Phi_2],[\Phi_4,\Phi_3]]-[[\Phi_1,\Phi_3],[\Phi_4,\Phi_2]]\big)+\mathcal{O}(\alpha'^3)\,,\end{aligned} \tag{3.19}$$

with the following shorthand for the derivatives:

$$\partial_{ij}\equiv(\partial_i\cdot\partial_j)\ . \tag{3.20}$$

The convention for the derivatives ∂_j is to only act on the position of Φ_j, e.g. the perturbiner expansion of $\partial_{12}[[\Phi_3,\Phi_2],\Phi_1]$ reproduces $\sum_{A=A_1A_2A_3}\sum_{B=B_1B_2B_3}(k_{A_1}\cdot k_{A_2})T^{B_1,B_2,B_3}_{A_3,A_2,A_1}$.

In view of the increasing number of Φ-factors at higher order in α', we will further lighten the notation and translate the commutators into multiparticle labels $\Phi_P\equiv\Phi_{i_1i_2\ldots i_p}$,

$$\Phi_{i_1i_2\ldots i_p}\equiv[[\ldots[[\Phi_{i_1},\Phi_{i_2}],\Phi_{i_3}],\ldots,\Phi_{i_{p-1}}],\Phi_{i_p}]\,, \tag{3.21}$$

which exhibit generalized Jacobi symmetries by construction.[12] Hence, any subset of the nested commutators of (3.19) can be separately expressed in terms of Φ_P; e.g. $[[\Phi_1,\Phi_2],[\Phi_3,\Phi_4]]=[\Phi_{12},\Phi_{34}]=\Phi_{1234}-\Phi_{1243}$. In this language, the Z-theory equation of motion (3.19) becomes

$$\frac{1}{2}\Box\Phi=[\Phi_1,\Phi_2]-\alpha'^2\zeta_2\Big(\partial_{23}[\Phi_{12},\Phi_3]-\partial_{12}[\Phi_1,\Phi_{32}]-[\Phi_{12},\Phi_{43}]+[\Phi_{13},\Phi_{42}]\Big)+\mathcal{O}(\alpha'^3)\,. \tag{3.22}$$

[12] These are the same symmetries in $P=i_1i_2\ldots i_p$ obeyed by contracted structure constants $f^{i_1i_2a}f^{ai_3b}\ldots f^{xi_py}$ as well as the local multiparticle superfields V_P [64] in pure spinor superspace.

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As will be explained below, this form of the Z-theory equation of motion provides the essential clue for proposing the Berends-Giele recursion to arbitrary orders of α'.

As a reformulation of (3.19) which does not rely on the notion of perturbiners, one can peel off the t^a generators[13] from the bi-colored fields $\Phi = \sum_A t^A \Phi_A$. The coefficients Φ_A are still Lie-algebra valued with respect to the $\tilde{t}^b$, and this is where the nested commutators act in the following rewriting of (3.19):

$$\begin{aligned}\frac{1}{2}\Box\Phi = &\sum_{A_1,A_2} t^{A_1A_2}[\Phi_{A_1},\Phi_{A_2}]\\ &-\sum_{A_1,A_2,A_3} t^{A_1A_2A_3}\alpha'^2\zeta_2\big(\partial_{23}[[\Phi_{A_1},\Phi_{A_2}],\Phi_{A_3}]-\partial_{12}[\Phi_{A_1},[\Phi_{A_3},\Phi_{A_2}]]\big)\\ &+\alpha'^2\zeta_2\sum_{A_1,A_2,A_3,A_4} t^{A_1A_2A_3A_4}\big([[\Phi_{A_1},\Phi_{A_2}],[\Phi_{A_4},\Phi_{A_3}]]-[[\Phi_{A_1},\Phi_{A_3}],[\Phi_{A_4},\Phi_{A_2}]]\big)\\ &+\mathcal{O}(\alpha'^3)\,.\end{aligned} \tag{3.23}$$

Upon comparison with (3.22), the notation in (3.21) can be understood as a compact way to track the relative multiplication orders of the t^a and $\tilde{t}^b$ generators.

3.3.1 Perturbiners at higher order in α'

The procedure of subsection 3.2 to determine the Berends-Giele recursion that reproduces the α'^2-corrections to the disk integrals was also applied to fix the recursion at the orders α'^3 and α'^4 (see appendix B for more details). Luckily, the analogous ansaetze at orders $\alpha'^{w\geq 5}$ could be bypassed since the general pattern of the field equations became apparent from the leading orders $\alpha'^{w\leq 4}$. To see this, it is instructive to spell out the Z-theory equation of motion up to the α'^3-order:

$$\begin{aligned}\frac{1}{2}\Box\Phi = &[\Phi_1,\Phi_2]+\Big(\alpha'^2\zeta_2\partial_{12}-\alpha'^3\zeta_3\partial_{12}(\partial_{12}+\partial_{23})\Big)[\Phi_1,\Phi_{32}]\\ &-\Big(\alpha'^2\zeta_2\partial_{23}-\alpha'^3\zeta_3\partial_{23}(\partial_{12}+\partial_{23})\Big)[\Phi_{12},\Phi_3]\\ &+\Big(\alpha'^2\zeta_2-\alpha'^3\zeta_3\big(\partial_{21}+2\partial_{31}+2\partial_{32}+2\partial_{42}+\partial_{43}\big)\Big)[\Phi_{12},\Phi_{43}]\\ &-\Big(\alpha'^2\zeta_2-\alpha'^3\zeta_3\big(2\partial_{21}+\partial_{31}+3\partial_{32}+\partial_{42}+2\partial_{43}\big)\Big)[\Phi_{13},\Phi_{42}]\\ &+2\alpha'^3\zeta_3\big(\partial_{42}+\partial_{43}\big)[\Phi_{123},\Phi_4]-\alpha'^3\zeta_3\big(3\partial_{42}+\partial_{43}\big)[\Phi_{132},\Phi_4]\\ &+2\alpha'^3\zeta_3\big(\partial_{31}+\partial_{21}\big)[\Phi_1,\Phi_{432}]-\alpha'^3\zeta_3\big(3\partial_{31}+\partial_{21}\big)[\Phi_1,\Phi_{423}]\\ &+\alpha'^3\zeta_3\Big(-[\Phi_{12},\Phi_{534}]+2[\Phi_{12},\Phi_{543}]-2[\Phi_{123},\Phi_{54}]-2[\Phi_{13},\Phi_{524}]+[\Phi_{132},\Phi_{54}]\\ &+2[\Phi_{134},\Phi_{52}]+3[\Phi_{14},\Phi_{523}]-2[\Phi_{14},\Phi_{532}]+2[\Phi_{142},\Phi_{53}]-3[\Phi_{143},\Phi_{52}]\Big)\\ &+\mathcal{O}(\alpha'^4)\,.\end{aligned} \tag{3.24}$$

[13] In view of the α'-corrections to KK relations from (2.12), the Z-theory scalar Φ is *not* Lie-algebra valued in the gauge group of the t^a but instead exhibits an expansion in the universal enveloping algebra spanned by $t^A = t^{a_1}t^{a_2}\ldots t^{a_{|A|}}$.

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After identifying $s_{ij} \leftrightarrow \partial_{ij}$, the coefficients of $[\Phi_{12}, \Phi_3]$ and $[\Phi_1, \Phi_{32}]$ in (3.24) are identical to the first *regular* terms in the expansion of the four-point disk integrals considered in [5]:

$$\begin{aligned}
&\text{reg} \int_0^1 \frac{\mathrm{d}z_2}{z_{12}} \sum_{m,n=0}^{\infty} \frac{(\alpha' s_{12} \ln|z_{12}|)^m}{m!} \frac{(\alpha' s_{23} \ln|z_{23}|)^n}{n!} = \alpha' \zeta_2 s_{23} - \alpha'^2 \zeta_3 s_{23}(s_{12}+s_{23}) + \mathcal{O}(\alpha'^3) \\
&\text{reg} \int_0^1 \frac{\mathrm{d}z_2}{z_{32}} \sum_{m,n=0}^{\infty} \frac{(\alpha' s_{12} \ln|z_{12}|)^m}{m!} \frac{(\alpha' s_{23} \ln|z_{23}|)^n}{n!} = -\alpha' \zeta_2 s_{12} + \alpha'^2 \zeta_3 s_{12}(s_{12}+s_{23}) + \mathcal{O}(\alpha'^3)
\end{aligned} \tag{3.25}$$

The endpoint divergences of these integrals as $z_2 \to z_1 = 0$ and $z_2 \to z_3 = 1$ require a regularization prescription denoted by "reg" and explained in section 4. The infinite sums in the above integrands arise from the Taylor expansion of a $\mathrm{SL}(2,\mathbb{R})$-fixed four-point Koba-Nielsen factor via

$$|z_{ij}|^{\alpha' s_{ij}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha' s_{ij} \ln|z_{ij}|)^n \,, \tag{3.26}$$

which removes the kinematic poles from the full disk integrals and yields their non-singular counterparts [5] upon regularization. Comparing the expansion of (3.25) at the next order in α' with the expression for the BG current obtained from an ansatz confirms the pattern, and we will later on see that the terms of order Φ^4 and Φ^5 in (3.24) can be traced back to regularized five- and six-point integrals.

3.4 All-order prediction for the BG recursion

From the observations in the previous subsection, we propose a closed form for the Φ^3 contributions to the Z-theory equations of motion for $\Box\Phi$, to all orders in α':

$$\begin{aligned}
\frac{1}{2}\Box\Phi = [\Phi_1, \Phi_2] - \alpha' \,\text{reg} \int_0^1 \mathrm{d}z_2 \sum_{m=0}^{\infty} \frac{(\alpha' \partial_{12} \ln|z_{12}|)^m}{m!} \sum_{n=0}^{\infty} \frac{(\alpha' \partial_{23} \ln|z_{23}|)^n}{n!} \\
\times \left(\frac{[\Phi_{12}, \Phi_3]}{z_{12}} + \frac{[\Phi_1, \Phi_{32}]}{z_{32}} \right) + \mathcal{O}(\Phi^4) \,.
\end{aligned} \tag{3.27}$$

The integrand in the second line bears a strong structural similarity to the correlation function in the four-point open string amplitude [11, 65]

$$A^{\text{open}}(1,2,3,4) = -\alpha' \int_0^1 \mathrm{d}z_2 \prod_{i<j}^{3} |z_{ij}|^{\alpha' s_{ij}} \left\langle \frac{V_{12} V_3 V_4}{z_{12}} + \frac{V_1 V_{32} V_4}{z_{32}} \right\rangle , \tag{3.28}$$

with $\langle V_P V_Q V_n \rangle$ denoting certain kinematic factors in pure spinor superspace. The precise correspondence between (3.27) and (3.28) maps multiparticle vertex operators V_P [64] to perturbiner commutators Φ_P defined in (3.21). Moreover, since V_P is fermionic and satisfies generalized Jacobi symmetries [64], the all-multiplicity mapping

$$\langle V_P V_Q V_n \rangle \longleftrightarrow [\Phi_P, \Phi_Q] \,, \quad |P| + |Q| = n - 1 \tag{3.29}$$

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preserves all the symmetry properties of its constituents. Finally, the Koba-Nielsen factor $\prod_{i<j}^{3}|z_{ij}|^{\alpha' s_{ij}}$ with $s_{ij} \to \partial_{ij}$ has been Taylor expanded according to (3.26) in converting (3.28) to (3.27). This projects out the kinematic poles of the integrals to ensure locality of the Z-theory equation of motion, but requires a regularization of the endpoint divergences at $z_2 \to 0$ and $z_2 \to 1$ as discussed in section 4.

It is easy to see that the correspondence (3.29) correctly "predicts" the first term in the right hand side of (3.27) from the well-known [38] expression $A^{\text{open}}(1,2,3) = \langle V_1 V_2 V_3 \rangle$ of the three-point massless disk amplitude under the mapping (3.29); $\langle V_1 V_2 V_3 \rangle \longleftrightarrow [\Phi_1, \Phi_2]$.

Extrapolating the above pattern, a natural candidate for the higher-order contributions $\Phi^4, \Phi^5, \ldots$ to the Z-theory equation of motion emerges from the integrand of the $(n-2)!$-term representation of the n-point disk amplitude [11],

$$\begin{aligned} A^{\text{open}}(1,2,\ldots,n) \,=\, (-\alpha')^{n-3} \int_{0\le z_2\le z_3\le\ldots\le z_{n-2}\le 1} \mathrm{d}z_2\,\mathrm{d}z_3\,\ldots\,\mathrm{d}z_n \prod_{i<j}^{n-1} |z_{ij}|^{\alpha' s_{ij}} \\ \times \bigg\langle \sum_{l=1}^{n-2} \frac{V_{12\ldots l} V_{n-1,n-2,\ldots,l+1} V_n}{(z_{12}z_{23}\ldots z_{l-1,l})(z_{n-1,n-2}z_{n-2,n-3}\ldots z_{l+2,l+1})} \\ +\text{perm}(2,3,\ldots,n-2) \bigg\rangle , \end{aligned} \tag{3.30}$$

which appeared in an intermediate step towards the minimal $(n-3)!$-term expression (2.13). This expression leads us to propose the following Z-theory equation of motion to all orders in the fields and their derivatives (with $\mathrm{SL}(2,\mathbb{R})$-fixing $z_1 = 0$ and $z_p = 1$):

$$\begin{aligned} \frac{1}{2}\Box\Phi \,=\, \sum_{p=2}^{\infty} (-\alpha')^{p-2} \int^{\text{eom}} \prod_{i<j}^{p} |z_{ij}|^{\alpha'\partial_{ij}} \\ \times \bigg(\sum_{l=1}^{p-1} \frac{[\Phi_{12\ldots l}, \Phi_{p,p-1\ldots l+1}]}{(z_{12}z_{23}\ldots z_{l-1,l})(z_{p,p-1}z_{p-1,p-2}\ldots z_{l+2,l+1})} + \text{perm}(2,3,\ldots,p-1) \bigg) . \end{aligned} \tag{3.31}$$

Apart from the correspondence (3.29) which settles the perturbiner commutators suggested by (3.30), we introduce a formal operator $\int^{\text{eom}}$ that maps the accompanying disk integrals to local expressions. The precise rules for the map $\int^{\text{eom}}$ to be explained in the next section include a Taylor expansion (3.26) of the Koba-Nielsen factor as seen in (3.27). Also, $\int^{\text{eom}}$ incorporates a regularization along with particular parameterization of the ubiquitous domain $0 \le z_2 \le z_3 \le \ldots \le z_{p-1} \le 1$ for the $p-2$ integration variables $z_2, z_3, \ldots, z_{p-1}$ which is left implicit in (3.31) for ease of notation. The shorthands $\Phi_{i_1 i_2 \ldots i_k}$ in (3.31) explained in section 3.3 compactly track the relative multiplication order of the gauge-group generators t^a and $\tilde{t}^b$ which govern the color structure of Φ.

