$C^{1,\beta}$-regularity at the boundary of two dimensional sliding almost minimal sets

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Abstract

In this paper, we will give a $C^{1,\beta}$-regularity result on the boundary for two dimensional sliding almost minimal sets. This effect may lead to the existence of a solution to the Plateau problem with sliding boundary conditions proposed by Guy David in [4].

1 Introduction

In [4], Guy David proposed to consider the Plateau Problem with sliding boundary conditions, since it is very natural to soap films and has some advantages to consider the boundary regularity.

Jean Taylor, in [10], proved a $C^{1,\beta}$-regularity theorem for Almgren almost minimal sets of dimensional two in an open set $U \subset \mathbb{R}^3$, and Guy David, in [2], given a new proof and generalized it to any codimension.

In [6], we proved a Hölder regularity of two dimensional sliding almost minimal set at the boundary. That is, suppose that $\Omega \subset \mathbb{R}^3$ is a closed domain with boundary $\partial\Omega$ a C^1 manifold of dimension 2, $E \subset \Omega$ is a 2 dimensional sliding almost minimal set with sliding boundary $\partial\Omega$, and that $\partial\Omega \subset E$. Then E, at the boundary, is locally biHölder equivalent to a sliding minimal cone in the upper half space Ω_0. In this paper, we will generalized the biHölder equivalence to a $C^{1,\beta}$ equivalence when the gauge function h satisfies that $h(t) \leq Ct^\alpha$. Let us refer to Theorem 1.2 for details. In our case, the list of sliding minimal cones is known: they are $\partial\Omega_0$ and $\partial\Omega_0 \cup Z$, where Z are half planes or cones of type $\mathbb{Y}_+$, which meet $\partial\Omega_0$ perpendicularly.

Let us introduce some notation and definitions before state our main theorem. A gauge function is a nondecreasing function $h : [0, \infty) \to [0, \infty]$ with $\lim_{t\to 0} h(t) = 0$. Let Ω be a closed domain of $\mathbb{R}^3$, L be a closed subset in $\mathbb{R}^3$, $E \subset \Omega$ be a given set. Let $U \subset \mathbb{R}^3$ be an open set. A family of mappings $\{\varphi_t\}_{0\leq t\leq 1}$, from E into Ω, is called a sliding deformation of E in U if following properties hold:

$$\varphi_t(x) = x \text{ for } x \in E \setminus U, \varphi_t(x) \subset U \text{ for } x \in E \cap U, 0 \leq t \leq 1,$$

$$\varphi_t(x) \in L \text{ for } x \in E \cap L,\ 0 \leq t \leq 1,$$

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the mapping

$$[0,1]\times E\to\Omega, (t,x)\mapsto\varphi_t(x)$$

is continuous, and

$$\varphi_1 \text{ is Lipschitz and } \psi_0=\mathrm{id}_E.$$

Definition 1.1. We say that an nonempty set $E\subset\Omega$ is locally sliding almost minimal at $x\in E$ with sliding boundary L and with gauge function h, called (Ω,L,h) locally sliding almost at $x\in E$ for short, if $\mathcal{H}^2\llcorner E$ is locally finite, and for any sliding deformation $\{\varphi_t\}_{0\le t\le 1}$ of E in $B(0,r)$, we have that

$$\mathcal{H}^2(E\cap B(x,r))\le\mathcal{H}^2(\varphi_1(E)\cap B(x,r))+h(r)r^2.$$

We say that E is sliding almost minimal with sliding boundary L and gauge function h, denote by $SAM(\Omega,L,h)$ the collection of all such sets, if E is locally sliding almost minimal at all points $x\in E$.

