Minimal conformal supergravity invariants in six dimensions

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Abstract. The off-shell superspace formulation for six-dimensional conformal supergravity given in arXiv:1606.02921 is presented here. We describe how the formulation may be used to describe two invariants for $\mathcal{N} = (1, 0)$ conformal supergravity in six dimensions. One of the invariants contains a C^3 term at the component level, while the other contains a $C\Box C$ term. It is demonstrated that any invariant for minimal conformal supergravity must be a linear combination of these two invariants.

1. Introduction

Conformal gravity invariants play an important role in conformal field theories, where on an arbitrary background anomalies appear in the Ward identities which follow from the conformal symmetry. These anomalies express the violation of the conservation and tracelessness of the stress-energy tensor in certain correlation functions. It is well known [1] that there are two main types of conformal anomalies in even dimensions: type A and type B. There is always exactly one type A anomaly but an increasing number of type B anomalies by increasing spacetime dimension D. The type B anomalies are associated with conformal gravity invariants, which are locally diffeomorphism and Weyl invariant functions of the metric and its derivatives. In four dimensions (4D) there is only one conformal gravity invariant (or type B anomaly), while in six dimensions (6D) there are three.

In 6D and in the presence of supersymmetry conformal anomalies acquire further significance. Supersymmetry places additional restrictions on the general structure of correlation functions. Remarkably, 6D proves to be the highest spacetime dimension in which superconformal field theories exist [2]. Furthermore, the only known non-trivial unitary conformal field theories in 6D are supersymmetric and arise in string theory, and realize either $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (1, 0)$ superconformal symmetry. This places 6D superconformal field theories at a special point of study in the space of conformal field theories.

An important step in the study of the general anomaly structure of 6D superconformal field theories is the construction of the conformal supergravity invariants, which are supersymmetric extensions of the conformal gravity invariants. For the minimally supersymmetric version, $\mathcal{N} = (1, 0)$, superconformal tensor calculus was developed over thirty years ago in [3] and later applied to describe certain 6D $\mathcal{N} = (1, 0)$ supergravity invariants, such as a supersymmetric Riemann curvature squared term [4]. However, it has not provided insight into the construction of further higher derivative invariants, in particular the minimal conformal supergravity invariants, which are important ingredients in the description of general supergravity-matter systems.

Apart from superconformal tensor calculus, there is another approach to $\mathcal{N}$-extended conformal supergravity for $D \leq 6$ based on using a curved $\mathcal{N}$-extended superspace $\mathcal{M}^{D|\delta}$, where δ is the number of fermionic dimensions. Within the superspace setting, one can choose to gauge only part of the superconformal algebra and use the structure group $\mathrm{SO(D-1,1)} \times G_R$, where G_R is some R-symmetry group. The superspace geometry is then constrained to describe conformal supergravity and thus permits additional transformations known as super-Weyl transformations, which are generated by a real unconstrained superfield parameter.[1] However, for applications involving the construction of super-Weyl invariant objects, it is often more useful to make use of another superspace formulation, known as conformal superspace [9, 10, 11, 12], which is based on gauging the entire superconformal group and is more general since the conventional formulation may be obtained by partially fixing the gauge freedom. Conformal superspace is technically easier to reduce to components and has already proved useful in the construction of higher derivative invariants, e.g. the 5D supersymmetric R^2 invariant in the standard Weyl multiplet [12]. For these reasons it is useful to have a 6D $\mathcal{N} = (1, 0)$ conformal superspace formulation for the construction of 6D minimal conformal supergravity invariants.

Before moving on to presenting the 6D conformal superspace formulation, we will first illustrate a conceptual difference in the 6D $\mathcal{N} = (1, 0)$ case from its 4D $\mathcal{N} = 2$ counterpart in the construction of the conformal supergravity invariants. The invariant for 4D $\mathcal{N} = 2$ conformal supergravity is a supersymmetric C^2 term and is remarkably simple.[2] It is given by a chiral integral of the form

$$I_{C^2} = \int \mathrm{d}^4x\, \mathrm{d}^{2\mathcal{N}}\theta\, \mathcal{E}\, \mathcal{L}_{\mathrm{c}} + \mathrm{c.c.} = \int \mathrm{d}^4x\, \mathrm{d}^{2\mathcal{N}}\theta\, \mathcal{E}\, W^{\alpha\beta} W_{\alpha\beta} + \mathrm{c.c.}\ , \tag{1.1}$$

where $\mathcal{E}$ is the chiral measure and $W^{\alpha\beta}$ is the super-Weyl tensor. However, the 6D $\mathcal{N} = (1, 0)$ case is conceptually different as there is no covariant analogue to the chiral action. Furthermore, although it is natural to consider a full superspace integral of the form $\int \mathrm{d}^6x \mathrm{d}^8\theta E \mathcal{L}$, where $\mathcal{L}$ is a real primary superfield of dimension 2, an appropriate superspace Lagrangian $\mathcal{L}$ cannot be constructed from the 6D $\mathcal{N} = (1, 0)$ super-Weyl tensor $W^{\alpha\beta}$. This makes addressing the 6D case non-trivial as one needs to find an appropriate action principle analogous to the chiral action.