For example, the equation of motion up to Φ^4-order following from (3.31) reads

$$\begin{aligned} \frac{1}{2}\Box\Phi \,=\, [\Phi_1,\Phi_2] - \alpha' \int^{\text{eom}} \prod_{i<j}^{3} |z_{ij}|^{\alpha'\partial_{ij}} \bigg(\frac{[\Phi_{12},\Phi_3]}{z_{12}} + \frac{[\Phi_1,\Phi_{32}]}{z_{32}} \bigg) \\ + \alpha'^2 \int^{\text{eom}} \prod_{i<j}^{4} |z_{ij}|^{\alpha'\partial_{ij}} \bigg(\frac{[\Phi_{123},\Phi_4]}{z_{12}z_{23}} + \frac{[\Phi_{12},\Phi_{43}]}{z_{12}z_{43}} + \frac{[\Phi_1,\Phi_{432}]}{z_{43}z_{32}} + (2\leftrightarrow 3) \bigg) + \mathcal{O}(\Phi^5) , \end{aligned} \tag{3.32}$$

and the low-energy expansion of the five-point integrals in the second line spelled out in appendix C reproduces the $\zeta_2\Phi^4$- and $\zeta_3\partial^2\Phi^4$-orders of the Z-theory equation of motion (3.24). Using the rules explained in the next section for obtaining the local terms indicated by $\int^{\text{eom}}$, we have made an explicit form of the Berends-Giele double current from (3.31) up to α'^7 publicly available on [29].

The Berends-Giele recursion for the α'-expansion of disk integrals is particularly advantageous over previous methods when computing the α'^w-order of disk integrals at *high multiplicities* $n > w+3$. That is because only a *finite* number of terms up to Φ^{w+2} in the field equation (3.31) is required to obtain terms of order α'^w in the disk integrals to *all multiplicities* (by simple deconcatenation of words as seen in (3.7)). This bypasses the manual pole subtractions in the polylogarithm-based method of [5] and the increasingly expensive matrix algebra involving matrices of dimension $(n-2)! \times (n-2)!$ in the Drinfeld associator method of [6].

4 Local disk integrals in the Z-theory equation of motion

In the previous section, we have proposed the Z-theory equation of motion (3.31) which determines the Berends-Giele double currents of disk integrals (3.1). The proposed field equations are inspired by the form (3.30) of the open-superstring disk amplitude and rely on a formal operator $\int^{\text{eom}}$ which converts the associated iterated integrals into local expressions. The purpose of this section is to give a precise definition of the map $\int^{\text{eom}}$ in (3.31) which incorporates a regularization of the disk integrals' endpoint divergences along with a prescription to settle the resulting ambiguities.

In section 4.1, we will briefly review the definition and properties of polylogarithms that are used to perform the integrals that appear in (3.31). Already the examples at the four-point level (3.25) will be seen to yield endpoint divergences, for which we will specify a suitable regularization scheme. Consequently, the results of iterated integrals at the $(n \geq 5)$-point level will depend on the order of integration.[14] The first non-trivial example is given by (4.12), where different regularized values may arise for the two orders of integration $\int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2$ and $\int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3$. A priori, it is not clear that any choice will lead to their regularized values required by the Z-theory equation of motion. But, on empirical grounds, we find a prescription that gives the correct answers: we identify a new basis for the integrands in (3.31) under partial fraction relations along with the integration orders for each of its elements. The recursive algorithms implementing these rules are described in sections 4.2 and 4.3.

The above prescription was derived by trial and error through comparison with known data for $Z(P|Q)$ at low number of points, and its consequences were extrapolated to arbitrary multiplicity. It remains an open question to find its rigorous mathematical justification.

[14] Note that Fubini's theorem stating the equivalence of integration orders for iterated integrals does not apply to the divergent integrals and their regularized values under discussion.

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4.1 Multiple polylogarithms and their regularization

In this section, we review selected aspects of the polylogarithm-based setup of [5] to extract local terms (also called regular terms) from the disk integrals[15] in (3.30). The requirement that the $\int^{\text{eom}}$ map must reproduce the correct Z-theory equation of motion induces systematic departures from [5] which will be highlighted in the subsequent discussion.

4.1.1 Polylogarithms and MZVs

We recall that multiple polylogarithms $G(A;z)$ with $A = a_1, a_2, \ldots, a_n$ and $a_j, z \in \mathbb{C}$ are defined by[16]

$$G(a_1, a_2, \ldots, a_n; z) \equiv \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t), \quad G(\emptyset; z) \equiv 1, \quad \forall z \neq 0\,, \tag{4.1}$$

setting $G(\emptyset; 0) \equiv 0$. The variables a_j and z on the left and right of the semicolon are referred to as the *labels* and the *argument* of the polylogarithm, respectively, and the number n of labels a_j is called the *weight*. Their recursive definition (4.1) as iterated integrals endows polylogarithms with a shuffle algebra

$$G(A;z)G(B;z) = G(A \sqcup B; z)\,, \tag{4.2}$$

and the regularization prescription discussed in the sequel is designed to preserve (4.2). After repeated application of the recursion (4.1), disk integrals ultimately boil down to $G(\ldots;1)$ at unit argument [5]. In the framework of the Z-theory equation of motion (3.31), this follows from the endpoint $z_p = 1$ for the uppermost integration variable z_{p-1} and reproduces the integral representation of MZVs (2.18),

$$\zeta_{n_1,n_2,\ldots,n_r} = (-1)^r G(\underbrace{0,0,\ldots,0,1}_{n_r}, \ldots, \underbrace{0,0,\ldots,0,1}_{n_1}; 1)\,, \tag{4.3}$$

see appendix D.1 for examples and extensions to regularized values of divergent integrals.

4.1.2 Polylogarithms and the Koba-Nielsen factor

Using the special cases of the multiple polylogarithms (4.1),

$$G(\underbrace{0,0,\ldots,0}_{w}; z) \equiv \frac{1}{w!}\big[\ln(z)\big]^w\,, \quad G(\underbrace{a,a,\ldots,a}_{w}; z) = \frac{1}{w!}\left[\ln\left(1 - \frac{z}{a}\right)\right]^w \quad a \neq 0\,, \tag{4.4}$$

see (4.8) for the regularization involved in the convention for $G(0,0,\ldots,0;z)$, the Taylor expansion (3.26) of the Koba-Nielsen factor with the $\mathrm{SL}(2,\mathbb{R})$-fixing $z_1 = 0$ and $z_p = 1$ can be written as [5]

$$\prod_{i<j}^{p} |z_{ij}|^{\alpha' \partial_{ij}} = \prod_{i=2}^{p-1} \sum_{n_i=0}^{\infty} \left(\sum_{l=1}^{i-1} \alpha' \partial_{il}\right)^{n_i} G(\underbrace{0,\ldots,0}_{n_i}; z_i) \prod_{2\leq j<k}^{p} \sum_{n_{jk}=0}^{\infty} (\alpha' \partial_{jk})^{n_{jk}} G(\underbrace{z_k,\ldots,z_k}_{n_{jk}}; z_j)\,. \tag{4.5}$$

[15] There is a vast body of literature related to iterated integrals on moduli spaces of genus-zero curves with n ordered marked points, see e.g. [50, 66–70] and references therein. Moreover, their symbolic computation have been recently implemented in computer programs [71, 72].

[16] Our conventions for polylogarithms agree with the work [73] of Goncharov as well as for instance reference [74]. See e.g. [75, 76] for other aspects of multiple polylogarithms.

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Therefore, the leading orders of the regularized four-point integrals in (3.25) can be traced back to

$$\begin{aligned}\int^{\text{eom}} \prod_{i<j}^{3} |z_{ij}|^{\alpha' \partial_{ij}} \frac{1}{z_2 - a} \\ = \text{reg} \int_1^0 \frac{\mathrm{d}z_2}{z_2 - a} \Big\{ 1 + \alpha' \Big(G(0; z_2)\partial_{12} + G(1; z_2)\partial_{23} \Big) \\ + \alpha'^2 \Big(G(0,0; z_2)\partial_{12}^2 + G(0 \sqcup\!\sqcup 1; z_2)\partial_{12}\partial_{23} + G(1,1; z_2)\partial_{23}^2 \Big) \\ + \mathcal{O}(\alpha'^3) \Big\} , \end{aligned} \tag{4.6}$$

with $a \in \{0, 1\}$, using (4.1) to perform the z_2-integral as well as (4.3) to convert the results to MZVs. Divergent cases as exemplified in (D.1) are addressed by the regularization scheme which is denoted by "reg" in (4.6) and will be the subject of the next subsection.

4.1.3 Regularization of endpoint divergences

It follows from their definition (4.1) that multiple polylogarithms diverge at the endpoints of the integration domains whenever $a_1 = z$ or $a_n = 0$, and therefore they need to be regularized. The convention for $G(0, 0, \ldots, 0; z)$ in (4.4) is part of the regularization procedure of interest to this work and can be understood in terms of a cutoff ϵ: the left hand side of

$$\int_\epsilon^z \frac{\mathrm{d}t}{t} = \ln|t| \,\big|_{t=\epsilon}^{t=z} = \ln|z| - \ln|\epsilon| \tag{4.7}$$

formally tends to $G(0; z)$ in the $\epsilon \to 0$ limit, and its regularized value $\ln|z|$ can be obtained from the right hand side by manually discarding (the source of divergences) $\ln|\epsilon|$. Together with a similar reasoning for divergences from the upper integration limit, we specify the following regularized values for divergent integrals at weight one:[17]

$$\text{reg} \int_0^z \frac{\mathrm{d}t}{t} = G(0; z) \equiv \ln|z| \,, \qquad \text{reg} \int_0^z \frac{\mathrm{d}t}{t - z} \equiv -\ln|z| = -G(0; z) \,, \qquad z > 0 \tag{4.8}$$

Further subtleties arise in situations where the endpoint divergence as $t \to z$ is approached from above. In this case, one defines

$$\text{reg} \int_z^w \frac{\mathrm{d}t}{t - z} \qquad \equiv G(z; w) + G(0; z) - i\pi \,, \qquad w > z \,, \tag{4.9}$$

where the occurrence of imaginary parts is an artifact of the decomposition of the integration domains in later sections. The choice of sign along with $i\pi$ in (4.9) is a convention, and the cancellation of imaginary parts in the Z-theory equation of motion serves as a consistency check of our integration setup.

One can combine (4.8) with (4.9) such as to define the regularized value of $G(z; z)$ via

$$G(z; z) \equiv -G(0; z) + i\pi\delta \,, \tag{4.10}$$

[17] We are indebted to Erik Panzer for suggesting this regularization to us.

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where $\delta = 0$ and $\delta = 1$ if $G(z;z)$ is obtained after integration over t such that $t < z$ and $t > z$, respectively.

Since the regularization scheme in this work is defined to preserve the shuffle algebra (4.2), the regularized values at weight one in (4.8) and (4.10) determine endpoint divergences at higher weight, see appendix D.2 for more details. For instance, the special cases $G(0;1) = G(1;1) = 0$ of (4.8) along with the shuffle algebra allow to extract finite linear combinations of MZVs from $G(1,\ldots;1)$ and $G(\ldots,0;1)$ with labels $\in \{0,1\}$, see (D.1).

In contrast to the regularizations (4.8) and (4.10) of this work which are selected by the Z-theory equation of motion, the regularization scheme of [5] preserves the scaling property of polylogarithms and implies a vanishing regularized value for $G(z;z)$.

4.1.4 Dependence on the integration order

As a subtle consequence of the shuffle-preserving regularization scheme based on (4.8) and (4.10), regularized values of disk integrals relevant to the Z-theory equation of motion (3.31) depend on the integration order. A simple example where the two integration orders $\int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2$ and $\int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3$ for the integration domain $0 \leq z_2 \leq z_3 \leq 1$ yield inequivalent results stems from the five-point integral over $\frac{\ln|z_{23}|}{z_{12}z_{13}}$ which arises from the partial-fraction identity

$$\frac{1}{z_{12}z_{13}} = \frac{1}{z_{12}z_{23}} + \frac{1}{z_{13}z_{32}} \tag{4.11}$$

along with the Koba-Nielsen expansion (4.5) at linear order in α'. Using $\ln|z_{23}| = G(0;z_3) + G(z_3;z_2)$ and $G(0,z_3;z_3) = -\zeta_2$ (see (D.10)) as well as (4.31) to render $G(0,z_2;1)$ suitable for integration over z_2, one finds the two different results[18]

$$\mathrm{reg}\int_0^1 \frac{\mathrm{d}z_3}{z_3}\int_0^{z_3}\frac{\mathrm{d}z_2}{z_2}\,\ln|z_{23}| = 0\,, \qquad \mathrm{reg}\int_0^1 \frac{\mathrm{d}z_2}{z_2}\int_{z_2}^1\frac{\mathrm{d}z_3}{z_3}\,\ln|z_{23}| = \zeta_3\,. \tag{4.12}$$

The task to rewrite $G(0,z;z)$ and $G(0,z;1)$ in a form suitable for integration over z via (4.1) is ubiquitous to regularized ($n \geq 5$)-point disk integrals [5]. The systematics of such "z-removal identities" is discussed in appendix D.3. It is worth noting that the symbolic program `HyperInt` [71] contains routines that automate this task.

It turns out that between the two orders of integration displayed in (4.12), the Z-theory equation of motion (3.24) (obtained from an ansatz for the equivalent BG recursion) is reproduced at the $\alpha'^3\zeta_3$ order only if the regularized integral of $\ln|z_{23}|/(z_{12}z_{13})$ vanishes. Therefore z_2 must be integrated prior to z_3 in presence of $(z_{12}z_{13})^{-1}$ in a five-point integrand. By worldsheet parity $z_j \to z_{5-j}$, the integral over $(z_{24}z_{34})^{-1}$ must then follow the converse order where z_3 is integrated first. The conclusion here is that different integrands

[18] In order to evaluate the second integral of (4.12) through the definition (4.1) of polylogarithms, the integration limits are rearranged according to

$$\int_{z_2}^1 \mathrm{d}z_3\, f(z_3) = \int_0^1 \mathrm{d}z_3\, f(z_3) - \int_0^{z_2} \mathrm{d}z_3\, f(z_3)\,.$$

require different orders of integration. Adapting the integration order to each integrand will be part of the map $\int^{\text{eom}}$ to be elaborated below.

Similarly, we identified the appropriate integration orders for the 4! six-point integrals in (3.31) at $p = 5$ by matching with the α'^3 and α'^4 order of the Berends-Giele recursion obtained from an ansatz. Moreover, an alternative method to determine the desired outcome of regularized integrals to arbitrary orders in α' is presented in appendix E which closely follows the handling of poles in [5]. An all-multiplicity algorithm to determine the integration orders which are observed to reproduce the Z-theory equation of motion will be described in section 4.3. As a preparation for this, however, a systematic change of integral bases via repeated use of partial-fraction identities will be introduced in the next section.