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^3$ be a closed connected set such that the boundary $\partial\Omega$ is a 2-dimensional manifold of class $C^{1,\alpha}$ for some $\alpha>0$. Let $E\subset\Omega$ be a closed set such that $E\supset\partial\Omega$ and E is a sliding almost minimal set with sliding boundary $\partial\Omega$ and with gauge function h satisfying that*

$$h(t)\le C_h t^{\alpha_1},\ 0<t\le t_0,\ \text{for some } C_h>0, \alpha_1>0 \text{ and } t_0>0.$$

Then for any $x_0\in\partial\Omega$, there is a unique tangent cone of E at x_0; moreover, there exist a radius $r>0$, a sliding minimal cone Z in Ω_0 with sliding boundary $\partial\Omega_0$, and a mapping $\Phi:\Omega_0\cap B(0,1)\to\Omega$ of class $C^{1,\beta}$, which is a diffeomorphism between its domain and image, such that $\Phi(0)=x_0$, $|\Phi(x)-x_0-x|\le 10^{-2}r$ for $x\in B(0,2r)$, and

$$E\cap B(x_0,r)=\Phi(Z)\cap B(x_0,r).$$

Theorem 1.2 and Jean Taylor's theorem imply that any set E as in above theorem is lipschitz neighborhood retract. This effect gives the existence of a solution to the Plateau problem with sliding boundary conditions in a special case, see Theorem 6.1.

2 Lower bound of the decay for the density

In this section, we will consider a simple case that Ω is a half space and L is its boundary; without loss of generality, we assume that Ω is the upper half space, and change the notation to be Ω_0 for convenience, i.e.

$$\Omega_0=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_3\ge 0\}, L_0=\partial\Omega_0.$$

It is well known that for, any 2-rectifiable set E, there exists an approximate tangent plane $\mathrm{Tan}^2(E,y)$ of E at y for $\mathcal{H}^2$-a.e. $y\in E$. We will denote by $\theta(y)\in[0,\pi/2]$ the angle between the segment $[0,y]$ and the plane $\mathrm{Tan}^2(E,y)$, by $\theta_x(y)\in[0,\pi/2]$ the angle between the segment $[x,y]$ and the plane $\mathrm{Tan}^2(E,y)$, for $x\in\mathbb{R}^3$.

In this section, we assume that there is a number $r_h > 0$ such that

$$\int_0^{r_h} \frac{h(2t)}{t} dt < \infty, \tag{2.1}$$

and put

$$h_1(t) = \int_0^t \frac{h(2s)}{s} ds, \text{ for } 0 \le t \le r_h.$$

Lemma 2.1. *Let $E \subset \Omega_0$ be any 2-rectifiable set. Then, by putting $u(r) = \mathcal{H}^2(E \cap B(x,r))$, we have that u is differentiable almost every $r > 0$, and that*

$$\mathcal{H}^1(E \cap \partial B(x,r)) \le u'(r)$$

for such r.

Proof. Considering the function $\psi : \mathbb{R}^3 \to \mathbb{R}$ defined by $\psi(y) = |y-x|$, we have that, for any $y \neq x$ and $v \in \mathbb{R}^3$,

$$D\psi(y)v = \left\langle \frac{y-x}{|y-x|}, v \right\rangle,$$

thus

$$\text{ap}\, J_1(\psi|_E)(y) = \sup\{|D\psi(y)v| : v \in T_x E, |v| = 1\} = \cos\theta_x(y). \tag{2.2}$$

Employing Theorem 3.2.22 in [7], we have that, for any $0 < r < R < \infty$,

$$\int_r^R \mathcal{H}^1(E \cap \partial B(x,t))dt = \int_{E\cap B(x,R)\setminus B(x,r)} \cos_x(y) d\mathcal{H}^2(y) \le u(R) - u(r),$$

we get so that, for almost every $r \in (0,\infty)$,

$$\mathcal{H}^1(E \cap \partial B(x,t)) \le u'(r)$$

□

Lemma 2.2. *Let E be a 2-rectifiable (Ω_0, L_0, h) locally sliding almost minimal at some point $x \in E$. If $x \in E \cap L_0$, then for $\mathcal{H}^1$-a.e. $r \in (0,\infty)$,*

$$\mathcal{H}^2(E \cap B(x,r)) \le \frac{r}{2}\mathcal{H}^1(E \cap \partial B(x,r)) + h(2r)(2r)^2. \tag{2.3}$$

If $x \in E \setminus L_0$, then inequality (2.2) holds for $\mathcal{H}^1$-a.e. $r \in (0, \operatorname{dist}(x, L_0))$.

