Collective Symmetry Breaking as a Solution to the Hierarchy Problem in the Standard Model

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Kollektive Symmetriebrechung als Lösung des Hierarchieproblems im Standardmodell:

Bei der Einbettung des Standardmodells in eine fundamentalere Theorie wird die elektroschwache Skala durch quadratisch divergente Strahlungskorrekturen destabilisiert. Damit die experimentell ermittelte Masse des Higgs-Bosons durch die Theorie reproduziert wird, müssen diverse Parameter mit extremer Genauigkeit gewählt werden. Dies wird als das Hierarchieproblem des Standardmodells bezeichnet.
In dieser Arbeit wird ein Lösungsansatz behandelt, der auf kollektiver Symmetriebrechung basiert. In der Implementierung in Little Higgs-Modellen ergibt sich das Higgs-Boson als pseudo-Nambu-Goldstone-Boson einer größeren globalen Symmetrie. Es werden zwei bekannte Little Higgs-Modelle präsentiert und bezüglich ihrer Konstruktion und Phänomenologie diskutiert. Im Weiteren werden Grundsätze zur erfolgreichen Implementation von kollektiver Symmetriebrechung erarbeitet. Da Little Higgs-Modelle nichtrenormierbar sind, werden abschließend mehrere Theorien zur Vervollständigung bei hohen Energien konstruiert.

Collective Symmetry Breaking as a Solution to the Hierarchy Problem in the Standard Model:

If the Standard Model is embedded into a more fundamental framework, the electroweak scale is destabilised due to radiative effects quadratically divergent in the embedding scale. In order to reproduce the observed mass of the Higgs boson, one needs to tune several parameters to an extreme accuracy. This is the gauge hierarchy problem of the Standard Model.
In this thesis, an approach to solve the hierarchy problem is presented: collective symmetry breaking as implemented in Little Higgs models. The Higgs boson is cast as a naturally light pseudo-Nambu-Goldstone boson of an enlarged global symmetry. The construction and phenomenology of two popular Little Higgs models is demonstrated, and a general guideline for the successful implementation of collective symmetry breaking is discussed. Furthermore, several possible ultraviolet completions to Little Higgs models are presented.

Contents

1 Introduction

The Standard Model of particle physics (SM) is a theory capable of explaining a vast amount of physical phenomena, ranging from the quark-gluon plasma stage of the early universe to the vacuum physics performed in experiments such as the Large Hadron Collider (LHC), where the Higgs boson was discovered in 2012 [Atl, Cms]. It describes three of the four known fundamental forces, the strong, weak and electromagnetic interactions. The input it requires are a mere 19 free parameters: the masses of the fundamental particles, gauge couplings, angles and the phase of the CKM matrix and the QCD vacuum angle as well as the vacuum expectation value (VEV) v of the Higgs field. The Standard Model is a renormalisable theory and as such does not a priori require an embedding into a high energy (ultraviolet, UV) completion. Since there is only the one scale v in the electroweak sector, there arises no problem of scale instability in the standalone Standard Model.

Before symmetry breaking (i.e. at high energies), all particles are massless due to gauge invariance under the product group $SU(3)_c \times SU(2)_L \times U(1)_Y$. In order to give masses to the elementary particles, the Higgs field is introduced. It is the only scalar field in the Standard Model. The so-called Higgs mechanism employs an idea borrowed from condensed matter physics: while the full theory (leptons, quarks and the Higgs field) obeys all gauge symmetries, the minimum of the scalar potential spontaneously breaks[1] the electroweak $SU(2)_L \times U(1)_Y$ symmetry down to $U(1)_{\text{em}}$. This minimal value is the VEV v of the Higgs field. After symmetry breaking, all particles transforming under the broken symmetries obtain masses proportional to their coupling to the Higgs and its VEV.

Albeit its immense success, there is evidence that the Standard Model is not a complete description of nature. For example, the observation of neutrino oscillation necessitates very small but non-zero neutrino masses. Further open questions include the nature of dark matter, the baryon asymmetry in the universe and the question how the Standard Model can be reconciled with a quantum description of the fundamental theory of gravity, General Relativity. A common feature of most solutions to these problems is the introduction of a new, heavy mass scale.

We conclude that the Standard Model needs to be embedded into a more fundamental theory at some energy scale, but then the stabilisation of the electroweak scale becomes an open problem. In the Standard Model the Higgs VEV is simply put in by hand. It must be measured and there is no prediction for the observed value $v = 246$ GeV. At the same time, the mass term in the scalar potential and thus the VEV are unprotected from quadratic divergences. Assume for example a cut-off placed at the Planck scale $\Lambda_{\text{Pl}} \approx 10^{19}$ GeV; i.e. our model is only valid up to this scale, where we expect effects of a theory of Quantum Gravity to become important. If we calculate the loop corrections to the Higgs mass, we note that there is no symmetry which forbids terms proportional to Λ_{Pl}^2. We thus conclude that these large contributions must cancel to extreme accuracy in order to produce a mass term of $\mathcal{O}(100\ \text{GeV})$. This is the *fine-tuning* or *gauge hierarchy problem* which occurs when embedding the Standard Model into another theory.

[1] An exact definition of *spontaneous symmetry breaking* and non-linear realisation of a symmetry will be given in Sec. 3.2.

In this thesis, a class of possible solutions to the problem is presented. The underlying mechanism is *collective symmetry breaking* as realised in *Little Higgs* theories [Lhr, Lit, Sim, Lsg]. The idea is to implement the Higgs as a pseudo-Nambu-Goldstone boson (PNGB) of a broken global symmetry. The model is constructed from a set of scalars ϕ, set in an effective theory with a cut-off $\Lambda = \mathcal{O}(\text{TeV})$. We require that ϕ transforms in some representation of a global symmetry. Subsequently, a proper subgroup of this global symmetry is gauged, where the global symmetry is broken explicitly in the process. Below Λ, we introduce a VEV f – for now, by hand – which spontaneously breaks the gauge symmetry. Due to Goldstone's theorem, we obtain a number of massless Goldstone bosons. In a normal procedure of spontaneous symmetry breaking, the massless modes would be eaten by the gauge bosons associated with the broken gauge symmetry (as is the case in the Standard Model).

The key observation here is that because we introduced a global symmetry which is explicitly broken by the gauge couplings, there are additional Goldstone bosons which obtain masses proportional to the couplings, making them pseudo-Nambu–Goldstone bosons. We identify four real degrees of freedom of these with the Standard Model Higgs field. By appropriate choice of the global and gauge groups, we can arrange for their masses to be loop-induced only and at most logarithmically sensitive to Λ, easily producing a small mass for the Higgs boson relative to the cut-off.

The aim of this thesis is to present the theory of Little Higgs models and how they can be implemented as UV extensions to the Standard Model. Note that as opposed to the literature [Lit, Sim] we require to generate the electroweak scale solely by this mechanism, i.e. we do not admit any mass term in the scalar Lagrangian.

The sections are structured as follows. In Sec. 2, some advanced techniques of the underlying mathematical foundation, Quantum Field Theory, are reviewed. In particular, this covers renormalisation and the one-loop effective potential, which will be used throughout this thesis. Sec. 3 gives a summary of the Standard Model, where spontaneous symmetry breaking and the hierarchy problem are explained in detail. We turn to Little Higgs models in Sec. 4, where the theory of Nambu-Goldstone bosons is recapitulated and collective symmetry breaking is implemented in a simple model. This is followed by the presentation of two popular Little Higgs models: the Simplest Little Higgs [Lsg, Sim] and the Littlest Higgs [Lit]. We check whether there exists a region in the parameter space in which the measured Standard Model VEV is reproduced; calculation of the one-loop effective potential will show that the Littlest Higgs is able to generate a sufficiently light Higgs, whereas some additional mechanism is required to achieve this in the Simplest Little Higgs. As Little Higgs models are effective field theories, they need to be embedded into a more fundamental theory in the UV. The main objective of a UV completion is to guarantee that the conditions for collective symmetry breaking below some scale are fulfilled, for which three approaches are presented in Sec. 5. The thesis closes with a summary of the conclusions and an outlook in Sec. 6. Derivations of the equations used are given in the Appendix.

2 Quantum Field Theory

Modern particle physics is built within the framework of Quantum Field Theory (QFT). It describes particles as excitations of fields which permeate spacetime. In this section we give a brief review of the advanced tools of QFT which we will need.

2.1 Constructing a Lagrangian

Usually, the basis of a model of particle physics is the Lagrangian density. When building a theory, symmetries and renormalisability serve as important guidelines. Take for example the Lagrangian of QED,

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 \tag{2.1.1}$$

with the gauge covariant derivative $D_\mu = \partial_\mu + ieA_\mu$, where the fields involved are the electron ψ and the photon A_μ. One can easily check that $\mathcal{L}_{\text{QED}}$ obeys a local $U(1)$ symmetry, and that it is renormalisable (see Sec. 2.2).

To assemble a model, we choose a gauge symmetry and then define the particle content. The interaction terms in the Lagrangian are constructed such that they form singlets under all symmetries (see also App. A.1). In the Standard Model for example, the Higgs H is an $SU(2)_L$ doublet and is coupled to the first lepton generation via the terms

$$-\lambda_e \overline{L} H e_R + \text{h.c.} \tag{2.1.2}$$

where L denotes the left-handed lepton doublet and e_R the right-handed electron.

2.2 Loop diagrams and renormalisation

Based on the Lagrangian, we set up perturbation theory, i.e. an expansion of the full quantum correlation functions in terms of small couplings in the theory. The perturbative expansion beyond classical order introduces divergences, which can be counteracted by renormalising the theory. In this section we briefly review the process for the example of ϕ^4-theory in d-dimensional spacetime as presented in Chapters 10 and 12 of [Ps].

A divergent diagram will contain an integral of the form

$$\lim_{\Lambda\to\infty} \int^{\Lambda} dk\, k^{D-1}. \tag{2.2.1}$$

where we establish the superficial degree of divergence D, which gives the highest power of divergence we expect of a diagram (given that it does not have any divergent subdiagrams). The Lagrangian always has mass dimension d, while the mass dimension of any scalar ϕ or fermion field ψ is fixed by the kinetic terms to $[\phi] = \frac{d-2}{2}$ and $[\psi] = \frac{d-1}{2}$. A diagram with n_ϕ external scalars and n_ψ external fermion lines has a coupling γ with mass dimension

$$[\gamma] = d - n_\phi \frac{d-2}{2} - n_\psi \frac{d-1}{2}. \tag{2.2.2}$$

At the same time, the diagram is constructed from V_i tree-level vertices of type i, each with a coupling λ_i. The graph diverges as $\prod_i \lambda_i^{V_i} \Lambda^D$, from which we obtain a general formula for the superficial degree of divergence,

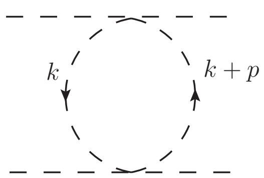
$$D = d - n_\phi \frac{d-2}{2} - n_\psi \frac{d-1}{2} - \sum_i V_i\,[\lambda_i]. \tag{2.2.3}$$

Note that the dimensions of the couplings λ_i define how the operator behaves at higher loop orders which simply contain more vertices. If all couplings have positive mass dimension, increasing orders have smaller superficial degree of divergence; thus, only a finite number of diagrams need to be renormalised. Such a theory is called *super-renormalisable*. If the couplings are dimensionless at most, the theory is *renormalisable*. Thus, divergences occur to all orders, but only a finite number of amplitudes needs to be renormalised. Contrarily, couplings with negative mass dimension lead to a *non-renormalisable* theory.

In four dimensions, the renormalisable Lagrangian of ϕ^4 theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4. \tag{2.2.4}$$

Note that the couplings are of non-negative mass dimension and thus the theory is renormalisable. We calculate the one-loop correction to the quartic vertex, which consists of the diagram



$$= \frac{(-i\lambda_0)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \tag{2.2.5}$$

and two other channels with exchanged momenta.

A diagram may be UV or IR divergent. In order to quantify the divergence of a loop integral, we need to regularise. Two common regularisation schemes for UV divergences are:

- *cut-off regularisation*, in which the scale Λ in (2.2.1) is kept as a variable, i.e.

  $$\int_0^\infty dk \to \int_0^\Lambda dk. \tag{2.2.6}$$

  We see that Lorentz invariance is broken by this regularisation method, and consequently gauge invariance is also affected. For example, a $U(1)$ gauge boson A_μ transforms as

  $$A_\mu(x) \mapsto A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x) \quad \Rightarrow \quad A_\mu(k) + \frac{i}{g}k_\mu\alpha(k). \tag{2.2.7}$$

  Nevertheless, cut-off regularisation is often employed, as the results give an intuitive interpretation; e.g. terms containing Λ^2 or $\log(\Lambda)$ are quadratically or logarithmically divergent, respectively.

- *dimensional regularisation*, where the degree of divergence is modified by going to $d - 2\epsilon$ dimensions, i.e.

  $$\int_0^\infty d^dk \to \mu^{2\epsilon} \int_0^\infty d^{d-2\epsilon}k. \tag{2.2.8}$$

  The divergences are restored for $\epsilon \to 0$. Dimensional regularisation preserves both Lorentz and gauge invariance.

The divergences are absorbed by redefining the couplings in the Lagrangian via counter-terms. In our example, these are

$$\delta\mathcal{L} = \frac{1}{2}\delta_Z\,\partial_\mu\phi\,\partial^\mu\phi - \frac{\delta_{m^2}}{2}\phi^2 - \frac{\delta_\lambda}{4!}\phi^4. \tag{2.2.9}$$

with $Z = (1+\delta_Z)$ defining the wavefunction renormalisation, $m^2 = (m_0^2 + \delta_{m^2})$ the mass and $\lambda = (\lambda_0 + \delta_\lambda)$ the quartic coupling renormalisation. The full quartic coupling λ encodes the bare and loop interactions of ϕ. In dimensional regularisation we expand in terms of ϵ,

$$\lambda = \lambda_0 + \delta_\lambda^{(1)}\epsilon^{-1} + \delta_\lambda^{(2)}\epsilon^{-2} + ... \tag{2.2.10}$$

and determine δ_λ order by order. The above one-loop diagram (2.2.5) implies the counter-term of the quartic coupling

$$\delta_\lambda^{(1)} = \frac{3}{16\pi^2}\frac{\lambda_0^2}{2}\frac{1}{\epsilon} + \text{finite} \tag{2.2.11}$$

with the renormalisation condition that the full quartic interaction is equal to $-i\lambda_0$ at some scale μ, the *renormalisation scale*. We note that the coupling λ changes with μ, and interpret μ as the scale at which an interaction in our theory takes place. Define the β-function $\beta_\alpha = \mu\frac{d\alpha}{d\mu}$ for any coupling $\alpha(\mu)$ which encodes the running of the couplings in the *renormalisation group* (RG). For the above example, the β-function (setting the mass to zero) is

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3). \tag{2.2.12}$$

The information on the running of all couplings is contained in the Callan-Symanzik or renormalisation group equation, which for an n-point correlation function is

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_{m^2}\frac{\partial}{\partial m^2} + n\,\gamma_\phi\right)G_n(\mu) = 0 \tag{2.2.13}$$

with the β-functions

$$\beta_\lambda = \mu\frac{d\lambda}{d\mu} \qquad \beta_{m^2} = \mu\frac{d}{d\mu}m^2 \qquad \gamma_\phi = \frac{\mu}{2}\frac{d}{d\mu}\log Z \tag{2.2.14}$$

where β_{m^2} is called the anomalous mass dimension, and γ_ϕ the anomalous dimension.

2.3 The one-loop effective potential

When constructing a model of particle physics, it is often necessary to investigate the radiative effects of the particle content on the scalar potential, as its ground state may be modified. This is summarised in the one-loop effective potential, which we derive in cut-off regularisation in App. A.2. For some scalar ϕ, it adds to the tree-level potential the contribution [Rsb, Eff]

$$V_{\text{eff}}[\phi_c] = \frac{1}{32\pi^2}\sum_i (-1)^{2s_i}\, n_i \left[m_i^2(\phi_c)\,\Lambda^2 + \frac{m_i^4(\phi_c)}{2}\left(\log\frac{m_i^2(\phi_c)}{\Lambda^2} - \frac{1}{2}\right)\right] \tag{2.3.1}$$

where i runs over all particles in the theory; m_i is the corresponding particle's mass, s_i denotes its spin and n_i are the number of degrees of freedom running in the loops. Note that we assume the potential to depend on a constant field value ϕ_c.

In many models some gauge symmetry is broken spontaneously (see also Sec. 3.2). This can be achieved by configuring the parameters in the scalar potential such that a minimum arises which does not obey the symmetry. In particular, one can set up a theory where the tree-level potential does not have such a non-zero vacuum expectation value (VEV), but it only emerges from radiative effects [Rsb]. In such a case, one can use the above formula to calculate the position of the VEV, which is what we will do in Sec. 4.

The application of the one-loop effective potential is as follows. We have assumed the scalar field to be constant throughout spacetime, as we are interested in the VEV. Thus, the dynamical field ϕ is replaced by ϕ_c in the Lagrangian and the field-dependent masses of the other particles calculated, e.g.

$$M_W^2(\phi_c) := \frac{\partial^2 \mathcal{L}(\phi_c)}{\partial W_\mu^a W_\mu^b} \qquad M_\psi(\phi_c) := \frac{\partial^2}{\partial\psi\,\partial\psi^\dagger}\mathcal{L}(\phi_c) \tag{2.3.2}$$

for the gauge bosons W_μ^a and a vector-like fermion ψ. This is then plugged into (2.3.1), which may obtain a minimum at some non-zero value of ϕ_c. But this is exactly the VEV, where the above equation acts as a self-consistency check.

3 The Standard Model of particle physics

3.1 Particle content

The Standard Model describes the currently established knowledge of particle physics. It explains the strong and electroweak forces, which are incorporated via the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. The fermion and scalar particle content is

$$\begin{aligned} Q^i &: (3,2)_{\frac{1}{6}} & L^i &: (1,2)_{-\frac{1}{2}} \\ u_R^i &: (3,1)_{\frac{2}{3}} & & \\ d_R^i &: (3,1)_{-\frac{1}{3}} & e_R^i &: (1,1)_{-1} \\ & & H : (1,2)_{\frac{1}{2}} & \end{aligned} \tag{3.1.1}$$

where the first and second number denote the representation under $SU(3)_c \times SU(2)_L$, respectively, and the subscript is the hypercharge Y. The index i runs over the three generations of leptons and quarks, which have identical internal quantum numbers. Note that there is no right-handed neutrino, as neutrinos are assumed to be massless in the Standard Model.

3.2 Spontaneous symmetry breaking

Because the left-handed fermions are arranged in doublets under $SU(2)_L$ and the right-handed fermions transform as singlets, mass terms cannot be written down without breaking gauge invariance. This is solved by the Higgs mechanism: we introduce the Higgs field $H = (h^+, h^0)$. The renormalisable potential reads

$$V(H) = -\mu^2 |H|^2 + \lambda |H|^4, \tag{3.2.1}$$

which for $\mu^2 > 0$ has a non-trivial minimum at $|H| = \sqrt{\mu^2/2\lambda}$. This defines the *vacuum expectation value* (VEV), which we choose to be $v/\sqrt{2}$ in the neutral component in order to preserve $U(1)_{\text{em}}$ as a local symmetry. In the Standard Model, the Higgs boson has a mass of about 125 GeV and its VEV is $v \approx 246$ GeV [Atl, Cms, Pdg].

We expand the field around the VEV, e.g. in the scalar gauge kinetic terms of the Lagrangian,

$$|D_\mu H|^2 = \left| \left(\partial_\mu + igW_\mu^a \frac{\sigma^a}{2} + i\frac{g_y}{2} B_\mu \right) \begin{pmatrix} h^+ \\ \frac{1}{\sqrt{2}}(v + h + i\eta) \end{pmatrix} \right|^2 \tag{3.2.2}$$

where the component h^0 has been divided into a real and an imaginary part (note that h^+ still incorporates two real degrees of freedom). We observe that the kinetic term is *not* invariant under the usual $SU(2)_L$ transformation, which sends

$$H \mapsto UH \qquad W_\mu \mapsto UW_\mu U^\dagger + \frac{i}{g} (\partial_\mu U)\, U^\dagger. \tag{3.2.3}$$

As evident from (3.2.2), the introduction of the VEV has obscured the symmetry: this is referred to as *spontaneous symmetry breaking* (SSB)[1].

[1] Even though the symmetry *appears* to be broken, this is a misnomer; the transformation behaviour has simply become non-linear. The term *hidden symmetry* has been proposed as a replacement. In the following however, we adapt the conventional nomenclature.

By Goldstone's theorem, the spontaneous breakdown of a continuous, internal symmetry with N broken generators yields the same number of massless Goldstone bosons. In the Standard Model, this corresponds to the three real degrees of freedom in h^+ and η. Due to gauge symmetry, we can choose U such that the Goldstone bosons are removed entirely from the theory. This corresponds to unitary gauge, in which the physical particle content is manifest. From the terms in (3.2.2) coupling the gauge bosons to v, the mass eigenstates are defined as the linear combinations

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \mp i\, W_\mu^2) && \text{with} \quad M_W = \frac{g\,v}{2} \\ Z_\mu &= \frac{1}{\sqrt{g^2+g_y^2}}(g\,W_\mu^3 - g_y B_\mu) && \text{with} \quad M_Z = \sqrt{g^2+g_y^2}\,\frac{v}{2} \\ A_\mu &= \frac{1}{\sqrt{g^2+g_y^2}}(g_y W_\mu^3 + g\,B_\mu) && \text{with} \quad M_A = 0. \end{aligned} \tag{3.2.4}$$

3.3 The hierarchy problem

The Standard Model is constructed as a renormalisable theory. If we ignore gravity, it describes the strong and electroweak interactions up to arbitrary energy scales. There are limits to its validity, e.g. the Landau pole of QED $\Lambda_{\text{QED}} = 10^{286}$ eV, which marks the breakdown of perturbation theory; but one can also reason that the Standard Model needs to be embedded into another theory much below this energy.

We know that nature features a fourth force, gravitation, which is explained by General Relativity. It is an inherently geometric theory, in which the spacetime metric $g(x)$ itself takes the role of a (classical) field. The dynamics are encoded in the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{16\pi G_N}\int d^4x \sqrt{|\det g(x)|}\, R(x) \tag{3.3.1}$$

where G_N is the gravitational constant and $R(x)$ the Ricci scalar. Much like an effective field theory, the coupling G_N gives a limit on the applicability of the classical theory. One conventionally defines the Planck scale

$$\Lambda_{\text{Pl}} := G_N^{-1/2} \approx 10^{19}\ \text{GeV} \tag{3.3.2}$$

at which gravitational interactions become of order one, and where we expect a theory of Quantum Gravity to become necessary. The Planck scale also sets a cut-off to the Standard Model, as gravitation couples to all particles in it. We will thus assume the Standard Model to be embedded into a greater framework below or at Λ_{Pl}.