4.2 Towards the simpset basis

Our investigations showed that the $(n-2)!$ integrals in the open-superstring amplitude (3.30) need to be rewritten in a very particular basis to define the $\int^{\text{eom}}$ prescription in the Z-theory equation of motion (3.31). In this section, we will introduce a basis where the $\int^{\text{eom}}$ prescription can be associated with appropriate integration orders for the regularized integrals such as to settle the ambiguity seen in (4.12). In order to explain this change of basis[19] it will be convenient to introduce the following chain of worldsheet propagators

$$\mathcal{Z}_A \equiv \frac{1}{z_{a_1a_2}z_{a_2a_3}\cdots z_{a_{|A|-1},a_{|A|}}}\,, \qquad |A| \geq 2\,, \tag{4.13}$$

in which two consecutive z_{ij} factors in the denominator always share a label, with a formal extension $\mathcal{Z}_A \equiv 1$ to words of length $|A| = 1$. One can check that $z_{ij} = -z_{ji}$ and partial-fraction identities (4.11) imply the shuffle symmetry [78]

$$\mathcal{Z}_{A\sqcup B} = 0\,, \quad \forall A, B \neq \emptyset\,. \tag{4.14}$$

Using the above definition, the $(n-2)!$ *chain basis* integrals in the amplitude (3.30) can be distinguished by their chain factors of $\mathcal{Z}_{1P}\mathcal{Z}_{(n-1)Q}$, with $|P|+|Q| = n-3$. As a part of the prescription for the map $\int^{\text{eom}}$, the integrals from the chain basis in the Z-theory equation of motion (3.31) are rewritten in another basis which is referred to as the *simpset basis*.

4.2.1 Description of the algorithm

At generic multiplicity, the elements of the simpset basis are obtained from the chain basis $\mathcal{Z}_{1P}\mathcal{Z}_{(n-1)Q}$ by recursively stripping off factors of $\mathcal{Z}_{ij} = z_{ij}^{-1}$. At each step, the shuffle symmetry (4.14) is applied to $\mathcal{Z}_{1P}$ and $\mathcal{Z}_{(n-1)Q}$ to factor out $\mathcal{Z}_{ij}$, where i and j are the labels in $1P$ which are maximally apart (i.e. at highest value of $|i-j|$). This procedure is repeated for the coefficient $\mathcal{Z}_R$ in the decomposition $\mathcal{Z}_{1P} = \mathcal{Z}_{ij}\mathcal{Z}_R$, leading to a recursive algorithm.

In a factor of $\mathcal{Z}_{1243}$ relevant at six points, the labels 1 and 4 constitute the pair which is maximally apart with a separation of $|1-4| = 3$. Therefore, to arrive at the elements in

[19] The "basis" of dimension $(n-2)!$ refers to the minimum elements under partial-fraction identities; integration by parts further reduce their number to $(n-3)!$ [11, 37]. The reduction of products of z_{ij}^{-1} via partial fractions to a $(n-2)!$-dimensional basis is also described in appendix A of [77].

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the simpset basis, one needs to rewrite $\mathcal{Z}_{1243}$ in such a way as to contain the factor z_{14}^{-1}. In this case it is easy to show using partial-fraction identities that

$$\mathcal{Z}_{1243} = -\mathcal{Z}_{14}\mathcal{Z}_{12}\mathcal{Z}_{34} - \mathcal{Z}_{14}\mathcal{Z}_{24}\mathcal{Z}_{34}\,, \tag{4.15}$$

in which the factor $\mathcal{Z}_{14} = z_{14}^{-1}$ has been stripped off from the chain $\mathcal{Z}_{1243}$. The two integrals on the right-hand side of (4.15) belong to the simpset basis since $\mathcal{Z}_{12}\mathcal{Z}_{34}$ cannot be written as a single chain factor $\mathcal{Z}_R$ and the maximally separated labels $2, 4$ in $\mathcal{Z}_{24}\mathcal{Z}_{34} = -\mathcal{Z}_{243}$ are already factored out.

The following recursive algorithm implements the change of basis required by the $\int^{\text{eom}}$ map. For each factor of $\mathcal{Z}_R$ one identifies the pair of labels i and j that are maximally separated and recursively applies the following corollaries of (4.14) and (4.13),

$$\mathcal{Z}_{iAaj} = -\mathcal{Z}_{(iA\sqcup j)a}, \qquad \mathcal{Z}_{iAjB} = \mathcal{Z}_{iAj}\mathcal{Z}_{jB}\,, \tag{4.16}$$

which eventually stops at $\mathcal{Z}_{ija} = \mathcal{Z}_{ij}\mathcal{Z}_{ja}$ where the factor $\mathcal{Z}_{ij}$ is singled out.

In order to illustrate the algorithm (4.16), consider the seven-point integral characterized by the factor $\mathcal{Z}_{63425}$ with five labels. Since the labels 2 and 6 are maximally separated, the second identity in (4.16) rewrites it as $\mathcal{Z}_{63425} = \mathcal{Z}_{6342}\mathcal{Z}_{25}$. The first factor now contains only four labels and iterating the application of the identities in (4.16) yields,

$$\begin{aligned}
\mathcal{Z}_{6342} &= -\mathcal{Z}_{6324} - \mathcal{Z}_{6234} - \mathcal{Z}_{2634} = -\mathcal{Z}_{632}\mathcal{Z}_{24} - \mathcal{Z}_{62}\mathcal{Z}_{234} - \mathcal{Z}_{26}\mathcal{Z}_{634} \\
&= (\mathcal{Z}_{623} + \mathcal{Z}_{263})\mathcal{Z}_{24} + \mathcal{Z}_{62}(\mathcal{Z}_{243} + \mathcal{Z}_{423}) - \mathcal{Z}_{26}\mathcal{Z}_{63}\mathcal{Z}_{34} \\
&= (\mathcal{Z}_{62}\mathcal{Z}_{23} + \mathcal{Z}_{26}\mathcal{Z}_{63})\mathcal{Z}_{24} + \mathcal{Z}_{62}(\mathcal{Z}_{24}\mathcal{Z}_{43} + \mathcal{Z}_{42}\mathcal{Z}_{23}) - \mathcal{Z}_{26}\mathcal{Z}_{63}\mathcal{Z}_{34} \\
&= \mathcal{Z}_{26}\mathcal{Z}_{63}\mathcal{Z}_{24} + \mathcal{Z}_{62}\mathcal{Z}_{24}\mathcal{Z}_{43} - \mathcal{Z}_{26}\mathcal{Z}_{63}\mathcal{Z}_{34}\,.
\end{aligned} \tag{4.17}$$

In order to arrive at the second line, the factor $\mathcal{Z}_{234}$ was manipulated w.r.t. the maximally-separated labels 2 and 4 (with similar considerations for the other factor $\mathcal{Z}_{632}$). Therefore,

$$\frac{1}{z_{63}z_{34}z_{42}z_{25}} = \frac{1}{z_{26}z_{63}z_{24}z_{25}} + \frac{1}{z_{62}z_{24}z_{43}z_{25}} - \frac{1}{z_{26}z_{63}z_{34}z_{25}}\,, \tag{4.18}$$

is the transformation from the chain to the simpset basis.

The first non-trivial application of the above algorithm leads the five-point simpset basis

$$\left\{\frac{1}{z_{12}z_{13}}, \frac{1}{z_{13}z_{23}}, \frac{1}{z_{12}z_{43}}, \frac{1}{z_{13}z_{42}}, \frac{1}{z_{42}z_{43}}, \frac{1}{z_{32}z_{42}}\right\}. \tag{4.19}$$

The complete set of denominators in the six-point simpset basis can be found in (4.26) (upon adjoining their parity images under $z_j \to z_{6-j}$), while the appendix F contains an overview of the seven-point simpset basis.

4.2.2 Back to the chain basis

For completeness, it is straightforward to exploit the shuffle symmetry (4.14) to obtain a recursive algorithm to expand the simpset basis elements back in the chain basis.[20] To

[20] This algorithm summarizes the discussion of the appendix A of [77] after noticing that simpset and chain basis elements can be described via Cayley graphs and Hamilton paths, respectively.

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motivate the algorithm below, consider the following example: to rewrite $\mathcal{Z}_{12}\mathcal{Z}_{13}\mathcal{Z}_{14}$ in the chain basis note that $\mathcal{Z}_{12}\mathcal{Z}_{13} = -\mathcal{Z}_{213}$. Next, to make a chain out of $\mathcal{Z}_{213}\mathcal{Z}_{14}$ one uses the identity $\mathcal{Z}_{A1B} = (-1)^{|A|}\mathcal{Z}_{1(\tilde{A}\sqcup B)}$ in the first factor to allow it to be prefixed by $\mathcal{Z}_{14} = -\mathcal{Z}_{41}$, yielding $\mathcal{Z}_{12}\mathcal{Z}_{13}\mathcal{Z}_{14} = -\mathcal{Z}_{4123} - \mathcal{Z}_{4132}$. Then, $\mathcal{Z}_{41ij} = -\mathcal{Z}_{14ij} - \mathcal{Z}_{1i4j} - \mathcal{Z}_{1ij4}$ completes the basis change to $\mathcal{Z}_{12}\mathcal{Z}_{13}\mathcal{Z}_{14} = \mathcal{Z}_{1234} + \mathrm{perm}(2,3,4)$.

Hence, the general algorithm to expand the simpset basis elements in the chain basis is based on the recursive application of the following two identities,

$$\mathcal{Z}_{Pi}\mathcal{Z}_{iQ} = \mathcal{Z}_{PiQ}\,, \qquad \mathcal{Z}_{AiB} = (-1)^{|A|}\mathcal{Z}_{i(\tilde{A}\sqcup B)}\,. \tag{4.20}$$

The second identity follows from (4.14) and implies that the basis dimension of Hamilton paths $\mathcal{Z}_P$ is $(|P|-1)!$.

4.3 Integration orders for the simpset elements

In the simpset basis of integrals attained through the algorithm (4.16), we can now complete the definition of the $\int^{\mathrm{eom}}$ map in (3.31). For each simpset element, the algorithm to be described in this section identifies at least one integration order for which the regularized[21] integrals involving the Koba-Nielsen factor (4.5) are observed to yield the correct Z-theory equation of motion.

It should be emphasized once more that the order of integration must not be confused with the integration domain in (3.31) which is always fixed to be $0 \le z_2 \le z_3 \le \ldots \le z_{p-1} \le 1$. Instead, "order of integration" refers to the decision whether an iterated integral over z_2, z_3 subject to $0 \le z_2 \le z_3 \le 1$ is represented as $\int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2$ or as $\int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_3$. In the first case, the integration over z_2 is performed first, and we will write 23, whereas the opposite integration order will be referred to through the shorthand 32, with obvious generalization to higher multiplicity.

4.3.1 Description of the algorithm

Let us introduce a formal operator "ord" that takes as input a product of z_{ij} from the denominators in the simpset basis and outputs a combination of words encoding the admissible integration orders. For example, $\mathrm{ord}(z_{12}z_{13}) = 23$ for the integrand in (4.12) means that $\int^{\mathrm{eom}}$ requires the integral over z_2 to be performed first, followed by z_3.

In order to describe a recursive algorithm to determine the order of integration, we associate a graph to each element in the simpset basis where each factor of z_{ij} contributes an edge between vertices i and j. Then, $\mathrm{ord}(\ldots)$ for a given element of the simpset basis can be obtained by repeated application of two steps:

1. If the graph of $z_{a_1a_2}\ldots z_{a_na_{n+1}} = (z_{b_1b_2}\ldots z_{b_pb_{p+1}})(z_{c_1c_2}\ldots z_{c_qc_{q+1}})$ is not connected (i.e. if $b_i \neq c_j \;\forall\; i,j$), apply $\mathrm{ord}(\ldots)$ to each of its connected subgraphs representing $z_{b_1b_2}\ldots z_{b_pb_{p+1}}$ as well as $z_{c_1c_2}\ldots z_{c_qc_{q+1}}$ and shuffle the resulting words,

$$\mathrm{ord}(z_{a_1a_2}\ldots z_{a_na_{n+1}}) = \mathrm{ord}(z_{b_1b_2}\ldots z_{b_pb_{p+1}})\sqcup\mathrm{ord}(z_{c_1c_2}\ldots z_{c_qc_{q+1}})\,. \tag{4.21}$$

[21] When the disk integrals do not contain any kinematic poles, the Taylor expansion of the Koba-Nielsen factor results in convergent integrals where all integration orders are equivalent.

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The shuffle between the ordered sequences $ijk\ldots$ generated by the individual ord$(\ldots)$ operators indicates that the associated integrations commute, e.g., 23⧢4 means that any integration order among $234, 243, 423$ is allowed.

2. If the element $z_{a_1a_2}\ldots z_{a_na_{n+1}}z_{ij}$ is represented by a connected graph where z_{ij} is the factor with maximal separation $|i-j|$, and j corresponds to the integration variable that has not yet been pulled out of ord$(\ldots)$, then

$$\text{ord}(z_{a_1a_2}\ldots z_{a_na_{n+1}}z_{ij}) = \text{ord}(z_{a_1a_2}\ldots z_{a_na_{n+1}})j\,. \tag{4.22}$$

By design of the algorithm, only one of i or j can correspond to an integration variable that has not yet been pulled out of ord$(\ldots)$.

For example, consider the element $z_{12}z_{35}z_{36}z_{46}$ from the seven-point simpset basis where z_{12} is associated with a disconnected subgraph. The first step splits ord$(\ldots)$ according to its connected components, and iterating the algorithm above yields,

$$\begin{aligned}\text{ord}(z_{12}z_{35}z_{36}z_{46}) &= \text{ord}(z_{12})⧢\big(\text{ord}(z_{35}z_{36}z_{46})\big) = 2⧢\big(\text{ord}(z_{35}z_{46})3\big)\\ &= 2⧢\Big(\big(\text{ord}(z_{35})⧢\text{ord}(z_{46})\big)3\Big) = 2⧢\big((5⧢4)3\big)\,.\end{aligned} \tag{4.23}$$

The above ordering means that any permutation of 2345 such that 4 and 5 appear before 3 (e.g. 5243) defines a viable integration order, while the position of 2 is arbitrary.