Proof. If $\mathcal{H}^2(E \cap \partial B(x,r)) > 0$, then $\mathcal{H}^1(E \cap \partial B(x,r)) = \infty$, and nothing need to do. We assume so that $\mathcal{H}^2(E \cap \partial B(x,r)) = 0$.

Let $f : [0,\infty) \to [0,\infty)$ be any Lipschitz function, we let $\phi : \Omega_0 \to \Omega_0$ be defined by

$$\phi(y) = f(|y-x|)\frac{y-x}{|y-x|}.$$

Then, for any $y \neq x$ and any $v \in \mathbb{R}^3$, by putting $\tilde{y} = y - x$, we have that

$$D\phi(y)v = \frac{f(|\tilde{y}|)}{|\tilde{y}|}v + \frac{|\tilde{y}|f'(|\tilde{y}|) - f(|\tilde{y}|)}{|\tilde{y}|^2}\left\langle \frac{\tilde{y}}{|\tilde{y}|}, v \right\rangle \tilde{y}$$

If the tangent plane $\mathrm{Tan}^2(E, y)$ of E at y exists, we take $v_1, v_2 \in \mathrm{Tan}^2(E, y)$ such that $|v_1| = |v_2| = 1$, v_1 is perpendicular to $y = x$, and that v_2 is perpendicular to v_1, let v_3 be a vector in $\mathbb{R}^3$ which is perpendicular to $\mathrm{Tan}^2(E, y)$ and $|v_3| = 1$, then

$$\tilde{y} = \langle \tilde{y}, v_2 \rangle v_2 + \langle \tilde{y}, v_3 \rangle v_3 = |\tilde{y}| \cos\theta_x(y) v_2 + |\tilde{y}| \sin\theta_x(y) v_3,$$

and

$$D\phi(y)v_1 \wedge D\phi(y)v_2 = \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2} v_1 \wedge v_2 + \frac{|\tilde{y}|f'(|\tilde{y}|)f(|\tilde{y}|) - f(|\tilde{y}|)^2}{|\tilde{y}|^3} \cos\theta_x(y) v_1 \wedge \tilde{y},$$

thus

$$\begin{aligned} \mathrm{ap}\, J_2(\phi|_E)(y) &= \|D\phi(y)v_1 \wedge D\phi(y)v_2\| \\ &= \frac{f(|\tilde{y}|)}{|\tilde{y}|}\left(f'(|\tilde{y}|)^2 \cos^2\theta_x(y) + \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2}\sin^2\theta_x(y)\right)^{1/2}. \end{aligned}$$

We consider the function $\psi : \mathbb{R}^3 \to \mathbb{R}$ defined by $\psi(y) = |y - x|$. Then, by (2), we have that

$$\mathrm{ap}\, J_1(\psi|_E)(y) = \cos\theta_x(y).$$

For any $\xi \in (0, r/2)$, we consider the function f defined by

$$f(t) = \begin{cases} 0, & 0 \leq t \leq r - \xi \\ \frac{r}{\xi}(t - r + \xi), & r - \xi < t \leq r \\ t, & t > r. \end{cases}$$

Then we have that

$$\mathrm{ap}\, J_2(\phi|_E)(y) \leq \frac{f(|\tilde{y}|)f'(|\tilde{y}|)}{|\tilde{y}|}\cos\theta_x(y) + \frac{f(|\tilde{y}|)^2}{|\tilde{y}|^2}\sin\theta_x(y).$$

Applying Theorem 3.2.22 in [7], by putting $A_\xi = E \cap B(0, r) \setminus B(0, r - \xi)$, we get that

$$\begin{aligned} \mathcal{H}^2(\phi(E \cap B(0, r))) &\leq \int_{A_\xi} \frac{r^2}{\xi^2} \cdot \frac{|\tilde{y}| - r + \xi}{|\tilde{y}|}\cos\theta_x(y) d\mathcal{H}^2(y) + \frac{r^2}{(r - \xi)^2}\mathcal{H}^2(A_\xi) \\ &= \int_{r-\xi}^{r} \frac{r^2(t - r + \xi)}{\xi^2 t}\mathcal{H}^1(E \cap \partial B(x, t))dt + 4\mathcal{H}^2(A_\xi), \end{aligned}$$

thus

$$\mathcal{H}^2(E \cap B(0, r)) \leq (2r)^2 h(2r) + \lim_{\xi \to 0+} r^2 \int_{r-\xi}^{r} \frac{t - r + \xi}{t\xi^2}\mathcal{H}^1(E \cap \partial B(x, t))dt.$$