The Standard Model has two scales on its own,

the electroweak scale:	$v = (\sqrt{2}G_F)^{-1/2}$	246 GeV
the QCD scale:	Λ_{QCD}	$\mathcal{O}(100\ \text{MeV})$

We can ask how the RG flow relates these scales to the Planck scale. Take the QCD sector, in which the strong gauge coupling depends on the renormalisation scale μ as

$$\alpha_s(\mu) = \frac{4\pi}{(11 - \frac{2n_f}{3})\log\frac{\mu^2}{\Lambda_{\text{QCD}}^2}} \tag{3.3.3}$$

where n_f denotes the number of kinematically available quark flavours; as a simplification, we set all six of them to be active down to Λ_{QCD}. We set $\mu = \Lambda_{\text{Pl}}$ and determine the gauge coupling at this scale,

$$\alpha_s(\Lambda_{\text{Pl}}) = 0.019 \quad \Rightarrow \quad g_s(\Lambda_{\text{Pl}}) = 0.5\,. \tag{3.3.4}$$

This is a natural value for a coupling, and we in particular note from the form of (3.3.3) that small variations to α_s at Λ_{Pl} *do not significantly change* Λ_{QCD}: the large hierarchy between the two scales does not rely on the fine-tuning of some parameters. Note that from the running of the dimensionless coupling $\alpha_s(\mu)$, the dimensionful scale Λ_{QCD} has emerged; this is known as *dynamical scale generation* or *dimensional transmutation.*

The electroweak scale on the other hand is determined by the mass parameter μ^2 and the quartic coupling λ in the scalar potential. In order to obtain the observed VEV $v = 246$ GeV, the quadratic coupling is required to be $\mu(M_Z) = \mathcal{O}(100\text{ GeV})$ [Pdg]. But the Higgs mass operator obtains radiative corrections which diverge quadratically. As an example, extend the Standard Model with a real singlet scalar S. The scalar potential reads

$$V(H,S) = -\mu^2|H|^2 + \lambda|H|^4 + \frac{1}{2}\mu_S^2 S^2 + \frac{\lambda_S}{4!}S^4 + \lambda_p|H|^2S^2 \tag{3.3.5}$$

where a $\mathbb{Z}_2$ is employed to forbid further terms in S. After inserting the VEV v of the Standard Model Higgs, the singlet obtains a mass $m_S^2 = \mu_S^2 + \lambda_p v^2$. The one-loop correction to the Higgs mass then reads [Sme]

$$\begin{aligned}\delta m_h^2 =& \frac{\Lambda^2}{16\pi^2}\left[12\lambda + 2\lambda_p - 12\lambda_t^2 + \frac{3}{2}g_y^2 + \frac{9}{2}g^2\right]\\ &- \frac{1}{16\pi^2}\left[6\lambda\, m_h^2 \log\left(\frac{m_h^2+\Lambda^2}{m_h^2}\right) + 2\lambda_p m_S^2 \log\left(\frac{m_S^2+\Lambda^2}{m_S^2}\right)\right].\end{aligned} \tag{3.3.6}$$

As evident from the last term, the Higgs mass m_h^2 is driven towards m_S^2 by radiative effects (for λ_p of order one).

Based on this, we generalise to any UV extension of the Standard Model where some new particle is coupled to the Higgs at tree-level: if a new mass scale is introduced to the theory, the Higgs mass is driven towards this scale. For an unknown UV theory, we assume the new scale as a cut-off to the Standard Model loop diagrams.

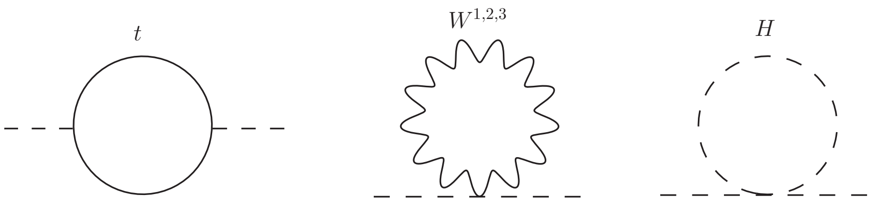


Figure 3.1: One-loop diagrams of the largest contributions to the quadratic divergence of the Higgs mass.

By the above argument, we regularise the momentum integrals for example with Λ_{Pl} as the cut-off. The contributions of these diagrams are [Lhr]

top loop:	$-\frac{3}{8\pi^2}\lambda_t^2\Lambda_{\text{Pl}}^2$
$SU(2)$ gauge boson loops:	$\frac{9}{64\pi^2}g^2\Lambda_{\text{Pl}}^2$
Higgs loop:	$\frac{1}{16\pi^2}\lambda^2\Lambda_{\text{Pl}}^2$

In order to obtain $\mu(M_Z) = \mathcal{O}(100\text{ GeV})$, these contributions are required to balance each other out at every loop order. Ignoring the $\mathcal{O}(\frac{1}{16\pi^2})$ factors, this means

$$\left[-\lambda_t^2 + g^2 + \lambda^2\right](10^{19}\text{ GeV})^2 \overset{!}{\approx} (100\text{ GeV})^2. \tag{3.3.7}$$

This constitutes the *fine-tuning problem* of the Standard Model: if the theory is defined to be valid only up to some high energy scale, the Higgs mass is sensitive to this scale and one needs to set the physical couplings with extreme accuracy in order to obtain a light Higgs boson.

This is a general feature of particle physics models, and it only affects the fundamental scalars in the theory. Fermion and gauge boson masses are protected by chiral and gauge symmetries, respectively.

4 Little Higgs models

As introduced in the previous section, the hierarchy problem is the necessity of fine-tuning several couplings in order to keep the Higgs mass at about 125 GeV when embedding the Standard Model into a greater framework. More generally, the question arises how the vacuum expectation value (VEV) v can be generated in a dynamical way. Many solutions to either problem have been proposed: for example, the VEV may be generated radiatively e.g. if there is no new physics beyond the Standard Model (note that then there is no hierarchy problem). Due to quantum effects, a non-trivial minimum in the Coleman-Weinberg potential can arise dynamically. In the Standard Model this is however excluded, as the top quark needs to be lighter than 85 GeV [Cw1, Cw2]. On the other hand, some popular theories beyond the Standard Model which address the hierarchy problem are Supersymmetry and Randall-Sundrum models.

An alternative approach is incorporated by Little Higgs models (see e.g. [Sim, Lit, Lsg, Lhr]). This class of theories employs a mechanism named *collective symmetry breaking*. Its main idea is to represent the Standard Model Higgs boson as a pseudo-Nambu-Goldstone boson (PNGBs) of an approximate global symmetry which is spontaneously broken at a scale in the TeV range.

We will review this procedure following [Lhr]. First, basic facts about the theory of Nambu-Goldstone bosons are presented. We will then investigate the phenomenon of collective symmetry breaking and construct a simple Little Higgs model.

While Little Higgs models are constructed in order to address the problem of fine-tuning, we are interested in generating the electroweak scale solely via collective symmetry breaking. We thus require the absence of *any* tree-level mass terms, for which the UV completion will need to provide an explanation (see Sec. 5).

4.1 Nambu-Goldstone bosons

Whenever a continuous symmetry is spontaneously broken, massless scalar particles arise [Gs]. These fields are called Nambu-Goldstone bosons (NGBs). Goldstone's theorem states that for internal symmetries, the number of NGBs is equal to the number of broken generators[1]. We shall demonstrate the features of NGBs for the case of a spontaneously broken $U(1)$ and an $SU(N)$ symmetry.

4.1.1 NGBs and shift symmetry

Consider a renormalisable theory which only contains a complex scalar ϕ (the *linear sigma model*, see also App. A.3). The Lagrangian reads

$$\mathcal{L} = |\partial_\mu \phi|^2 + m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 =: |\partial_\mu \phi|^2 - V(\phi) \tag{4.1.1}$$

and has a global $U(1)$ symmetry under which ϕ transforms as $\phi \mapsto e^{i\alpha}\phi$ for any constant $\alpha \in \mathbb{R}$. For $m^2 > 0$ and $\lambda > 0$, the potential $V(\phi)$ is minimised at $|\phi| = \sqrt{2m^2/\lambda}$, i.e. there is an infinitely degenerate vacuum in the phase direction of ϕ. We choose $f/\sqrt{2} \in \mathbb{R}$ to be the vacuum expectation value (VEV) of ϕ. By reinterpreting the two real degrees of freedom as a radial mode and a phase,

[1] Several general proofs can be found in [Gsp].

we can parametrise the radial mode as fluctuations around the VEV,

$$\phi = \frac{1}{\sqrt{2}}\,(\rho(x) + f)\, e^{i/f\,\theta(x)} \qquad \text{with } \rho(x),\, \theta(x) \in \mathbb{R}. \tag{4.1.2}$$

Checking with the Lagrangian reveals that we have spontaneously broken the $U(1)$ symmetry, which has one generator. The associated massless NGB is $\theta(x)$ whereas the radial mode $\rho(x)$ has become massive. An important observation is that the $U(1)$ symmetry is now hidden and realized as a shift symmetry of the NGB[2],

$$\theta(x) \mapsto \theta(x) + \alpha. \tag{4.1.3}$$

This implies that the NGB may only couple via derivatives and cannot develop a potential – neither at classical nor at quantum level – which in particular protects it from acquiring any mass term. This will become a crucial ingredient to constructing Little Higgs theories.

A more general breaking pattern of an $SU(N)$ symmetry further illustrates this argument. Assume ϕ to be in the fundamental representation and let one of its degrees of freedom obtain a VEV such that the symmetry is spontaneously broken to the subgroup $SU(N-1)$. The number of broken generators and thus NGBs is

$$(N^2 - 1) - [(N-1)^2 - 1] = 2N - 1. \tag{4.1.4}$$

We parametrise ϕ as

$$\phi = \exp\left\{\frac{i}{f}\,\pi(x)\right\}\phi_0, \qquad \text{with} \quad \pi = \left(\begin{array}{ccc|c} & & & \pi_1 \\ & 0 & & \vdots \\ & & & \pi_{N-1} \\ \hline \pi_1^* & \cdots & \pi_{N-1}^* & \pi_0/\sqrt{2} \end{array}\right) \text{ and } \phi_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f + \frac{\rho(x)}{\sqrt{2}} \end{pmatrix}, \tag{4.1.5}$$

where π_0 is real and the remaining components in $(\pi_1, \ldots, \pi_{N-1}) =: \vec{\pi}$ are complex scalars. The normalisation has been chosen such that the kinetic terms in the Lagrangian have canonical prefactors. This vector notation visualises the $SU(N-1)$ symmetry obeyed by the $N-1$ components of $\vec{\pi}$.

Under application of the $SU(N-1)$ transformation matrix embedded in $SU(N)$ space

$$U_{N-1} = \begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1.6}$$

where $\hat{U}_{N-1}$ denotes the transformation matrix corresponding to the $SU(N-1)$ subspace, ϕ transforms as

$$\phi \overset{SU(N-1)}{\longmapsto} U_{N-1}\,\phi = \left(U_{N-1}\, e^{i/f\,\pi}\, U_{N-1}^\dagger\right) U_{N-1}\,\phi_0 = e^{i/f\,(U_{N-1}\,\pi\, U_{N-1}^\dagger)}\,\phi_0. \tag{4.1.7}$$

This yields the transformation behaviour of the components of π,

$$\begin{aligned} \pi &= \begin{pmatrix} 0 & \vec{\pi} \\ \vec{\pi}^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \pi_0/\sqrt{2} \end{pmatrix} \\ &\overset{SU(N-1)}{\longmapsto} \begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & \vec{\pi} \\ \vec{\pi}^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \pi_0/\sqrt{2} \end{pmatrix}\right] \begin{pmatrix} \hat{U}^\dagger{}_{N-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \hat{U}_{N-1}\vec{\pi} \\ \vec{\pi}^\dagger\, \hat{U}^\dagger{}_{N-1} & 0 \end{pmatrix} + \frac{\pi_0}{\sqrt{2}}, \end{aligned} \tag{4.1.8}$$

[2]This is also referred to as a non-linearly realized symmetry.

i.e. $\vec{\pi}$ transforms in the fundamental representation of $SU(N-1)$ and π_0 is a singlet. Using the fact that any $SU(N)$ matrix can be decomposed into a product of transformations in $SU(N)/SU(N-1)$ and $SU(N-1)$ space [Spl], we find the transformation property of π under the broken $SU(N)$ symmetry,

$$\begin{aligned} \phi \overset{SU(N)}{\longmapsto} U_N\,\phi &= U_{N/N-1}\,U_{N-1}\,e^{i/f\,\pi}\,\phi_0 \\ &= \left(U_{N/N-1}\,U_{N-1}\,e^{i/f\,\pi}\,U^\dagger_{N-1}\right)U_{N-1}\,\phi_0 \\ &= \left(U_{N/N-1}\,U_{N-1}\,e^{i/f\,\pi}\,U^\dagger_{N-1}\right)\phi_0 \\ &=: e^{i/f\,\pi'}\,\phi_0. \end{aligned} \tag{4.1.9}$$

This is the same transformation behaviour we have found above with an additional factor $U_{N/N-1}$. The transformed π' is a complicated function of $\vec{\pi}$ and some rotation angle $\vec{\alpha}$. Recalling that π is hermitian, we can however absorb the $SU(N-1)$ transformation by setting $\pi \mapsto U^\dagger_{N-1}\pi\,U_{N-1}$. We also specify the $U_{N/N-1}$ transformation matrix

$$U_{N/N-1} = \exp\left\{i\begin{pmatrix} 0 & \vec{\alpha} \\ \vec{\alpha}^\dagger & 0\end{pmatrix}\right\}. \tag{4.1.10}$$

and expand the transformed fields to linear order in α and π,

$$\begin{aligned} e^{i/f\,\pi'} &\doteq \left[\mathbb{1} + i\begin{pmatrix} 0 & \vec{\alpha} \\ \vec{\alpha}^\dagger & 0\end{pmatrix} + \ldots\right]\left[\mathbb{1} + \frac{i}{f}\begin{pmatrix} 0 & \vec{\pi} \\ \vec{\pi}^\dagger & 0\end{pmatrix} + \ldots\right] \\ &= \frac{i}{f}\begin{pmatrix} 0 & \vec{\pi} \\ \vec{\pi}^\dagger & 0\end{pmatrix} + i\begin{pmatrix} 0 & \vec{\alpha} \\ \vec{\alpha}^\dagger & 0\end{pmatrix} + \ldots \end{aligned} \tag{4.1.11}$$

We thus conclude that the π fields comprise the NGBs and have again found the shift symmetry which protects the NGBs from obtaining a potential.

If the symmetry which is broken spontaneously is not exact but violated by some term in the Lagrangian, the emerging scalars are massive and thus called pseudo-Nambu-Goldstone bosons (PNGBs). The mass terms of the PNGBs are proportional to the parameters in the symmetry breaking terms. This is easily seen as in the limit for these couplings going to zero, the original symmetry is restored and the scalars become massless again [We2].

4.2 Constructing a Little Higgs model

We will now put together a simple model where scalars emerge as PNGBs of an approximate global symmetry and introduce the concept of collective symmetry breaking.

Let ϕ_1, ϕ_2 denote two sets of scalars, transforming as triplets under the gauge group $SU(3)_V$. The two scalars each comprise six real degrees of freedom. We assume that the theory is renormalisable and thus the scalar part of the Lagrangian density reads

$$\mathcal{L} = |D_\mu\phi_1|^2 + |D_\mu\phi_2|^2 - V(\phi_1,\phi_2) \tag{4.2.1}$$

with the potential

$$V(\phi_1,\phi_2) = \lambda_1|\phi_1|^4 + \lambda_2|\phi_2|^4 + \lambda_p|\phi_1|^2|\phi_2|^2 \tag{4.2.2}$$

and gauge covariant derivative (for $i = 1, 2$)

$$D_\mu\phi_i = (\partial_\mu + igW_\mu)\phi_i, \quad W_\mu = W^a_\mu\,T^a_V \;\text{ for } a = 1, ..., 8 \tag{4.2.3}$$

where W^a_μ denote the eight real gauge fields of $SU(3)_V$ and $T^a_V = \lambda^a/2$ its hermitian generators. Note that several terms are absent from the potential. Those containing odd powers of ϕ_1 and ϕ_2

can be forbidden by a $\mathbb{Z}_2$ symmetry, while avoiding mixing terms of type $|\phi_1^\dagger\phi_2|^2$ is a more intricate matter which will be addressed in Sec. 5.

Applying our knowledge about NGBs, we spontaneously break the gauge group $SU(3)_V$ down to $SU(2)_V$. In order to do so, we introduce a VEV f and parametrise non-linearly as[3]

$$\phi_i = \exp\left\{\frac{i}{f}\,\pi_i(x)\right\}\begin{pmatrix}0\\0\\f\end{pmatrix} \qquad \text{with} \quad \pi_i = \frac{\eta_i}{\sqrt{2}}\,\mathbb{1} + \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & h_i \\ \hline & h_i^\dagger & 0 \end{array}\right) \tag{4.2.4}$$

where compared to (4.1.5) the radial degree of freedom $\rho(x)$ has been integrated out (see App. A.4). Note that now $|\phi_i|^2 = f^2/2 = \text{const.}$ and thus the potential V can be dropped as a whole. The fields h_i each have four degrees of freedom and are in the fundamental representation of $SU(2)_V$, while η_i are singlets. As all of them are NGBs of the spontaneously broken symmetry, we expect them to be massless.

We can rewrite the scalar fields as

$$\begin{aligned} \phi_1 &= \exp\left\{\frac{i}{f}\,\Pi(x)\right\}\exp\left\{+\frac{i}{f}\,\pi(x)\right\}\begin{pmatrix}0\\0\\f\end{pmatrix} \\ \phi_2 &= \exp\left\{\frac{i}{f}\,\Pi(x)\right\}\exp\left\{-\frac{i}{f}\,\pi(x)\right\}\begin{pmatrix}0\\0\\f\end{pmatrix} \end{aligned} \tag{4.2.5}$$

and choose unitary gauge, i.e. absorb five scalar degrees of freedom $\Pi(x)$ by an appropriate $SU(3)$ transformation. These are *eaten* by the gauge bosons of the broken $SU(3)_V$ symmetry, meaning that they provide the missing massive degrees of freedom. Thus there are now five massive gauge bosons of $SU(3)_V/SU(2)_V$, three massless gauge bosons of $SU(2)_V$ and five remaining scalar degrees of freedom, parametrised as

$$\phi_i = \exp\left\{\pm\frac{i}{f}\,\pi(x)\right\}\begin{pmatrix}0\\0\\f\end{pmatrix} \qquad \text{with} \quad \pi = \frac{\eta}{\sqrt{2}}\,\mathbb{1} + \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & h \\ \hline & h^\dagger & 0 \end{array}\right) \tag{4.2.6}$$

where the positive (negative) contribution in the exponential corresponds to the first (second) set of scalars.

Note that the above parametrisation is only valid up to some scale Λ, i.e. we have constructed an *effective field theory.* We determine the cut-off Λ by expanding the kinetic term of the scalars,

$$|\partial_\mu\phi_i|^2 \doteq |\partial_\mu h|^2 + \frac{1}{f^2}|\partial_\mu h|^2 h^\dagger h + \ldots \tag{4.2.7}$$

We can compute a loop diagram from the second term in the expansion by contracting the operator $h^\dagger h$. The derivative coupling translates to a Feynman rule which is proportional to the momentum. The integral comes with a loop factor of $\frac{1}{16\pi^2}$ and is cut off by Λ. Collecting all parts, we find that the diagram has a contribution proportional to

$$\frac{1}{16\pi^2}\frac{\Lambda^2}{f^2}. \tag{4.2.8}$$

By the limit of perturbativity, this defines the Little Higgs scale $\Lambda \lesssim 4\pi f$.

[3] Note that for simplicity the two VEVs have been aligned and set to $f_1 = f_2 =: f$. In the realistic Simplest Little Higgs model we will not assume equal VEVs.

4.2.1 Symmetry breaking pattern

The introduction of the VEV f appears to imply a sole symmetry breaking $SU(3)_V \to SU(2)_V$. However, if we look closely at the Lagrangian (4.2.1), we observe that sending the gauge coupling g to zero enhances the symmetry[4] by a *global* $SU(3)_1 \times SU(3)_2$, each corresponding to one ϕ_i in the respective fundamental representation, i.e.

$$\phi_i \mapsto U_i \phi_i \qquad \text{with } U_i \in SU(3)_i. \tag{4.2.9}$$

Once we gauge the Lagrangian, both global symmetries are explicitly broken. We see this by checking the transformation behaviour under $SU(3)_1$:

$$\begin{aligned} \phi_1 \mapsto U_1\phi_1, \qquad \phi_2 \mapsto \phi_2, \qquad W_\mu \mapsto W_\mu \\ \Rightarrow |D_\mu \phi_1|^2 \mapsto |U_1 \partial_\mu \phi_1 + ig \underbrace{W_\mu U_1}_{\neq U_1 W_\mu} \phi_1|^2 \neq |U_1 D_\mu \phi_1|^2 \end{aligned} \tag{4.2.10}$$

where we used $U_1 = \exp\{i\theta^b T_1^b\}$ and $[T_V^a, T_1^b] \neq 0$ in general. Under the gauged symmetry, the fields transform in the usual form,

$$\phi_1 \mapsto U\phi_1, \qquad \phi_2 \mapsto U\phi_2, \qquad W_\mu \mapsto U W_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger \tag{4.2.11}$$

for $U \in SU(3)_V$. The kinetic term of ϕ_1 transforms as $|D_\mu\phi_1|^2 \mapsto |UD_\mu\phi_1|^2$. But this also transforms the gauge fields in the kinetic term of ϕ_2, which in turn is compensated by the transformation of ϕ_2. We see that we have fixed $U_1 = U_2$ in the process of gauging and thus only the diagonal subgroup $SU(3)_V$ remains as an exact symmetry. To illustrate this further, we can rewrite the group product into a product of the vector group – which is exactly the diagonal subgroup – and the axial group[5],

$$SU(3)_1 \times SU(3)_2 \cong SU(3)_V \times SU(3)_A. \tag{4.2.12}$$

That is, the ϕ_i now transform as 3 under $SU(3)_V$ and as $\bar{3}$ under the approximate $SU(3)_A$ [Bep].

It is now evident that while the spontaneous breaking of $SU(3)_V$ yields five massless NGBs, the additional ungauged $SU(3)_A$ symmetry is violated by the gauge coupling parameter g and thus produces five light PNGBs.

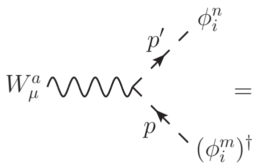
In summary, spontaneous symmetry breaking has been implemented as follows. We have defined two sets of scalars which each transform under their own global $SU(3)$ symmetry. Then, the diagonal subgroup $SU(3)_V$ was gauged, breaking the global symmetries explicitly. Via the VEV f, the gauge symmetry was broken to $SU(2)_V$. By Goldstone's theorem, there are $(3^2-1)-(2^2-1) = 5$ massless Goldstone bosons due to the broken gauge symmetry; in unitary gauge, these five modes provide the longitudinal degree of freedom of the massive gauge bosons, i.e. they cannot comprise the Higgs doublet. The key observation is the following: if we set the gauge coupling to zero, the global symmetry is restored, which means that there would be five *additional* Goldstone bosons. We conclude that these are in fact PNGBs which obtain masses only via gauge boson loop diagrams.