4.3.2 Examples

Let us list the outcomes of the above algorithm for a few elements. At four points, the two-dimensional basis has a unique order:

$$\text{ord}(z_{12}) = 2\,,\quad \text{ord}(z_{23}) = 2\,. \tag{4.24}$$

At five points, the six-dimensional simpset basis requires the following integration orders:

$$\begin{aligned}&\text{ord}(z_{12}z_{13}) = 23, && \text{ord}(z_{12}z_{34}) = 2⧢3, && \text{ord}(z_{23}z_{24}) = 32,\\ &\text{ord}(z_{13}z_{23}) = 23, && \text{ord}(z_{13}z_{24}) = 2⧢3, && \text{ord}(z_{24}z_{34}) = 32.\end{aligned} \tag{4.25}$$

At six points, the order for twelve simpset basis elements is given by

$$\begin{aligned}&\text{ord}(z_{12}z_{13}z_{14}) = 234, && \text{ord}(z_{12}z_{34}z_{14}) = (2⧢3)4, && \text{ord}(z_{23}z_{24}z_{14}) = 324,\\ &\text{ord}(z_{13}z_{23}z_{14}) = 234, && \text{ord}(z_{13}z_{24}z_{14}) = (2⧢3)4, && \text{ord}(z_{24}z_{34}z_{14}) = 324,\\ &\text{ord}(z_{12}z_{13}z_{45}) = 23⧢4, && \text{ord}(z_{14}z_{24}z_{35}) = 24⧢3, && \text{ord}(z_{13}z_{14}z_{25}) = 34⧢2,\\ &\text{ord}(z_{13}z_{23}z_{45}) = 23⧢4, && \text{ord}(z_{12}z_{14}z_{35}) = 24⧢3, && \text{ord}(z_{14}z_{34}z_{25}) = 34⧢2,\end{aligned} \tag{4.26}$$

while the integration order for the remaining twelve integrals are obtained from worldsheet parity $z_j \to z_{6-j}$, e.g. ord$(z_{25}z_{35}z_{45}) = 432$. The integration orders of the simpset basis at seven points are explicitly listed in appendix F.

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4.3.3 Iterated integrals and integration order

The above algorithm generates the allowed integration orders for all the $(n-2)!$ elements in the simpset basis, with $p = n-1$ in the Z-theory equation of motion (3.31). Since the integration domain is always $0 \leq z_2 \leq z_3 \ldots \leq z_{p-1} \leq 1$, one can show that the resulting words $\mathrm{ord}(z_{i_1 j_1} z_{i_2 j_2} \ldots z_{i_k j_k}) = a_1 a_2 \ldots a_k$ translate into the following iterated integrals

$$\int^{\mathrm{eom}} \frac{1}{z_{i_1 j_1} z_{i_2 j_2} \ldots z_{i_k j_k}} = \mathrm{reg} \int_0^1 \mathrm{d}z_{a_k} \int_{b_{k-1}}^{c_{k-1}} \mathrm{d}z_{a_{k-1}} \ldots \int_{b_2}^{c_2} \mathrm{d}z_{a_2} \int_{b_1}^{c_1} \mathrm{d}z_{a_1} \frac{1}{z_{i_1 j_1} z_{i_2 j_2} \ldots z_{i_k j_k}} , \tag{4.27}$$

with lower limits $b_j \equiv \max\{x \in \{0, z_{a_{j+1}}, z_{a_{j+2}}, \ldots, z_{a_k}\} \,|\, x \leq z_{a_j}\}$ as well as upper limits $c_j \equiv \min\{x \in \{1, z_{a_{j+1}}, \ldots, z_{a_k}\} \,|\, x \geq z_{a_j}\}$.

4.4 Summary and overview example

As discussed in the previous subsections, the $\int^{\mathrm{eom}}$ map converts the integrands in the Z-theory equation of motion (3.31) to series expansions in derivatives and MZVs by:

(i) changing the basis of integrals to the simpset basis through the algorithm in (4.16)

(ii) determining the integration orders $\mathrm{ord}(\ldots)$ for simpset denominators through the algorithm in (4.21) and (4.22)

(iii) applying the regularization techniques of section 4.1 to perform the integrals (4.27) with Koba-Nielsen insertions (4.5)

The above steps will be illustrated through a simple yet representative example

$$\prod_{i<j}^{5} |z_{ij}|^{\alpha' s_{ij}} \langle V_1 V_{5243} V_6 \rangle \, \mathcal{Z}_{5243} \;\longleftrightarrow\; \prod_{i<j}^{5} |z_{ij}|^{\alpha' \partial_{ij}} [\Phi_1, \Phi_{5243}] \, \mathcal{Z}_{5243} \tag{4.28}$$

taken from the six-point open superstring amplitude and the Φ^5-order of the Z-theory equation of motion (3.31), respectively. We focus on the term proportional to $\alpha' \partial_{12} G(0; z_2)$ in the expansion (4.5) of the Koba-Nielsen factor to order α'. This example was chosen because it touches all the subtle points of the regularization prescription in section 4.1. The calculations are long and tedious to perform by hand, but they are straightforward to automate in a computer.[22]

The chain basis element $\mathcal{Z}_{5243}$ under discussion also belongs to the simpset basis with $\mathrm{ord}(z_{52} z_{24} z_{43}) = 342$. Hence, the $\int^{\mathrm{eom}}$ map instructs to evaluate the regularized integral

$$\alpha'^4 \partial_{12} [\Phi_1, \Phi_{5243}] \;\longleftrightarrow\; \mathrm{reg} \int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_4 \int_{z_2}^{z_4} \mathrm{d}z_3 \frac{G(0; z_2)}{z_{52} z_{24} z_{43}} , \tag{4.29}$$

where the reference to the shuffle regularization scheme (4.8) and (4.10) via "reg" will be left implicit in the remainder of this section. The integration limits in (4.29) associated to

[22] We are releasing our code that performs this task via [29]. The evaluation of the $8! = 40.320$ integrals in the 10-point simpset basis to their leading order $\sim \zeta_7, \zeta_2\zeta_5, \zeta_2^2\zeta_3$ takes about two hours on a laptop. The program is written in FORM [79, 80], and improvements to the code are certainly possible and highly welcomed.

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the order $\mathrm{ord}(z_{52}z_{24}z_{43}) = 342$ follow from (4.27). Rewriting $\int_{z_i}^{z_j} = \int_0^{z_j} - \int_0^{z_i}$ yields four integrals, where integration over z_3 leads to

$$\begin{aligned}
+\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^1 \frac{\mathrm{d}z_4}{z_{24}} \int_0^{z_4} \mathrm{d}z_3 \frac{G(0;z_2)}{z_{43}} &= -\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^1 \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_4) \\
-\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^1 \frac{\mathrm{d}z_4}{z_{24}} \int_0^{z_2} \mathrm{d}z_3 \frac{G(0;z_2)}{z_{43}} &= +\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^1 \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_2) \\
-\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^{z_2} \frac{\mathrm{d}z_4}{z_{24}} \int_0^{z_4} \mathrm{d}z_3 \frac{G(0;z_2)}{z_{43}} &= +\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^{z_2} \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_4) \\
+\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^{z_2} \frac{\mathrm{d}z_4}{z_{24}} \int_0^{z_2} \mathrm{d}z_3 \frac{G(0;z_2)}{z_{43}} &= -\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^{z_2} \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_2)\,. \qquad (4.30)
\end{aligned}$$

We stress that the shuffle regularization to use in the first and third integrals is (4.10) with $\delta = 0$ since $G(z_4;z_4)$ is obtained after integration over z_3 (subject to $z_3 < z_4$),

$$G(z_4;z_4) = -G(0;z_4)\,.$$

In addition, in order to integrate over z_4, the polylogarithm $G(z_4;z_2)$ needs to be rewritten using the general z-removal identities, in particular (D.11),

$$G(z_4;z_2) = G(z_2;z_4) + G(0;z_2) - G(0;z_4) - i\pi\,. \qquad (4.31)$$

After the above considerations, the integrals over z_4 yield

$$\begin{aligned}
&-\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^1 \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_4) \qquad (4.32)\\
&\qquad = -\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} G(0;z_2)G(z_2,0;1) \\
&+\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^1 \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_2) \qquad (4.33)\\
&\qquad = \int_0^1 \frac{\mathrm{d}z_2}{z_{52}} G(0;z_2) \times \big(G(z_2,0;1) - G(z_2,z_2;1) + i\pi G(z_2;1) - G(0;z_2)G(z_2;1)\big) \\
&+\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^{z_2} \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_4) \qquad (4.34)\\
&\qquad = \int_0^1 \frac{\mathrm{d}z_2}{z_{52}} G(0;z_2) \times \big(i\pi\, G(0;z_2) - G(0;z_2)G(0;z_2) - G(0,z_2;z_2)\big) \\
&-\int_0^1 \frac{\mathrm{d}z_2}{z_{52}} \int_0^{z_2} \frac{\mathrm{d}z_4}{z_{24}} G(0;z_2)G(z_4;z_2) \qquad (4.35)\\
&\qquad = \int_0^1 \frac{\mathrm{d}z_2}{z_{52}} G(0;z_2) \times \big(G(z_2,z_2;z_2) - i\pi\, G(z_2;z_2) - G(z_2,0;z_2) + G(0;z_2)G(z_2;z_2)\big)\,.
\end{aligned}$$

As an important distinction from the previous integration (over z_3), the present divergent polylogarithms of the form $G(z_2,\ldots;z_2)$ were generated after integration over z_4, where the endpoint divergence is approached from above by $z_4 > z_2$. Hence, the shuffle regularization

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in this case requires $\delta = 1$ in (4.10), and the techniques of appendix D.3 imply

$$\begin{aligned} G(z_2, z_2; z_2) &= \frac{1}{2}\big(-G(0; z_2) + i\pi\big)\big(-G(0; z_2) + i\pi\big) \\ G(z_2, 0; z_2) &= \big(-G(0; z_2) + i\pi\big)G(0; z_2) + \zeta_2 \\ G(z_2; z_2) &= -G(0; z_2) + i\pi\,, \end{aligned} \tag{4.36}$$

using $G(0, z_2; z_2) = -\zeta_2$ by (D.10). It is interesting to observe that the last line of (4.35) becomes $\frac{1}{2}G(0; z_2)^2 + G(0, z_2; z_2) + \frac{1}{2}\pi^2$, where the term π^2 can be traced back to an interplay between two subtle factors of $i\pi$ from very distinct sources: one from the general z-removal identity (4.31) and the other from the $\delta = 1$ shuffle regularization (4.10).[23]

In addition to the above shuffle regularizations, the following z-removal identities based on $G(0; 1) = G(0, 0; 1) = 0$ are needed to perform the final integration over z_2:

$$\begin{aligned} G(z_2, z_2; 1) &= \frac{1}{2}\Big(G(0; z_2)^2 + G(1; z_2)^2 - \pi^2\Big) - G(0; z_2)G(1; z_2) + i\pi\Big(G(1; z_2) - G(0; z_2)\Big) \\ G(z_2, 0; 1) &= 2\zeta_2 + i\pi G(0; z_2) - G(0, 0; z_2) + G(0, 1; z_2) \\ G(0, z_2; 1) &= -2\zeta_2 - i\pi G(0; z_2) + G(0, 0; z_2) - G(0, 1; z_2) \\ G(z_2; 1) &= G(1; z_2) - G(0; z_2) + i\pi\ . \end{aligned} \tag{4.37}$$

In combination with the shuffle algebra (4.2), the identities in (4.37) yield the following results for the remaining integral over z_2 (setting $z_5 = 1$):

$$\begin{aligned} (4.32) &= \frac{1}{2}\zeta_2^2 - 2i\pi\zeta_3\,, & (4.33) &= \frac{17}{10}\zeta_2^2\,, \\ (4.34) &= \frac{7}{5}\zeta_2^2 + 2i\pi\zeta_3\,, & (4.35) &= -\frac{16}{5}\zeta_2^2\,. \end{aligned} \tag{4.38}$$

Finally, summing the above results yields the regularized value of the integral (4.29),

$$\mathrm{reg}\int_0^1 \mathrm{d}z_2 \int_{z_2}^1 \mathrm{d}z_4 \int_{z_2}^{z_4} \mathrm{d}z_3 \frac{G(0; z_2)}{z_{52}z_{24}z_{43}} = \frac{2}{5}\zeta_2^2 = \zeta_4\,. \tag{4.39}$$

Using the prescription (3.31), this implies that the Z-theory equation of motion contains the term $-\alpha'^4\zeta_4\partial_{12}[\Phi_1, \Phi_{5243}]$, in agreement with the Berends-Giele recursion at order α'^4 previously obtained from an ansatz.

5 Closed-string integrals

Our results have a natural counterpart for closed-string scattering, where tree-level amplitudes involve integrals over worldsheets of sphere topology. Similar to the characterization

[23] Fortunately, the independent proposal for the regularized value for the integral (4.29) inspired by the methods of [5] and described in the appendix E allowed us to fix all these subtleties. This ultimately led us to our final regularization prescription that has ever since passed many tests at much higher order in α'.

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of disk integrals (2.2) via two cycles P and Q, any sphere integral in tree-level amplitudes of the type II superstring[24] boils down to

$$W(P|Q) \equiv \left(\frac{\alpha'}{\pi}\right)^{n-3} \int_{\mathbb{C}^n} \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \cdots \mathrm{d}^2 z_n}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C}))} \prod_{i<j}^{n} |z_{ij}|^{\alpha' s_{ij}}\, C(P)\, \overline{C}(Q)\,. \tag{5.1}$$

The inverse volume of the conformal Killing group $\mathrm{SL}(2,\mathbb{C})$ of the sphere generalizes (2.5) in an obvious manner, and $\overline{C}(Q)$ denotes the complex conjugate of the chain (2.1) of worldsheet propagators with $z_{ij} \to \overline{z}_{ij}$.

While the field-theory limit of the sphere integrals (5.1) yields the same doubly partial amplitudes as the corresponding disk integrals [61],

$$m(A,n|B,n) = \lim_{\alpha'\to 0} W(A,n|B,n)\,, \tag{5.2}$$

only a subset of the α'-corrections in $Z(P|Q)$ can be found in the closed string (5.1). These selection rules obscured by the KLT relations [14] have been identified to all orders in [47] and realize the single-valued projection "sv" [81, 82] of the MZVs in the disk integrals [61, 83]

$$W(P|Q) = \mathrm{sv}\big[Z(P|Q)\big]\ . \tag{5.3}$$

The single-valued map projects Riemann zeta values to their representatives of odd weights, $\mathrm{sv}(\zeta_{2n}) = 0$ and $\mathrm{sv}(\zeta_{2n+1}) = 2\zeta_{2n+1}$, and acts on MZVs (2.18) of depth $r \geq 2$ in a manner explained in [81, 82]. As an immediate consequence of (5.3), the Berends-Giele representation

$$W(A,n|B,n) = s_A\, \mathrm{sv}\big[\phi_{A|B}\big]\,, \tag{5.4}$$

of closed-string integrals can be derived from the same currents $\phi_{A|B}$ which govern the disk integrals via (3.1). Hence, any tentative "single-valued Z-theory" defined by reproducing the closed-string integrals (5.1) as its doubly partial amplitudes is necessarily contained in the non-abelian Z-theory of this paper.

Note that reality of the sphere integrals $W(P|Q)$ along with the phase-space constraint $s_A = 0$ for n on-shell particles with $P = (A,n)$ implies that single-valued currents obey the following on-shell properties

$$\mathrm{sv}\big[\phi_{A|B}\big] = \mathrm{sv}\big[\phi_{B|A}\big] + \mathcal{O}(s_A)\,, \qquad \mathrm{sv}\big[\phi_{P\sqcup\!\sqcup Q|B}\big] = \mathcal{O}(s_A)\,. \tag{5.5}$$

Hence, one can perform field redefinitions such as to render the associated perturbiner $\mathrm{sv}[\Phi]$ Lie-algebra valued in both gauge groups.