Since the function $g(t) = \mathcal{H}^1(E \cap B(x, t))/t$ is a measurable function, we have that, for almost every r,

$$\lim_{\xi \to 0+} \int_0^\xi \frac{tg(t - r + \xi)}{\xi^2}dt = \frac{1}{2}g(r),$$

thus for such r,

$$\mathcal{H}^2(E \cap B(x,r)) \le (2r)^2 h(2r) + \frac{r}{2}\mathcal{H}^1(E \cap B(x,r)).$$

$\square$

For any set $E \subset \mathbb{R}^3$, we set

$$\Theta_E(x,r) = \frac{1}{r^2}\mathcal{H}^2(E \cap B(x,r)), \text{ for any } r > 0,$$

and denote by $\Theta_E(x) = \lim_{r\to 0+}\Theta_E(x,r)$ if the limit exist, we may drop the script E if there is no danger of confusion.

Theorem 2.3. *Let E be a 2-rectifiable (Ω_0, L_0, h) locally sliding almost minimal at $x \in E$. If $x \in L_0$, then $\Theta(x,r) + 8h_1(r)$ is nondecreasing as $r \in (0, r_h)$; if $x \notin L_0$, then $\Theta(x,r) + 8h_1(r)$ is nondecreasing as $r \in (0, \min\{r_h, \operatorname{dist}(x,L)\})$.*

Proof. From Lemma 2.2 and Lemma 2.1, by putting $u(r) = \mathcal{H}^2(E \cap B(x,r))$, we get that, if $x \in L$,

$$u(r) \le \frac{r}{2}u'(r) + h(2r)(2r)^2, \tag{2.4}$$

for almost every $r \in (0,\infty)$; if $x \notin L$, then (2) holds for almost every $r \in (0, \min\{r_h, \operatorname{dist}(x,L)\})$.

We put $v(r) = r^{-2}u(r)$, then $v'(r) \ge -8r^{-2}h(2r)$, we get that $\Theta(x,r) + 8h_1(r)$ is nondecreasing. $\square$

Remark 2.4. *Let E be a 2-rectifiable (Ω_0, L_0, h) locally sliding almost minimal at some point $x \in E$. Then by Theorem 2.3, we get that $\Theta_E(x)$ exists.*

3 Estimation of upper bound

Let $\mathcal{Z}$ be a collection of cones. We say that a set $E \subset \mathbb{R}^3$ is locally $C^{k,\alpha}$-equivalent (resp. C^k-equivalent) to a cone in $\mathcal{Z}$ at $x \in E$ for some nonnegative integer k and some number $\alpha \in (0,1]$, if there exist $\varrho_0 > 0$ and $\tau_0 > 0$ such that for any $\tau \in (0,\tau_0)$ there is $\varrho \in (0,\varrho_0)$, a cone $Z \in \mathcal{Z}$ and a mapping $\Phi : B(0,2\varrho) \to \mathbb{R}^3$, which is a homeomorphism of class $C^{k,\alpha}$ (resp. C^k) between $B(0,2\varrho)$ and its image $\Phi(B(0,2\varrho))$ with $\Phi(0) = x$, satisfying that

$$\|\Phi - \mathrm{id} - \Phi(0)\|_\infty \le \varrho\tau \tag{3.1}$$

and

$$E \cap B(x,\varrho) \subset \Phi\left(Z \cap B\left(0, 2\varrho\right)\right) \subset E \cap B(x, 3\varrho). \tag{3.2}$$

Similarly, if $\Omega \subset \mathbb{R}^3$ is a closed domain with the boundary $\partial\Omega$ is a 2-dimensional smooth manifold, a set $E \subset \Omega$ is called locally $C^{k,\alpha}$-equivalent to a sliding minimal cone Z in Ω_0 at $x \in E \cap \partial\Omega$, if there exist $\varrho_0 > 0$ and $\tau_0 > 0$ such that for any $\tau \in (0,\tau_0)$ there is $\varrho \in (0,\varrho_0)$ and a mapping $\Phi : B(0,2\varrho) \cap \Omega_0 \to \Omega$, which is a diffeomorphism of class $C^{k,\alpha}$ between $B(0,2\varrho) \cap \Omega_0$ and its image $\Phi(B(0,2\varrho) \cap \Omega_0)$ with $\Phi(0) = x$ satisfying that $\Phi(L_0 \cap B(0,2\varrho)) \subset \partial\Omega$ and (3) and (3).