[4] Keeping in mind that this global symmetry would be violated by the additional portal terms we chose to forbid, one can also argue that we directly assumed the potential to be $SU(3)_1 \times SU(3)_2$ symmetric.

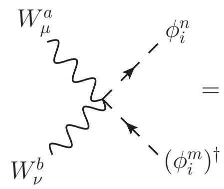
[5] This terminology is adopted from the realization of chiral symmetry breaking in QCD.

4.2.2 Absence of quadratic divergences at one-loop order

We will explicitly verify that all remaining scalars are PNGBs and determine the superficial degree of divergence of their mass terms. Recalling that the fields π_i obey a shift symmetry which is exact for the five NGBs and broken only by the gauge coupling g for the PNGBs, we know that there is no tree level potential. However, loop corrections to the PNGBs in π_i can produce mass terms. As laid out in Sec. 3.3, we hope to find that there are no quadratically divergent diagrams. The Feynman rules extracted from the Lagrangian (4.2.1) are



$$= -ig(p_\mu + p'_\mu)(\tfrac{\lambda^a}{2})^m{}_n$$



$$= i\, g^2\, g^{\mu\nu} \left\{\tfrac{\lambda^a}{2}, \tfrac{\lambda^b}{2}\right\}^m{}_n \tag{4.2.13}$$

where $i = 1, 2$ denote the two sets of scalars and $m, n = 1, 2, 3$ are indices in the corresponding $SU(3)_V$ space.

There are two types of diagrams which can potentially produce a mass term for the π_i: (case a) corrections to the propagators and couplings of the form $|\phi_i^\dagger \phi_i|^n$ and (case b) cross-terms proportional to $|\phi_1^\dagger \phi_2|^n$. The relevant kinds of one-loop diagrams are shown in Fig. 4.1.

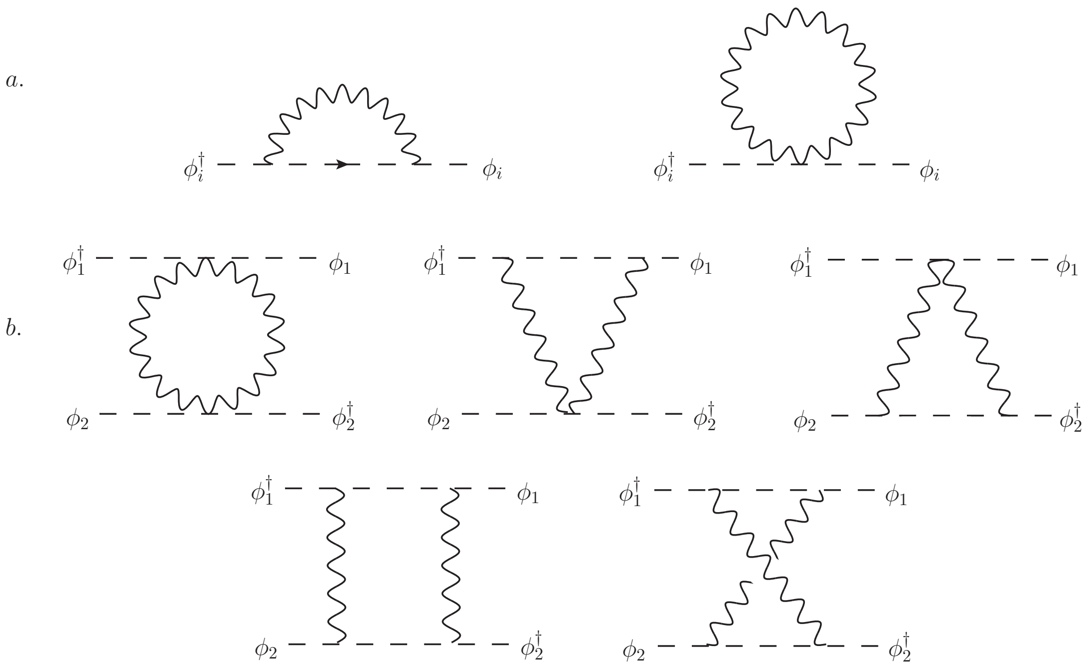


Figure 4.1: One-loop diagrams for the scalar fields. The diagrams in a give no contribution to the scalar potential while diagrams of type b induce a logarithmic divergence. Note that the gauge boson lines mark both the five massive fields of the broken $SU(3)_V/SU(2)_V$ as well as the three massless fields of the unbroken $SU(2)_V$. Depending on the choice of gauge, there are additional diagrams of both types which also include massless Goldstone bosons.

We are interested in the dependency of their divergence on the cut-off $\Lambda \lesssim 4\pi f$. However, without explicit calculation one can argue that diagrams of type a do not give any contribution. This is immediately apparent in the previously defined parametrisation (4.2.6), where we see that $|\phi_i^\dagger \phi_i|^n = f^{2n}$. The constant term merely results in a shift of the zero-point energy and thus can be renormalised away fully. We conclude that diagrams of type a imply no mass term.

Type b diagrams however induce terms proportional to

$$\begin{aligned} |\phi_1^\dagger \phi_2|^2 &= f^4 \exp\left\{-\frac{2\,i}{f}\pi\right\}_{33} \exp\left\{+\frac{2\,i}{f}\pi\right\}_{33} \\ &= f^4 \left\{1 - \frac{2\,i}{f}\frac{\eta}{\sqrt{2}} + \frac{1}{2!}\left(-\frac{2\,i}{f}\right)^2 \left[\frac{\eta^2}{2} + |h|^2\right]\right\} \left\{1 + \frac{2\,i}{f}\frac{\eta}{\sqrt{2}} + \frac{1}{2!}\left(\frac{2\,i}{f}\right)^2 \left[\frac{\eta^2}{2} + |h|^2\right]\right\} + \ldots \end{aligned} \tag{4.2.14}$$

which is non-constant and contains a mass term for the Higgs field[6]. We thus determine the superficial degree of divergence for type b diagrams, which we will do by naive power counting; this corresponds to the worst degree of divergence one expects of the proper calculation. As opposed to the previous chapters, we now employ a more general parametrisation for the ϕ_i and choose an arbitrary R_ξ gauge. This simplifies the analysis of the loop diagrams, taking into account the correct number of degrees of freedom in light of the symmetry breaking.

We thus reinstate all twelve real degrees of freedom of the fields $\phi_{1,2}$, i.e. no degrees of freedom are eaten yet. Assigning a VEV to one field of each ϕ_i leaves ten massless Goldstone bosons. We can again integrate out the massive modes and are left with the Goldstone bosons as well as the eight gauge bosons. In R_ξ gauge, the propagators are [Ps]

$$= \left(\frac{-i}{k^2 - g^2 F F^T}\left[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi g^2 F F^T}(1-\xi)\right]\right)_{ab}$$

$$= \left(\frac{i}{k^2 - \xi g^2 F^T F}\right)_{ij}$$

where ${F^a}_i = T^a_{ij}\,(\phi_0)_j$ is a matrix defined by the symmetry generators T^a and the vector of the vacuum ϕ_0. We read off that every gauge boson or scalar propagator gives a maximum of (-2) powers of k. From the Feynman rules (4.2.13) we see that every $W\phi\phi$ vertex with at least one leg attached to an internal line contributes with one power of k, while the $WW\phi\phi$ coupling gives no contribution.

Counting the powers of k in the diagrams of type b in Fig. 4.1, we see that all of them are suppressed by k^{-4} before integration. We thus conclude that after integrating over $\int d^4k$, there can only be a logarithmic divergence at most.

[6] Note that performing the loop calculations with the fields ϕ_i and then extracting terms in h is possible due to the Equivalence theorem for scalars. It states that reparametrisation and renormalisation does not change their S-Matrix elements [Frd, Eqt].

4.3 The Simplest Little Higgs

Based on the previous findings, we are now able to construct a realistic model including fermions. The approach we will take is the Simplest Little Higgs proposed by Kaplan and Schmaltz [Lsg, Sim].

We assume the electroweak sector[7] of the Standard Model gauge group $SU(2)_L \times U(1)_Y$ to be embedded into the group product $SU(3)_V \times U(1)_X$. Again we require two sets of complex scalars ϕ_1, ϕ_2 in the fundamental representation of $\mathrm{SU}(3)_V$ and additionally assume that they are charged under $U(1)_X$ transformations. The scalars each obtain a VEV and in this break the symmetry down to the Standard Model.

Furthermore, we expect the one-loop effective potential of the remaining scalar degrees of freedom to develop a VEV. Note that the $SU(2)_L$ doublet h takes on the role of the familiar Higgs field, and we aim to find a valid parameter space for the Standard Model value $v = 246$ GeV. It will however show that the Simplest Little Higgs model is not able to produce a VEV which is compatible with experimental observations. The full symmetry breaking pattern and how the Simplest Little Higgs is embedded into a UV theory is illustrated in Fig. 4.2.

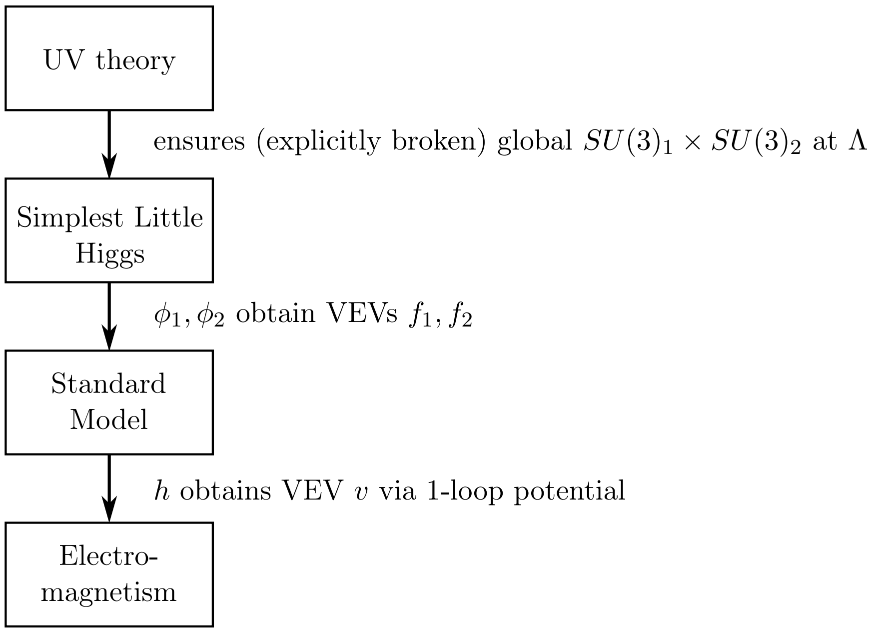


Figure 4.2: Schematic visualisation of the embedding and symmetry breaking pattern in the Simplest Little Higgs model.

4.3.1 Particle content and symmetry breaking

Fermions are implemented by embedding the Standard Model fermion $SU(2)_L$ doublets into $SU(3)_V$ triplets. To this end we introduce the new heavy quark fields U, C, T and leptons n_e, n_μ, n_τ and thus the left-handed triplets are now formed by $Q = (u, d, U)_L$ and $L = (\nu_e, e, n_e)_L$ and analogously for the other two generations. The heavy fields will obtain masses of order $f = \sqrt{f_1^2 + f_2^2}$. The new particles are assumed to be Dirac fermions and thus each one is partnered by a right-handed field. For now, we will assume that the three generations are exact copies of each other. The particle content and the corresponding $(SU(3)_C, SU(3)_V)_{U(1)_X}$ transformation behaviour reads:

[7] The QCD sector of the Standard Model remains unaltered by any of the implications of this model.

$$\begin{aligned} Q &: (3,3)_{\frac{1}{3}} & L &: (1,3)_{-\frac{1}{3}} \\ u_R^1 &: (3,1)_{\frac{2}{3}} & n_R^e &: (1,1)_0 \\ u_R^2 &: (3,1)_{\frac{2}{3}} & & \\ d_R &: (3,1)_{-\frac{1}{3}} & e_R &: (1,1)_{-1} \\ & & \phi_1, \phi_2 &: (1,3)_{-\frac{1}{3}} \end{aligned} \tag{4.3.1}$$

All three generations have the same quantum numbers[8]. Note that as the u_R fields carry the same charges, they generally mix. One linear combination corresponds to the Standard Model and the other one to the heavy up quark.

The new hypercharge X is fixed by relating it to the Standard Model hypercharge Y. For this we need to find a linear combination of the generators of $SU(3)_V$ and $U(1)_X$ under which the vacuum vector is invariant. This is fulfilled by

$$T_Y := \frac{1}{\sqrt{3}}\frac{\lambda^8}{2} + \frac{1}{3}T_X \tag{4.3.2}$$

where λ^8 is the corresponding Gell-Mann matrix, T_Y denotes the generator of the Standard Model hypercharge and $T_X \propto \mathbb{1}$ generates the new hypercharge [Lsg]. Indeed,

$$T_Y \begin{pmatrix} 0 \\ 0 \\ f_i \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ f_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.3.3}$$

This sets e.g. the charges of the scalars under the new hypercharge to $-1/3$.

As in the previous section, symmetry breaking occurs by giving one component of each scalar triplet a VEV. Similarly to (4.2.6), we parametrise the scalar fields non-linearly and integrate out the radial degrees of freedom, but keep the VEVs as two different values. The fields now read

$$\begin{aligned} \phi_1 &= \exp\left\{+\frac{i}{f}\frac{f_2}{f_1}\pi(x)\right\}\begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix} \\ \phi_2 &= \exp\left\{-\frac{i}{f}\frac{f_1}{f_2}\pi(x)\right\}\begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix} \end{aligned} \qquad \text{with} \quad \pi = \frac{\eta}{\sqrt{2}}\mathbb{1} + \left(\begin{array}{cc|c} 0 & 0 & \multirow{2}{*}{h} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{h^\dagger} & 0 \end{array}\right) \tag{4.3.4}$$

Note that in this convention the Higgs field h carries hypercharge $Y = -1/2$ and thus the *upper* component acquires the VEV v.

By the same analysis as in Sec. 4.2.2, we expect there to be no quadratic divergences for scalar self-couplings (see also Sec. 4.3.5). Note that we ignore $\eta(x)$ from now on, as it is a total gauge singlet.

4.3.2 Yukawa couplings

The Yukawa couplings are found by contracting the fermion fields with the scalar sets into singlets in all possible ways. For the quark sector, the only possible terms of dimension 4 are [Sim]

$$\left[\lambda_1^t\,\bar{t}_R^1\,\phi_1^\dagger + \lambda_2^t\,\bar{t}_R^2\,\phi_2^\dagger\right] Q + \left[\lambda_3^t\,\bar{t}_R^1\,\phi_2^\dagger + \lambda_4^t\,\bar{t}_R^2\,\phi_1^\dagger\right] Q, \tag{4.3.5}$$

[8] This corresponds to *Model 1* of [Sim].

where we have turned to the third generation due to its dominant Yukawa coupling; the other families are treated analogously. For simplicity, we forbid the terms mixing particles of type 1 and 2 (second bracket) by assigning charges of a global $\mathbb{Z}_2$ symmetry to the scalars as well as right-handed top-like quarks and leptons,

$$t_R^1, n_R^e, \phi_1 : -1 \qquad t_R^2, \phi_2 : 1. \tag{4.3.6}$$

In order to relate the first term to the Standard Model couplings, we expand the scalar fields to second order in h,

$$\phi_1^\dagger Q = (0\; 0\; f_1) \exp\left\{-\frac{i}{f}\frac{f_2}{f_1}\pi\right\} \begin{pmatrix} t \\ b \\ T \end{pmatrix}_L \doteq f_1 \left(1 - \frac{1}{2f^2}\frac{f_2^2}{f_1^2}|h|^2\right) T_L - i\,\frac{f_2}{f} h^\dagger q + ... \tag{4.3.7}$$

and equally for ϕ_2, where $q = (t,b)_L$ denotes the Standard Model left-handed quark doublet. Plugging this into (4.3.5) gives

$$\underbrace{-\frac{i}{f}\left[\lambda_1^t\, f_1\, \bar{t}_R^1 - \lambda_2^t\, f_2\, \bar{t}_R^2\right]}_{:=\lambda_t \bar{t}_R} h^\dagger q + \left[\lambda_1^t \left(1 - \frac{1}{2f^2}\frac{f_2^2}{f_1^2}|h|^2\right) f_1\, \bar{t}_R^1 + \lambda_2^t \left(1 - \frac{1}{2f^2}\frac{f_1^2}{f_2^2}|h|^2\right) f_2\, \bar{t}_R^2\right] T_L \tag{4.3.8}$$

where we can identify the first term as the Standard Model top quark coupling. This also defines the right-handed heavy top quark T_R,

$$m_T T_R = \sqrt{(\lambda_1^t f_1)^2 + (\lambda_2^t f_2)^2}\, T_R := \lambda_1^t\, f_1\, t_R^1 + \lambda_2^t\, f_2\, t_R^2 \tag{4.3.9}$$

and we obtain the coupling terms

$$\lambda_t\, \bar{t}_R\, h^\dagger q + i\lambda_t \frac{1}{4f}\left(\frac{f_1^2}{f_2^2} - \frac{f_2^2}{f_1^2}\right) |h|^2\, \bar{t}_R\, T_L + m_T \left[1 - \frac{1}{4f^2}\left(\frac{f_1^2}{f_2^2} + \frac{f_2^2}{f_1^2}\right) |h|^2\right] \overline{T}_R T_L. \tag{4.3.10}$$

We can read off the tree level masses

$$m_t = \lambda_t \frac{v}{\sqrt{2}}, \qquad m_T = \sqrt{(\lambda_1^t f_1)^2 + (\lambda_2^t f_2)^2} \tag{4.3.11}$$

where h has acquired a VEV $v/\sqrt{2}$.

The second term in (4.3.10) induces a mixing of t and T at one-loop level, but this effect is suppressed by v/f compared to the tree level masses and due to both VEVs being in the TeV range.

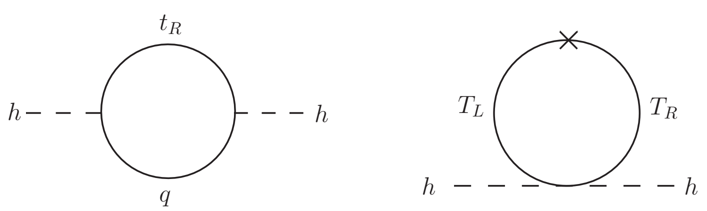


Figure 4.3: The quadratically divergent one-loop diagrams contributing to the Higgs mass.

We can now evaluate the quadratic divergence due to the top quark loop. The corresponding one-loop diagrams are depicted in Fig. 4.3 and in cut-off regularisation add up to

$$\frac{1}{16\pi^2}\left[\lambda_t^2 - m_T^2 \frac{1}{2f^2}\left(\frac{f_1^2}{f_2^2} + \frac{f_2^2}{f_1^2}\right)\right] \Lambda^2\, |h|^2. \tag{4.3.12}$$

Suppressing this divergent contribution gives us a constraint on the mass of the heavy top quark and the two VEVs.

Masses of bottom-type quarks are implemented via higher-dimensional operators. We thus add the term [Sim]

$$\frac{1}{\Lambda}\,\lambda^d\, d_R\, \epsilon_{ijk}\, \phi_1^i \phi_2^j\, Q^k. \tag{4.3.13}$$

Similarly, lepton masses arise from the couplings

$$\lambda^n\, n_R^e\, \phi_1^\dagger L + \frac{1}{\Lambda}\,\lambda^e\, e_R\, \epsilon_{ijk}\, \phi_1^i \phi_2^j\, L^k \tag{4.3.14}$$

where again the $\mathbb{Z}^2$ symmetry eliminates the cross term.

4.3.3 Gauge anomalies

Before it is broken by the VEVs f_1, f_2, the $SU(3)_V \times U(1)_X$ symmetry is assumed to be exact. However, introducing chiral fermions may cause gauge anomalies, i.e. breaking of the symmetry at quantum level[9]. In order for the $U(1)_X^3$ anomalies to cancel, the corresponding charges of the fermions need to fulfil [Qsm]

$$\sum_{\text{left-handed}} X^3 - \sum_{\text{right-handed}} X^3 \stackrel{!}{=} 0. \tag{4.3.15}$$

Plugging in the values of our model, we find that this relation is violated. There are two remedies suggested in [Sim]: the addition of heavy spectator particles from a UV-completion or modification of the charges of the available particles.

The second alternative is realised by changing the quantum numbers of the quarks in the two lighter generations (u, c) to

$$\begin{aligned} Q &: (3,\bar{3})_0 & \qquad d_R^1 &: (3,1)_{-\frac{1}{3}} \\ u_R &: (3,1)_{\frac{2}{3}} & \qquad d_R^2 &: (3,1)_{-\frac{1}{3}} \end{aligned} \tag{4.3.16}$$

and leave the leptons and the third quark generation as they are. Indeed all gauge anomalies cancel with this assignment, which now yields the heavy quarks (D, S, T) as partners to the Standard Model quarks. As the third family remains unaltered and has the greatest effect on the electroweak potential, choice of either charge assignment does not affect the following implications.

4.3.4 Gauge interactions

In the symmetric phase, our model has nine massless gauge bosons. By going to lower energies, we expect five of them obtain masses of order of the VEVs f_1, f_2. We will for now also assume that h acquires a suitable VEV v and breaks the Standard Model gauge group in the usual way. I.e., both steps in the symmetry breaking procedure are performed successively.

The relevant term for the gauge boson masses is extracted from the kinetic part of the Lagrangian,

$$\left|\left(\partial_\mu + i\, g\, A_\mu^a T^a - \frac{i}{3}\, g_x A_\mu^x\right)\phi_i\right|^2 \quad\rightarrow\quad \mathrm{tr}\left[\left(g\, A_\mu^a T^a - \frac{1}{3}\, g_x A_\mu^x\right)^2 \phi_i\, \phi_i^\dagger\right] \tag{4.3.17}$$

[9] Anomaly cancellation is usually required in electroweak theories beyond the Standard Model, as one expects to retrieve the Standard Model gauge symmetries intact and we know that electromagnetism is conserved in the low energy limit.

where the trace runs over the index of the fundamental representation of the scalars. Plugging in the VEVs, this defines the matrix

$$\langle\phi_1\,\phi_1^\dagger\,+\,\phi_2\,\phi_2^\dagger\rangle = \begin{pmatrix} f_1^2\sin^2\left(\frac{f_2 v}{\sqrt{2}ff_1}\right)+f_2^2\sin^2\left(\frac{f_1 v}{\sqrt{2}ff_2}\right) & 0 & \frac{i}{2}\left[f_1^2\sin\left(\frac{\sqrt{2}f_2 v}{ff_1}\right)-f_2^2\sin\left(\frac{\sqrt{2}f_1 v}{ff_2}\right)\right] \\ 0 & 0 & 0 \\ -\frac{i}{2}\left[f_1^2\sin\left(\frac{\sqrt{2}f_2 v}{ff_1}\right)-f_2^2\sin\left(\frac{\sqrt{2}f_1 v}{ff_2}\right)\right] & 0 & f_1^2\cos^2\left(\frac{f_2 v}{\sqrt{2}ff_1}\right)+f_2^2\cos^2\left(\frac{f_1 v}{\sqrt{2}ff_2}\right) \end{pmatrix}. \tag{4.3.18}$$

Inserting this into (4.3.17), we perform the trace and diagonalise the gauge boson mass matrix. However, in order to relate the hypercharge X and $SU(3)_V$ gauge couplings to the Standard Model counterparts, we first look at the intermediate unbroken $SU(2)_L \times U(1)_Y$ phase, i.e. set $v = 0$. Diagonalisation mixes the neutral fields A^3, A^8 and A^x associated to diagonal generators to the Standard Model bosons B and W^3 and the heavy Z' and we define

$$\begin{aligned} W^3_\mu &= A^3_\mu \\ B_\mu &= \frac{-g_x A^8_\mu + \sqrt{3} g A^x_\mu}{\sqrt{3g^2+g_x^2}} \qquad\qquad g_y = \frac{\sqrt{3}\,g\,g_x}{\sqrt{3g^2+g_x^2}}. \\ Z'_\mu &= \frac{\sqrt{3} g A^8_\mu + g_x A^x_\mu}{\sqrt{3g^2+g_x^2}} \end{aligned} \tag{4.3.19}$$

We note that the $SU(2)_L$ gauge bosons due to the symmetry breaking structure carry over as $A^i = W^i$ for $i = 1, 2, 3$. The remaining four degrees of freedom become heavy and are rearranged into a complex $SU(2)_L \times U(1)_Y$ doublet $(W'_\pm, W'_0)$ with hypercharge 1/2.