In this paper we present the results of [13] in which the 6D minimal conformal supergravity invariants were constructed. The invariants were constructed by achieving the following: (i) developing the 6D $\mathcal{N} = (1, 0)$ conformal superspace; (ii) deriving action principles capable of supporting the Weyl invariants; and (iii) applying the action principles to describe the Weyl invariants. We will describe each of these points in the main body of this paper.

This paper is organized as follows. In section 2 we give a brief review of 6D conformal gravity highlighting important points of relevance for section 3. In section 3 we present the conformal superspace formulation of conformal supergravity and demonstrate that there should only be two invariants for minimal conformal supergravity. Section 4 is devoted to a review on the superform approach to the construction of locally supersymmetric invariants and how it can be used to construct the invariants for minimal conformal supergravity. Finally, conclusions are presented in section 5.

2. Conformal gravity in six dimensions

Before diving into superspace it is useful to briefly review 6D conformal gravity and the conformal gravity invariants emphasising some important points for the subsequent presentation.

[1] This is known as the conventional superspace approach. It was developed for the 4D $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases in [5, 6] and presented for the 5D $\mathcal{N} = 1$ case in [7] which was extended to the 6D $\mathcal{N} = (1, 0)$ case in [8].

[2] Here C schematically represents the Weyl tensor C_{abcd}.

Conformal gravity in 6D may be viewed as being based on gauging the entire conformal group SO(6,2), which contains the generators $X_{\underline{A}} = \{P_a, M_{ab}, \mathbb{D}, K_a\}$ corresponding to translation, Lorentz, dilatation and special conformal transformations, respectively.[3] To gauge the conformal group one begins with a 6D manifold parametrised by local coordinates x^m, $m = 0, 1, \cdots 5$. Then one introduces the inverse vielbein $e_a = e_a{}^m \partial_m$ associated with P_a, and the Lorentz connection $\omega_a{}^{bc}$, the dilatation connection b_a and the special conformal connection f_{ab} associated with each of their respective generators. The transformations laws for the vielbein and the connections are chosen such that

$$\nabla_a = e_a - \frac{1}{2}\omega_a{}^{bc} M_{bc} - b_a \mathbb{D} - f_{ab} K^b \tag{2.1}$$

transform covariantly. Their transformation laws under the gravity gauge group may be summarized by[4]

$$\delta_{\mathcal{K}} \nabla_a = [\mathcal{K}, \nabla_a] \ , \quad \mathcal{K} := \xi^a \nabla_a + \Lambda^{\underline{a}} X_{\underline{a}} := \xi^a \nabla_a + \frac{1}{2}\Lambda(M)^{ab} M_{ab} + \sigma \mathbb{D} + \Lambda(K)_a K^a \ . \tag{2.2}$$

It is worth mentioning that the transformation laws of the connections are such that the generators $X_{\underline{a}}$ act on the covariant derivatives in the same way as they do on P_a, except with P_a replaced by ∇_a.

In order to describe conformal gravity it is necessary to impose some constraints on the geometry. These constraints fix the entire covariant derivative algebra in terms of the Weyl tensor as follows

$$[\nabla_a, \nabla_b] \equiv -\mathcal{F}_{ab} = -\frac{1}{2} C_{ab}{}^{cd} M_{cd} - \frac{1}{6} \nabla^d C_{abcd} K^c \ , \tag{2.3}$$

where C_{abcd} is the Weyl tensor which is primary,[5] $K_f C_{abcd} = 0$, and satisfies the symmetry properties, $C_{abcd} = C_{[ab][cd]}$ and $C_{[abc]d} = 0$. The Bianchi identity $[\nabla_{[a}, \mathcal{F}_{bc]}] = 0$ requires the following differential constraint on the Weyl tensor

$$\nabla_{[a} C_{bc]}{}^{de} = -\frac{2}{3} \nabla_f C_{[ab}{}^{f[d} \delta^{e]}_{c]} \ . \tag{2.4}$$

It is important to note that the covariant derivative algebra is expressed entirely in terms of the Weyl tensor, which provides a very simple way of seeing that a vanishing Weyl tensor implies a conformally flat geometry.

All conformal gravity invariants may be written down as integrals of the form

$$I = \int \mathrm{d}^6 x\, e\, L \ , \quad K_a L = 0 \ , \quad \mathbb{D} L = 6L \ , \tag{2.5}$$

where the Lagrangian L is a dimension 6 primary field constructed out of the Weyl tensor. There are exactly three possible Lagrangians that one may construct:

$$L^{(1)}_{C^3} = C_{abcd} C^{aefd} C_e{}^{bc}{}_f \ , \tag{2.6a}$$

$$L^{(2)}_{C^3} = C_{abcd} C^{cdef} C_{ef}{}^{ab} \ , \tag{2.6b}$$

$$L_{C\Box C} = C^{abcd} \Box C_{abcd} + \frac{1}{2} (\nabla_e C_{abcd}) \nabla^e C^{abcd} + \frac{8}{9} (\nabla^d C_{abcd}) \nabla_e C^{abce} \ , \tag{2.6c}$$

where $\Box := \nabla^a \nabla_a$.