[24] The same kind of organization in terms of (5.1) is expected to be possible in tree-level amplitudes of the heterotic string and the bosonic string. This would imply the universality of gravitational tree-level interactions in these theories whenever their order of α' ties in with the weight of the accompanying MZV [44, 47].

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6 Conclusions and outlook

We have proposed a recursive method to calculate the α'-expansion of disk integrals present in the massless n-point tree-level amplitudes of the open superstring [11, 37]. As a backbone of this method, the disk integrals themselves are interpreted as the tree amplitudes in an effective field theory of bi-colored scalars Φ, dubbed as Z-theory in previous work [7]. Its equation of motion (3.31) furnishes the central result of this work and compactly encodes the Berends-Giele recursions that elegantly compute the α'-expansions of the disk integrals at arbitrary multiplicity. More precisely, the Z-theory equation of motion (3.31) is satisfied by the perturbiner series of the Berends-Giele currents, and its structure is shared by an $(n-2)!$-term representation of the n-point open-string tree amplitude derived in [11].

As a practical result of this work, the BG recursion relations for disk integrals $Z(P|Q)$ with any given words P and Q of arbitrary multiplicity is made publicly available up to order α'^7 in a `FORM` [79, 80] program called `BGap`. In order to ease replication, the auxiliary computer programs used in the derivation of the BG recursion via regularized polylogarithms are also available to download on the website [29].

As a conceptual benefit of this computational achievement, the Berends-Giele description of disk integrals sheds new light on the double-copy structure of the open-string tree-level S-matrix [5]. As manifested by (2.17), disk amplitudes exhibit a KLT-like factorization into SYM amplitudes and disk integrals $Z(P|Q)$. Following the interpretation of $Z(P|Q)$ as Z-theory amplitudes [7], the perturbiner description of the Berends-Giele recursion for disk integrals pinpoints the field equation (3.31) of Z-theory. Hence, our results give a more precise definition of Z-theory, the second double-copy component of open superstrings.

6.1 Further directions

To conclude, we would like to mention an incomplete selection of the numerous open questions raised by the results of this work.

The non-linear equation of motion (3.31) of Z-theory gives rise to wonder about a Lagrangian origin. Moreover, the form of (3.31) is suitable for (partial) specialization to abelian generators in gauge group of the integration domain. Hence, we will explore the implications of our results for the α'-corrections to the NLSM [7] as well as mixed Z-theory amplitudes involving both bi-colored scalars and NLSM pions in future work [84].

Do worldsheet integrals over higher-genus surfaces admit a similar interpretation as Z-theory amplitudes? It might be rewarding to approach the low-energy expansion of superstring loop amplitudes at higher multiplicity with Berends-Giele methods. At the one-loop order, this concerns annulus integrals involving elliptic multiple zeta values [85–87] and torus integrals involving modular graph functions [88–96].

Is there an efficient BCFW description of Z-theory amplitudes? Given that BCFW on-shell recursions [97] can in principle be applied string amplitudes [98–100], it would be interesting to relate the Berends-Giele recursion for Z-theory amplitudes to BCFW methods.

Furthermore, what are the non-perturbative solutions to the full Z-theory equation of motion (3.31)? A non-perturbative solution to the field equation $\Box\Phi = \Phi^2$ of bi-adjoint

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scalars (obtained from the field-theory limit $\alpha' \to 0$) has been recently found [101] in an attempt to understand the non-perturbative regime of the double-copy construction.

In addition, is it possible to obtain field equations or effective actions for massless open- or closed-superstring states along similar lines of (3.31)? In order to approach the α'-corrections to the SYM action, the resemblance of such an equation of motion with the Berends-Giele description of superfields in pure spinor superspace [9, 64] is intriguing. This parallel might for instance be useful in generating the α'-corrections to the on-shell constraint $\{\nabla_\alpha, \nabla_\beta\} - \gamma^m_{\alpha\beta}\nabla_m = 0$ of ten-dimensional SYM [102–107].

Related to this, it would be desirable to express the Z-theory equation of motion and tentative corollaries for superstring effective actions in terms of the Drinfeld associator. Given that disk integrals in a basis (2.14) of ${F_P}^Q$ have been recursively computed from the associator [6], we expect that suitable representations of its arguments allow to cast the α'-expansion of the Berends-Giele recursion into a similarly elegant form. One could even envision to generate the tree-level effective action of the open superstring from the SYM action by acting with appropriate operator-valued arguments of the associator.

Finally, a rigorous mathematical justification for the various prescriptions used in "converting" the open string amplitude (3.30) to the Z-theory equation of motion was not the subject of this paper but clearly deserves further investigation. In particular, it seems mysterious to us at this point why the Z-theory setup selects the regularization scheme for $G(0;z)$, $G(z;z)$, the integration orders, and the change of basis presented in section 4.

Acknowledgments

We are grateful to Johannes Brödel, John Joseph Carrasco and Ellis Yuan for combinations of valuable discussions and fruitful collaboration on related topics. We are indebted to Erik Panzer for indispensable email exchange and numerous enlightening discussions, in particular for guidance on the subtleties of polylog regularization and different orders of integration. Also, we would like to thank Johannes Brödel and Erik Panzer for helpful comments on an initial draft. The authors would like to thank IAS at Princeton where this work was initiated as well as Nordita and in particular Paolo Di Vecchia and Henrik Johansson for providing stimulating atmosphere, support and hospitality through the "Aspects of Amplitudes" program. CRM thanks the Albert-Einstein-Institut in Potsdam for hospitality during the final stages of this work. CRM is supported by a University Research Fellowship from the Royal Society, and gratefully acknowledges support from NSF grant number PHY 1314311 and the Paul Dirac Fund during the initial phase of this work.

A Symmetries of Berends-Giele double currents

In this appendix we discuss the symmetries obeyed by the Berends-Giele double currents.

A.1 Shuffle symmetry

In order to make sure that our ansaetze for BG currents (3.1) for disk integrals satisfy the shuffle-symmetry $\phi_{A|P\sqcup Q} = 0$, we will need the generalization of the result proven in the

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appendix of [9]. That is, in a deconcatenation (into non-empty words X_i) of the form

$$\phi_P = \sum_{X_1X_2=P} H_{X_1,X_2} + \sum_{X_1X_2X_3=P} H_{X_1,X_2,X_3} + \sum_{X_1X_2X_3X_4=P} H_{X_1,X_2,X_3,X_4} + \cdots , \tag{A.1}$$

if $H_{X_1,X_2,\ldots,X_n}$ satisfies shuffle symmetries within each individual slot and collectively on all the slots (treating each X_i as a single letter)

$$H_{X_1,X_2,\ldots,A\sqcup B,\ldots,X_n} = 0\,, \qquad H_{(X_1,X_2,\ldots,X_j)\sqcup(X_{j+1},\ldots,X_n)} = 0\,, \qquad j = 1,2,\ldots,n-1\,, \tag{A.2}$$

then ϕ_P in (A.1) is expected to satisfy the shuffle symmetry for words of arbitrary length,

$$\phi_{R\sqcup S} = 0, \quad \forall\ R,S \neq \emptyset\,. \tag{A.3}$$

It would be interesting to rigorously derive the symmetry in (A.3) from the properties (A.2) of the deconcatenations in (A.1), possibly along the lines of the appendix of [9].

A.2 Generalized Jacobi symmetry

The definition of $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ in (3.13) implies the shuffle symmetries (3.14) in the B_j-slots at fixed ordering of the A_j-slots. This raises the question about the dual symmetry properties when the A_j-slots are permuted at a fixed ordering of the B_j-slots. For this purpose it is convenient to use the left-to-right Dynkin bracket mapping ℓ defined by $\ell(A_1) = A_1$ and [31, 33],

$$\ell(A_1,A_2,\ldots,A_n) = \ell(A_1,A_2,\ldots,A_{n-1}),A_n - A_n,\ell(A_1,A_2,\ldots,A_{n-1}) \tag{A.4}$$

such as $\ell(A_1,A_2) = (A_1,A_2) - (A_2,A_1)$ and $\ell(A_1,A_2,A_3) = (A_1,A_2,A_3) - (A_2,A_1,A_3) - (A_3,A_1,A_2) + (A_3,A_2,A_1)$. One can show that (A.4) projects to the symmetries of nested commutators with

$$\ell([[\ldots[[A_1,A_2],A_3]\ldots],A_n]) = n[[\ldots[[A_1,A_2],A_3]\ldots],A_n]\,. \tag{A.5}$$

Lemma 1. *The object $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ defined by (3.13) satisfies the generalized Jacobi symmetries in the A_j-slots, i.e. the symmetries of nested commutators*

$$T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n} \quad \longleftrightarrow \quad [\ldots[[A_1,A_2],A_3],\ldots,A_n] \tag{A.6}$$

such as $T^{B_1,B_2}_{A_1,A_2} = -T^{B_1,B_2}_{A_2,A_1}$ and $T^{B_1,B_2,B_3}_{A_1,A_2,A_3} + T^{B_1,B_2,B_3}_{A_2,A_3,A_1} + T^{B_1,B_2,B_3}_{A_3,A_1,A_2} = 0$.

Proof. According to (A.5) it suffices to show that

$$T^{B_1,B_2,\ldots,B_n}_{\ell(A_1,A_2,\ldots,A_n)} = nT^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}\,, \tag{A.7}$$

which in turn follows from

$$T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n} = T^{\rm dom}_{\ell(A_1,A_2,\ldots,A_n)} \otimes T^{\rm int}_{B_1,B_2,\ldots,B_n}\,, \tag{A.8}$$

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since the Dynkin bracket satisfies $\ell^2(A_1,\ldots,A_n) = n\ell(A_1,\ldots,A_n)$ [33]. One can conveniently verify (A.8) by induction:

$$\begin{aligned} T^{\rm dom}_{\ell(A_1,A_2,\ldots,A_n)} \otimes T^{\rm int}_{B_1,B_2,\ldots,B_n} &= \big(T^{\rm dom}_{\ell(A_1,A_2,\ldots,A_{n-1}),A_n} - T^{\rm dom}_{A_n,\ell(A_1,A_2,\ldots A_{n-1})}\big) \otimes T^{\rm int}_{B_1,B_2,\ldots,B_n} \\ &= \phi_{A_n|B_n} T^{\rm dom}_{\ell(A_1,A_2,\ldots,A_{n-1})} \otimes T^{\rm int}_{B_1,B_2,\ldots,B_{n-1}} \\ &\quad -\phi_{A_n|B_1} T^{\rm dom}_{\ell(A_1,A_2,\ldots A_{n-1})} \otimes T^{\rm int}_{B_2,\ldots,B_n} \\ &= \phi_{A_n|B_n} T^{B_1,B_2,\ldots,B_{n-1}}_{A_1,A_2,\ldots,A_{n-1}} - \phi_{A_n|B_1} T^{B_2,B_3,\ldots,B_n}_{A_1,A_2,\ldots,A_{n-1}} \,. \end{aligned} \tag{A.9}$$

In the first line, we apply the recursive definition (A.4) of the Dynkin bracket operator, followed by the definition (3.12) of the tensor product $T^{\rm dom}_{\ldots} \otimes T^{\rm int}_{\ldots}$ in the second line. In passing to the third line, we have used the inductive assumption, i.e. (A.8) at $n \to n-1$, and the resulting expression can be identified with the recursive definition (3.13) of $T^{B_1,B_2,\ldots,B_n}_{A_1,A_2,\ldots,A_n}$ which finishes the proof. □

Note that $\rho^2(A_1,\ldots,A_n) = n\rho(A_1,\ldots,A_n)$ [31] and (A.5) imply a duality between the shuffle symmetry of the B_j slots and the generalized Jacobi symmetry of the A_j slots,

$$T^{\rho(B_1,B_2,\ldots,B_n)}_{A_1,A_2,\ldots,A_n} = T^{B_1,B_2,\ldots,B_n}_{\ell(A_1,A_2,\ldots,A_n)} \,. \tag{A.10}$$

A.3 Berends-Giele double current and nested commutators

As discussed above, the BG double current satisfies generalized Jacobi symmetries within the A_j slots. This means that its expansion in terms of products of $\phi_{A_i|B_j}$ can be written as linear combinations of $T^{B_1,\ldots,B_n}_{A_1,\ldots,A_n}$ as, according to Lemma 1, they encode the symmetries of nested commutators. For example, the following terms of order α'^2 that multiply the factor $(k_{A_1}\cdot k_{A_2})$ in (3.7)

$$\phi_{A_1|B_1}\phi_{A_2|B_3}\phi_{A_3|B_2} - \phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3} + \phi_{A_1|B_3}\phi_{A_2|B_1}\phi_{A_3|B_2} - \phi_{A_1|B_3}\phi_{A_2|B_2}\phi_{A_3|B_1} \tag{A.11}$$

are equal to $T^{B_1,B_2,B_3}_{A_2,A_3,A_1}$. This is easy to verify but hard to obtain when the expressions are large. Fortunately, one can use an efficient algorithm due to Dynkin, Specht and Wever (for a pedagogical account, see [63]) to find the linear combinations of $T^{B_1,\ldots,B_n}_{A_1,\ldots,A_n}$ that capture the products of $\phi_{A_i|B_j}$. The solution exploits the fact that the Dynkin bracket ℓ gives rise to a Lie idempotent; $\theta_n \equiv \frac{1}{n}\ell(A_1,\ldots,A_n)$. Therefore, rewriting each word of length n within a Lie polynomial as $\frac{1}{n}\ell(P)$ leads to the answer, e.g., $ab-ba = \frac{1}{2}\ell(ab) - \frac{1}{2}\ell(ba) = \ell(ab)$.

In order to apply this algorithm to products of $\phi_{A_i|B_j}$, first rewrite its products such that the B_j labels are always in the same order $B_1B_2B_3$. For example, (A.11) becomes,

$$\begin{aligned} \phi_{A_1|B_1}\phi_{A_3|B_2}\phi_{A_2|B_3} - \phi_{A_1|B_1}\phi_{A_2|B_2}\phi_{A_3|B_3} + \phi_{A_2|B_1}\phi_{A_3|B_2}\phi_{A_1|B_3} - \phi_{A_3|B_1}\phi_{A_2|B_2}\phi_{A_1|B_3} \\ \equiv L_1L_3L_2 - L_1L_2L_3 + L_2L_3L_1 - L_3L_2L_1 \,, \end{aligned} \tag{A.12}$$

where in the second line we used the shorthand notation $\phi_{A_i|B_1}\phi_{A_j|B_2}\phi_{A_k|B_3} \equiv L_iL_jL_k$ with non-commutative variables $L_{\ldots}$. Applying the idempotent operator θ_n one obtains

$$\begin{aligned} (\text{A.12}) &= \frac{1}{3}\ell(L_1,L_3,L_2) - \frac{1}{3}\ell(L_1,L_2,L_3) + \frac{1}{3}\ell(L_2,L_3,L_1) - \frac{1}{3}\ell(L_3,L_2,L_1) \\ &= -\frac{1}{3}\ell(L_1,\ell(L_2,L_3)) - \frac{1}{3}\ell(L_1,\ell(L_2,L_3)) + \frac{1}{3}\ell(L_1,\ell(L_3,L_2)) \\ &= -\ell(L_1,\ell(L_2,L_3)) = \ell(L_2,L_3,L_1) \equiv T^{B_1,B_2,B_3}_{A_2,A_3,A_1} \,, \end{aligned}$$

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where we used the property $\ell(a_1,a_2,i) = -\ell(i,\ell(a_1,a_2))$ [33]. This algorithm has been used to cast the α'-expansion of the BG double current in terms of the definition (3.13).