Jean Taylor proved, in [10], that if E is a 2-dimensional almost minimal set in an open set $U \subset \mathbb{R}^3$ with gauge function h satisfying $h(r) \le cr^\alpha$, then E is locally $C^{1,\beta}$-equivalent to a minimal cone at each point $x \in E$ for some $\beta > 0$. In [2, Theorem 1.15], Guy David gave a different proof of this result and generalized it to high dimensional ambient space. In [6], we got that, when $\Omega \subset \mathbb{R}^3$ is a closed domain with the boundary $\partial\Omega$ is a 2-dimensional smooth manifold, any sliding almost minimal set $E \supset \partial\Omega$ in Ω with sliding boundary $\partial\Omega$ and with gauge function h satisfying (2), is locally $C^{0,\beta}$-equivalent to a sliding minimal cone in Ω_0 at $x \in E \cap \partial\Omega$.

3.1 Approximation of $E \cap \partial B(0, r)$ by rectifiable curves

For any sets $X, Y \subset \mathbb{R}^3$, any $z \in \mathbb{R}^3$ and any $r > 0$, we denote by $d_{z,r}$ a normalized local Hausdorff distances defined by

$$d_{z,r}(X,Y) = \frac{1}{r} \sup_{x \in X \cap \overline{B(z,r)}} \operatorname{dist}(x, Y) + \frac{1}{r} \sup_{y \in Y \cap \overline{B(z,r)}} \operatorname{dist}(y, X).$$

A cone $Z \subset \Omega_0$ is called of type $\mathbb{P}_+$ is if it is a half plane perpendicular to L_0; a cone $Z \subset \Omega_0$ is called of type $\mathbb{Y}_+$ is if $Z = \Omega_0 \cap Y$, where Y is a cone of type $\mathbb{Y}_+$ perpendicular to L_0; for convenient, we will also use the notation $\mathbb{P}_+$, to denote the collection of all of cones of type $\mathbb{P}_+$, and $\mathbb{Y}_+$ to denote the collection of all of cones of type $\mathbb{Y}_+$.

For any set $E \subset \Omega_0$ with $0 \in E$ and $r > 0$, we set

$$\varepsilon_P(r) = \inf\{d_{0,r}(E, Z) : Z \in \mathbb{P}_+\}$$
$$\varepsilon_Y(r) = \inf\{d_{0,r}(E, Z) : Z \in \mathbb{Y}_+\}.$$

and set $K_r = E \cap \partial B(0, r)$. If E is 2-rectifiable and $\mathcal{H}^2(E) < \infty$, then K_r is 1-rectifiable and $\mathcal{H}^1(K_r) < \infty$ for $\mathcal{H}^1$-a.e. $r \in (0, \infty)$; we consider the function $u : (0, \infty) \to \mathbb{R}$ defined by $u(r) = \mathcal{H}^2(E \cap B(0, r))$, then u is nondecreasing, and u is differentiable for $\mathcal{H}^1$-a.e.; we will denote by $\mathscr{R}$ the set $r \in (0, \infty)$ such that $\mathcal{H}^1(K_r) < \infty$ and u is differentiable at r.

Lemma 3.1. *Let $E \subset \mathbb{R}^3$ be a connected set. If $\mathcal{H}^1(E) < \infty$, then E is path connected.*

For a proof, see for example Lemma 3.12 in [5]

Lemma 3.2. *Let $\mathbb{X}$ be a locally connected and simply connected compact metric space. Let A and B be two connected subsets of $\mathbb{X}$. If F is a closed subset of $\mathbb{X}$ such that A and B are contained in two different connected components of $\mathbb{X} \setminus F$, then there exists a connected closed set $F_0 \subset F$ such that A and B still lie in two different connected components of $\mathbb{X} \setminus F_0$.*

Proof. See for example 52.III.1 on page 335 in [9]. □

For any $r > 0$, we put $\mathfrak{z}_r = (0, 0, r) \in \mathbb{R}^3$.