[3] We refer the reader to [13] for the conformal algebra and conventions.
[4] A tensor field U transforms as $\delta_K U = \mathcal{K} U$.
[5] A superfield U is primary if $K_a U = 0$.

We highlight that once we have gauged the entire conformal algebra the curvature and torsion tensors are constrained such that they are expressed entirely terms of the Weyl tensor and its covariant derivatives. Furthermore, the conformal gravity invariants are all described by primary Lagrangians. In the remaining sections we seek to find an analogous description for the supersymmetric case.

3. $\mathcal{N} = (1, 0)$ conformal superspace

In lower dimensions [9, 10, 11, 12] conformal superspace possesses the following key properties: (i) the entire superconformal algebra is gauged in superspace; (ii) the torsion and curvature tensors are built out of covariant derivatives of a single primary superfield; and (iii) the covariant derivative algebra resembles that of supersymmetric Yang-Mills theory. In this section we present the 6D $\mathcal{N} = (1, 0)$ conformal superspace developed in [13], which satisfies the three aforementioned properties. For the sake of brevity we refer the reader to [13] for our conventions and the superconformal algebra, which now contains, in addition to the generators in the bosonic case, the fermionic generators Q^i_α and S^α_i, denoting the Q-supersymmetry and S-supersymmetry generators, as well as J^{ij} the SU(2) R-symmetry generator.

3.1. Gauging the superconformal algebra

To gauge the superconformal algebra one takes a $\mathcal{N} = (1, 0)$ curved superspace $\mathcal{M}^{6|8}$ parametrised by coordinates $z^M = (x^m, \theta^\mu_{\mathtt{i}})$, where $m = 0, 1, 2, 3, 4, 5$, $\mu = 1, 2, 3, 4$ and $\mathtt{i} = \underline{1}, \underline{2}$. One now associates with each generator $X_{\underline{a}} = (M_{ab}, J_{ij}, \mathbb{D}, S^\gamma_k, K^c)$ a connection one-form $\omega^{\underline{a}} = (\Omega^{ab}, \Phi^{ij}, B, \mathfrak{F}^k_\gamma, \mathfrak{F}_c) = \mathrm{d}z^M \omega_M{}^{\underline{a}}$ and with P_A the vielbein $E^A = (E^\alpha_i, E^a)$. They may be used to construct the covariant derivatives

$$\nabla_A = E_A{}^M \partial_M - \frac{1}{2}\Omega_A{}^{ab} M_{ab} - \Phi_A{}^{ij} J_{ij} - B_A \mathbb{D} - \mathfrak{F}_A{}^j_\beta S^\beta_j - \mathfrak{F}_{Ab} K^b \ . \tag{3.1}$$

It is important to note that the generators appearing in the structure group act on the covariant derivatives in the same way as they do on P_A except with P_A replaced by ∇_A.

The supergravity gauge transformations of the covariant derivatives may be summarised as

$$\delta_\mathcal{K} \nabla_A = [\mathcal{K}, \nabla_A] \ , \quad \mathcal{K} := \xi^A \nabla_A + \frac{1}{2}\Lambda^{bc} M_{bc} + \Lambda^{ij} N_{ij} + \sigma \mathbb{D} + \Lambda^i_\alpha S^\alpha_i + \Lambda_a K^a \ , \tag{3.2}$$

where the gauge parameters associated with $\mathcal{K}$ satisfy natural reality properties. It should be mentioned that the supergravity transformations act on a tensor superfield U as $\delta_\mathcal{K} U = \mathcal{K} U$. The superfield U is said to be *primary* and of dimension Δ if $K_A U = 0$ and $\mathbb{D} U = \Delta U$.

The torsion and curvatures appear in the the (anti-)commutation relations

$$\begin{aligned}[\nabla_A, \nabla_B\} \equiv -\mathcal{F}_{AB} = -T_{AB}{}^C \nabla_C - \frac{1}{2} R(M)_{AB}{}^{cd} M_{cd} - R(N)_{AB}{}^{kl} J_{kl} \\ - R(\mathbb{D})_{AB} \mathbb{D} - R(S)_{AB}{}^k_\gamma S^\gamma_k - R(K)_{ABc} K^c \ ,\end{aligned} \tag{3.3}$$

where the torsion and curvatures are constrained to satisfy the Bianchi identities

$$[\nabla_{[A}, \mathcal{F}_{BC\}}\} = 0 \ . \tag{3.4}$$

3.2. Conformal supergravity

It is clear that the geometric setup of the previous subsection contains too many fields to describe conformal supergravity. In order to describe conformal supergravity it is necessary to impose further constraints on the covariant derivative algebra. As in lower dimensions, suitable constraints prove to resemble those of the Yang-Mills multiplet. Here we explicitly present the consequences in the 6D $\mathcal{N}=(1,0)$ case.