B Ansatz for the Berends-Giele recursion at higher order in α'

As explicitly tested up to and including order α'^4, one arrives at a unique recursion for the Berends-Giele double current $\phi_{A|B}$ that reproduces, via (3.1), the disk integrals at various $\alpha'^{w\geq 2}$-orders by imposing the following constraints on an ansatz of the form in (3.6):

1. adjusting the powers of momenta and fields to the mass dimensions of the α'^w-order
2. reflection symmetry in both slots A and B as well as shuffle symmetry in the B slot
3. absence of dot products $(k_{A_i}\cdot k_{B_j})$, $(k_{B_i}\cdot k_{B_j})$ and $k_{A_i}^2$
4. absence of dot products $(k_{A_1}\cdot k_{A_p})$ referring to the outermost slots in $\sum_{A=A_1A_2\ldots A_p}$
5. matching the order-α'^w recursion with known n-point disk integrals for all $n\leq w+3$

By dimensional analysis and triviality of the three-point integral, the BG recursion of the disk integrals at a given order is captured by the following number of fields and derivatives,

$$(\text{order } \alpha'^w) \;\leftrightarrow\; (k_{A_i}\cdot k_{A_j})^p \phi_{A_1|B_{i_1}}\phi_{A_2|B_{i_2}}\cdots\phi_{A_{w+2-p}|B_{i_{w+2-p}}}\,, \quad p=0,1,\ldots,w-1\,,$$

e.g. the ansatz of the form (3.6) for the $\alpha'^2\zeta_2$-order generalizes to three types of terms with schematic form $k^4\phi^3$, $k^2\phi^4$, ϕ^5 along with $\alpha'^3\zeta_3$,

$$(\text{order } \alpha'^3) \;\leftrightarrow\; (k_{A_p}\cdot k_{A_q})(k_{A_r}\cdot k_{A_s})\prod_{j=1}^{3}\phi_{A_j|B_{i_j}}\,, \quad (k_{A_p}\cdot k_{A_q})\prod_{j=1}^{4}\phi_{A_j|B_{i_j}}\,, \quad \prod_{j=1}^{5}\phi_{A_j|B_{i_j}}\,.$$

C Regular parts of five-point integrals

The contributions to $\Box\Phi$ of order Φ^4 in the fields are governed by the α'-expansion of regularized five-point integrals, see (3.33). In the regularization scheme explained in section 4, the relevant leading orders are given by

$$\int^{\text{eom}}\prod_{i<j}^{4}|z_{ij}|^{\alpha'\partial_{ij}}\frac{1}{z_{12}z_{23}} = 2\alpha'\zeta_3(\partial_{24}+\partial_{34})+\mathcal{O}(\alpha'^2) \tag{C.1}$$

$$\int^{\text{eom}}\prod_{i<j}^{4}|z_{ij}|^{\alpha'\partial_{ij}}\frac{1}{z_{13}z_{32}} = -\alpha'\zeta_3(3\partial_{24}+\partial_{34})+\mathcal{O}(\alpha'^2)$$

$$\int^{\text{eom}}\prod_{i<j}^{4}|z_{ij}|^{\alpha'\partial_{ij}}\frac{1}{z_{12}z_{34}} = -\zeta_2+\alpha'\zeta_3(\partial_{12}+2\partial_{13}+2\partial_{23}+2\partial_{24}+\partial_{34})+\mathcal{O}(\alpha'^2)$$

$$\int^{\text{eom}}\prod_{i<j}^{4}|z_{ij}|^{\alpha'\partial_{ij}}\frac{1}{z_{13}z_{24}} = \zeta_2+\alpha'\zeta_3(-2\partial_{12}-\partial_{13}-3\partial_{23}-\partial_{24}-2\partial_{34})+\mathcal{O}(\alpha'^2)$$

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$$\int^{\text{eom}} \prod_{i<j}^{4} |z_{ij}|^{\alpha' \partial_{ij}} \frac{1}{z_{42} z_{23}} = -\alpha' \zeta_3 (\partial_{12} + 3\partial_{13}) + \mathcal{O}(\alpha'^2)$$
$$\int^{\text{eom}} \prod_{i<j}^{4} |z_{ij}|^{\alpha' \partial_{ij}} \frac{1}{z_{43} z_{32}} = 2\alpha' \zeta_3 (\partial_{12} + \partial_{13}) + \mathcal{O}(\alpha'^2)\,,$$

while the terms at higher orders in α' can be found in the ancillary files. Note that the integrals over $(z_{12}z_{23})^{-1}$ and $(z_{43}z_{32})^{-1}$ have been assembled from the simpset basis (4.19).

D Multiple polylogarithm techniques

D.1 Polylogarithms and MZVs

Polylogarithms $G(a_1, a_2, \ldots, a_n; 1)$ at unit argument with labels $a_i \in \{0, 1\}$ can be converted to MZVs via (4.3) provided that $a_1 = 0$ and $a_n = 1$ prevent endpoint divergences. Divergent iterated integrals $G(1, \ldots; 1)$ and $G(\ldots, 0; 1)$ in this work will be shuffle-regularized based on the special cases $G(1; 1) = G(0; 1) = 0$ of (4.8). At weight two and three, the appearance of ζ_2 and ζ_3 in (3.25) can be traced back to

$$\begin{aligned}
G(1,0;1) &= +\zeta_2, & G(0,1;1) &= -\zeta_2 & & \\
G(1,0,0;1) &= -\zeta_3, & G(0,1,0;1) &= +2\zeta_3, & G(0,0,1;1) &= -\zeta_3 \\
G(1,1,0;1) &= +\zeta_3, & G(1,0,1;1) &= -2\zeta_3, & G(0,1,1;1) &= +\zeta_3\,.
\end{aligned} \tag{D.1}$$

The analogous higher-weight relations follow from (4.3), while several identities among MZVs can be found in [108] (obtained using harmonic polylogarithms [109]).

D.2 Methods for shuffle regularization

By the shuffle algebra (4.2), the regularized values (4.8) and (4.10) for weight-one cases $G(0; z)$ and $G(z; z)$ propagate to divergent multiple polylogarithms at higher weight, e.g.

$$G(A, a_{n-1}, 0; z) = G(A, a_{n-1}; z) G(0; z) - G(A \sqcup\!\sqcup 0, a_{n-1}; z)\,, \qquad a_{n-1} \neq 0 \tag{D.2}$$
$$G(z, a_2, A; z) = G(z; z) G(a_2, A; z) - G(a_2, z \sqcup\!\sqcup A; z)\,, \qquad a_2 \neq z\,. \tag{D.3}$$

In case of (D.2), $a_{n-1} \neq 0$ implies that $G(0; z) \equiv \ln|z|$ captures the entire endpoint divergence from the lower integration limit. The same kind of shuffle operations including $G(0, 0; z) = \frac{1}{2} G(0; z)^2$ allows to reduce cases with multiple terminal labels 0 such as $G(A, a_{n-2}, 0, 0; z)$ with $a_{n-2} \neq 0$ to convergent polylogarithms and polynomials in $G(0; z)$ [69]. Analogous statements based on a regularization prescription for $G(z; z)$ can be made for upper-endpoint divergences in integrals like $G(z, z, \ldots, z, a_k, \ldots, a_n; z)$ with $a_k \neq z$.

D.3 z-removal identities

The definition (4.1) of polylogarithms applies to situations where the integration variable z only appears on the right of the semicolon in $G(a_1, a_2, \ldots, a_n; z)$, i.e. to labels $a_j \neq z$.

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This appendix is devoted to integration techniques for polylogarithms with more general arguments, i.e. with multiple appearances of the integration variable z as $G(\ldots,z,\ldots;z)$ or $G(\ldots,z,\ldots;b)$ with $b\neq z$. These techniques rely on rewritings such as [5],

$$G(a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_n;z) = \int_0^z \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\,G(a_1,\ldots,a_{i-1},t,a_{i+1},\ldots,a_n;t) + c(a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n)\,, \tag{D.4}$$

with appropriate initial value $c(a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n)$ at $z=0$. The total derivative in (D.4) can be evaluated through the differential equations ($\hat{a}_j$ means that a_j is omitted)

$$\begin{aligned}
\frac{\partial}{\partial z}G(\vec{a};z) &= \frac{1}{z-a_1}G(a_2,\ldots,a_n;z) \qquad (D.5)\\
\frac{\partial}{\partial a_i}G(\vec{a};z) &= \frac{1}{a_{i-1}-a_i}G(\ldots,\hat{a}_{i-1},\ldots;z)+\frac{1}{a_i-a_{i+1}}G(\ldots,\hat{a}_{i+1},\ldots;z)\\
&\quad+\left(\frac{1}{a_i-a_{i-1}}-\frac{1}{a_i-a_{i+1}}\right)G(\ldots,\hat{a}_i,\ldots;z)\,,\qquad i\neq 1,n\\
\frac{\partial}{\partial a_n}G(\vec{a};z) &= \frac{1}{a_{n-1}-a_n}G(\ldots,\hat{a}_{n-1},a_n;z)+\left(\frac{1}{a_n-a_{n-1}}-\frac{1}{a_n}\right)G(\ldots,a_{n-1};z)\,.
\end{aligned}$$

D.3.1 Simple z-removal identities

Let us first address the simpler subset of z-removal identities, where the integration variable is present on both sides of the semicolon, i.e. cases of the schematic form $G(\ldots,z,\ldots;z)$. Inserting the differential equations (D.5) into (D.4) recursively eliminates the variable z from the labels [5],

$$\begin{aligned}
G(a_1,\ldots,a_{i-1},z,a_{i+1},\ldots,a_n;z) &= c(a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n) \qquad (D.6)\\
&\quad+G(a_{i-1},a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n;z)\\
&\quad-\int_0^z\frac{\mathrm{d}t}{t-a_{i-1}}\,G(a_1,\ldots,\hat{a}_{i-1},t,a_{i+1},\ldots,a_n;t)\\
&\quad-G(a_{i+1},a_1,\ldots,a_{i-1},\hat{z},a_{i+1},\ldots,a_n;z)\\
&\quad+\int_0^z\frac{\mathrm{d}t}{t-a_{i+1}}\,G(a_1,\ldots,a_{i-1},t,\hat{a}_{i+1},\ldots,a_n;t)\\
&\quad+\int_0^z\frac{\mathrm{d}t}{t-a_1}\,G(a_2,\ldots,a_{i-1},t,a_{i+1},\ldots,a_n;t),\qquad i\neq 1,n\,,
\end{aligned}$$

with the following specialization for when z is the rightmost label (with $n\neq 1$):

$$\begin{aligned}
G(a_1,\ldots,a_{n-1},z;z) &= c(a_1,\ldots,a_{n-1},\hat{z})+G(a_{n-1},a_1,\ldots,a_{n-1};z)-G(0,a_1,\ldots,a_{n-1};z)\\
&\quad-\int_0^z\frac{\mathrm{d}t}{t-a_{n-1}}\,G(a_1,\ldots,a_{n-2},t;t)+\int_0^z\frac{\mathrm{d}t}{t-a_1}\,G(a_2,\ldots,a_{n-1},t;t)\,.
\end{aligned} \tag{D.7}$$

Similar recursions for repeated appearance of z among the labels as in $G(\ldots,z,z,\ldots;z)$ can be derived from (D.5) and (D.4) in exactly the same manner.

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The integration constants $c(\ldots,\hat{z},\ldots)$ in (D.4) are generically zero unless the labels are exclusively formed from letters $a_j \in \{0,\hat{z}\}$, in which case they yield MZVs (4.3):

$$c(a_1,a_2,\ldots,a_n) = \begin{cases} 0 & : \exists\ a_j \notin \{0,\hat{z}\} \\ G(\frac{a_1}{\hat{z}},\frac{a_2}{\hat{z}},\ldots,\frac{a_n}{\hat{z}};1) & : a_j \in \{0,\hat{z}\} \end{cases} \tag{D.8}$$

The simplest nonzero applications of (D.8) at weight two and three are

$$c(0,\hat{z}) = -\zeta_2, \quad c(\hat{z},0) = +\zeta_2, \quad c(0,0,\hat{z}) = c(\hat{z},0,0) = -\zeta_3, \quad c(0,\hat{z},0) = 2\zeta_3 \tag{D.9}$$

and follow from (D.1). For example, the above steps lead to the z-removal identities[25]

$$\begin{aligned}
G(a_1,z;z) &= G(a_1,a_1;z) - G(0,a_1;z) - \delta_{a_1,0}\zeta_2\,, \\
G(a_1,a_2,z;z) &= G(a_2,0,a_1;z) - G(a_2,a_1,a_1;z) + G(a_1,a_2,a_2;z) \\
&\quad -G(a_1,0,a_2;z) + G(a_2,a_1,a_2;z) - G(0,a_1,a_2;z) \\
&\quad -\delta_{a_2,0}G(a_1;z)\zeta_2 + \delta_{a_1,0}G(a_2;z)\zeta_2 - \delta_{a_1,0}\delta_{a_2,0}\zeta_3 \\
G(a_1,z,a_2;z) &= G(a_1,a_1,a_2;z) - G(a_2,0,a_1;z) + G(a_2,a_1,a_1;z) \\
&\quad -G(a_2,a_1,a_2;z) - \delta_{a_1,0}G(a_2;z)\zeta_2 + 2\delta_{a_1,0}\delta_{a_2,0}\zeta_3 \\
G(a,z,z;z) &= G(0,0,a;z) - G(0,a,a;z) - G(a,0,a;z) + G(a,a,a;z) + \delta_{a,0}\zeta_3\,.
\end{aligned} \tag{D.10}$$

Note that analogous z-removal identities for $G(z,a_1;z)$, $G(z,a_1,a_2;z)$ and other divergent cases follow from the shuffle relation (4.2), see (4.10) for the regularized values of $G(z;z)$ that differ from the choice in [5].