Lemma 3.3. *Let $E \subset \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$. Suppose that $0 \in E$, and that E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0. Then there exist $\mathfrak{r} = \mathfrak{r}(\tau) > 0$ such that, for any $r \in (0, \mathfrak{r})$ and $\varepsilon > \varepsilon_P(r)$, we can find $y_r \in E \cap \partial B(0, r) \setminus L$, $\mathfrak{X}_{r,1}, \mathfrak{X}_{r,2} \in E \cap L \cap \partial B(0, r)$ and two simple curves $\gamma_{r,1}, \gamma_{r,2} \subset E \cap \partial B(0, r)$ satisfying that*

(1) $|y_r - \mathfrak{z}_r| \leq \varepsilon r$ *and* $|z_{r,1} - z_{r,2}| \geq (2-2\varepsilon)r$;

(2) $\gamma_{r,i}$ *joins* y_r *and* $\mathfrak{X}_{r,i}$, $i=1,2$;

(3) $\gamma_{r,1}$ *and* $\gamma_{r,2}$ *are disjoint except for point* y_r.

Proof. Since E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ at 0, there exist $\tau > 0$, $\varrho > 0$, sliding minimal cone Z of type $\mathbb{P}_+$, and a mapping $\Phi : \Omega_0 \cap B(0,2\varrho) \to \Omega_0$ which is a homeomorphism between $\Omega_0 \cap B(0,2\varrho)$ and $\Phi(\Omega_0 \cap B(0,2\varrho))$ with $\Phi(0)=0$ and $\Phi(\partial\Omega_0 \cap B(0,2\varrho)) \subset \partial\Omega_0$ such that (3) and (3) hold. We new take $\mathfrak{r} = \varrho$. Then for any $r \in (0,\mathfrak{r})$,

$$\Phi^{-1}\left[E \cap \partial B(0,r)\right] \subset Z \cap B(0,3\varrho).$$

Without loss of generality, we assume that $Z = \{(x_1,0,x_3) \mid x_1 \in \mathbb{R}, x_3 \geq 0\}$. Applying Lemma 3.2 with $\mathbb{X} = Z \cap \overline{B(0,3\varrho)}$, $F = \Phi^{-1}\left[E \cap \partial B(0,r)\right]$, $A=\{0\}$ and $B = Z \cap \partial B(0,3\varrho)$, we get that there is a connected closed set $F_0 \subset F$ such that A and B lie in two different connected components of $A \setminus F_0$, thus $\phi(F_0) \subset E \cap \partial B(0,r)$ is connected. We put $a_1 = \{(x_1,0,0) \mid x_1 < 0\}$ and $a_2 = \{(x_1,0,0) \mid x_1 > 0\}$. Then $F_0 \cap a_i \neq \emptyset$, $i=1,2$; otherwise A and B are contained in a same connected component of $X \setminus F_0$. We take $z_{r,i} \in F_0 \cap a_i$, and let $\mathfrak{X}_{r,i} = \phi(z_{r,i}) \in E \cap \partial B(0,r)$. Then $|\mathfrak{X}_{r,1} - \mathfrak{X}_{r,2}| \geq (2-2\varepsilon)r$.

Since F_0 is connected and $\mathcal{H}^1(F_0) < \infty$, by Lemma 3.1, F_0 is path connected. Let γ be a simple curve which joins $z_{r,1}$ and $z_{r,2}$. We see that $B(\mathfrak{z}_r, \varepsilon r) \cap \gamma \neq \emptyset$, because $\varepsilon_P(r) < \varepsilon$ and $\mathfrak{z}_r \in Z$ for sliding minimal cone Z of type $\mathbb{P}_+$. We take $y_r \in B(\mathfrak{z}_r, \varepsilon r) \cap \gamma$. □