The covariant derivative algebra is constrained to resemble that of Yang-Mills theory:

$$\begin{aligned} \{\nabla_\alpha^i, \nabla_\beta^j\} &= -2\mathrm{i}\varepsilon^{ij}(\gamma^c)_{\alpha\beta}\nabla_c \ , \\ [\nabla_a, \nabla_\beta^j] &= -\mathcal{F}_{a\beta}{}^j = (\gamma_a)_{\alpha\beta}\mathcal{W}^{\beta i} \ , \\ [\nabla_a, \nabla_b] &= -\mathcal{F}_{ab} = -\frac{\mathrm{i}}{8}(\gamma_{ab})_\alpha{}^\beta\{\nabla_\beta^k, \mathcal{W}_k^\alpha\} \ , \end{aligned} \tag{3.5}$$

where the operator $\mathcal{W}^{\alpha i}$ is constrained by the Bianchi identity

$$[\nabla_{[A}, \mathcal{F}_{BC\}}\} = 0 \quad \Longrightarrow \quad \{\nabla_\alpha^{(i}, \mathcal{W}^{\beta j)}\} = \frac{1}{4}\delta_\alpha^\beta\{\nabla_\gamma^{(i}, \mathcal{W}^{\gamma j)}\} \ . \tag{3.6}$$

The operator $\mathcal{W}^{\alpha i}$ is further constrained to be built out of the super-Weyl tensor $W^{\alpha\beta}$ [8] as follows

$$\mathcal{W}^{\alpha i} = W^{\alpha\beta}\nabla_\beta^i + \frac{1}{2}\mathcal{W}(W)^{\alpha i ab}M_{ab} + \mathcal{W}(W)^{\alpha i}\mathbb{D} + \mathcal{W}(W)^{\alpha i}{}_B K^B \ , \tag{3.7}$$

where super-Weyl tensor $W^{\alpha\beta}$ is a symmetric primary superfield of dimension 1,

$$W^{\alpha\beta} = W^{\beta\alpha} \ , \quad K^A W^{\beta\gamma} = 0 \ , \quad \mathbb{D}W^{\alpha\beta} = W^{\alpha\beta} \ . \tag{3.8}$$

The Bianchi identity (3.6) is now equivalent to the following differential constraints on the super-Weyl tensor:

$$\nabla_\alpha^{(i}\nabla_\beta^{j)}W^{\gamma\delta} = -\delta_{[\alpha}^{(\gamma}\nabla_{\beta]}^{(i}\nabla_\rho^{j)}W^{\delta)\rho} \ , \tag{3.9a}$$

$$\nabla_\alpha^k\nabla_{\gamma k}W^{\beta\gamma} - \frac{1}{4}\delta_\alpha^\beta\nabla_\gamma^k\nabla_{\delta k}W^{\gamma\delta} = 8\mathrm{i}\nabla_{\alpha\gamma}W^{\gamma\beta} \ . \tag{3.9b}$$

In order to give the operator $\mathcal{W}^{\alpha i}$ more explicitly it is useful to introduce the dimension 3/2 superfields

$$X_\gamma^{k\alpha\beta} = -\frac{\mathrm{i}}{4}\nabla_\gamma^k W^{\alpha\beta} - \delta_\gamma^{(\alpha}X^{\beta)k} \ , \quad X^{\alpha i} := -\frac{\mathrm{i}}{10}\nabla_\beta^i W^{\alpha\beta} \tag{3.10}$$

together with the dimension 2 descendant superfields:

$$Y_\alpha{}^{\beta ij} := -\frac{5}{2}\Big(\nabla_\alpha^{(i}X^{\beta j)} - \frac{1}{4}\delta_\alpha^\beta\nabla_\gamma^{(i}X^{\gamma j)}\Big) = -\frac{5}{2}\nabla_\alpha^{(i}X^{\beta j)} \ , \tag{3.11a}$$

$$Y := \frac{1}{4}\nabla_\gamma^k X_k^\gamma \ , \tag{3.11b}$$

$$Y_{\alpha\beta}{}^{\gamma\delta} := \nabla_{(\alpha}^k X_{\beta)k}{}^{\gamma\delta} - \frac{1}{6}\delta_\beta^{(\gamma}\nabla_\rho^k X_{\alpha k}{}^{\delta)\rho} - \frac{1}{6}\delta_\alpha^{(\gamma}\nabla_\rho^k X_{\beta k}{}^{\delta)\rho} \ . \tag{3.11c}$$