D.3.2 General z-removal identities

As exemplified by (4.12), some of the regularized integrals require different orders of integration over the variables $z_2,z_3,\ldots,z_{n-2}$. In these situations it can happen that polylogarithms such as $G(0,z_4;z_3)$ need to be converted to $G(\ldots;z_4)$ with no additional instance of z_4 in the ellipsis in order to integrate over z_4 first. This requires a generalization of the techniques in the previous subsection. As before, the starting point for a recursion is the differential equation (D.4) for derivatives in the labels of polylogarithms. The recursion is supplemented by the initial condition

$$G(z_1;z_2) = G(z_2;z_1) + G(0;z_2) - G(0;z_1) - i\pi\,\mathrm{sign}(z_2,z_1)\,, \tag{D.11}$$

where

$$\mathrm{sign}(z_i,z_j) \equiv \begin{cases} 1 & : z_i < z_j \\ -1 & : z_i > z_j \end{cases}\,. \tag{D.12}$$

For example, the first identity in (D.10) generalizes to

$$G(a_1,z_1;z_2) = G(a_1,0;z_2) - G(a_1,z_2;z_1) - G(0;z_2)G(a_1;z_1) + G(a_1,0;z_1)$$

[25] Note that the identities (E.1) in reference [5] exclude $a_j = 0$ and therefore do not exhibit the constant terms of (D.10) in the analogous identities.

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n-pts	MZVs	BG current	z-removal	Koba-Nielsen
5	ζ_2	ϕ^4	$w=1$	$\ell=0$
	ζ_3	$k^2\phi^4$	$w=2$	$\ell=1$
	ζ_4	$k^4\phi^4$	$w=3$	$\ell=2$
	ζ_5	$k^6\phi^4$	$w=4$	$\ell=3$
	ζ_6	$k^8\phi^4$	$w=5$	$\ell=4$
	ζ_7	$k^{10}\phi^4$	$w=6$	$\ell=5$
6	ζ_3	ϕ^5	$w=2$	$\ell=0$
	ζ_4	$k^2\phi^5$	$w=3$	$\ell=1$
	ζ_5	$k^4\phi^5$	$w=4$	$\ell=2$
	ζ_6	$k^6\phi^5$	$w=5$	$\ell=3$
	ζ_7	$k^8\phi^5$	$w=6$	$\ell=4$
7	ζ_4	ϕ^6	$w=3$	$\ell=0$
	ζ_5	$k^2\phi^6$	$w=4$	$\ell=1$
	ζ_6	$k^4\phi^6$	$w=5$	$\ell=2$
	ζ_7	$k^6\phi^6$	$w=6$	$\ell=3$
8	ζ_5	ϕ^7	$w=4$	$\ell=0$
	ζ_6	$k^2\phi^7$	$w=5$	$\ell=1$
	ζ_7	$k^4\phi^7$	$w=6$	$\ell=2$
9	ζ_6	ϕ^8	$w=5$	$\ell=0$
	ζ_7	$k^2\phi^8$	$w=6$	$\ell=1$
10	ζ_7	ϕ^9	$w=6$	$\ell=0$

Table 1. Summary of the contributions from regularized n-point integrals, the order of MZVs, the schematic form of the Berends-Giele double current, the required weight w of z-removal identities ($G(a_1,\ldots,a_w;z)$) and the order α'^ℓ of the Koba-Nielsen expansion (4.5).

$$
\begin{aligned}
&+G(a_1;z_2)\big[G(a_1;z_1)-G(0;z_1)\big]-2\delta_{a_1,0}\zeta_2\\
&+i\pi\,\mathrm{sign}(z_2,z_1)(G(a_1;z_1)-G(a_1;z_2))\,.
\end{aligned}
\tag{D.13}
$$

Note that the polylogarithms on the right hand side are suitable for integration over z_1 since there are no instances of z_1 among their labels.

The use of z-removal identities represents the most expensive step in the computation of regularized integrals as they tend to increase the number of terms considerably. An overview of the weights of the identities required at a given order of the Berends-Giele recursion is given in table 1. For example, terms at the order of $\alpha'^6\zeta_6\Phi^5$ in the Z-theory equation of motion (3.31) arise from integrating the third subleading order $\sim\alpha'^3$ of the Koba-Nielsen factor (4.5) — the offset is due to the factor $(-\alpha')^{(n-3)}$ in (3.30) — and require z-removal identities for $G(P;z)$ at weight $|P|=5$.

E Alternative description of regularized disk integrals

In this appendix, we present a method to determine the α'-expansions for regularized disk integrals selected by the Z-theory equation of motion from the $(n-3)!\times(n-3)!$ basis

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${F_P}^Q$ defined in (2.14). This approach has been very useful to constrain the required regularization scheme via explicit data at high orders of α', without the need to obtain the Berends-Giele recursion from an ansatz at these orders. However, we only understand this method as an intermediate tool to determine the appropriate regularization scheme selected by the Z-theory equation of motion: the ultimate goal and achievement of this work is to compute α'-expansions of disk integrals at multiplicities and orders where no prior knowledge of ${F_P}^Q$ is available.

Closely following the lines of [5], the basic idea is to divide disk integrals[26] $Z(I|P)$ into a singular and a regular part with respect to region variables $s_{i,i+1\ldots j}$ in (2.4). The singular parts associated with the propagators of the field-theory limits can be subtracted with residues given by lower-multiplicity data, and the leftover local expression is identified with the regularized integrals in (3.31). However, there are ambiguities in the subtraction scheme by shifting the numerator $N \to N + \mathcal{O}(s)$ in the subtracted singular expression N/s by polynomials in the associated Mandelstam invariant $s \equiv s_{i,i+1\ldots j}$. Five-point examples suggest that changes in the regularization scheme or the integration order can be compensated by the choice of subtraction scheme when reproducing the associated local expressions from regularized integrals over Taylor-expanded Koba-Nielsen factors.

In the setup of [5], the regularization scheme for divergent integrals was fixed and designed to preserve the shuffle algebra and scaling relations of polylogarithms such that $G(z;z) \equiv 0$ instead of (4.10). Moreover, the integration orders were globally chosen as $23\ldots n{-}2$ (i.e. integrating over z_2 first and over z_{n-2} in the last step). In all examples under consideration in [5], it was possible to choose a scheme for pole subtraction such that the resulting regular parts could be reproduced by integration in the canonical order $23\ldots n{-}2$ within the given scaling-preserving regularization. In these adjustments of the subtraction scheme, certain regular admixtures were incorporated by systematically shifting the arguments of the lower-point integrals in the above numerators N.

Here, by contrast, we work with a fixed (or "minimal") subtraction scheme for the poles of $Z(I|P)$. The resulting regular parts — to be denoted by $J^{\text{reg}}_{\ldots}(\ldots)$ in the sequel — turn out to exactly reproduce the desired Z-theory equation of motion upon insertion into (3.31). As will become clear from the following examples, this subtraction scheme is canonical in the sense that the aforementioned regular admixtures of [5] are completely avoided, reflecting the different choices of regularization scheme and integration orders between this work and [5].

We will regard $\mathrm{SL}(2,\mathbb{R})$-fixed combinations of disk integrals $Z(P|Q)$ in the notation

$$J_{u_1v_1,u_2v_2,\ldots,u_{n-3}v_{n-3}}(k_1,k_2,\ldots,k_{n-1}) \equiv \alpha'^{n-3} \int_{0\leq z_2\leq z_3\leq\ldots\leq z_{n-2}\leq 1} \frac{\mathrm{d}z_2\,\mathrm{d}z_3\,\ldots\,\mathrm{d}z_{n-2}\,\prod_{i<j}^{n-1}|z_{ij}|^{\alpha' s_{ij}}}{z_{u_1,v_1}z_{u_2,v_2}\ldots z_{u_{n-3},v_{n-3}}} \tag{E.1}$$

[26] For the sake of simplicity, the discussion of [5] and the current appendix is restricted to linear combinations of disk integrals $Z(I|P)$ with the canonical domain $I = 12\ldots n$, where the choices of P only leave a single pole channel in the field-theory limit.

as functions of $n-1$ massless momenta k_j which determine the s_{ij} on the right hand side through their independent dot products. The product $k_1 \cdot k_{n-1}$ can be eliminated by momentum conservation and is absent in (E.1) by the $\mathrm{SL}(2,\mathbb{R})$-fixing $z_1 = 0$ and $z_{n-1} = 1$. This reflects the choice of ansatz in appendix B, where $(k_{A_1} \cdot k_{A_p})$ referring to the outermost slots A_1, A_p in a deconcatenation $\sum_{A=A_1 A_2 \ldots A_p}$ is excluded.

In the four-point case, the field-theory limit of (E.1), which follows from the rules in section 4 of [5] or from (2.28), already exhausts the singular part. Hence, the expressions

$$J_{21}^{\mathrm{reg}}(k_1,k_2,k_3) = J_{21}(k_1,k_2,k_3) - \frac{1}{s_{12}}\,, \qquad J_{32}^{\mathrm{reg}}(k_1,k_2,k_3) = J_{32}(k_1,k_2,k_3) - \frac{1}{s_{23}} \tag{E.2}$$

are analytic in s_{ij} and coincide with the regularized integrals (3.25) [5] in any regularization scheme of our awareness. Their α'-expansion is straightforwardly determined by ${F_2}^2$ in (2.19) (also see [110] for a neat representation in terms of $G(0,\ldots,0,1,\ldots,1;1)$),

$$J_{21}(k_1,k_2,k_3) = \frac{{F_2}^2}{s_{12}}\,, \qquad J_{32}(k_1,k_2,k_3) = \frac{{F_2}^2}{s_{23}}\,. \tag{E.3}$$

The regular parts $J_{ij}^{\mathrm{reg}}(\ldots)$ in (E.2) are by themselves functions of three light-like momenta under $s_{pq} \to k_p \cdot k_q$ and can later on be promoted to massive momenta k_P provided that no reference to k_P^2 is expected.

E.1 Five-point pole subtraction

At five points, generic field-theory limits of $Z(P|Q)$ yield two simultaneous propagators, and by factorization on four-point integrals, the residue on single poles in s_{ij} still involves all orders in α'. As elaborated in [5], the α'-dependence of the singular pieces can be removed using the regular four-point expressions in (E.2) with composite momenta $k_{ij} \equiv k_i + k_j$,

$$\begin{aligned}
J_{21,43}^{\mathrm{reg}}(k_1,k_2,k_3,k_4) &= J_{21,43}(k_1,k_2,k_3,k_4) - \frac{J_{21}^{\mathrm{reg}}(k_1,k_2,k_{34})}{s_{34}} - \frac{J_{32}^{\mathrm{reg}}(k_{12},k_3,k_4)}{s_{12}} - \frac{1}{s_{12}s_{34}} \\
J_{31,42}^{\mathrm{reg}}(k_1,k_2,k_3,k_4) &= J_{31,42}(k_1,k_2,k_3,k_4) \\
J_{21,31}^{\mathrm{reg}}(k_1,k_2,k_3,k_4) &= J_{21,31}(k_1,k_2,k_3,k_4) - \frac{J_{21}^{\mathrm{reg}}(k_1,k_2,k_3)}{s_{123}} - \frac{J_{21}^{\mathrm{reg}}(k_{12},k_3,k_4)}{s_{12}} - \frac{1}{s_{12}s_{123}} \\
J_{32,31}^{\mathrm{reg}}(k_1,k_2,k_3,k_4) &= J_{32,31}(k_1,k_2,k_3,k_4) - \frac{J_{32}^{\mathrm{reg}}(k_1,k_2,k_3)}{s_{123}} - \frac{J_{21}^{\mathrm{reg}}(k_1,k_{23},k_4)}{s_{23}} - \frac{1}{s_{23}s_{123}}\,.
\end{aligned} \tag{E.4}$$

Following the dot products of momenta, arguments k_{12}, k_3, k_4 in the above J_{ij}^{reg} instruct to replace any s_{12} and s_{23} in their expansion from (E.2) and (E.3) by $s_{13} + s_{23}$ and s_{34}, respectively [5]. Note that the counterpart of $J_{21}^{\mathrm{reg}}(k_1,k_{23},k_4)$ in [5] required a different replacement $s_{12} \to s_{123}$ instead of the prescription $s_{12} \to s_{12} + s_{13}$ in (E.4). This kind of dependence on $k_{23}^2 = 2s_{23}$ was inevitable to accommodate with the regularization scheme of the [5] with $G(z;z) \equiv 0$.

In the same way as the α'-dependence of the local four-point expressions $J_{ij}^{\mathrm{reg}}(\ldots)$ is accessible from ${F_2}^2$, their five-point counterparts $J_{ij,pq}^{\mathrm{reg}}(\ldots)$ can be expanded as soon as the

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right hand side of (E.4) is expressed in terms of the basis functions $\{F_{23}{}^{23}, F_{23}{}^{32}\}$,

$$\begin{aligned} J_{21,43}(k_1,k_2,k_3,k_4) &= \frac{F_{23}{}^{23}}{s_{12}s_{34}}\,, & J_{21,31}(k_1,k_2,k_3,k_4) &= \frac{F_{23}{}^{23}}{s_{12}s_{123}} + \frac{F_{23}{}^{32}}{s_{13}s_{123}} \\ J_{31,42}(k_1,k_2,k_3,k_4) &= \frac{F_{23}{}^{32}}{s_{13}s_{24}}\,, & J_{32,31}(k_1,k_2,k_3,k_4) &= \frac{F_{23}{}^{23}}{s_{23}s_{123}} - \left(\frac{1}{s_{13}}+\frac{1}{s_{23}}\right)\frac{F_{23}{}^{32}}{s_{123}}\,. \end{aligned} \tag{E.5}$$

Explicit results on the α'-expansion of $\{F_{23}{}^{23}, F_{23}{}^{32}\}$ as pioneered in [51–54] are available from the all-multiplicity methods based on polylogarithms [5] and the Drinfeld associator [6]. Moreover, recent advances based on their hypergeometric-function representation [45, 110] render even higher orders in α' accessible, also see [45] for a closed-form solution. Once we adjoin the parity images

$$\begin{aligned} J^{\rm reg}_{43,42}(k_1,k_2,k_3,k_4) &= J^{\rm reg}_{21,31}(k_1,k_2,k_3,k_4)\big|_{k_j\to k_{5-j}} \\ J^{\rm reg}_{42,32}(k_1,k_2,k_3,k_4) &= J^{\rm reg}_{32,31}(k_1,k_2,k_3,k_4)\big|_{k_j\to k_{5-j}}\,, \end{aligned} \tag{E.6}$$

one can extract valuable all-weight information on the regularization scheme for five-point integrals in (3.31) by demanding the α'-expansion of (E.4) and (E.6) to match with

$$J^{\rm reg}_{pq,rs}(k_1,k_2,k_3,k_4) = \alpha'^2 \int^{\rm eom} \prod_{i<j}^{4} |z_{ij}|^{\alpha' s_{ij}} \frac{1}{z_{pq}z_{rs}}\,. \tag{E.7}$$

Again, the arguments $s_{ij} \to k_i\cdot k_j$ of $J^{\rm reg}_{pq,rs}$ can be promoted to massive momenta $k_i \to k_P$ as we will now see in the pole subtractions at higher-multiplicity.