Furthermore, we break the electroweak symmetry by reinstating v and diagonalise anew. The neutral fields now form the Z and γ bosons plus the Z', while W^1, W^2 mix as usual to the $W_\pm$ bosons. The masses are

$$\begin{aligned} M^2_\gamma &= 0 \\ M^2_Z &= \frac{g^2 v^2}{4}(1+t^2)\left[1+\frac{v^2}{2f^2}\left(1-\frac{1}{3}\frac{f^4}{f_1^2 f_2^2}\right)-\frac{v^2}{8f^2}(1-t^2)^2\right]+\mathcal{O}\left(\frac{v^6}{f^4}\right) \\ M^2_{W_\pm} &= \frac{g^2}{4}\left[f^2-\sqrt{f^4-4f_1^2 f_2^2\sin^2\left(\frac{v}{\sqrt{2}}\frac{f}{f_1 f_2}\right)}\right] = \frac{g^2 v^2}{4}\left[1+\frac{v^2}{2f^2}\left(1-\frac{1}{3}\frac{f^4}{f_1^2 f_2^2}\right)\right]+\mathcal{O}\left(\frac{v^6}{f^4}\right) \\ M^2_{W'_\pm} &= \frac{g^2}{4}\left[f^2+\sqrt{f^4-4f_1^2 f_2^2\sin^2\left(\frac{v}{\sqrt{2}}\frac{f}{f_1 f_2}\right)}\right] = \frac{g^2}{2}f^2+\mathcal{O}(v^2) \\ M^2_{W'_0} &= \frac{g^2}{2}f^2 \\ M^2_{Z'} &= \frac{g^2}{2}f^2\frac{4}{3-t^2}+\mathcal{O}(v^2) \end{aligned} \tag{4.3.20}$$

where we set $t = \tan\theta_W = g_y/g$ and recall the definition $f^2 = f_1^2 + f_2^2$. Note that in all masses, v only appears in the argument of trigonometric functions, even though these are expanded here for brevity. As expected, the five heavy gauge bosons have acquired masses of $\mathcal{O}(f)$, the $SU(2)_L\times U(1)_Y$ gauge bosons obtained masses $\mathcal{O}(v)$ and there remains the massless photon. However, comparing the $W_\pm$ and Z masses to the Standard Model terms, we see modifications suppressed by v^2/f^2.

4.3.5 The electroweak VEV

While the VEVs f_1, f_2 are for now put in by hand (which we attempt to explain in Sec. 5), the $SU(2)_V \times U(1)_Y$ breaking VEV v is expected to be generated radiatively. We thus calculate the one-loop effective potential and investigate whether there is a viable parameter region for which $v = 246$ GeV emerges.

We assume the neutral component of the Higgs field to obtain a background value h_b^0 and first calculate the gauge boson contribution to the Coleman-Weinberg potential[10],

$$V_{\text{gauge}}^{\text{eff}} = \frac{3}{32\pi^2} \sum_i \left[M_i^2(h_b^0)\,\Lambda^2 + \frac{M_i^4(h_b^0)}{2} \left(\log \frac{M_i^2(h_b^0)}{\Lambda^2} - \frac{1}{2} \right) \right] \tag{4.3.21}$$

where i runs over the eight massive gauge bosons. As we did before by naive power counting, we need to evaluate whether the potential contains quadratically divergent mass terms of the Higgs. The first term appears to contribute with such a divergence, it however evaluates to

$$\frac{1}{3}\left(12\, g^2 + g_x^2\right) f^4 = \text{const.} \tag{4.3.22}$$

We conclude that the Higgs mass is at most logarithmically sensitive to the cut-off Λ.

For the fermion contribution to the effective potential, we diagonalise the Yukawa couplings (4.3.5) analogously to the gauge sector. Due to the small Yukawa coupling however, we only consider the contribution of the top-like quarks. The one-loop masses are

$$\begin{aligned} M_t^2 &= \frac{m_T^2}{2} - \frac{1}{2}\sqrt{m_T^4 - 4 f_1^2 f_2^2 (\lambda_1^t)^2 (\lambda_2^t)^2 \sin^2\left(\frac{v}{\sqrt{2}} \frac{f}{f_1 f_2}\right)} = \lambda_t^2 \frac{v^2}{2} + \mathcal{O}\left(\frac{v^4}{f^2}\right) \\ M_T^2 &= \frac{m_T^2}{2} + \frac{1}{2}\sqrt{m_T^4 - 4 f_1^2 f_2^2 (\lambda_1^t)^2 (\lambda_2^t)^2 \sin^2\left(\frac{v}{\sqrt{2}} \frac{f}{f_1 f_2}\right)} = m_T^2 + \mathcal{O}\left(v^2\right) \end{aligned} \tag{4.3.23}$$

with $m_T^2 = f_1^2\,(\lambda_1^t)^2 + f_2^2\,(\lambda_2^t)^2$. The effective potential reads

$$V_{\text{fermion}}^{\text{eff}} = -\frac{6}{32\pi^2} \sum_k \left[M_k^2(h_b^0)\,\Lambda^2 + \frac{M_k^4(h_b^0)}{2} \left(\log \frac{M_k^2(h_b^0)}{\Lambda^2} - \frac{1}{2} \right) \right] \tag{4.3.24}$$

and again presents no quadratic divergence.

We can now assemble the full one-loop effective potential and check for a parameter space that fulfils the conditions: the VEVs f_1, f_2 need to be large enough to ensure the new gauge bosons to be sufficiently heavy, the top loop diagrams should cancel and the top quark mass as well as the Higgs VEV need to come out correctly. This gives us the constraints $f_i \geq \mathcal{O}(\text{TeV})$ and $M_T^2 \approx 2\,\lambda_t^2 f^2 \left(f_1^2/f_2^2 + f_2^2/f_1^2\right)^{-1}$.

Exploring the parameter space, one finds that the model cannot fulfill all of these conditions at once. In particular, the VEVs f_1, f_2 come out too small: choosing e.g. f_1 above $\mathcal{O}(100$ GeV) fixes f_2 to an unsustainably small value such that the heavy gauge boson are too light (see Fig. 4.5). The heavy top mass however does not impact the VEV and can be chosen to accommodate for the loop cancellation. We conclude that without modification, experimental constraints rule out this model.

We have seen that the Simplest Little Higgs is able to reproduce the Standard Model electroweak sector and its symmetry breaking. Scale separation is – in theory – achieved and one can push the fine tuning limit to the TeV range. However, the model severely violates experimental bounds, predicting new gauge bosons with masses $\mathcal{O}(100$ GeV). Note that the symmetry breaking pattern requires $f_i > v$ which also cannot be fulfilled.

[10] We normalise $h = \frac{1}{\sqrt{2}}\left(h^0, h^-\right)$ so that $h_b^0 =: v$.

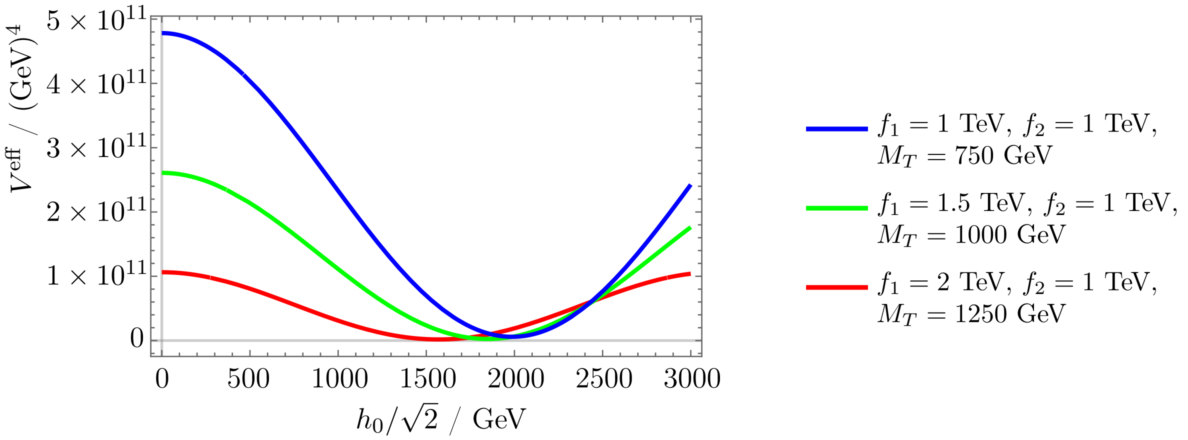


Figure 4.4: One-loop effective potential of the neutral component of the Higgs field for different combinations of f_1, f_2 and M_T in the low TeV range. The electroweak minimum v lies between 1.5 and 2 TeV.

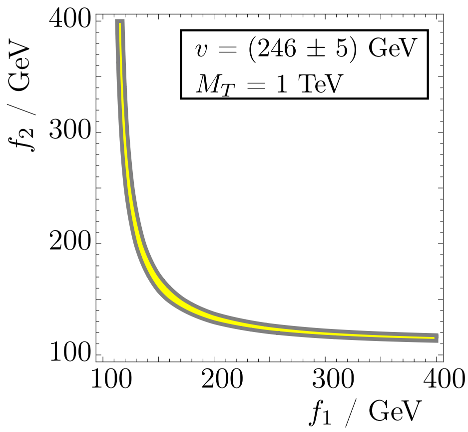
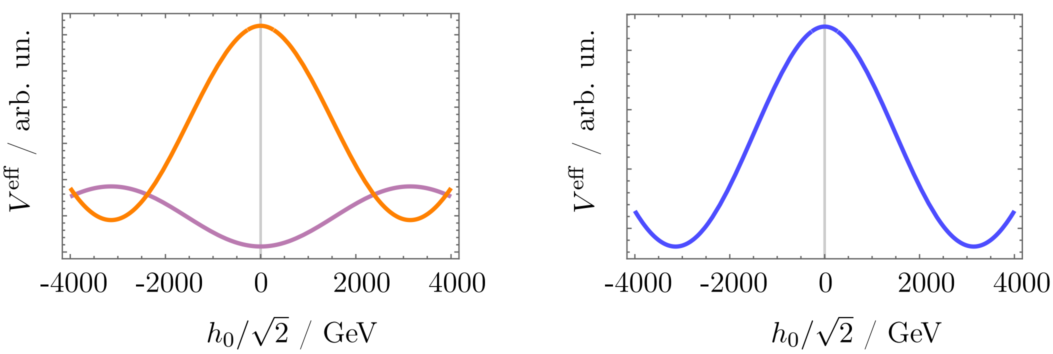


Figure 4.5: Parameter region of the VEVs f_1, f_2 for which the correct Standard Model VEV emerges from the 1-loop effective potential. A deviation by $v \pm 5$ GeV is employed for ease of visibility only. There are no valid pairs at higher energies. The results heavily clash with electroweak precision data.



(a) Gauge boson (violet) and fermion contribution (orange) to V^{eff}, normalised by subtracting/adding a common constant.

(b) Full one-loop effective potential obtained by addition of both contributions.

Figure 4.6: One-loop effective potential of the neutral component of the Higgs field for f_1 and $f_2 = 2$ TeV and $M_T = 750$ GeV. Note that the gauge boson and fermionic contributions do not add up to an extremum at smaller h_0 than the individual parts.

4.4 The Littlest Higgs

Building on the previous experience, we now construct another model and check its parameter space: the Littlest Higgs due to Arkani-Hamed, Cohen, Katz and Nelson [Lit]. It is based on the global symmetry breaking pattern $SU(5) \to SO(5)$. The procedure is analogous to the Simplest Little Higgs: we introduce a set of scalars in some representation of the global $SU(5)$ and fit the Standard Model particles into representations of an enlarged gauge group.

We choose to gauge two copies of the electroweak gauge group, $[SU(2) \times U(1)]^2$. These are embedded in a representation of a global $SU(5)$. As before, the scalar potential is required to obey the full $SU(5)$. We then break this global symmetry down to $SO(5)$ and recover the electroweak sector of the Standard Model (see Fig. 4.7).

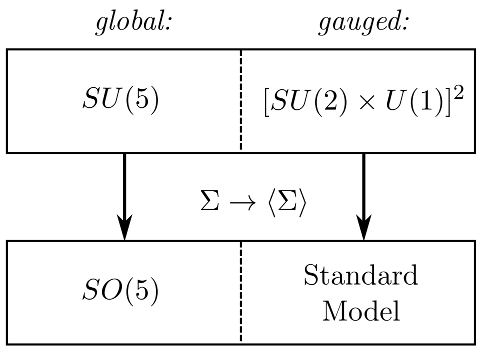


Figure 4.7: Schematic visualisation of the embedding and symmetry breaking pattern as mediated by the scalar Σ obtaining a VEV $\langle\Sigma\rangle$ in the Littlest Higgs model.

4.4.1 Particle content and symmetry breaking

Assume the complex scalar Σ to be a two-index tensor in the symmetric representation 15 of $SU(5)$. It transforms as $\Sigma \mapsto U\Sigma U^T$ for $U \in SU(5)$. Assigning to it a VEV $\langle\Sigma\rangle$ proportional to the unit matrix breaks $SU(5) \to SO(5)$, whereby 14 Goldstone bosons emerge. Of the 24 generators T_a of $SU(5)$, ten carry over as the generators of $SO(5)$. This is checked via

$$\begin{aligned} \Sigma \mapsto U\langle\Sigma\rangle U^T &\doteq (\mathbb{1} + i\,\theta_a T_a)\,\langle\Sigma\rangle\,\left(\mathbb{1} + i\,\theta_a T_a^T\right) + ... \stackrel{!}{=} \langle\Sigma\rangle \\ \mapsto \quad & T_a\langle\Sigma\rangle + \langle\Sigma\rangle T_a^T = 0 \end{aligned} \tag{4.4.1}$$

which is fulfilled only by the antisymmetric generators (in the usual basis of generalised Pauli matrices) of $SO(5)$. The broken generators $\hat{T}_a$ are the symmetric ones and thus obey

$$\hat{T}_a\langle\Sigma\rangle - \langle\Sigma\rangle\hat{T}_a^T = 0. \tag{4.4.2}$$

For convenience in the process of gauging however, we instead use an equivalent basis related by a transformation matrix,

$$\langle\Sigma\rangle = fS = f\begin{pmatrix} & & \mathbb{1} \\ & 1 & \\ \mathbb{1} & & \end{pmatrix}, \tag{4.4.3}$$

which also redefines the generators.

The form of Σ is dictated by its transformation properties: as it is a matrix, there need to be two free indices. We assume

$$\Sigma_{ij} = \xi_{in}\langle\Sigma\rangle_{nk}\,\tilde{\xi}_{kj} \tag{4.4.4}$$

and apply the symmetry condition

$$\begin{aligned}\Sigma_{ji} &= \xi_{jk}\langle\Sigma\rangle_{kn}\,\tilde{\xi}_{ni} \\ &= \tilde{\xi}_{ni}\langle\Sigma\rangle_{nk}\,\xi_{jk} \\ &= (\tilde{\xi}^T)_{in}\langle\Sigma\rangle_{nk}\,(\xi^T)_{kj} \stackrel{!}{=} \Sigma_{ij},\end{aligned} \tag{4.4.5}$$

from which we conclude $\tilde{\xi} = \xi^T$. We thus define the usual non-linear parametrisation in the broken phase,

$$\Sigma(x) = e^{i\pi(x)/f}\,\langle\Sigma\rangle\,e^{i\pi^T(x)/f} = e^{2i\pi(x)/f}\,\langle\Sigma\rangle \tag{4.4.6}$$

with the Goldstone bosons arranged as [Lhr]

$$\pi = \pi^a T_a = \begin{pmatrix} \chi + \eta/(2\sqrt{5}) & h^*/\sqrt{2} & \Delta^\dagger \\ h^T/\sqrt{2} & -2\,\eta/\sqrt{5} & h^\dagger/\sqrt{2} \\ \Delta & h/\sqrt{2} & \chi^T + \eta/(2\sqrt{5}) \end{pmatrix}. \tag{4.4.7}$$

Here $\chi = \chi_a\sigma^a/2$, comprising three real degrees of freedom, η is a real singlet, $h = (h^+, h^0)$ is the Higgs doublet[11] and Δ is a complex triplet parametrised as a symmetric 2×2 matrix. Note that in the second step of (4.4.6) we used the Goldstone bosons are defined as fluctuations around the VEV in the broken directions, and thus were able to use (4.4.2).

As is characteristic for Little Higgs models, we now gauge a subgroup of the global $SU(5)$, thus breaking it explicitly[12]. This is achieved by embedding the generators of two copies of $SU(2)\times U(1)$ into the $SU(5)$ space,

$$\begin{aligned} Q_1^a &= \begin{pmatrix} -(\sigma^a/2)^* & & \\ & & \\ & & \end{pmatrix} \quad \text{and} \quad Y_1 = \mathrm{diag}(-3,-3,2,2,2)/10, \\ Q_2^a &= \begin{pmatrix} & & \\ & & \\ & & \sigma^a/2 \end{pmatrix} \quad \text{and} \quad Y_2 = \mathrm{diag}(-2,-2,-2,3,3)/10. \end{aligned} \tag{4.4.8}$$

After symmetry breaking, the electroweak gauge group of the Standard Model is generated by the linear combinations $Q^a := (Q_1^a + Q_2^a)$ and $Y := (Y_1 + Y_2)$, as evident by plugging into (4.4.1). The orthogonal combinations correspond to the broken $SU(2)\times U(1)$, whose gauge bosons obtain masses of order f. Of the 14 Goldstone bosons, four are eaten: the fields χ and η, which we eliminate by going to unitary gauge. We check the charges of h and Δ by computing the commutators of π with the unbroken generators:

$$\begin{aligned} [Q^a, \pi] &= \begin{pmatrix} 0 & -\left(\frac{\sigma^a}{2}h/\sqrt{2}\right)^* & -(\frac{\sigma^a}{2}\Delta - \Delta\frac{\sigma^a}{2})^\dagger \\ \left(\frac{\sigma^a}{2}h/\sqrt{2}\right)^T & 0 & -\left(\frac{\sigma^a}{2}h/\sqrt{2}\right)^\dagger \\ \frac{\sigma^a}{2}\Delta - \Delta\frac{\sigma^a}{2} & \left(\frac{\sigma^a}{2}h/\sqrt{2}\right) & 0 \end{pmatrix} \\ [Y, \pi] &= \begin{pmatrix} 0 & -\frac{1}{2}\,h^*/\sqrt{2} & -\Delta^\dagger \\ \frac{1}{2}\,h^T/\sqrt{2} & 0 & -\frac{1}{2}\,h^\dagger/\sqrt{2} \\ \Delta & \frac{1}{2}\,h/\sqrt{2} & 0 \end{pmatrix}, \end{aligned} \tag{4.4.9}$$

[11] Note that in this convention the Standard Model hypercharge of the Higgs comes out as $Y = 1/2$.

[12] To understand this, divide the kinetic part of the scalar Lagrangian into the derivative and gauge boson parts. The derivative term commutes with a global $SU(5)$ transformation, but the terms containing gauge bosons do not. See also Sec. 4.2.1.

and confirm the $SU(2)_L \times U(1)_Y$ charges $h : 2_{\frac{1}{2}}$ and $\Delta : 3_1$. The scalar part of the Lagrangian reads

$$\begin{aligned} &\frac{1}{8}\,\mathrm{tr}\,|D_\mu \Sigma|^2 \\ &\quad = \frac{1}{8}\,\mathrm{tr}\,|\partial_\mu \Sigma - \sum_{k=1,2} [ig_k W_k^a (Q_k^a \Sigma + \Sigma (Q_k^a)^T + ig_k' B_k (Y_k \Sigma + \Sigma (Y_k)^T]\,|^2, \end{aligned} \tag{4.4.10}$$

where $W_{1,2}$ and $B_{1,2}$ respectively denote the gauge bosons of each of the $SU(2)$ and $U(1)$ symmetries and $g_{1,2}, g'_{1,2}$ the corresponding gauge couplings.

In summary, the symmetry breaking occurs as follows. Breaking the global $SU(5) \to SO(5)$ through $\langle \Sigma \rangle$ produces 14 Goldstone bosons. By the same VEV, $[SU(2) \times U(1)]^2$ is broken down to its diagonal subgroup, the electroweak gauge group of the Standard Model, while four real modes are eaten. This leaves only the Higgs and a complex scalar triplet.

Note that when turning off one of the $SU(2)$ gauge interactions, there is an additional global $SU(3)$ symmetry in either the lower right or upper left 3×3 block of the gauge covariant derivative. We label the symmetry corresponding to $g_2 = 0$ as $SU(3)_1$ and vice versa. Either of them induces a shift symmetry of the Higgs, which in turn can only be broken by terms proportional to both g_1 and g_2, i.e. the Higgs is indeed a PNGB with loop-order mass.

4.4.2 Top Yukawa couplings

For the fermions in this model, we copy the assignments of the Standard Model, but add a vector-like quark pair T_L, t_R^1. We arrange (only) the third generation doublet into a vector $Q = (b, -t, T)_L$ of the global $SU(3)_1$. As in the Simplest Little Higgs, t_R^1 and the right-handed top quark t_R^2 mix to produce the Standard Model and the heavy top quark.

This allows us to write down Yukawa couplings without reintroducing quadratic divergences: we require that the terms which couple Σ to t_R^1 obey the global $SU(3)_1$ and respectively for t_R^2 and $SU(3)_2$. Thus, mass terms may only arise with logarithmic degree of divergence. This is achieved by the couplings [Lit]

$$\frac{\lambda_1}{f} \sum_{\substack{i,j,k \\ =1,2,3}} \sum_{x,y=4,5} \epsilon_{ijk} \epsilon_{xy}\, \bar{t}_R^1 Q_i \Sigma_{jx} \Sigma_{ky} + \lambda_2 f\, \bar{t}_R^2 T_L + \text{h.c.} \tag{4.4.11}$$

Indeed, we find that the vectors Σ_{i4} and Σ_{i5} with $i = 1, 2, 3$ both transform as triplets under $SU(3)_1$. Thus the antisymmetric product of the λ_1 term forms a singlet of $SU(3)_1$, but breaks $SU(3)_2$. The second term conversely respects $SU(3)_2$ but not $SU(3)_1$. The $(U(1)_1, U(1)_2)$ charges can be chosen such that the Yukawa terms are neutral. A valid assignment is $q = (t, b)_L : (\frac{1}{6}, 0)$, $T_L : (\frac{2}{3}, 0)$ and $t_R^1 : (\frac{7}{15}, \frac{1}{5})$ [Lhr].