The action of the spinor covariant derivatives, as well as with the S-supersymmetry generators, on the defined superfields may be found in [13]. In terms of these fields one finds

$$\begin{aligned} \mathcal{W}^{\alpha i} = \ & W^{\alpha\beta}\nabla_\beta^i + X_\gamma^{i\,\alpha\beta}M_\beta{}^\gamma - \frac{1}{8}X^{\beta i}M_\beta{}^\alpha - \frac{5}{4}X_j^\alpha J^{ij} + \frac{5}{16}X^{\alpha i}\mathbb{D} \\ & + \frac{1}{16}Y_\beta{}^{\alpha ij}S_j^\beta + \frac{\mathrm{i}}{4}\nabla_{\beta\gamma}W^{\gamma\alpha}S^{\beta i} - \frac{5}{64}YS^{\alpha i} \\ & - \frac{1}{12}(\gamma^{ab})_\beta{}^\gamma\big(\nabla_b X_\gamma^{i\,\beta\alpha} - \frac{3}{4}\delta_\gamma^\alpha\nabla_b X^{\beta i}\big)K_a \ . \end{aligned} \tag{3.12}$$

The remaining covariant derivative algebra is constrained in terms of the super-Weyl tensor, its descendents and their vector covariant derivatives [13].

It is important to stress that the entire covariant derivative algebra is expressed directly in terms of the super-Weyl tensor and its descendents. In particular, it is trivial to see that when $W^{\alpha\beta}$ vanishes the supergeometry is superconformally flat. Furthermore, the standard Weyl multiplet of $6D$ $\mathcal{N} = (1,0)$ conformal supergravity is encoded in the superspace geometry. The component fields can be readily identified as certain $\theta = 0$ parts of the superspace gauge one-forms and descendants of $W^{\alpha\beta}$.

3.3. An immediate application: The number of independent invariants

Now that we have presented the conformal superspace formulation, it is possible to determine what the maximum number of invariants for conformal supergravity is. The maximum number of invariants was determined in [13] by making use of an argument based on analysing possible supercurrents. It is illustrative to reproduce the argument below. Before moving on to the constructive proof it is worth first recalling the definition of the supercurrent in 6D $\mathcal{N} = (1,0)$ supersymmetry.

The supercurrent is a supermultiplet described by a scalar superfield $\mathcal{J}$ containing, amongst its component content, the energy momentum tensor and the supersymmetry current(s), together with additional components, e.g. R-symmetry current. In the curved case it is described by a scalar primary superfield $\mathcal{J}$ satisfying the differential constraint

$$\nabla^{(i}_{[\alpha} \nabla^{j}_{\beta} \nabla^{k)}_{\gamma]} \mathcal{J} = 0 \ . \tag{3.13}$$

Since this constraint must be annihilated by the S-supersymmetry generator to be consistent with the superconformal symmetry, $\mathcal{J}$ must be of dimension 4.

Each of the invariants for conformal supergravity have an associated non-trivial supercurrent. Thus a bound on the number of possible invariants for conformal supergravity is given by the maximum number of possible supercurrents (up to irrelevant normalizations) one can construct out of the Weyl tensor and its covariant derivatives. To determine the number of supercurrents possible, one simply writes down the most general possible Ansatz which is

$$\begin{aligned} \mathcal{J} \ = \ & c_1 \nabla^a \nabla_a Y + c_2 Y^2 + \mathrm{i}\, c_3 \, X^{\alpha i} \nabla_{\alpha\beta} X^{\beta}_i + \mathrm{i}\, c_4 \, X^{i}_{\alpha}{}^{\beta\gamma} \nabla_{\gamma\delta} X_{\beta i}{}^{\alpha\delta} + c_5 \, Y_\alpha{}^{\beta ij} Y_\beta{}^{\alpha}{}_{ij} \\ & + c_6 \, Y_{\alpha\beta}{}^{\gamma\delta} Y_{\gamma\delta}{}^{\alpha\beta} + c_7 \, W^{\alpha\gamma} \nabla_{\alpha\beta} \nabla_{\gamma\delta} W^{\delta\beta} + c_8 \nabla_{\beta\alpha} W^{\alpha\gamma} \nabla_{\gamma\delta} W^{\delta\beta} \\ & + c_9 \varepsilon_{\alpha_1 \cdots \alpha_4} \varepsilon_{\beta_1 \cdots \beta_4} W^{\alpha_1\beta_1} \cdots W^{\alpha_4\beta_4} \ , \end{aligned} \tag{3.14}$$

where c_n are arbitrary real coeffcients. Requiring that $\mathcal{J}$ be primary and satisfy the differential constraint yields

$$\begin{aligned} & c_3 = -\frac{8}{3} c_2 - 5 c_1 \ , \qquad c_4 = -\frac{32}{15} c_2 - 16 c_1 \ , \quad c_5 = \frac{2}{15} c_2 + \frac{6}{5} c_1 \ , \\ & c_6 = \frac{2}{45} c_2 + \frac{1}{3} c_1 \ , \qquad c_7 = -\frac{2}{15} c_2 - \frac{1}{5} c_1 \ , \quad c_8 = \frac{1}{2} c_7 = -\frac{1}{15} c_2 - \frac{1}{10} c_1 \ , \\ & c_9 = 0 \ , \end{aligned} \tag{3.15}$$

which tells us that we have a two parameter family of solutions and at most two invariants for conformal supergravity.