E.2 Six and seven-point pole subtraction

The above five-point examples shed light on various aspects of the regularization scheme selected by the Z-theory equation of motion including the integration orderings and the z-removal identities in appendix D.3. However, the appearance of $i\pi$ in (4.10) cannot be seen from integrals below multiplicity six, so the $J^{\rm reg}_{\cdots}(\ldots)$ at $(n\geq 6)$-points have been an instrumental window to infer these particularly subtle ingredients of the regularization scheme. In this section, we present one example each at multiplicity six and seven:

$$\begin{aligned} J^{\rm reg}_{31,32,54}(k_1,k_2,\ldots,k_5) = J_{31,32,54}(k_1,k_2,\ldots,k_5) &- \frac{J^{\rm reg}_{31,32}(k_1,k_2,k_3,k_{45})}{s_{45}} \\ &- \frac{J^{\rm reg}_{32}(k_1,k_2,k_3)J^{\rm reg}_{32}(k_{123},k_4,k_5)}{s_{123}} - \frac{J^{\rm reg}_{21,43}(k_1,k_{23},k_4,k_5)}{s_{23}} \\ &- \frac{J^{\rm reg}_{32}(k_1,k_2,k_3)}{s_{123}s_{45}} - \frac{J^{\rm reg}_{21}(k_1,k_{23},k_{45})}{s_{23}s_{45}} - \frac{J^{\rm reg}_{32}(k_{123},k_4,k_5)}{s_{23}s_{123}} \\ &- \frac{1}{s_{23}s_{123}s_{45}} \end{aligned} \tag{E.8}$$

Note that also the counterparts of $J^{\rm reg}_{21}(k_1,k_{23},k_{45})$ and $J^{\rm reg}_{21,43}(k_1,k_{23},k_4,k_5)$ seen in [5] exhibit additional contributions $\sim s_{23}$ in their arguments. In the $J^{\rm reg}_{\cdots}(\ldots)$ under discussion, however, the argument $s_{23} = \frac{1}{2}k_{23}^2$ is by construction absent in $k_1\cdot k_{23} = s_{12}+s_{13}$.

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At seven points, the local integral used in [5] to generate the expansion of $F_P{}^Q$ up to and including the α'^7-order stored on the website [56] matches with

$$\begin{aligned}
J^{\rm reg}_{21,31,41,65} = &-\frac{1}{s_{12}s_{123}s_{1234}s_{56}} - \frac{J^{\rm reg}_{21}(k_1,k_2,k_3)}{s_{123}s_{1234}s_{56}} - \frac{J^{\rm reg}_{21}(k_{12},k_3,k_4)}{s_{12}s_{1234}s_{56}} - \frac{J^{\rm reg}_{21}(k_{123},k_4,k_{56})}{s_{12}s_{123}s_{56}} \\
&- \frac{J^{\rm reg}_{32}(k_{1234},k_5,k_6)}{s_{12}s_{123}s_{1234}} - \frac{J^{\rm reg}_{21,31}(k_1,k_2,k_3,k_4)}{s_{56}s_{1234}} - \frac{J^{\rm reg}_{21}(k_1,k_2,k_3)J^{\rm reg}_{21}(k_{123},k_4,k_{56})}{s_{123}s_{56}} \\
&- \frac{J^{\rm reg}_{21}(k_1,k_2,k_3)J^{\rm reg}_{32}(k_{1234},k_5,k_6)}{s_{123}s_{1234}} - \frac{J^{\rm reg}_{21,31}(k_{12},k_3,k_4,k_{56})}{s_{12}s_{56}} \\
&- \frac{J^{\rm reg}_{21,43}(k_{123},k_4,k_5,k_6)}{s_{12}s_{123}} - \frac{J^{\rm reg}_{21}(k_{12},k_3,k_4)J^{\rm reg}_{32}(k_{1234},k_5,k_6)}{s_{12}s_{1234}} \\
&- \frac{J^{\rm reg}_{21,31,41}(k_1,k_2,k_3,k_4,k_{56})}{s_{56}} - \frac{J^{\rm reg}_{21,31,54}(k_{12},k_3,k_4,k_5,k_6)}{s_{12}} \\
&- \frac{J^{\rm reg}_{21}(k_1,k_2,k_3)J^{\rm reg}_{21,43}(k_{123},k_4,k_5,k_6)}{s_{123}} - \frac{J^{\rm reg}_{21,31}(k_1,k_2,k_3,k_4)J^{\rm reg}_{32}(k_{1234},k_5,k_6)}{s_{1234}} \\
&+ J_{21,31,41,65} \ . \qquad \text{(E.9)}
\end{aligned}$$

The α'-expansion of the right hand sides of (E.8) and (E.9) is available from the following decompositions into basis functions $F_P{}^Q$:

$$J_{31,32,54} = \frac{F_{234}{}^{234}}{s_{23}s_{45}s_{123}} - \frac{F_{234}{}^{324}}{s_{123}s_{45}}\left(\frac{1}{s_{13}}+\frac{1}{s_{23}}\right) \qquad \text{(E.10)}$$

$$J_{21,31,41,65} = \frac{1}{s_{1234}s_{56}}\left(\frac{F^{2345}_{2345}}{s_{12}s_{123}} + \frac{F^{2435}_{2345}}{s_{12}s_{124}} + \frac{F^{3245}_{2345}}{s_{13}s_{123}} + \frac{F^{3425}_{2345}}{s_{13}s_{134}} + \frac{F^{4235}_{2345}}{s_{14}s_{124}} + \frac{F^{4325}_{2345}}{s_{14}s_{134}}\right)$$

E.3 The general strategy

The choice of labels and momenta for the $J^{\rm reg}_{\cdots}(k_{A_1},k_{A_2},\ldots,k_{A_{m-1}})$ in the above pole subtractions follows from an algorithm explained in section 4.3 of [5]. This algorithm applies to integrals $J_{\cdots}(\ldots)$ of the form (E.1) with a single cubic diagram in their field-theory limit. Each factor of z_{ij}^{-1} in the integrand is associated with one of the $n-3$ propagators of the field-theory diagram, and the pole subtraction exhausts all 2^{n-3} possibilities to relax a subset of these propagators. The residue of diagrams with less than $n-3$ propagators is a $J^{\rm reg}_{\cdots}(\ldots)$ labeled by the z_{ij}^{-1}-factors associated with the relaxed propagators, i.e. each relaxed propagator increases the multiplicity of the associated $J^{\rm reg}_{\cdots}(\ldots)$ by one. The massive momenta in its arguments can be read off from the structure of the leftover propagators in the diagram. The reader is referred to [5] for further details, examples and diagrammatic illustrations.

From these rules, it is straightforward to extract the local parts of integrals at arbitrary multiplicity. We have checked up to and including the 5! integrals at seven points that these $J^{\rm reg}_{\cdots}(\ldots)$ at $s_{ij} \leftrightarrow k_i{\cdot}k_j$ are compatible with the integrals (3.31) in the regularization scheme and integration orders of this work,

$$J^{\rm reg}_{u_1v_1,u_2v_2,\ldots,u_{p-2}v_{p-2}}(k_1,k_2,\ldots,k_p) = (\alpha')^{p-2}\int^{\rm eom}\frac{\prod_{i<j}^p |z_{ij}|^{\alpha' s_{ij}}}{z_{u_1,v_1}z_{u_2,v_2}\ldots z_{u_{p-2},v_{p-2}}} \ . \qquad \text{(E.11)}$$

A variety of alternative regularization schemes and integration orders including those of [5] are expected to correspond to a modified choice of arguments for $J^{\text{reg}}_{\dots}(k_{A_1}, k_{A_2}, \dots, k_{A_{n-1}})$, where selected dot products $k_{A_p} \cdot k_{A_q}$ are shifted by (half of) $k^2_{A_i}$.

F Integration orders for the seven-point integrals

In this appendix, we explicitly list the results of section 4.3 on the integration orders for regularized seven-point integrals in the simpset basis (see section 4.2). The first topology of seven-point integrals is spanned by single factors of $\mathcal{Z}_{1P}$ in (4.13) with $|P| = 4$:

$$
\begin{aligned}
&z_{15}z_{12}z_{13}z_{14} \to 2345, && z_{15}z_{12}z_{34}z_{14} \to (2\sqcup 3)45, && z_{15}z_{23}z_{24}z_{14} \to 3245,\\
&z_{15}z_{13}z_{23}z_{14} \to 2345, && z_{15}z_{13}z_{24}z_{14} \to (2\sqcup 3)45, && z_{15}z_{24}z_{34}z_{14} \to 3245,\\
&z_{15}z_{12}z_{13}z_{45} \to (23\sqcup 4)5, && z_{15}z_{14}z_{24}z_{35} \to (24\sqcup 3)5, && z_{15}z_{13}z_{14}z_{25} \to (34\sqcup 2)5,\\
&z_{15}z_{13}z_{23}z_{45} \to (23\sqcup 4)5, && z_{15}z_{12}z_{14}z_{35} \to (24\sqcup 3)5, && z_{15}z_{14}z_{34}z_{25} \to (34\sqcup 2)5,\\
&z_{15}z_{12}z_{35}z_{45} \to (43\sqcup 2)5, && z_{15}z_{13}z_{24}z_{25} \to (42\sqcup 3)5, && z_{15}z_{14}z_{25}z_{35} \to (32\sqcup 4)5,\\
&z_{15}z_{12}z_{34}z_{35} \to (43\sqcup 2)5, && z_{15}z_{13}z_{25}z_{45} \to (42\sqcup 3)5, && z_{15}z_{14}z_{23}z_{25} \to (32\sqcup 2)5,\\
&z_{15}z_{25}z_{35}z_{45} \to 4325, && z_{15}z_{25}z_{23}z_{45} \to (3\sqcup 4)25, && z_{15}z_{25}z_{24}z_{34} \to 3425,\\
&z_{15}z_{25}z_{34}z_{35} \to 4325, && z_{15}z_{25}z_{24}z_{35} \to (3\sqcup 4)25, && z_{15}z_{25}z_{24}z_{23} \to 3425.
\end{aligned}
\tag{F.1}
$$

Another seven-point topology can be derived from products $\mathcal{Z}_{1P}\mathcal{Z}_{6Q}$ with $|P| = 3,\ |Q| = 1$:

$$
\begin{aligned}
&z_{12}z_{13}z_{14}z_{56} \to 234\sqcup 5, && z_{12}z_{34}z_{14}z_{56} \to ((2\sqcup 3)4)\sqcup 5, && z_{23}z_{24}z_{14}z_{56} \to 324\sqcup 5,\\
&z_{13}z_{23}z_{14}z_{56} \to 234\sqcup 5, && z_{13}z_{24}z_{14}z_{56} \to ((2\sqcup 3)4)\sqcup 5, && z_{24}z_{34}z_{14}z_{56} \to 324\sqcup 5,\\
&z_{12}z_{13}z_{15}z_{46} \to 235\sqcup 4, && z_{12}z_{35}z_{15}z_{46} \to ((2\sqcup 3)5)\sqcup 4, && z_{23}z_{25}z_{15}z_{46} \to 325\sqcup 4,\\
&z_{13}z_{23}z_{15}z_{46} \to 235\sqcup 4, && z_{13}z_{25}z_{15}z_{46} \to ((2\sqcup 3)5)\sqcup 4, && z_{25}z_{35}z_{15}z_{46} \to 325\sqcup 4,\\
&z_{12}z_{14}z_{15}z_{36} \to 245\sqcup 3, && z_{12}z_{45}z_{15}z_{36} \to ((2\sqcup 4)5)\sqcup 3, && z_{24}z_{25}z_{15}z_{36} \to 425\sqcup 3,\\
&z_{14}z_{24}z_{15}z_{36} \to 245\sqcup 3, && z_{14}z_{25}z_{15}z_{36} \to ((2\sqcup 4)5)\sqcup 3, && z_{25}z_{45}z_{15}z_{36} \to 425\sqcup 3,\\
&z_{13}z_{14}z_{15}z_{26} \to 345\sqcup 2, && z_{13}z_{45}z_{15}z_{26} \to ((3\sqcup 4)5)\sqcup 2, && z_{34}z_{35}z_{15}z_{26} \to 435\sqcup 2,\\
&z_{14}z_{34}z_{15}z_{26} \to 345\sqcup 2, && z_{14}z_{35}z_{15}z_{26} \to ((3\sqcup 4)5)\sqcup 2, && z_{35}z_{45}z_{15}z_{26} \to 435\sqcup 2.
\end{aligned}
\tag{F.2}
$$

The seven-point topology of $\mathcal{Z}_{1P}\mathcal{Z}_{6Q}$ with $|P| = |Q| = 2$ for both factors gives rise to the following integration orders,

$$
\begin{aligned}
&z_{12}z_{13}z_{46}z_{56} \to 23\sqcup 54, && z_{13}z_{23}z_{46}z_{56} \to 23\sqcup 54, && z_{12}z_{13}z_{45}z_{46} \to 23\sqcup 54,\\
&z_{13}z_{23}z_{45}z_{46} \to 23\sqcup 54, && z_{12}z_{14}z_{36}z_{56} \to 24\sqcup 53, && z_{14}z_{24}z_{36}z_{56} \to 24\sqcup 53,\\
&z_{12}z_{14}z_{35}z_{36} \to 24\sqcup 53, && z_{14}z_{24}z_{35}z_{36} \to 24\sqcup 53, && z_{13}z_{14}z_{26}z_{56} \to 34\sqcup 52,\\
&z_{14}z_{34}z_{26}z_{56} \to 34\sqcup 52, && z_{13}z_{14}z_{25}z_{26} \to 34\sqcup 52, && z_{14}z_{34}z_{25}z_{26} \to 34\sqcup 52,\\
&z_{12}z_{15}z_{36}z_{46} \to 25\sqcup 43, && z_{15}z_{25}z_{36}z_{46} \to 25\sqcup 43, && z_{12}z_{15}z_{34}z_{36} \to 25\sqcup 43,\\
&z_{15}z_{25}z_{34}z_{36} \to 25\sqcup 43, && z_{13}z_{15}z_{26}z_{46} \to 35\sqcup 42, && z_{15}z_{35}z_{26}z_{46} \to 35\sqcup 42,\\
&z_{13}z_{15}z_{24}z_{26} \to 35\sqcup 42, && z_{15}z_{35}z_{24}z_{26} \to 35\sqcup 42, && z_{14}z_{15}z_{26}z_{36} \to 45\sqcup 32,\\
&z_{15}z_{45}z_{26}z_{36} \to 45\sqcup 32, && z_{14}z_{15}z_{23}z_{26} \to 45\sqcup 32, && z_{15}z_{45}z_{23}z_{26} \to 45\sqcup 32,
\end{aligned}
\tag{F.3}
$$

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and the remaining topologies of the simpset basis at seven points follow from (F.1) and (F.2) via parity $z_j \to z_{7-j}$. The seven-point $\int^{\text{eom}}$-integrals are sufficient to determine the Φ^6-order of the Z-theory equation of motion (3.31) to any order in α' and the α'^4-order of disk integrals at any multiplicity.

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