As in the Simplest Little Higgs, we define t_R and T_R as linear combinations of t_R^1 and t_R^2. Abbreviating $\phi_i = \epsilon_{ijk} \Sigma_{j4} \Sigma_{k5} / f^2$ and neglecting Δ, we can rewrite (4.4.11) as

$$\overline{Q}\, M_f(\Sigma)\, \chi + \text{h.c.} \qquad \text{with} \quad M_f(\Sigma) = f \begin{pmatrix} & 0 & 0 \\ \lambda_1 \phi & 0 & 0 \\ & 0 & \lambda_2 \end{pmatrix} \tag{4.4.12}$$

where we have arranged the right-handed quarks into a vector $\chi = (t^1, b, t^2)_R$. Expanding to linear order in h, we find the Higgs and top-like couplings

$$i\sqrt{2}\lambda_1\, \bar{t}_R^1\, q\, h + f(\lambda_1 \bar{t}_R^1 + \lambda_2 \bar{t}_R^2) T_L + \text{h.c.} \tag{4.4.13}$$

We define the linear combinations which become the Dirac partner of T_L and the Standard Model top quark,

$$t_R = \frac{\lambda_2 t_R^1 - \lambda_1 t_R^2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \qquad T_R = \frac{\lambda_2 t_R^2 + \lambda_1 t_R^1}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \tag{4.4.14}$$

We find the tree level mass $m_T = f\sqrt{\lambda_1^2 + \lambda_2^2} =: f\lambda_T$ and recover the Standard Model top quark Yukawa term [Lhr]

$$\lambda_t \, \bar{t}_R \, q \, h \qquad \text{where} \quad \lambda_t = i\frac{\sqrt{2}\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \tag{4.4.15}$$

For the Yukawa couplings of the two lighter generations, we can reuse the above terms but need not introduce heavy up- and charm-like quark pairs [Lit]. Down quarks and charged leptons are taken care of by replacing Σ with Σ^*. Leaving out the corresponding heavy partners induces quadratic divergences, but these are unproblematic due to the small Yukawa couplings [Lhr].

4.4.3 Gauge interactions

As in the Simplest Little Higgs, we calculate the gauge boson masses after both breaking procedures $[SU(2) \times U(1)]^2 \to SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$ by assuming suitable VEVs f and v for Σ and h, respectively. We again require f to be in the TeV range and will later check whether $v = 246$ GeV can be arranged for.

After breaking the symmetry down to the Standard Model gauge group, the six bosons of the $SU(2) \times SU(2)$ symmetry mix to form three massive W' and the W bosons of the unbroken $SU(2)_L$. Equally, the $U(1) \times U(1)$ gauge bosons form one massive linear combination B' and the massless B boson of the Standard Model. From their coefficients, we can read off the gauge couplings

$$g_y = \frac{g_1' g_2'}{\sqrt{g_1'^2 + g_2'^2}} \qquad g = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \tag{4.4.16}$$

of the Standard Model hypercharge and weak couplings. We also define the gauge couplings $\tilde{g}_y$ and $\tilde{g}$ of the broken $SU(2) \times U(1)$,

$$\tilde{g}_y = \frac{g_1'^2 + g_2'^2}{\sqrt{g_1'^2 + g_2'^2}} \qquad \tilde{g} = \frac{g_1^2 + g_2^2}{\sqrt{g_1^2 + g_2^2}} \tag{4.4.17}$$

Plugging in v, we break the Standard Model down to $U(1)_Y$ in the usual way. We diagonalise the terms coupling Σ to the gauge bosons in (4.4.10) and obtain the masses

$$
\begin{aligned}
M_\gamma^2 &= 0 \\
M_Z^2 &= \frac{g^2 v^2}{4}(1+t^2) + \mathcal{O}\left(\frac{v^4}{f^2}\right) \\
M_{W^\pm}^2 &= \frac{f^2}{16}\left[2g_1^2 + 2g_2^2 - \sqrt{8g_1^2 g_2^2 \cos\left(\frac{\sqrt{2}v}{f}\right) + 2g_1^2 g_2^2 \cos\left(\frac{2\sqrt{2}v}{f}\right) + 4g_1^4 - 2g_1^2 g_2^2 + 4g_2^4}\right] \\
&= \frac{g^2 v^2}{4} + \mathcal{O}\left(\frac{v^4}{f^2}\right) \\
M_{B'}^2 &= \frac{\tilde{g}_y^2 f^2}{20} + \mathcal{O}\left(v^2\right) \\
M_{W'_{1,2}}^2 &= \frac{f^2}{16}\left[2g_1^2 + 2g_2^2 + \sqrt{8g_1^2 g_2^2 \cos\left(\frac{\sqrt{2}v}{f}\right) + 2g_1^2 g_2^2 \cos\left(\frac{2\sqrt{2}v}{f}\right) + 4g_1^4 - 2g_1^2 g_2^2 + 4g_2^4}\right] \\
&= \frac{\tilde{g}^2 f^2}{4} + \mathcal{O}\left(v^2\right) \\
M_{W'_3}^2 &= \frac{\tilde{g}^2 f^2}{4} + \mathcal{O}\left(v^2\right)
\end{aligned}
\tag{4.4.18}
$$

where again $t = g_y/g$. As in the Simplest Little Higgs, the VEV v only appears inside of trigonometric functions, which we expanded here. We see that the gauge bosons corresponding to the first broken $SU(2) \times U(1)$ indeed obtain masses of order f, while the Standard Model masses are recovered with corrections suppressed by v/f.

4.4.4 The electroweak VEV

We proceed to compute the contribution of the gauge bosons and the top-like quarks to the one-loop effective potential by diagonalising $M_f(\Sigma)$. Assuming a background value for h^0, we calculate the one-loop masses of the quarks by taking the eigenvalues of $M_f(\Sigma)$,

$$
\begin{aligned}
M_t^2 =& \frac{f^2}{16}\left[-\sqrt{\left(\lambda_1^2\left(4\cos(x) - \cos(2x) + 5\right) + 8\lambda_2^2\right)^2 - 128\lambda_1^2\lambda_2^2 \sin^2(x)}\right. \\
&\left. - 4\lambda_1^2 \cos^2(x/2)\left(\cos(x) - 3\right) + 8\lambda_2^2\right] = \lambda_t^2 \frac{v^2}{2} + \mathcal{O}\left(\frac{v^4}{f^2}\right) \\
M_T^2 =& \frac{f^2}{16}\left[\sqrt{\left(\lambda_1^2\left(4\cos(x) - \cos(2x) + 5\right) + 8\lambda_2^2\right)^2 - 128\lambda_1^2\lambda_2^2 \sin^2(x)}\right. \\
&\left. - 4\lambda_1^2 \cos^2(x/2)\left(\cos(x) - 3\right) + 8\lambda_2^2\right] = m_T^2 + \mathcal{O}\left(v^2\right)
\end{aligned}
\tag{4.4.19}
$$

where we defined $x = \sqrt{2}\,v/f$. We plug these masses into the fermionic part of the potential as given in (4.3.24). Equally, the gauge bosons masses given in (4.4.18) contribute via the terms of (4.3.21). As expected, we observe that there is no quadratically divergent part.

We can now check whether the full one-loop effective potential of the Littlest Higgs is able to develop a VEV v compatible with experimental evidence. In particular, we need to check whether the Littlest Higgs model is compatible with electroweak precision tests (EWPT) and give lower bounds on the masses of the new particles M_T, M'_B and M'_{W_i} (for $i = 1, 2, 3$). From a wide range of analyses of data acquired at LEP1 & 2 and the LHC running at 2, 7 and 8 TeV, we extract a lower bound on the VEV $f \gtrsim 3.5$ TeV at a confidence level of 95% [Li1]–[Li6].

Plugging in the values for the Standard Model couplings g, g_y and the top Yukawa coupling λ_t, we thus perform a parameter scan with the free parameters. Excerpts of the viable parameter regions are shown in Fig. 4.8. The analysis is simplified by fixing the heavy top mass and using the tree-level masses of the new heavy gauge bosons, where $M'_{W_i} =: M'_W$. We observe that the Littlest Higgs is indeed able to radiatively induce the correct VEV v while avoiding conflicts with EWPT. The masses of the neutral heavy boson and the top are pushed to $\mathcal{O}(\text{TeV})$, while the charged bosons are required to be an order of magnitude heavier.

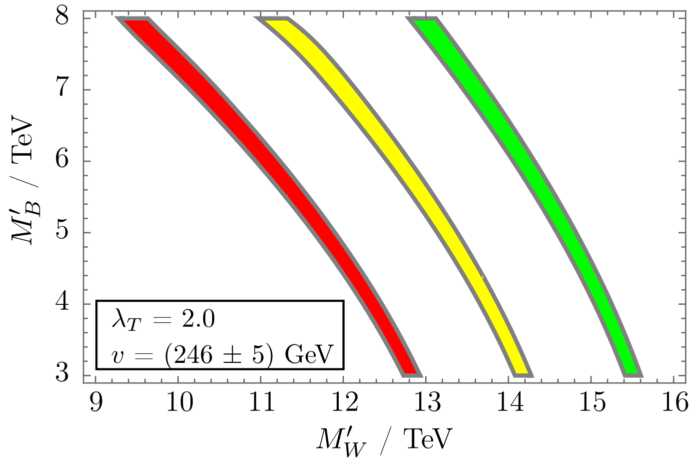
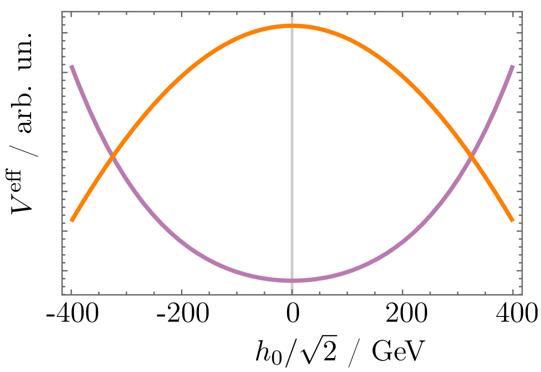
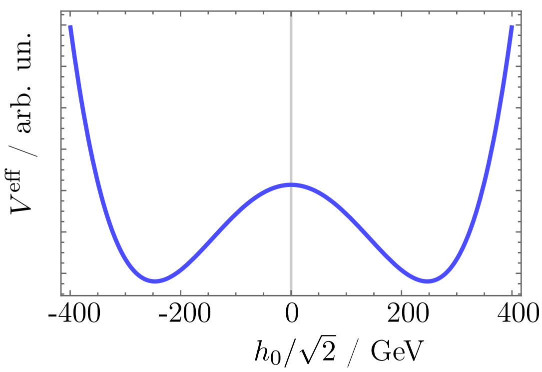


Figure 4.8: Viable parameter regions for which the correct Standard Model VEV emerges from the 1-loop effective potential as functions of the tree-level masses of the new gauge bosons. The Little Higgs VEV is chosen as $f = 3.5$ TeV, shown in red, 3.75 TeV in yellow and 4 TeV in green. The heavy top coupling has been set to $\lambda_T = 2$, i.e. $M_T \approx 2f$. The bands show regions in which the correct Standard Model VEV emerges. A deviation by $v \pm 5$ GeV is employed for ease of visibility only.



(a) Gauge boson (violet) and fermion contribution (orange) to V^{eff}, normalised by subtracting/adding a common constant.



(b) Full one-loop effective potential obtained by addition of both contributions.

Figure 4.9: One-loop effective potential of the neutral component of the Higgs field for $f = 3.5$ TeV, $M'_B = 6$ TeV, $M'_W = 11$ TeV and $M_T = 7$ TeV. We observe that the full potential has the correct form and the VEV $v = 246$ GeV emerges.

4.4.5 T-parity

The bound on f can be lowered by considering the Littlest Higgs model with T-parity (LHT). Under this new discrete symmetry, we assign odd charge to the new, heavy particles and even charge to the Standard Model fields [Lhr]. Higher dimensional operators coupling Standard Model particles via tree-level exchange of one heavy field are thus forbidden, as each vertex contains an odd number of heavy fields. At the same time, loop processes remain allowed. This relaxes constraints from EWPT similarly to how R-parity acts in phenomenological models of Supersymmetry.

Results extracted from LHC Run 1 data show that f may be as low as 700 GeV (at 95% confidence level) in the LHT [Li1, Li7]. Analysis of data from the second run may lead to higher boundaries or hint at a discovery.

4.5 Constructing Little Higgs models with viable parameter space

The results of the previous two sections have shown that models which employ collective symmetry breaking to realise the Higgs as a pseudo-Nambu-Goldstone boson exhibit rich phenomenology, which is testable by current and near-future experiments. In both models, we observe that all masses and thus the effective potential include trigonometric functions periodic in v, cf. (4.3.20) and (4.4.18). This is implied by the non-linear parametrisation of the scalars, as the Higgs field itself has a discrete symmetry in this parametrisation; note however that only the first period is physical, as the parametrisation breaks down for large field values $\pi(x) = \mathcal{O}(f)$. We have seen that in the Littlest Higgs model, the radiative generation of the Standard Model VEV via the one-loop effective potential is generally possible and there exists a parameter region compatible with electroweak precision tests. In the Simplest Little Higgs however, it is not possible to induce $v = 246$ GeV without lowering the scales f_1, f_2 to values inconsistent with electroweak precision tests and the symmetry breaking pattern.

This raises the question which requirements need to be met for the construction of feasible Little Higgs models. We now collect indicators for the compatibility of a generic Little Higgs model with electroweak observations by noting the differences between the Simplest Little Higgs and the Littlest Higgs model[13]:

1. *The Simplest Little Higgs features two sets of scalars, while the Littlest Higgs only requires one.*

   This is a result of the implementation of collective symmetry breaking, as the scalars need to obey a global symmetry larger than the gauged symmetry if all gauge and Yukawa couplings are set to zero. The two models apply different solutions: either gauge only a subgroup (Littlest Higgs) or introduce multiple sets of scalars (Simplest Little Higgs). Collective symmetry breaking with one fully gauged set is not possible, as all Goldstone modes are eaten by the gauge bosons of the broken symmetry (see e.g. [Lhr]). We come to the conclusion that none of the additional degrees of freedom enter the effective potential and thus they cannot influence the magnitude of the VEV v.

2. *In the Littlest Higgs, we have gauged $[SU(2) \times U(1)]^2$ by embedding their generators into the larger $SU(5)$. This is opposed to the process of gauging $SU(3)_V$ applied in the Simplest Little Higgs scenario, under which the scalars ϕ_i transform in the fundamental representation.*

   However, the embedding of generators as found in the Littlest Higgs can be replicated in the Simplest Little Higgs by rearranging the scalars into a column vector $(\phi_1, \phi_2)^T$. Now the upper and lower components each obey one part of the global symmetry. We gauge two copies of the generators acting on the subspaces[14], i.e.

   $$T^a = \begin{pmatrix} \lambda^a/2 & \\ & \lambda^a/2 \end{pmatrix}, \quad T_X \propto \mathbb{1}_{6\times 6}. \tag{4.5.1}$$

   As in both models we now gauge a subgroup yet phenomenological implications have not changed, we reason that this cannot be the cause of the different VEV ranges.

[13] This list is non-exhaustive, and thus the following implications are only propositions.

[14] Note that we avoid gauging $[SU(3) \times U(1)]^2$ by assigning the same transformation properties to both ϕ_1 and ϕ_2.

3. *In the first stage of symmetry breaking, caused by the VEVs f_1, f_2 and f, a global $[SU(3)]^2$ symmetry is broken down to $[SU(2)]^2$ in one model versus $SU(5)$ to $SO(5)$ in the other.*

   This is caused by the form of the VEVs, $(\phi_i)_0 = (0, 0, f_i)^T$ and $\langle\Sigma\rangle \propto \mathbb{1}$. Recall however that the global symmetry is explicitly broken by gauge and Yukawa interactions, but the background field value of the Higgs implies the breaking of a *gauged* symmetry. There is no reason for the second step in the symmetry breaking pattern to depend on the exact global symmetry; it is only required to be large enough to ensure four real degrees of freedom for the Higgs doublet to become pseudo-Nambu-Goldstone bosons.

4. *Leaving out the $U(1)$ symmetries, in the Littlest Higgs we break a gauged $SU(N) \times SU(N)$ down to its diagonal subgroup. In the Simplest Little Higgs, the breaking pattern of the gauge symmetry is $SU(N)$ to $SU(N-1)$.*

   We investigate the difference of these patterns via two toy models. In the *bitriplet model*, we assume a set of scalars Σ to transform as $(3, 3)$ under a gauged $SU(2)_1 \times SU(2)_2$ symmetry. Conversely, the *sextet model* features ϕ as a complex sextet of the gauged $SU(3)$. Both scalars acquire a VEV, $\langle\Sigma\rangle$ and ϕ_0 respectively, such that the gauge symmetry is broken down to $SU(2)$. This is then fully broken by assigning another VEV to the set of Goldstone bosons π. The symmetry breaking pattern is sketched in Fig. 4.10.

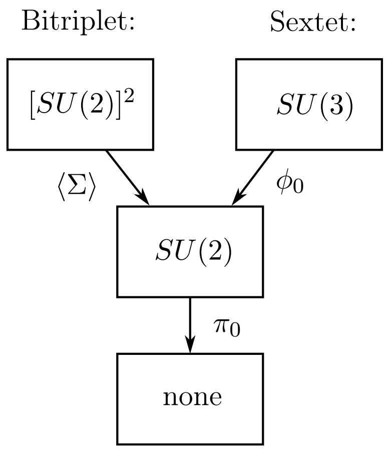


Figure 4.10: Schematic visualisation of the symmetry breaking pattern in both toy models.

Note that neither of these toy models implements collective symmetry breaking, as there is no larger global symmetry. Our aim is instead to calculate the gauge boson masses in a background field of π, as this determines the position of the VEV.

In the bitriplet model, the set of scalars transforms under the $SU(2)_1 \times SU(2)_2$ as

$$\Sigma \mapsto U_1\,\Sigma\,U_2^\dagger \doteq \left(\mathbb{1} + i\theta^a T_1^a\right)\Sigma\left(\mathbb{1} - i\theta^a (T_2^a)^\dagger\right) + \ldots \tag{4.5.2}$$

where the generators have been expressed as 3×3 matrices. We are able to read off the gauge coupling and construct the scalar Lagrangian

$$\mathcal{L} = \frac{1}{8}\mathrm{tr}\,|\partial_\mu\Sigma + ig_1 W_1^a\,T_1^a\,\Sigma - ig_2 W_2^a\,\Sigma\,T_2^a|^2\,. \tag{4.5.3}$$

From (4.5.2) we see that assigning $\langle\Sigma\rangle = f \cdot \mathbb{1}$ breaks the symmetry down to the diagonal subgroup $SU(2)_V$. Of the twelve real degrees of freedom, three obtain masses of order f, while another three are eaten. The remaining six degrees of freedom form the set π and are denoted in the usual non-linear parametrisation; note that after symmetry breaking, Σ can be expressed in the same symmetric form as the scalars in the Littlest Higgs, see (4.4.6). We assume a background field value h_0 for one of its real components,

$$\langle\Sigma\rangle = f\exp\left\{2\frac{i}{f}\,\pi_0\right\}\mathbb{1}_{3\times 3} \qquad \text{with} \quad \pi_0 = h_0\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}. \tag{4.5.4}$$

We are now able to calculate the masses of the six gauge bosons in terms of h_0 and plug them into the one-loop effective potential.

As we set out to compare the effect of the symmetry breaking pattern, we perform the same steps for the sextet model. ϕ is a symmetric matrix of $SU(3)$ [Sla], and thus we can copy the kinetic term from the Littlest Higgs,

$$\mathcal{L} = \frac{1}{8}\mathrm{tr}\left|\partial_\mu\phi + igW^a\left[\frac{\lambda^a}{2}\,\phi + \phi\left(\frac{\lambda^a}{2}\right)^T\right]\right|^2, \tag{4.5.5}$$

were λ^a denote the eight Gell-Mann matrices. The full symmetry breaking VEV reads

$$\phi_0 = \exp\left\{2\frac{i}{f}\,\hat{\pi}_0\right\}\begin{pmatrix}0&0&0\\0&0&0\\0&0&f\end{pmatrix} \qquad \text{with} \quad \hat{\pi}_0 = h_0\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \tag{4.5.6}$$

and we end up with eight massive gauge bosons. We again use this to calculate the effective potential.

In order to analyse the potentials of both models, we normalise the masses of the heavy gauge bosons such that they are equal for $h_0 = 0$. This is achieved by setting the gauge couplings $g_1, g_2 = g/\sqrt{2}$. Plugging in generic values for g and f, we can now compare the potentials, see Fig. 4.11.

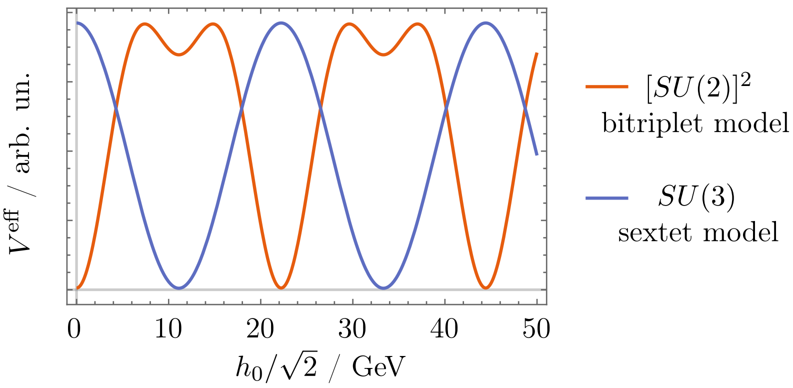


Figure 4.11: One-loop effective potential from gauge boson contributions in both toy models. The couplings have been set to $g_{1,2} = 1/\sqrt{2}$ and $g = 1$, and $f = 10$ GeV is assumed. Vertical normalisation has been adjusted arbitrarily in order to compare both potentials.

We observe that albeit their different origins, the periods of the potentials are identical. Recalling that the bitriplet model is constructed in resemblance to the Littlest Higgs, one might have expected it to feature a more rapid oscillation, indicating a smaller value for the background field.

We however find that the potential of the bitriplet model has its phase shifted by half a period compared to the sextet model and exhibits a positive slope around the origin. Keeping in mind that fermions contribute to V^{eff} with a negative sign, one might hope that the two contributions balance each other such that a small, non-zero VEV emerges. Generalising to Little Higgs model, this suggests that product gauge groups in the vein of the bitriplet model boast a smaller electroweak minimum. However, comparison with the gauge boson contributions in both original models (see Fig. 4.6 and 4.9) contradicts this hypothesis: both gauge boson potentials in the Simplest and the Littlest Higgs feature a positive slope around the minimum, i.e. the expected disparity between the product and the single group model does not appear. The same argument holds for the substructure seen in the amplitude of the bitriplet model.