4. Superform action principles and the invariants

Now that we have demonstrated that there can only be at most two invariants for conformal supergravity, we would like to describe how they were constructed in [13] in this section. It turns out that there are exactly two invariants, one of which contains a certain combination of C^3 terms at the component level, while the other contains a $C\Box C$ term at the component level. Their construction relies heavily on the use of the superform approach to constructing supersymmetric invariants [14, 15, 16], which we now turn to briefly reviewing in the context of 6D $\mathcal{N} = (1, 0)$ conformal supergravity.

4.1. The superform approach to constructing supersymmetric invariants

The superform approach to constructing supersymmetric invariants is a general approach based on the idea that a closed super D-form automatically leads to a supersymmetric invariant. In six dimensions, one introduces a six-form $J = \frac{1}{6!}\mathrm{d}z^{M_6} \wedge \cdots \wedge \mathrm{d}z^{M_1} J_{M_1\cdots M_6}$ in 6D $\mathcal{N} = (1, 0)$ superspace satisfying the closure condition,

$$\mathrm{d}J = \frac{1}{6!}\mathrm{d}z^{M_6} \wedge \cdots \wedge \mathrm{d}z^{M_0} \partial_{M_0} J_{M_1\cdots M_6} = 0 \ . \tag{4.1}$$

Such a superform immediately leads to the action principle

$$S = \int_{\mathcal{M}^6} i^* J = \int \mathrm{d}^6 x \, e \, {}^*J|_{\theta=0} \ , \qquad {}^*J := \frac{1}{6!}\varepsilon^{mnpqrs} J_{mnpqrs} \ , \tag{4.2}$$

where $i : \mathcal{M}^6 \to \mathcal{M}^{6|8}$ is the inclusion map and i^* is its pullback, the effect of which is to project $\theta^\mu_i = \mathrm{d}\theta^\mu_i = 0$. Invariance under supersymmetry follows since under a superdiffeomorphism with $\xi = \xi^A E_A = \xi^M \partial_M$, the superform J transforms as

$$\delta_\xi J = \mathcal{L}_\xi J \equiv i_\xi \mathrm{d}J + \mathrm{d}i_\xi J = \mathrm{d}i_\xi J \ , \tag{4.3}$$

which corresponds to a total derivative. It is important to note that the component action may be obtained by expressing the action in terms of the tangent frame

$$\begin{aligned} S &= \int \mathrm{d}^6 x \, \frac{1}{6!}\varepsilon^{m_1\cdots m_6} E_{m_6}{}^{A_6} \cdots E_{m_1}{}^{A_1} J_{A_1\cdots A_6}|_{\theta=0} \ , \\ &= \frac{1}{6!}\int \mathrm{d}^6 x \, \mathrm{e} \, \varepsilon^{a_1\cdots a_6} \Big[J_{a_1\cdots a_6} + 3\psi_{a_1}{}^\alpha_i J^i_{\alpha a_2\cdots a_6} + \frac{15}{4}\psi_{a_2}{}^\beta_j \psi_{a_1}{}^\alpha_i J^{i\,j}_{\alpha\beta a_3\cdots a_6} \\ &\quad + \frac{5}{2}\psi_{a_3}{}^\gamma_k \psi_{a_2}{}^\beta_j \psi_{a_1}{}^\alpha_i J^{i\,j\,k}_{\alpha\beta\gamma a_4 a_5 a_6} + \mathcal{O}(\psi^4) \Big] |_{\theta=0} \ . \end{aligned} \tag{4.4}$$

In order to have a sensible action it must be invariant under all other gauge transformations, which form the subgroup $\mathcal{H}$. Therefore, we require J to also transform by (at most) an exact form under these transformations, $\delta_\mathcal{H} J = \mathrm{d}\Theta(\Lambda^{\underline{a}})$, where Θ is some five-form. For conformal supergravity this requires invariance under the standard superconformal transformations. Under the superconformal transformations, it is worth distinguishing two cases: (i) $\Theta = 0$; and (ii) $\Theta \neq 0$. The first case corresponds to when the closed six-form is itself invariant, which is of relevance to the construction of the C^3 invariant, while the second case is of relevance to the $C\Box C$ invariant. We now move on to describing the first case and the construction of the C^3 invariant.

4.2. A primary superform and the C^3 invariant

A special case is given by when the superform J is itself closed. This implies that the components $J_{A_1\cdots A_6}$ of $J = \frac{1}{6!}E^{A_6} \cdots E^{A_1} J_{A_1\cdots A_6}$ transform covariantly. In particular, with respect to the special conformal transformations we have the conditions

$$S^\beta_j J_{a_1\cdots a_n}{}^{i_1}_{\alpha_1}{}^{\cdots}_{\cdots}{}^{i_{6-n}}_{\alpha_{6-n}} = -\mathrm{i}\, n \, (\tilde\gamma_{[a_1})^{\beta\gamma} J_{\gamma j a_2\cdots a_n]}{}^{i_1}_{\alpha_1}{}^{\cdots}_{\cdots}{}^{i_{6-n}}_{\alpha_{6-n}} \ , \qquad K^b J_{A_1\cdots A_6} = 0 \ , \tag{4.5}$$

which allows us to express the closure condition (4.1) in terms of the tangent frame as follows

$$\nabla_{[A_1} J_{A_2\cdots A_7\}} + 3T_{[A_1A_2}{}^B J_{|B|A_3\cdots A_7\}} = 0 \ . \tag{4.6}$$

We define *primary* superforms as those superforms satisfying the condition (4.5).

Under certain weak assumptions one can show that a closed primary six-form in 6D $\mathcal{N}=(1,0)$ superspace is unique [13], and is given in [17] in the context of SU(2) superspace. In conformal superspace, the six-form corresponds to choosing the lower dimension components of J to vanish and the dimension 9/2 component as follows:

$$J_{abc}{}_{\alpha\beta\gamma}^{\,i\,j\,k} = 3(\gamma_{abc})_{(\alpha\beta}A_{\gamma)}{}^{ijk} \ , \tag{4.7}$$

where, due to eq. (4.5), $A_\alpha{}^{ijk}$ is primary $K^B A_\alpha{}^{ijk} = 0$. The closure condition (4.6) can be shown to hold identically once one imposes the following differential condition on $A_\alpha{}^{ijk}$:

$$\nabla_{(\alpha}^{(i} A_{\beta)}{}^{jkl)} = 0 \ . \tag{4.8}$$

Closure (4.6) also fixes the remaining components of the superform, e.g. the top component of the superform is

$$J_{abcdef} = -\frac{\mathrm{i}}{2^4 4!}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{abcdef}\nabla_{\alpha i}\nabla_{\beta j}\nabla_{\gamma k}A_\delta{}^{ijk} \ . \tag{4.9}$$

The remaining components of the superform are given in [13].

The superform constructed in terms of $A_\alpha{}^{ijk}$ can be used to describe supersymmetric invariants in an analogous way as the chiral action principle in the 4D $\mathcal{N}=1$ and $\mathcal{N}=2$ cases. In particular, all that one must do is search for a composite expression for $A_\alpha{}^{ijk}$ such that it is built out of the superfields of the theory in consideration. This very approach can be used to describe the supersymmetric C^3 invariant. By allowing $A_\alpha{}^{ijk}$ to be built out of the super-Weyl tensor and its descendents yields the following unique solution [13]

$$\begin{aligned} A_\alpha{}^{ijk} \ = \ & 5\mathrm{i}\varepsilon_{\alpha\beta\gamma\delta}X^{\beta(i}X^{\gamma j}X^{\delta k)} - 8\mathrm{i}\varepsilon_{\alpha\beta\gamma\delta}X^{\beta(i}X_{\alpha'}^{j}{}^{\gamma\beta'}X_{\beta'}^{k)\delta\alpha'} \\ & + \frac{64\mathrm{i}}{3}\varepsilon_{\alpha\beta\gamma\delta}X_{\alpha'}^{(i}{}^{\beta\beta'}X_{\beta'}^{j}{}^{\gamma\gamma'}X_{\gamma'}^{k)\delta\alpha'} \\ & + 4\varepsilon_{\alpha\beta\gamma\delta}Y_\rho{}^{\beta(ij}X_\eta^{k)\rho\gamma}W^{\eta\delta} - 3\varepsilon_{\alpha\beta\gamma\delta}Y_\rho{}^{\beta(ij}X^{\gamma k)}W^{\rho\delta} \ . \end{aligned} \tag{4.10}$$

The solution (4.10) automatically describes a supersymmetric invariant. By plugging the result (4.10) into eq. (4.9) and keeping in mind the component reduction formula (4.4), one can show that the component action contains a C^3 term proportional to

$$-\frac{1}{8}\varepsilon^{abcdef}\varepsilon_{a'b'c'd'e'f'}C_{ab}{}^{a'b'}C_{cd}{}^{c'd'}C_{ef}{}^{e'f'} = 4L_{C^3}^{(2)} - 8L_{C^3}^{(1)} \ . \tag{4.11}$$

4.3. A non-primary superform and the $C\Box C$ invariant

We have presented a primary superform that may be used as an action principle in constructing invariants. However, it becomes apparent that the primary superfield $A_\alpha{}^{ijk}$ cannot be used to describe a supersymmetric $C\Box C$ invariant due to its uniqueness. In particular, one can construct another object of the right index structure as $A_\alpha{}^{ijk}$ that satisfies the right differential constraint, but it can be shown to not be primary. This suggests that we should generalise our approach to look for non-primary closed superforms.