Gathering the information extracted from comparing the toy models, we speculate that the difference in the symmetry breaking pattern – single group versus product group – can not be the solemn reason for the viable parameter space of the Littlest Higgs as compared to the Simplest Little Higgs. However, one must keep in mind that the toy models represent a stark simplification of the original models, omitting the $U(1)$ symmetries, collective symmetry breaking and fermions altogether.

5. *The scalars $\phi_{1,2}$ of the Simplest and Σ of the Littlest Higgs are in different representations of the underlying global symmetries.*

   The two ϕ_i are each in one of the fundamental representations of $SU(3)_1 \times SU(3)_2$. On the other side, Σ is a symmetric matrix with 15 complex degrees of freedom. In non-linear parametrisation, the fields read

$$\phi_i(x) = e^{\pm i\pi(x)/\tilde{f}}\,\phi_0 \quad \text{and} \quad \Sigma(x) = e^{2i\pi(x)/f}\,\langle\Sigma\rangle \tag{4.5.7}$$

   where $f_1 = f_2 = \tilde{f}$ has been set for simplicity. It is clear that the factor 2 in the exponent of Σ will have an impact on all later calculations; in particular, it modifies the ratio h_0/f found in the trigonometric functions of all masses. This easily allows for a smaller VEV.

   Two conditions were used in the derivation of the parametrisation of Σ: that it is a symmetric matrix and that its Goldstone bosons commute with the generators of the broken symmetry. The second requirement is met in any Little Higgs model, as we define the remaining scalar modes as fluctuations in the broken directions. We are however not able to cast the vectors ϕ_i into a symmetric matrix form with the same transformation behaviour as in (4.4.4).

   We conclude that a generic Little Higgs model is more likely to feature a small electroweak VEV if the sets of scalars are not in the fundamental, but a higher dimensional symmetric matrix representation[15].

[15] This can possibly be extended to non-symmetric representations, but one needs to check the commutator of the broken generators with the VEV. This highly depends on the symmetry breaking pattern one chooses.

In summary, it is reasonable to attribute the feasibility of Little Higgs models to the choice of the gauge group structure (dimension, single versus product group) and the representations in the scalar sector.

It is worth noting that some applications of Little Higgs models extend the acceptable parameter space by admitting quadratic μ^2-term to the scalar potential by hand [Sim]. Setting aside the problem of generating this scale, it enables us to push f to higher values, e.g. opening up a viable parameter space for the Simplest Little Higgs. But then the μ^2-term needs to be set to a precise value in order to cancel large contributions by heavy particles; i.e. we have reintroduced fine-tuning. Allowing for 1% of fine-tuning, the Simplest Little Higgs can be realised with $f_1^2 + f_2^2 \gtrsim 4$ TeV [Li1].

5 Completing Little Higgs models in the UV

So far we have seen that collective symmetry breaking can indeed be implemented into specific Little Higgs models in such a way that electroweak precision tests are not contradicted and the Standard Model VEV $v = 246$ GeV is reproduced. However, Little Higgs theories are only valid up to a cut-off Λ_{LH} (cf. Sec. 4.2). Above this scale they must be embedded into a grander framework, which we name the *ultraviolet (UV) completion*. In this section, we review the conditions which need to be met by candidates for UV completing theories and develop and present several executions.

The most important ingredient to collective symmetry breaking is the symmetry structure: the scalars are required to be representations of a global symmetry H, of which only a proper subgroup G is gauged. Symmetry transformations in H are explicitly broken by gauging G, as demonstrated in Sec. 4.2.1. The scalar potential is not affected by the gauging process. In fact, it is required that the scalar potential obeys H symmetry in order to forbid terms which introduce quadratic divergencies (Sec. 4.2.2).

The construction of Little Higgs models relies on the global symmetry to be manifest (albeit explicitly broken) down to the electroweak scale; i.e. there must be no terms in the scalar potential that obey H at Λ_{LH} but break it at lower energies. The reason is that breaking the global symmetry at tree-level invalidates the application of Goldstone's theorem, thus undoing the construction of collective symmetry breaking as shown in Sec. 4.2.1.

To summarise, the scalar Lagrangian in a generic Little Higgs model needs to decompose (below Λ_{LH}) as

$$\mathcal{L} = \underbrace{\mathcal{L}_{\text{gauge-kin.}}}_{G \subset H \text{ invariant}} - \underbrace{V}_{H \text{ invariant}} . \tag{5.0.1}$$

The UV completion is required to provide this form at the scale Λ_{LH}. It must also generate the Little Higgs VEV f (or f_1, f_2, depending on the implementation) without reintroducing fine-tuning. For sake of computability, we require the embedding model to be perturbative up to the Planck scale[1]. Keep in mind that the cut-off $\Lambda_{\text{LH}} \lesssim 4\pi f$ is set by non-perturbativity as shown in Sec. 4.2; these two properties need to be reconciled in some way. Finally, we aim to find UV completions which do not introduce fields with masses below $\mathcal{O}(f)$, i.e. in the low energy limit, physics are described by the Little Higgs theory alone.

In the construction of both Little Higgs models we have not only approached the hierarchy problem but also omitted any tree-level mass term for the scalars, thus addressing the question how the electroweak scale can be generated dynamically. Therefore, the UV completions we investigate will need to provide a reason for the absence of any dimensionful parameters. This can be achieved by promoting scale invariance to a classical symmetry [Si1, Si2, Si3], e.g. as a result of a decoupled theory of Quantum Gravity. In the following, we assume this to be the case.

Several proposals to UV complete Little Higgs models exist in the literature[2]. A large number of them employ Supersymmetry, another class of theories which may be used to alleviate the hierarchy problem [Sup, Luv]. There are implementations which are based on extra dimensions [Luv], and a

[1] The embedding theory may equally only be valid up to some scale between the Little Higgs and Planck scale. In the following, the Planck scale serves as a stand-in for the cut-off of the UV theory, as the Little Higgs UV theory needs to be replaced at Λ_{Pl} *at the latest* due to the inclusion of quantum gravity.

[2] An extensive list of Little Higgs UV completions can be found in [Coh].

large class of theories which implement the Little Higgs scalar as a composite particle under strong dynamics, many of which also build on Supersymmetry [Co1, Co2].

One approach to the construction of a *weakly* coupled UV completion is to assume that the scalar set of the Little Higgs model itself emerges from collective symmetry breaking [Lht, Lhu]. To achieve this, another Little Higgs theory with a raised cut-off of $\mathcal{O}(10\,\text{TeV})$ is stacked on the basic Little Higgs model. By iteration of this process one can build up a *Little Higgs tower* and reach up to the Planck scale.

In the following, we investigate obstacles of UV completions and attempt some implementations. When constructing a UV completion to Little Higgs models, the first objective is to explain how the global symmetry H comes about. When setting all gauge and Yukawa couplings in the theory to zero, H must be restored at and below the scale Λ_{LH}. Two approaches come to mind: H can be generated dynamically, or one can gauge the full H and then break it down to the proper subgroup G. Implementing either path can be done in a variety of ways.

We go on to propose and investigate three attempts to UV complete Little Higgs models: dynamical generation of a global symmetry H, breaking of the fully gauged H symmetry with a scalar and with a fermion condensate.

5.1 Dynamical generation of the global symmetry

As the name suggests, we now attempt to generate the global symmetry H radiatively. This means that we make use of the running of the couplings due to renormalisation: we check whether the parameters in the Lagrangian can be chosen such that certain terms become zero at Λ_{LH} so that H emerges without any further mechanism.

We investigate this idea on basis of the Simplest Little Higgs model. Recall that it features two sets of scalars ϕ_1, ϕ_2 which respectively transform as complex triplets under one of the global $H = SU(3)_1 \times SU(3)_2$ symmetries,. The diagonal subgroup $G = SU(3)_V$ is gauged along with the new hypercharge $U(1)_X$. The symmetry is then broken down to the electroweak sector of the Standard Model by the VEV

$$\langle \phi_i \rangle = \begin{pmatrix} 0 \\ 0 \\ f_i \end{pmatrix} \qquad \text{for } i = 1, 2. \tag{5.1.1}$$

The UV completion is required to generate the global symmetry H and the VEVs $\langle \phi_i \rangle$ below Λ_{LH}.

In the vein of $\phi_{1,2}$, let $\Phi_{1,2}$ denote two sets of scalars with six real degrees of freedom. We arrange them into triplet gauge eigenstates under $SU(3)_V$, which is gauged in the usual way. The scalar part of the renormalisable Lagrangian reads

$$\begin{aligned}
\mathcal{L}(\Phi_1, \Phi_2) &= |D_\mu \Phi_1|^2 + |D_\mu \Phi_2|^2 - \Big[V(\Phi_1, \Phi_2) + \widetilde{V}(\Phi_1, \Phi_2) \Big] \\
V(\Phi_1, \Phi_2) &= \lambda_1 |\Phi_1|^4 + \lambda_2 |\Phi_2|^4 + \lambda_p |\Phi_1|^2 |\Phi_2|^2 \\
\widetilde{V}(\Phi_1, \Phi_2) &= \lambda_3 |\Phi_1^\dagger \Phi_2|^2 + \Big[\lambda_4 |\Phi_1|^2 (\Phi_1^\dagger \Phi_2) + \lambda_5 |\Phi_2|^2 (\Phi_2^\dagger \Phi_1) + \lambda_6 (\Phi_1^\dagger \Phi_2)^2 + \text{h.c.} \Big]
\end{aligned} \tag{5.1.2}$$

with the gauge covariant derivative $D_\mu \Phi_i = (\partial_\mu + i g W_\mu^a T^a) \Phi_i$. The potential has been divided into two parts: $V(\Phi_1, \Phi_2)$ is H-symmetric and does not contain any couplings which introduce quadratic divergencies (cf. Sec. 4.2), while $\widetilde{V}(\Phi_1, \Phi_2)$ only obeys the G symmetry and thus features all of the terms we need to eliminate in order to obtain a small Higgs mass. Note that mass terms have been forbidden by classical scale invariance, while mixing terms with odd powers of Φ_1 and Φ_2 are disallowed by a $\mathbb{Z}_2$ symmetry[3].

The premise of dynamical symmetry generation requires all of the couplings in $\widetilde{V}(\Phi_1, \Phi_2)$ to be zero at and below Λ_{LH}. We calculate the β-functions of the potential in order to check whether this

[3] This symmetry remains to be explained by a more complete UV theory.

can be fulfilled. The running of λ_5 for example is given by[4]

$$(4\pi)^2\beta_{\lambda_5} = (112\lambda_2 + 96\lambda_6 + 96\lambda_3 + 24\lambda_p - 16g^2)\lambda_5 + (32\lambda_p + 16\lambda_3 + 16\lambda_6)\lambda_4. \tag{5.1.3}$$

Several problems of our approach become evident: firstly, the β-function contains terms that do not depend on λ_5. This means that setting $\beta_{\lambda_5} = 0$ at some scale does not guarantee that λ_5 will remain constant, as the contributions in the second bracket drive it away from any such fixed point. This can be remedied by setting *all* β-functions to zero at some scale, i.e. assuming an exact fixed point. But this cannot be reconciled with the couplings in the Standard Model, which clearly change with the renormalisation scale. Alternatively, one may assume that the contributions of the couplings balance each other out such that all couplings in $\widetilde{V}(\Phi_1, \Phi_2)$ are zero below Λ_{LH}, but this is exactly the fine-tuning problem; additionally, one also needs to make sure that the running of the gauge couplings (and the Yukawa couplings, which we have not taken into account) leads to the correct Standard Model values.

We conclude that dynamical generation of a global symmetry is not a viable method to UV complete the Simplest Little Higgs model without fine-tuning the couplings. This statement can be generalised: a UV completion to any Little Higgs model would have to balance many contributing couplings due to the enlarged particle content. Without a new mechanism to stabilise the couplings, this approach will always reintroduce fine-tuning.

5.2 Dynamical breaking of a gauged symmetry

In our first attempt, we have noted the difficulty of generating the global symmetry H radiatively. We conclude that there must be some mechanism that introduces H naturally. One solution is to fully gauge H (not only its proper subgroup G) and spontaneously break H down to G. We now attempt an implementation of this procedure.

As a basis for the discussion, we take the Littlest Higgs model. The UV completion needs to provide the global symmetry $H = SU(5)$ below the Little Higgs scale Λ_{LH}. Starting at the Planck scale Λ_{Pl}, we thus gauge H fully. The usual scalar Σ of the Littlest Higgs is now charged under the $SU(5)$. In order to break $SU(5) \to [SU(2) \times U(1)]^2$, we introduce another set of scalars Δ, which we assume to be in some matrix representation. We assign a VEV of the form

$$\langle \Delta \rangle = \Delta_0 \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}. \tag{5.2.1}$$

Ultimately, the UV completion will need to provide a mechanism so that this VEV emerges at an intermediate scale Λ_{int}, which lies between Λ_{Pl} and the Little Higgs scale Λ_{LH}. From the transformation behaviour under the gauged $SU(5)$, we know that $[T^a, \Delta] = 0$ must hold for any unbroken symmetry, where T^a denote the generators of $SU(5)$. Indeed, plugging in the generators (4.4.8) of the Littlest Higgs shows that $\langle \Delta \rangle$ leaves the subgroup $[SU(2) \times U(1)]^2$ invariant[5].

In the UV theory, the scalar potential of Σ and Δ reads

$$V(\Sigma, \Delta) = \lambda_\Sigma^1 \operatorname{tr}(\Sigma^\dagger\Sigma)^2 + \lambda_\Sigma^2 (\operatorname{tr}\Sigma^\dagger\Sigma)^2 + \lambda_\Delta^1 \operatorname{tr}(\Delta^\dagger\Delta)^2 + \lambda_\Delta^2 (\operatorname{tr}\Delta^\dagger\Delta)^2 + \lambda_p \operatorname{tr}(\Sigma^\dagger\Sigma)\operatorname{tr}(\Delta^\dagger\Delta) \tag{5.2.2}$$

[4] The β-function was calculated with PyR@te 1.2.7 [Pyr].

[5] The remaining symmetry is in fact larger than $[SU(2) \times U(1)]^2$; for example, one can identify two additional $SU(2)$ symmetries with their generators in the top right and bottom left corner of a 5×5 matrix. For simplicity, we omit the additional gauge bosons in this discussion. A realistic implementation would need to feature a different symmetry breaking pattern.

where mass terms again are forbidden by classical scale invariance. The number of terms has also been reduced by restricting the representation of Δ: any term in the Lagrangian must be a total gauge singlet, i.e. one can only write down terms in which products of fields yield a one-dimensional representation (see Sec. A.1). Turning this argument around, we set Δ and Σ to transform in representations such that no gauge singlets can be formed from their product.

Assume for now that there exists some mechanism from which $\langle\Delta\rangle$ emerges. Then, the potential contains the terms

$$\lambda_\Sigma^1 \operatorname{tr}(\Sigma^\dagger\Sigma)^2 + \lambda_\Sigma^2 (\operatorname{tr}\Sigma^\dagger\Sigma)^2 + \lambda_p \operatorname{tr}(\Sigma^\dagger\Sigma)\operatorname{tr}\langle\Delta\rangle^2 \tag{5.2.3}$$

which resembles the usual Standard Model scalar potential, only in a more complicated matrix form (see Sec. 3). Plugging $\langle\Delta\rangle$ into the portal term λ_p creates mass terms for Σ, which in turn induce a minimum $\langle\Sigma\rangle$ for the Little Higgs scalars. As in the Standard Model, the mass of Σ naturally lies in the same order as $\langle\Delta\rangle$.

The promising results of this symmetry breaking cascade are however outweighed by the requirement of the global symmetry at Λ_{LH}. The potential can be divided into three parts based on their dependence on either Σ or Δ. Before introducing the VEV $\langle\Delta\rangle$, the whole potential features the $SU(5)$ symmetry; after symmetry breaking, the parts obey different symmetries:

$$V(\Sigma,\Delta) = \underbrace{V(\Sigma)}_{SU(5)} + \underbrace{V(\Delta) + V_p(\Sigma,\Delta)}_{[SU(2)\times U(1)]^2}. \tag{5.2.4}$$

It is evident that the required global symmetry under which Σ transforms is broken by $\langle\Delta\rangle$ in $V_p(\Sigma,\Delta)$. Thus, the conditions for collective symmetry breaking are not fulfilled and the Higgs field is not protected from quadratic divergencies. One might argue that $\lambda_p = 0$ can be set, but this reiterates the problem of fine-tuning the β-functions as acknowledged in the previous attempt.

The breaking of the global symmetry by portal terms is a feature common to all UV completions of this type, independent of the respective Little Higgs model. We come to the conclusion that simply breaking $H \to G$ by an additional set of scalars does not recover the Little Higgs model, and thus this approach cannot act as a UV completing theory.

5.3 Symmetry breaking via a fermion condensate

In the last section, we have seen that the Little Higgs scalars Σ do not obey the global symmetry H at Λ_{LH} due to the portal term in the potential (5.2.4). We note that the terms containing only Σ are not affected by the symmetry breaking and conclude that the global symmetry can be restored if we provide a reason for the absence of terms coupling Σ and Δ in the potential.

On the basis of the previous attempt, we try another realisation of a UV completion: assume that the symmetry breaking is driven by condensation of a new heavy fermion F into a scalar bound state S, which takes the role of Δ in this model. The composite scalar obtains a VEV $\langle S\rangle$, which breaks the fully gauged H symmetry. Condensation is expected to happen at some intermediate scale Λ_{int} with $\Lambda_{\text{Pl}} \gg \Lambda_{\text{int}} > \Lambda_{\text{LH}}$ in order to ensure the global symmetry at Λ_{LH}. Similar to QCD, one expects the VEV $\langle S\rangle$ and scale Λ_{int} to be of the same order; for simplicity, we equate them in this analysis.

In the UV, Yukawa terms between the Little Higgs scalar Σ and F can be forbidden by gauge symmetry. After symmetry breaking, the scalar potential at tree-level is thus expected to be strictly separated into terms containing either S or Σ. The condensation scale will be determined by the *most attractive channel* analysis [Ma1]–[Ma5]. We will also need to implement a mechanism to explain the Little Higgs VEV $\langle\Sigma\rangle$, which we choose to generate via radiative effects. The full breaking pattern of the UV theory is sketched in Fig. 5.1.

As before we construct an exemplary UV completion on the basis of the Littlest Higgs model. Let F be a vector-like fermion pair in some representation of the gauged $SU(5)$. The Lagrangian

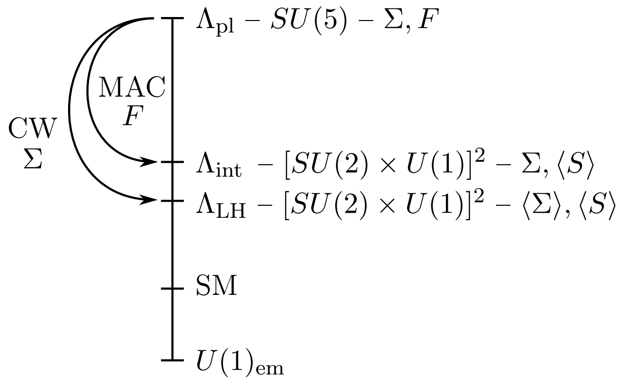


Figure 5.1: Schematic visualisation of the symmetry breaking pattern in the UV completion and the mechanisms that drives it. At the UV scale $\Lambda_{\rm Pl}$, the Little Higgs scalar Σ and the new heavy fermion F transform under the fully gauged $SU(5)$. Via most attractive channel (MAC) analysis, the condensation scale $\Lambda_{\rm int}$ is set, at which F forms the scalar bound state S and the local symmetry is broken down to $[SU(2)\times U(1)]^2$. Radiative effects also induce the VEV $\langle\Sigma\rangle$ at $\Lambda_{\rm LH}$, which is described by the Coleman-Weinberg mechanism (CW). Below this scale, the Littlest Higgs model is recovered.

of Σ and F in the UV complete theory reads

$$\begin{aligned}\mathcal{L} &= \frac{1}{8}\,\mathrm{tr}\,|D_\mu\Sigma|^2 - V(\Sigma) + i\,\overline{F}\not{D}F \\ V(\Sigma) &= \lambda^1_\Sigma\,\mathrm{tr}(\Sigma^\dagger\Sigma)^2 + \lambda^2_\Sigma\,(\mathrm{tr}\,\Sigma^\dagger\Sigma)^2\end{aligned} \tag{5.3.1}$$

where mass terms for Σ and F are absent due to the assumption of classical scale invariance. The gauge covariant derivative of the fermion term is $\not{D} = \gamma^\mu(\partial_\mu + igW^a_\mu T^a)$. We choose F to be in a representation such that Yukawa terms of the form $\overline{F}\Sigma F$ cannot be formed as gauge singlets. In the Littlest Higgs model, Σ is the representation 15_s of $SU(5)$, i.e. a symmetric 5×5 matrix. With the help of Young tableaux we find that the antisymmetric tensor 10_a is also a representation of $SU(5)$ (see App. A.1). As an example, we define F to be in either one of the two representations. The Yukawa term then decomposes into the product representation[6]

$$\begin{aligned}\overline{F}\,\Sigma\,F: \qquad & \overline{10}_a \otimes 15_s \otimes 10_a \\ &= (24 \oplus 126) \otimes 10_a \\ &= 10 \oplus 2\,(15) \oplus 40 \oplus 160 \oplus 2\,(175) \oplus 210 \oplus 700 \\ & \overline{15}_s \otimes 15_s \otimes 15_s \\ &= (1 \oplus 24 \oplus 200) \otimes 15_s \\ &= 10 \oplus 3\,(15) \oplus 2\,(160) \oplus 2\,(175) \oplus 560 \oplus 875 \oplus 1215.\end{aligned} \tag{5.3.2}$$

We note that there is no singlet representation, i.e. it is not possible to contract all $SU(5)$ indices into a gauge singlet. This is a general feature of this type of UV completion: by a good choice of the representation of F we can always eliminate direct couplings of new vector-like fermions to the Little Higgs scalar Σ.

The UV completion requires that F forms a bound state which then condenses at some scale $\Lambda_{\rm int}$. This is triggered by the running of the gauge coupling. The phenomenon is identical to the formation of hadrons in QCD: if the coupling increases when lowering the scale, low energy bound states can form below some scale. This goes along with the breakdown of perturbation theory, as loop effects are not longer suppressed by a small gauge coupling.

[6] Calculation of the product representations has been performed with LieART 1.1.5 [Lie].

By the most attractive channel analysis, we identify the scale below which perturbation theory fails with the condensation scale Λ_{int}. The interaction of the heavy fermion F with the gauge bosons is given by

$$i\,\overline{F}\not{D}F \supset -g\overline{F}\not{W}F = -g\gamma^{\mu}\overline{F}W^{a}_{\mu}T^{a}_{F}F. \tag{5.3.3}$$

The subscript to the generators T^a_F has been added to emphasize that their form depends on the representation of F. We read off that the coupling of $\overline{F}$ to F is proportional to $g\,T^a_F$. Assuming that the fermions bind in the state $\overline{F}F$, the tree-level interaction is

$$\overline{F}F \propto \sum_a g^2\,(T^a_F)^2 = 4\pi\,\alpha_5\,C_2(F)\,\mathbb{1} \tag{5.3.4}$$

with $\alpha_5 := g^2/(4\pi)$ and the quadratic Casimir operator $C_2(F)$. Calculations with the one-loop effective potential [Ma4] show that perturbation theory breaks down if $C_2(F)\,\alpha_5 \gtrsim 1$ (for more references, see [16] and [17] in [Ma1]).

This requirement allows for an important feature: by choosing the representation of F such that $C_2(F)$ becomes large, we can realise a bound state *without the need for a large coupling*. This is particularly important as the Little Higgs gauge group descends from the gauged $SU(5)$, and thus large values of g would not lead to the correct Standard Model couplings strengths.

With the use of (A.1.7), we calculate the value of α_5 at which binding occurs for both the 10_a and 15_s representations of F:

$$\begin{aligned} C_2(10_a) &= \frac{3/2\cdot(5^2-1)}{10} = \frac{36}{10} \qquad \Rightarrow \qquad \alpha_5 \approx \frac{10}{36} = 0.278 \\ C_2(15_s) &= \frac{7/2\cdot(5^2-1)}{15} = \frac{84}{15} \qquad \Rightarrow \qquad \alpha_5 \approx \frac{15}{84} = 0.179 \end{aligned} \tag{5.3.5}$$

We note that the couplings strengths are within the limit of perturbativity.

Following [Hbs], we now sketch an approach to obtain an effective theory of the fermion bound state. Below Λ_{int}, the relevant degree of freedom is the bound state $\overline{F}F =: S$. The coupling of the fermion to itself becomes large in this low energy limit. We contract the four-fermion diagrams by integrating out the gauge boson (and ghost) fields. The resulting point-interaction is the basis for a Hubbard-Stratonovich transform [Hs1, Hs2], which introduces the scalar state S. The process is depicetd in Fig 5.2.

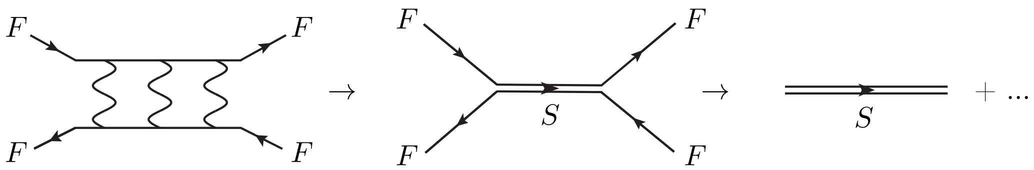
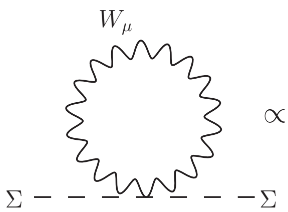


Figure 5.2: Diagrammatic visualisation of bosonisation. At low energies, the self-interaction of the fundamental fermion F becomes strong. The Hubbard-Stratonovich transform is then applied, which introduces the scalar bound state S. After F becomes heavy it is integrated out, which leaves the propagator and coupling terms for S.

In this UV completion, the VEV of the fundamental scalar $\langle\Sigma\rangle$ can be induced by radiative effects. In this case, the analysis of the Coleman-Weinberg potential (2.3.1) is simplified in the Gildener-Weinberg formalism [Gw]: assume that the tree-level scalar potential of the UV complete theory due to RG running develops some flat direction, i.e. a degenerate line of vacua. Quantum corrections then distort this flat direction, and by the Gildener-Weinberg method we determine whether and at which scale μ_{GW} a VEV arises. Without an explicit mass scale, the gauge couplings run over large energy scales before inducing a VEV due to the form of their β-functions (see Sec. 3.3). This means that there can be a large gap between $\Lambda_{\rm Pl}$ and the scale $\mu_{\rm GW}$ at which $\langle\Sigma\rangle$ emerges. This is a general feature; we do not perform an explicit calculation, as the results are highly model-dependent and do not affect the general symmetry breaking pattern of the UV completing theory.

We thus assume a set of values of the couplings at $\Lambda_{\rm Pl}$ for which the Gildener-Weinberg formalism implies a VEV $\langle\Sigma\rangle$ at some scale $\mu_{\rm GW}$, the Gildener-Weinberg scale. How are $\mu_{\rm GW}$, the fermion condensation scale $\Lambda_{\rm int}$ and the Little Higgs scale $\Lambda_{\rm LH}$ related? The UV completion is required to establish the global symmetry H and the VEV $\langle\Sigma\rangle$ at $\Lambda_{\rm LH}$. The Little Higgs scale is thus set at the energy at which both conditions are fulfilled. If the Gildener-Weinberg scale is set above the condensation scale, the VEV breaks the gauged $SU(5)$ down to the Standard Model without the intermediate stage of an additional global symmetry; this means that collective symmetry breaking does not occur[7]. We can thus identify the Gildener-Weinberg and Little Higgs scale, $\mu_{\rm GW} = \Lambda_{\rm LH}$.

The opposite case is $\mu_{\rm GW} < \Lambda_{\rm int}$. Consider the implication for the scalar between the two scales: the fermion F couples to the gauge bosons via (5.3.3). Once the bound state has formed and obtained a VEV, the $SU(5)$ symmetry is broken down to $[SU(2)\times U(1)]^2$, a process in which the gauge bosons of the broken symmetry obtain a mass. But the same gauge bosons couple to Σ via diagrams such as



$$\propto \frac{g^2}{16\pi^2}(T^a)^2\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-M_W^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{M_W^2}\right)$$
$$\propto \frac{g^2}{16\pi^2}(T^a)^2 M_W^2.$$

This carries the symmetry breaking to the scalar part of the Lagrangian, and the 16 radial modes of Σ obtain a mass proportional to the gauge boson masses. Without fine-tuning, we expect these to be of order $\langle S\rangle$. Thus, diagrams of the above type induce a one-loop suppressed mass of some or all components of Σ to arise below $\Lambda_{\rm int}$. This influences the radiative generation of $\langle\Sigma\rangle$, which relies on the absence of any mass scale coupled to the scalars. The resulting mechanism for the generation of $\langle S\rangle$ either consists solely of the fermion condensation (if $\mu_{\rm GW}$ is well below $\Lambda_{\rm int}$) or a combination of effects of the Coleman-Weinberg mechanism and the VEV $\langle S\rangle$. Either implementation drives both scales close to each other, $\Lambda_{\rm LH} \approx \Lambda_{\rm int}$.

We conclude that this set-up fulfils the requirements of a UV completion to Little Higgs models: at the scale $\Lambda_{\rm LH}$, the global symmetry has been generated by the fermion condensate and the VEV $\langle\Sigma\rangle$ emerges by radiative effects.

A realistic model may be constructed from this, but one needs to consider several pitfalls: as an example, the formation of the fermion condensate is a non-perturbative effect. One needs to carefully investigate which interactions arise from the new formulation. This holds as well for the radiative generation of $\langle\Sigma\rangle$, where one needs to reconcile the Gildener-Weinberg formalism with the effect of having massive gauge bosons below $\Lambda_{\rm int}$. Also, the representations we chose imply that binding occurs for $\alpha_5 \approx 0.2$. But the coupling must be matched with those of the low energy theory. For example, the weak gauge coupling of the Standard Model is $\alpha_{\rm ew} = 0.034$ at the mass of the Z boson [Pdg]. Another issue is the effect of the gauge boson masses on the Little Higgs scalar: even though the mass terms are suppressed by a loop factor, they will influence the masses of the scalars in the low-energy theory, i.e. the Higgs boson. Careful examination of which degrees of freedom of Σ couple to the massive gauge bosons needs to be performed. A problem we have not discussed so far is the cancellation of gauge anomalies, which for the Littlest Higgs is treated in [Luv].

[7] This kind of symmetry breaking can however be implemented in a model with a composite Little Higgs, i.e. S posing as the Little Higgs scalar. See [Co1].

6 Conclusions & Outlook

In this thesis, collective symmetry breaking and its implementation in Little Higgs models has been presented as a means to treat the hierarchy problem of the embedded Standard Model and reproduce the vacuum expectation value of the Higgs boson. The Little Higgs theory was cast as an UV extension to the Standard Model. Furthermore, we required the absence of any mass term in the symmetric phase of the Little Higgs theory[1]. We have seen that this successfully eliminates diagrams which contribute with a quadratic divergence to the mass of the Higgs boson. Two implementations have been investigated in particular: the Simplest Little Higgs and the Littlest Higgs model. The phenomenology of both models has shown to differ greatly: the Littlest Higgs complies with current data from electroweak precision tests, while the Simplest Little Higgs is excluded if no modifications are made.

Based on this analysis, an attempt has been made to predict the success or failure of any given Little Higgs model to arrange for a viable electroweak minimum. While speculative, the investigation of differences between the two given models can serve as a guideline for the construction of new Little Higgs-like models. Future work should be dedicated to an analytic and group theory based study of a generic Little Higgs model, which may reveal additional conditions for successful model building.

Fundamental particle physics strives to describe nature as a whole, i.e. up to arbitrary energy scales; but Little Higgs theories are effective field theories and come with a cut-off in the TeV-range. We have studied different ways of embedding Little Higgs models into a theory with extended validity, e.g. up to the Planck scale. Three approaches have been proposed and checked for their feasibility. While radiative generation of the global symmetry and breaking of a larger, fully gauged symmetry with a scalar can in principle be arranged such that a Little Higgs theory emerges in the TeV-range, a realistic model without drastic fine-tuning or violation of electroweak constraints appears impossible. Replacing the scalar in the second approach with a fermion bound state shows more promise, as direct couplings to the Little Higgs scalar can be forbidden. However, there remain several challenges to a real world implementation; most notably, the bosonisation of the new, heavy fermion introduces couplings to the other fields in the theory which we have ignored for now. Furthermore, some mechanism needs to be introduced to explain the emergence of the vacuum expectation value of the condensate.

The idea of collective symmetry breaking and its implementation in Little Higgs models remain of interest as LHC run 2 and other experiments are ongoing. Current searches focus on the Littlest Higgs with T-parity [Rd1, Rd2], where the lightest T-odd particle is stable and thus constitutes a good dark matter candidate. At the same time, recent works have used collective symmetry breaking for applications in other types of models or even different fields. An example found in cosmology features the inflaton field as a pseudo-Nambu-Goldstone boson in the vein of the Little Higgs [Re1]. A model which is more closely related is presented in [Re2], where the corrections to the Higgs mass via top loops are compensated by new gauge bosons.

[1] One can argue more precisely that this approach takes up the question, *'how can the mass scale in the Standard Model be generated?'*, as a small degree of fine-tuning is not a priori disallowed.

In conclusion, we have seen that collective symmetry breaking enables the stabilisation of the electroweak scale against radiative corrections with a cut-off up to $\mathcal{O}(10 \text{ TeV})$. If a Little Higgs model is indeed realised in nature, currently ongoing experiments like the LHC may observe the new heavy particles it introduces. Finding a UV extension valid up to the Planck scale remains a difficult challenge, but the results found in the literature and in this thesis motivate further investigation.

A Appendix

A.1 Group theory in particle physics

Modern fundamental physics is based on the principle of symmetries. Every relativistic theory is formulated to be invariant under Lorentz transformations, where the particles e.g. of spin 0, $\frac{1}{2}$ and 1 form different representations of the Lorentz group. Furthermore, particle charges are defined as specific representations of internal symmetries. In this chapter, we will go trough the vocabulary of group theory applications in particle physics as presented in [Qsm].

A *group* is a set $G = \{g_i\}$ together with a rule $g_i \times g_j = g_k$. It is required to be associative, and that there exists an identity as well as an inverse element to each g_i. We can construct operators from the elements by letting them act on a vector space V. The concrete form of the g_i depends on V; a specific embedding into operators is called the *representation*. We call the dimension of V the *dimension* of the representation. Finite-dimensional representations are expressed in the form of matrices.

As an example, consider the group $G = \mathbb{Z}_2 = \{e, x\}$ where e abstractly denotes the identity and x a non-identity element. If we set G to act on $V = \mathbb{R}$, we would effectively use the operators $\{1, -1\}$. Note however that in particle physics, we often say that an element in V is *in a representation* of G, but technically the matrix operator is the representation (a usual expression is e.g. *'the Higgs field is a doublet under $SU(2)_L$'*).

For the construction of the Standard Model and both Little Higgs models presented here, the internal symmetries are all chosen to be *Lie groups*. A Lie group has an infinite number of elements, and all group elements U which are continuously connected to the identity can be expressed as

$$U = \exp\{i\, \alpha^a T^a\}\ \mathbb{1}, \tag{A.1.1}$$

where α^a are parameters and T^a are the *generators* of the Lie group. In practice, the generators are obtained by expanding U around $\mathbb{1}$. They form a basis of the *Lie algebra*, which is defined through the commutator

$$[T^a, T^b] = i f^{abc} T^c, \tag{A.1.2}$$

with the structure constant f^{abc}. A Lie algebra is useful to study the elements of the corresponding Lie group: for example, a Lie group is *abelian* if all its elements commute, i.e. $f^{abc} = 0$, else it is *non-abelian*.

The groups of internal symmetries are usually chosen to be either special unitary or orthogonal groups, which are defined as

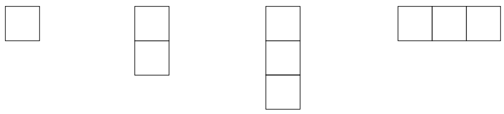
$$\begin{aligned} SU(N) &= \{U \in GL(N, \mathbb{C}) : U^\dagger U = \mathbb{1},\ \det U = 1\} \\ SO(N) &= \{O \in GL(N, \mathbb{R}) : O^T O = \mathbb{1},\ \det O = 1\}. \end{aligned} \tag{A.1.3}$$

While in general one can construct infinitely many representations of these groups, there are several special types. The *fundamental* representation is the non-trivial representation of lowest dimension; for $SU(N)$, it is the set of $N \times N$ hermitian matrices with determinant 1. A field ϕ in the fundamental representation is mapped under infinitesimal group transformations to

$$\phi_i \mapsto \phi_i + i\, \alpha^a (T^a_{\text{fund}})_{ij} \phi_j. \tag{A.1.4}$$

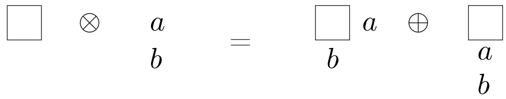
If the gauge bosons are added to a model via minimal coupling, they are in the *adjoint* representation [Qsm]. The vector space of this representation is spanned by the generators themselves. Its dimension is thus given by the number of generators, which is $N^2 - 1$ for $SU(N)$ and $\frac{N(N-1)}{2}$ for $SO(N)$. This is why the Standard Model has eight gluons and three weak bosons.

Other representations can be constructed using Young tableaux. For $SU(N)$, the procedure is as follows [Pdg]: a Young diagram is a set of boxes arranged in left-justified rows, where each row must not be longer than the one above it. Each diagram corresponds to a *Dynkin label* $(n_1, n_2, n_3, ..., n_{N-1})$, where the n_i count the number of boxes in the i-the row minus the boxes in the $(i+1)$-th row. Some example diagrams and labels for $SU(3)$ are:



$(1,0)$ $(0,1)$ $(0,0)$ $(3,0)$

We can form the direct product of two representations by *'multiplying'* the diagrams: replace the boxes of one representation with letters $a, b, ...$, where the first row is filled with a's, the second with b's, etc. Add the a's to the first diagram such that it is still a Young diagram and there is at most one a per column; the unlettered diagram always forms the upper left part of the enlarged diagram. Repeat for the b's, and further letters. Read the letters in the diagram from right to left, top to bottom, and discard any diagram where more b's occur than a's, etc., *at any point.* For example, multiplying the first two diagrams in the above example gives:



$(1,0)$ $(0,1)$ $(1,1)$ $(0,0)$

The Dynkin label is translated to the dimension of the representation via the formula

$$\dim(R) = \prod_{i=1}^{N-1} \frac{n_i + 1}{1} \prod_{j=1}^{N-2} \frac{n_j + n_{j+1} + 2}{2} \prod_{k=1}^{N-3} \frac{n_k + n_{k+1} + n_{k+2} + 3}{3} \cdot ... \tag{A.1.5}$$

This translates $(1,1)$ to an octet and $(0,0)$ to a singlet of $SU(3)$. As a general formula, we can check our result by calculating the dimension of the original product and comparing it to the dimension of the sum of representations. Here $(1,0)$ and $(0,1)$ both have dimension 3, which correctly gives $3 \cdot 3 = 8 + 1$. In general, a column of N boxes is a singlet of $SU(N)$, while a single box symbolizes the fundamental representation and a column of $N-1$ boxes is the antifundamental representation.

On the basis of the generators T_R^a of a representation R, we define the quadratic Casimir operator,

$$\sum_a (T_R^a)^2 =: C_2(R)\, \mathbb{1}. \tag{A.1.6}$$

In Sec. 5.3, we make use of an alternative way to calculate the quadratic Casimir,

$$C_2(R) = \frac{T(R)\, \dim[SU(N)]}{\dim[R]} \tag{A.1.7}$$

where $T(R)$ is the index of the representation, defined as $T(R)\, \delta^{ab} := \mathrm{tr}\, (T^a T^b)$. For further reading, see also [Sla].

A.2 One-loop effective potential and Coleman-Weinberg mechanism

Adding scalar degrees of freedom to a theory generally introduces a tree-level potential to the Lagrangian. It can be modified and new terms may be added due to quantum effects. A practical tool to analyse all contributions of tree and one-loop diagrams is the one-loop effective potential due to Jackiw [Fep].

We now derive the one-loop effective potential in a simple model based on [Eff]. As we will be interested in the dependence of the Higgs mass on some UV-scale, we regularise with the momentum cut-off Λ.

A.2.1 Generating functional

The dynamics of a quantum field theory are defined by its action S. For a theory with one scalar field φ, it reads

$$S[\varphi] = \int d^4x\, \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)]. \tag{A.2.1}$$

Using an arbitrary source $J(x)$, we define the generating functional

$$Z[J] := \int \mathcal{D}\varphi \exp\left\{ i\, S + i \int d^4x\, J(x)\, \varphi(x) \right\}. \tag{A.2.2}$$

We define the one particle irreducible action Γ_{eff} by the Legendre transform

$$\begin{aligned} \Gamma_{\text{eff}}[\phi] &:= W[J] - \int d^4x \phi(x) J(x) \\ &\text{with} \quad Z[J] =: \exp\{i\, W[J]\} \quad \text{and} \quad \phi := \frac{\partial W[J]}{\partial J(x)}. \end{aligned} \tag{A.2.3}$$

As its name suggests, Γ_{eff} generates 1-particle irreducible correlation functions (without external propagators), i.e. Feynman diagrams where all particles take part in the scattering and which cannot be divided into two separate diagrams by cutting one line. We express it in terms of these Green's functions Γ_n with n external points [Ps],

$$\Gamma_{\text{eff}}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, \phi(x_1) \ldots \phi(x_n)\, \Gamma_n(x_1, \ldots, x_n). \tag{A.2.4}$$

We go to momentum space by plugging in the Fourier transform

$$\phi(x) =: \int \frac{d^4p}{(2\pi)^4}\, e^{ipx} \phi(p) \tag{A.2.5}$$

and obtain

$$\Gamma_{\text{eff}}[\phi] = \sum_{n=0}^{\infty} \int \prod_{i=1}^{n} \left[\frac{d^4p_i}{(2\pi)^4} \phi(-p_i) \right] (2\pi)^4 \delta^4(p_1 + \ldots + p_n)\, \Gamma_n(p_1, \ldots, p_n). \tag{A.2.6}$$

A.2.2 Background field method and one-loop effective potential

As we are interested in the constant background field which induces spontaneous symmetry breaking, we set $\phi =: \phi_c = \text{const}$. The effective action then reads in position space [Eff]

$$\Gamma_{\text{eff}}[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_c^n \, \Gamma_n(p_i = 0) \int d^4x. \tag{A.2.7}$$

Comparing this to the potential terms in the Lagrangian, we divide by the spacetime volume and obtain the effective potential

$$V_{\text{eff}}[\phi_c] := -\frac{1}{\text{Vol}_{\mathbb{R}^{1,3}}} \Gamma_{\text{eff}}[\phi_c] = -\sum_{n=0}^{\infty} \frac{1}{n!} \phi_c^n \, \Gamma_n(p_i = 0). \tag{A.2.8}$$

We can now calculate the one-loop effective potential in a simple model: consider a massless real scalar ϕ. The renormalisable Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda_0}{4!}\phi^4. \tag{A.2.9}$$

In order to cancel the upcoming divergences due to loops, we define the counter-terms

$$\delta\mathcal{L} = \frac{1}{2}\delta Z (\partial_\mu \phi)^2 - \frac{\delta m^2}{2}\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \tag{A.2.10}$$

where $(1 + \delta Z)$ is the wave-function renormalisation, δm^2 and $\delta\lambda$ are the mass and quartic coupling counter-terms respectively. In order to define the physical normalisation, mass and coupling, we choose to impose the renormalisation conditions

$$\begin{aligned} Z(0) &= 1 \\ m^2 &= -\Gamma_2(p = 0) = \left.\frac{d^2 V_{\text{eff}}}{d\phi_c^2}\right|_{\phi_c = 0} \\ \lambda &= -\Gamma_4(p = 0) = \left.\frac{d^4 V_{\text{eff}}}{d\phi_c^4}\right|_{\phi_c = 0}. \end{aligned} \tag{A.2.11}$$

In order to evaluate (A.2.8) at one loop order, we need to sum all diagrams with one loop and external momenta set to zero. As dictated by the Feynman rules of this theory, the correlation function Γ_{2n} has n internal propagators, n vertices and $2n$ external legs (see Fig. A.1).

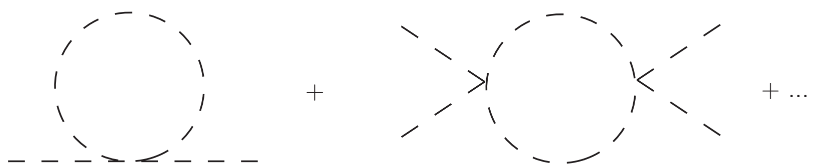


Figure A.1: One particle irreducible diagrams contributing to the effective potential.