The obstruction we encounter in using the previous action principle for the construction of another conformal supergravity invariant is the special conformal transformations. To remedy this we introduce the special conformal connections $\mathfrak{F}_A$ into our ansatz for the closed superform as follows:

$$J = J_0 + \mathfrak{F}_\alpha^i \wedge J_{S}{}_i^\alpha + \mathfrak{F}_a \wedge J_K{}^a \ , \tag{4.12}$$

where J_0 is a six-form, and $J_{S}{}_i^\alpha$ and $J_K{}^a$ are some 5-forms. The purpose of the five-forms $J_{S}{}_i^\alpha$ and $J_K{}^a$ is to capture the "anomalous terms" arising from the transformations of J under S-supersymmetry and special conformal transformations such that

$$\delta_S J = -\mathrm{d}(\Lambda_S{}_\alpha^i J_S{}_i^\alpha) \ , \qquad \delta_K J = -\mathrm{d}(\Lambda_{Ka} J_K{}^a) \ . \tag{4.13}$$

Given our ansatz (4.12), it is still necessary to know how to constrain the components of the superform. A clue comes from considering the full superspace integral

$$S = \int \mathrm{d}^6 x \, \mathrm{d}^8\theta \, E \, \mathcal{L} \ , \tag{4.14}$$

where $\mathcal{L}$ is a primary superfield of dimension 2. If we replace $\mathcal{L}$ by a tensor superfield Φ, which satisfies

$$B^{\alpha\beta ij} = \frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta\gamma\delta}\nabla_\gamma^{(i}\nabla_\delta^{j)}\Phi = 0 \ , \tag{4.15}$$

the full superspace integral (4.14) vanishes since Φ has a prepotential formulation [18]. This suggests that there should be an action principle based on a primary superfield of the form $B^{\alpha\beta ij} \leftrightarrow B_a{}^{ij}$, which provides a way of describing the full superspace integral as a closed superform.

A solution based on the ansatz (4.12) was found in [13] such that all components of J_0, $J_S{}_i^\alpha$ and $J_K{}^a$ are constructed in terms of a primary superfield of dimension 3 satisfying the constraint

$$\nabla_\alpha^{(i} B^{\beta\gamma jk)} = -\frac{2}{3}\delta_\alpha^{[\beta}\nabla_\delta^{(i} B^{\gamma]\delta jk)} \ . \tag{4.16}$$

The superfield $B^{\alpha\beta ij}$ appears in the component of the J_K with lowest mass dimension.

$$J_K{}_{bcd}{}_{\alpha}^{i}{}_{\beta}^{j}{}^{a} = -64\mathrm{i}(\gamma_{bcd})_{\alpha\beta} \, B^{a\,ij} \ , \tag{4.17}$$

while all other higher dimension components are expressed in terms of spinor derivatives of $B^{\alpha\beta ij}$, e.g.

$$J_0{}_{a_1a_2a_3a_4a_5a_6} = -\varepsilon_{a_1a_2a_3a_4a_5a_6}\Big(-\frac{1}{5}\nabla^4_{ijkl}C^{ijkl} + \cdots \Big) \ , \tag{4.18}$$

where $C^{ijkl} := -\frac{\mathrm{i}}{12}\nabla_\alpha^{(i}\nabla_\beta^{j}B^{\alpha\beta kl)}$ and the ellipses represents terms that directly involve the Weyl tensor. We refer the reader to [13] for the remaining components and details.

The supeform based on $B_{\alpha\beta}{}^{ij}$ provides an action principle analogous to the one in the previous subsection, in the sense that an invariant action may be constructed by allowing $B^{\alpha\beta ij}$ to be composed of the superfields involved in the theory of interest. For example, if one requires $B^{\alpha\beta ij}$ to be expressed in terms of the super-Weyl tensor and its covariant derivatives, one finds the following unique solution:

$$B^{\alpha\beta\,ij} = W^{\gamma[\alpha}Y_\gamma{}^{\beta]ij} + 8\mathrm{i}\,X_\gamma{}^{\delta[\alpha(i}X_\delta{}^{\beta]\gamma j)} - \frac{5\mathrm{i}}{2}X^{[\alpha(i}X^{\beta]j)} \ . \tag{4.19}$$

One can check that by plugging this result into the superform (4.18), one finds (upon integration by parts) a $C_{abcd}\Box C^{abcd}$ term at the component level.

5. Conclusion

In conclusion, we presented the 6D $\mathcal{N} = (1,0)$ conformal superspace constructed in [13] and demonstrated it is well adapted to describing the invariants for conformal supergravity. The formulation was used to give a simple proof that there can only be a maximum of two invariants for conformal supergravity. We described a primary superform that can be used as an action principle and showed how it can be used to describe the supersymmetric C^3 invariant. We also showed that a non-primary superform action principle was required to construct the supersymmetric $C\Box C$ invariant. Due to the supercurrent analysis presented here, all invariants for minimal conformal supergravity must be a linear combination of the two invariants. The complete component actions for the invariants are given by their component projections and the bosonic sectors will be explicitly given in a forthcoming publication.

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