Plugging in the scalar propagator

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{A.2.12}$$

and taking into account the symmetry factors of the diagrams, (A.2.8) evaluates to

$$\begin{aligned} V_{\text{eff}}[\phi_c] &= i \sum_{n=1}^{\infty} \int \frac{d^4p}{(2\pi)^4} \frac{1}{2n} \left[\frac{\lambda_0}{2} \frac{\phi_c^2}{p^2 - m^2 + i\epsilon} \right]^n \\ &= -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \log \left[1 - \frac{\lambda_0}{2} \frac{\phi_c^2}{p^2 - m^2 + i\epsilon} \right]. \end{aligned} \tag{A.2.13}$$

We perform a Wick rotation $p^0 \to ip^0$ and use the renormalisation condition of the mass to write

$$V_{\text{eff}}[\phi_c] = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \log \left[p^2 + m^2(\phi_c)\right] \tag{A.2.14}$$

where a field-independent term has been dropped. We cut off the momentum at the scale Λ as $p^2 \leq \Lambda^2$ and perform the integral, which yields [Eff]

$$V_{\text{eff}}[\phi_c] = \frac{1}{32\pi^2} \left[m^2(\phi_c)\,\Lambda^2 + \frac{m^4(\phi_c)}{2} \left(\log \frac{m^2(\phi_c)}{\Lambda^2} - \frac{1}{2}\right)\right]. \tag{A.2.15}$$

The effective potential can be generalised to include gauge boson and fermion contributions. One must pay attention to identify all contributing degrees of freedom and recall that fermionic loops yield an overall minus sign. The *master formula* for the one-loop effective potential reads [Eff]:

$$V_{\text{eff}}[\phi_c] = \frac{1}{32\pi^2} \sum_i (-1)^{2s_i}\, n_i \left[m_i^2(\phi_c)\,\Lambda^2 + \frac{m_i^4(\phi_c)}{2} \left(\log \frac{m_i^2(\phi_c)}{\Lambda^2} - \frac{1}{2}\right)\right] \tag{A.2.16}$$

where i runs over all particles in the theory; m_i is the corresponding particle's mass in terms of the background field value, s_i denotes its spin and n_i are the number of degrees of freedom running in the loops. The $W^\pm$ bosons for example have $n_W = 6$ and $s_W = 1$ while the top quark is a coloured Dirac spinor and thus has $n_t = 12$ and $s_t = 1/2$. Note that the gauge boson terms have been calculated in Landau gauge.

A.2.3 The Coleman-Weinberg mechanism

The one-loop effective potential finds an application in the Coleman-Weinberg model [Rsb]. It describes scalar electrodynamics of one complex scalar degree of freedom ϕ. As we will see, loop effects will induce spontaneous symmetry breaking without the need of a tree level vacuum expectation value; this is named the Coleman-Weinberg mechanism.

The Lagrangian of this model has a gauged $U(1)$ symmetry and reads

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 + |D_\mu\phi|^2 + \mu^2|\phi|^2 - \frac{\lambda}{4!}|\phi|^4 \tag{A.2.17}$$

with the gauge covariant derivative $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$. Note that for $\mu^2 > 0$, the minimum of ϕ is not at zero and thus, the symmetry is broken spontaneously (see also Sec. 4.1.1). The counter-terms for the scalar are

$$\delta\mathcal{L} = \delta Z|\partial_\mu\phi|^2 + \delta\mu^2|\phi|^2 - \frac{\delta\lambda}{4!}|\phi|^4. \tag{A.2.18}$$

We now set $\mu^2 = 0$ and ask whether loop effects can reintroduce symmetry breaking. By the background field method, we assume for ϕ some real, constant background field ϕ_c. The scalar to gauge boson coupling reads

$$e^2\phi_c^2 A_\mu^2 =: \frac{M_A^2(\phi_c)}{2}A_\mu^2 \tag{A.2.19}$$

where we have defined a background field dependent mass for the gauge field A. We now plug this into the one-loop effective potential and check for self-consistency: if indeed a stable minimum $\phi_c =: v/\sqrt{2}$ emerges, the assumption of a background field is justified.

Plugging into (A.2.16), we obtain

$$V_{\text{eff}}[\phi_c] = \frac{3}{32\pi^2}\left[2e^2\phi_c^2\,\Lambda^2 + 2e^4\phi_c^4\left(\log\frac{2e^2\phi_c^2}{\Lambda^2} - \frac{1}{2}\right)\right] + \delta\mu^2\phi_c^2 - \frac{\lambda+\delta\lambda}{4!}\phi_c^4. \tag{A.2.20}$$

Setting the renormalisation conditions

$$\begin{aligned} 0 &= \left.\frac{d^2V_{\text{eff}}}{d\phi_c^2}\right|_{\phi_c=0} \\ \lambda &= \left.\frac{d^4V_{\text{eff}}}{d\phi_c^4}\right|_{\phi_c=M} \end{aligned} \tag{A.2.21}$$

where M is some renormalisation scale, we absorb the terms quadratic in Λ and arrive at the effective one-loop potential [Rsb]

$$V_{\text{eff}}[\phi_c] = \frac{\lambda}{4!}\phi_c^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\phi_c^4\left[\log\frac{\phi_c^2}{\mu^2} - \frac{25}{6}\right]. \tag{A.2.22}$$

We see that for small ϕ_c, the second term carries a negative sign and thus a minimum at $\phi_c \neq 0$ can be induced, see Fig. A.2.

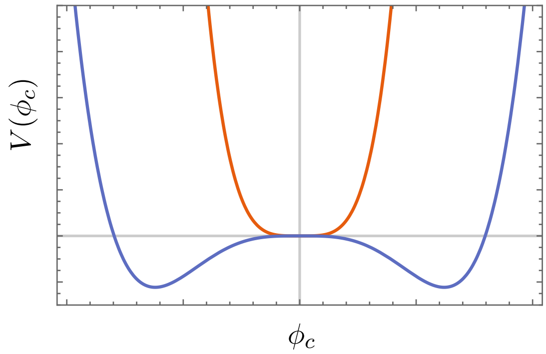


Figure A.2: Tree-level potential (red) and one-loop effective potential (blue) in the Coleman-Weinberg model. Note that the loop effects induce a non-zero minimum.

A.3 Sigma models

An essential ingredient to all Little Higgs theories is the non-linear parametrisation of the scalar(s). It is based on non-linear sigma models[1], which we will review in this section.

A sigma model

Consider the linear sigma model as presented in Sec. 4.1.1: the complex scalar ϕ has the Lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 + m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \tag{A.3.1}$$

which has a global $U(1)$ symmetry. ϕ obtains a vacuum expectation value $f = 2\sqrt{m^2/\lambda}$, which breaks the symmetry. We write

$$\phi = \frac{1}{\sqrt{2}}\left(\rho(x) + f\right) e^{i/f\,\theta(x)}. \tag{A.3.2}$$

However, the symmetry is still realised non-linearly as

$$\theta(x) \mapsto \theta(x) + \alpha \tag{A.3.3}$$

for some $\alpha \in \mathbb{R}$, which forbids the phase of ϕ to obtain a mass. The radial mode ρ however becomes massive with $m_\rho = \sqrt{2}\,m$. As we are interested in the low energy theory, we aim to remove this degree of freedom from the theory[2]. This is achieved by taking the decoupling limit in which m^2, $\lambda \to \infty$ while holding f at a constant value [Qsm]; a more rigorous and general procedure is presented in App. A.4.

After dropping ρ, the Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \theta)^2. \tag{A.3.4}$$

This is an example of a *non-linear sigma model*. Its key ingredient is that all remaining degrees of freedom obey a shift symmetry. Alternatively, one defines it as a theory in which the scalar part of the Lagrangian can be written as [Qcp]

$$\mathcal{L} = G_{ij}[S(x)]\, D_\mu S_i(x) D^\mu S_j(x) \tag{A.3.5}$$

with the constraint $|S(x)|^2 = 1$. Here S denotes the scalars and G_{ij} is some function of S. For example in the previous model we have $S = e^{i/f\,\theta(x)}$, $G = f^2/2$ and replace the gauge covariant derivative with the ordinary one.

[1] Some authors refer to the theory of pions (see below) as *the* non-linear sigma model. We adapt a broader definition here.

[2] Removing a heavy degree of freedom is referred to as *decoupling* or *integrating out* the mode.

A more interesting case of such a model can be found in QCD [Qsm]. In strong interactions at low energies, pions mediate the nuclear force which binds the protons and neutrons in the nucleus. But pions are not fundamental particles; their constituents are the up and down quark.

We can write down a theory where pions emerge as quark bound states. The Lagrangian of up and down quarks interacting with gluons reads

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + i\bar{u}\not{D}u + i\bar{d}\not{D}d - m_u\bar{u}u - m_d\bar{d}d \tag{A.3.6}$$

where the field strength tensor of the eight gluons is $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu$ with f^{abc} the structure constant of $SU(3)_c$. We set the quark masses to zero as they are small compared to the QCD scale Λ_{QCD}, which we will discuss later. Recall that each quark has a left- and right-handed component. We can separate them by replacing $q_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\, q$. Then the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + i\bar{u}_R\not{D}u_R + i\bar{u}_L\not{D}u_L + i\bar{d}_R\not{D}d_R + i\bar{d}_L\not{D}d_L. \tag{A.3.7}$$

We note that now there are two independent global $SU(2)$ symmetries:

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \end{pmatrix} \tag{A.3.8}$$

for $U_L \in SU(2)_L$ and $U_R \in SU(2)_R$. This is the global $SU(2)_L \times SU(2)_R$ symmetry of QCD. It is a chiral symmetry as it acts independently on left- and right-handed components.

In the early universe, this symmetry was broken as the average temperature dropped below $T \approx \Lambda_{\text{QCD}}$ [Qsm]. The quarks formed bound states and hadrons emerged. We implement this into the theory by assigning a vacuum expectation value to the quark bilinears

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle =: V^3. \tag{A.3.9}$$

Note that both bilinears obtain the same background field value; we will see that this is necessary to retrieve isospin symmetry which is approximately conserved in QCD.

As we are interested in the composition of up and down quarks into pions, we may worry that calculating bound states in QCD is for now reserved to lattice calculations. The reason is that the coupling α_S becomes non-perturbative at the scale $\Lambda_{\text{QCD}} = \mathcal{O}(100 \text{ MeV})$ [Qsm].

There is however an alternative, more simple route by the name of *chiral perturbation theory*. It is an effective theory which treats the composite, pseudoscalar pions as excitations of a set of scalars Σ. By assigning the vacuum expectation values according to (A.3.9), we break $SU(2)_L \times SU(2)_R$ down to the diagonal subgroup $SU(2)_V$. The big advantage is that all of this goes through without explicit knowledge of the fundamental interactions.

Let Σ transform as a bidoublet, i.e. $\Sigma \mapsto U_L \Sigma U_R^\dagger$. The effective Lagrangian reads

$$\mathcal{L} = |\partial_\mu \Sigma|^2 + m^2|\Sigma|^2 - \frac{\lambda}{4}|\Sigma|^4 \tag{A.3.10}$$

and Σ has its minimum at[3]

$$\langle \Sigma \rangle = \frac{v}{\sqrt{2}}\,\mathbb{1}_{2\times 2} \qquad \text{with} \qquad v = \frac{2m}{\sqrt{\lambda}}. \tag{A.3.11}$$

Under the global transformations, the vacuum is mapped to $\langle \Sigma \rangle \mapsto U_L \langle \Sigma \rangle U_R^\dagger$, which is only invariant for $U_L = U_R$. We conclude that indeed the symmetry is broken down to $SU(2)_V$.

[3] We expect the minimum V of QCD and v of the effective theory to be approximately equal [Qsm].

Below the scale v, we parametrise the four real degrees of freedom as

$$\Sigma(x) = \frac{v + \sigma(x)}{\sqrt{2}} \exp\left\{2i\frac{\pi(x)}{v}\right\} \qquad \text{with} \quad \pi(x) = \pi^a \frac{\sigma^a}{2} \tag{A.3.12}$$

where σ^a are the three Pauli matrices. We integrate out the massive mode $\sigma(x)$ and use the complexified generators $\sigma^{\pm} = \frac{1}{\sqrt{2}}(\sigma^1 \pm i\sigma^2)$ of $SU(2)$ to write

$$\Sigma(x) = \frac{v}{\sqrt{2}} \exp\left\{\frac{i}{v}\begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^- \\ \sqrt{2}\,\pi^+ & -\pi^0 \end{pmatrix}\right\} \tag{A.3.13}$$

with $\pi^0 = \pi^3$. We read off that there is one real and one complex pion, in line with phenomenology. Their dynamics are dictated by the *chiral Lagrangian*, which contains the leading order term in the derivative expansion

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}\left[(D_\mu\Sigma)(D^\mu\Sigma)^\dagger\right] \tag{A.3.14}$$

where the covariant derivative contains electroweak interactions, but no gluons. The chiral Lagrangian can be used to study pion decay and other phenomena, but we are now only interested in the mass of the three lightest mesons.

As we set the quark masses to zero, the $SU(2)_L \times SU(2)_R$ symmetry was made exact in the fundamental theory and thus the Goldstone bosons – the three pions – remain massless until now. We restore a small mass term by adding [Qsm]

$$\frac{V^3}{\sqrt{2}\,v}\,\mathrm{tr}\left[M\Sigma + M\Sigma^\dagger\right] \qquad \text{with} \quad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d. \end{pmatrix} \tag{A.3.15}$$

The prefactor V^3 has been inserted because the quark masses stem from the fundamental theory, whose vacuum expectation value is V (instead of v). Expanding Σ to quadratic order in the pion fields, we retrieve the mass terms

$$-\frac{V^3}{2v^2}(m_u + m_d)(\pi_0^2 + \pi_1^2 + \pi_2^2). \tag{A.3.16}$$

We find the Gell-Mann-Oakes-Renner relation $m_\pi^2 = \frac{V^3}{v^2}(m_u + m_d)$ [Qsm].

A.4 Integrating out heavy fields

If a theory contains heavy degrees of freedom as well as much lighter ones, the structure of the path integral allows to expand the Lagrangian in terms of effective higher dimensional operators. In this process, the heavy degrees of freedom are eliminated from the theory. The procedure is called *integrating out* the heavy fields, and the effect is summarized in the *decoupling theorems*: for small external momenta, any Feynman diagram with heavy internal degrees of freedom is suppressed by powers of their masses [Irs, Dec]. We present here the procedure for a heavy scalar mode following [Ep1, Ep2]. The idea is then applied to the parametrisation of the scalars found in the Simplest and Littlest Higgs models.

Consider a real scalar φ. The Lagrangian can be written as

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m_\varphi^2}{2}\varphi^2}_{=:\mathcal{L}_0} + \underbrace{J(x)\,\varphi}_{=:\mathcal{L}_{\text{source}}} \tag{A.4.1}$$

wherein all interaction terms have been absorbed into an external source $J(x)$. The equation of motion reads

$$(\partial^2 + m_\varphi^2)\,\varphi(x) = J(x). \tag{A.4.2}$$

The generating functional is given by the path integral

$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi \exp\left\{ i \int d^4x \left[\frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m_\varphi^2}{2}\varphi^2 + J(x)\,\varphi \right] \right\} \\ &\overset{PI}{=} \int \mathcal{D}\varphi \exp\left\{ -i \int d^4x \left[\frac{1}{2}\,\varphi\,(\partial^2 + m_\varphi^2)\,\varphi - J(x)\,\varphi \right] \right\}, \end{aligned} \tag{A.4.3}$$

where the surface term has been dropped by assumption. As the path integral obeys shift invariance, we are free to add a generic term to φ,

$$\varphi(x) \to \varphi(x) + \varphi_b(x). \tag{A.4.4}$$

The generating functional is then

$$\begin{aligned} Z[J] = \int \mathcal{D}\varphi \exp\Big\{ -i \int d^4x \Big\{ &\frac{1}{2}\,\varphi\,(\partial^2 + m_\varphi^2)\,\varphi + \frac{1}{2}\,\varphi_b\,(\partial^2 + m_\varphi^2)\,\varphi \\ &\frac{1}{2}\,\varphi\,(\partial^2 + m_\varphi^2)\,\varphi_b + \frac{1}{2}\,\varphi_b\,(\partial^2 + m_\varphi^2)\,\varphi_b - J(x)\,(\varphi + \varphi_b) \Big\}. \end{aligned} \tag{A.4.5}$$

Performing integration by parts twice and plugging in the Klein-Gordon equation (A.4.2) which holds for both φ and φ_b this reduces to

$$\begin{aligned} Z[J] &= \int \mathcal{D}\varphi \exp\Big\{ -i \int d^4x \Big\{ \frac{1}{2}\,\varphi\,(\partial^2 + m_\varphi^2)\,\varphi + \varphi\,(\partial^2 + m_\varphi^2)\,\varphi_b \\ &\qquad\qquad + \frac{1}{2}\,\varphi_b\,(\partial^2 + m_\varphi^2)\,\varphi_b - J(x)\,(\varphi + \varphi_b) \Big\} \\ &= \int \mathcal{D}\varphi \exp\Big\{ -i \int d^4x \Big\{ \frac{1}{2}\,\varphi\,(\partial^2 + m_\varphi^2)\,\varphi - \frac{1}{2}\,J(x)\,\varphi_b \Big\} \\ &=: Z[0]\,\exp\Big\{ \frac{i}{2} \int d^4x\,J(x)\,\varphi_b(x) \Big\}. \end{aligned} \tag{A.4.6}$$

Here $Z[0]$ has been defined without external sources (i.e. as a free theory) and thus will drop out when calculating correlation functions. Note that the last step requires the source $J(x)$ to be independent of φ. The generic shift φ_b can be chosen as

$$\varphi_b(x) = -\int d^4y\,\Delta(x-y)\,J(y) \tag{A.4.7}$$

and the generating functional reads

$$Z[J] = Z[0]\,\exp\bigg\{-\frac{i}{2}\int d^4x \int d^4y\, J(x)\,\Delta(x-y)J(y)\bigg\}. \tag{A.4.8}$$

This holds for any $\Delta(x-y)$, i.e. we are free to insert the free scalar propagator (divided by a factor of i). In momentum space this is

$$\Delta(k) = \frac{1}{k^2 - m_\varphi^2} \tag{A.4.9}$$

which we expand in terms of low momenta,

$$\Delta(k) \doteq -\frac{1}{m_\varphi^2}\left[1 + \left(\frac{k^2}{m_\varphi^2}\right) + \left(\frac{k^2}{m_\varphi^2}\right)^2 + \mathcal{O}(k^4)\right]. \tag{A.4.10}$$

To first order, $\Delta = -\frac{1}{m_\varphi^2}$. Fourier transforming to position space, we obtain

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\left(-\frac{1}{m_\varphi^2}\right) e^{-ik(x-y)} = \left(-\frac{1}{m_\varphi^2}\right)\delta^4(x-y) \tag{A.4.11}$$

and thus

$$Z[J] = Z[0]\,\exp\bigg\{\frac{i}{2m_\varphi^2}\int d^4x\, J(x)\,J(x)\bigg\}. \tag{A.4.12}$$

Including higher orders in the expansion involves Fourier transforming power of the momentum k,

$$\int \frac{d^4k}{(2\pi)^4}\, k^{2n}\, e^{-ik(x-y)} = (-1)^n \frac{\partial^{2n}}{\partial(x-y)^{2n}}\delta^4(x-y) \tag{A.4.13}$$

with $n \in \mathbb{N}$. Via integration by parts the derivatives acting on the delta distribution can be moved to the sources,

$$\begin{aligned}
&\int d^4x \int d^4y\; J(x)\left[\frac{\partial^{2n}}{\partial(x-y)^{2n}}\delta^4(x-y)\right] J(y) \\
&= \int d^4x \int d^4y\; \left[\frac{\partial^n}{\partial(x-y)^n}J(x)\right]\delta^4(x-y)\left[\frac{\partial^n}{\partial(x-y)^n}J(y)\right] + \text{boundary terms} \\
&= (-1)^n \int d^4x\; \left[\frac{\partial^n}{\partial x^n}J(x)\right]^2 + \text{boundary terms}
\end{aligned} \tag{A.4.14}$$

where the boundary terms are dropped. Plugging the expansion into the generating function yields

$$Z[J] = Z[0]\,\exp\bigg\{\frac{i}{2m_\varphi^2}\int d^4x \sum_{n=0}^{\infty}\left[\frac{\partial^n}{\partial x^n}J(x)\right]^2\left(\frac{1}{m_\varphi^2}\right)^n\bigg\}. \tag{A.4.15}$$

We conclude that we have found a prescription to integrate out heavy scalar fields: first, the free scalar part $\mathcal{L}_0$ of the Lagrangian is dropped. The coupling term linear in φ is then replaced,

$$\mathcal{L}_{\text{source}} \to \frac{1}{2m_\varphi^2}\sum_{n=0}^{\infty}\left[\frac{\partial^n}{\partial x^n}J(x)\right]^2\left(\frac{1}{m_\varphi^2}\right)^n. \tag{A.4.16}$$

This contracts the interaction vertex $J \cdot \phi$ into operators of the type $J \cdot J$ and in the process introduces higher dimensional couplings. The mechanism can be expanded to gauge boson and fermion operators. A well known example is given in Fermi theory, where the heavy W boson is integrated out and replaced by a dimension-six vertex of four fermions [Ep1].

It is important to note that while this replacement is exact on the classical level, its derivation required the Euler-Lagrange equation (A.4.2) to hold and thus disregards any loop effects.

We now apply this replacement rule to the parametrisation of a set of scalars as employed in the Simplest Little Higgs (4.3.4) and the Littlest Higgs (4.4.6). Consider a generic scalar representation $\Omega(x)$. Below $\Lambda = 4\pi f$ we assign the vacuum expectation value f to $\omega(x)$, one of its degrees of freedom. The set of scalars is then denoted by

$$\Omega(x) = \frac{1}{\sqrt{2}}[f + \omega(x)]\, U(x) \tag{A.4.17}$$

where $U(x) = \exp\{{}^{i}\!/_{f}\pi(x)\}$ contains the remaining degrees of freedom in some way. The kinetic term of Ω reads

$$\begin{aligned} \operatorname{tr}|\partial_\mu\Omega(x)|^2 &= \frac{1}{2}(\partial_\mu\omega)^2 + \frac{1}{2}[f + \omega(x)]^2 \operatorname{tr}|\partial_\mu U(x)|^2 \\ &= \frac{1}{2}(\partial_\mu\omega)^2 + \frac{1}{2}[f^2 + 2f\omega(x) + \omega(x)^2] \operatorname{tr}|\partial_\mu U(x)|^2 \end{aligned} \tag{A.4.18}$$

The second line contains terms coupling ω and ω^2 to U. The term quadratic in the radial mode implies diagrams which modify the coefficients of the following expansion [Ep2]; they are neglected in this discussion. We apply the replacement rule (A.4.16) to the linear coupling and obtain

$$\omega(x)\cdot f \operatorname{tr}|\partial_\mu U(x)|^2 \;\to\; \frac{f^2}{2m_\omega^2}\sum_{n=0}^{\infty}\left[\frac{\partial^n}{\partial x^n}\operatorname{tr}|\partial_\mu U(x)|^2\right]^2\left(\frac{1}{m_\omega^2}\right)^n. \tag{A.4.19}$$

Expanding to first order and including the term lead by f^2 in (A.4.18), the non-linear parametrisation of Ω after integrating out the radial mode is

$$\Omega(x) = \frac{f^2}{2}\operatorname{tr}|\partial_\mu U(x)|^2 + \frac{f^2}{2m_\omega^2}\left[\operatorname{tr}|\partial_\mu U(x)|^2\right]^2 + \ldots \tag{A.4.20}$$

We go to Fourier space by replacing ∂_μ with the momentum $-i\,p_\mu$ and note that we have expanded in powers of p^2/m_ω^2. This means that higher order terms are suppressed; if the momentum is sufficiently small, all terms beyond leading order can be neglected.

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 29. September 2016 ....