Lemma 3.4. *Let $E \subset \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$. Suppose that $0 \in E$, and that E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$ at 0. Then there exist $\mathfrak{r} = \mathfrak{r}(\tau) > 0$ such that, for any $r \in (0,\mathfrak{r})$ and $\varepsilon > \varepsilon_Y(r)$, we can find $y_r \in E \cap \partial B(0,r) \setminus L$, $\mathfrak{X}_{r,1}, \mathfrak{X}_{r,2}, \mathfrak{X}_{r,3} \in E \cap L \cap \partial B(0,r)$ and three simple curves $\gamma_{r,1}, \gamma_{r,2}, \gamma_{r,3} \subset E \cap \partial B(0,r)$ satisfying that*

(1) $|\mathfrak{z}_r - y_r| \leq \pi r/6$, *and there exists* $Z \in \mathbb{Y}_+$ *with* $\operatorname{dist}(x,Z) \leq \varepsilon r$ *for* $x \in \gamma$;

(2) $\gamma_{r,i}$ *join* y_r *and* $\mathfrak{X}_{r,i}$;

(3) $\gamma_{r,i}$ *and* $\gamma_{r,j}$ *are disjoint except for point* y_r.

Proof. Since E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$ at 0, there exist $\tau > 0$, $\varrho > 0$, sliding minimal cone Z of type $\mathbb{Y}_+$, and a mapping $\Phi : \Omega_0 \cap B(0,2\varrho) \to \Omega_0$ which is a homeomorphism between $\Omega_0 \cap B(0,2\varrho)$ and $\Phi(\Omega_0 \cap B(0,2\varrho))$ with $\Phi(0)=0$ and $\Phi(\partial\Omega_0 \cap B(0,2\varrho)) \subset \partial\Omega_0$ such that (3) and (3) hold. We new take $\mathfrak{r} = \varrho$. Then for any $r \in (0,\mathfrak{r})$,

$$\Phi^{-1}\left[E \cap \partial B(0,r)\right] \subset Z \cap B(0,3\varrho).$$

Applying Lemma 3.2 with $\mathbb{X} = Z \cap \overline{B(0,3\varrho)}$, $F = \Phi^{-1}\left[E \cap \partial B(0,r)\right]$, $A=\{0\}$ and $B = Z \cap \partial B(0,3\varrho)$, we get that there is a connected closed set $F_0 \subset F$ such that A and B lie in two different connected components of $A \setminus F_0$, thus $\phi(F_0) \subset E \cap \partial B(0,r)$ is connected. We let a_i, $i=1,2,3$, be the there component of $Z \cap L_0 \setminus A$. Then $F_0 \cap a_i \neq \emptyset$, $i=1,2,3$; otherwise A and B are contained in a same connected component of $X \setminus F_0$. We take $z_{r,i} \in F_0 \cap a_i$, and let $\mathfrak{X}_{r,i} = \phi(z_{r,i}) \in E \cap \partial B(0,r)$. Then $|\mathfrak{X}_{r,1} - \mathfrak{X}_{r,2}| \geq (\sqrt{3} - 2\varepsilon)r$.

Since F_0 is connected and $\mathcal{H}^1(F_0) < \infty$, by Lemma 3.1, F_0 is path connected. □

3.2 Approximation of rectifiable curves in $\mathbb{S}^2$ by Lipschitz graph

We denote by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. We say that a simple rectifiable curve $\gamma \subset \mathbb{S}^2$ is a Lipschitz graph with constant at most η, if it can be parametrized by

$$z(t) = \left(\sqrt{1 - v(t)^2} \cos \theta(t), \sqrt{1 - v(t)^2} \sin \theta(t), v(t) \right),$$

where v is Lipschitz with $\mathrm{Lip}(v) \le \eta$.

Lemma 3.5. *Let $T \in [\pi/3, 2\pi/3]$ be a number, and $\gamma : [0, T] \to \mathbb{S}^2$ a simple rectifiable curve given by*

$$\gamma(t) = \left(\sqrt{1 - v(t)^2} \cos \theta(t), \sqrt{1 - v(t)^2} \sin \theta(t), v(t) \right),$$

where v is a continuous function with $v(0) = v(T) = 0$, θ is a continuous function with $\theta(0) = 0$ and $\theta(T) = T$. Then there is a small number $\tau_0 \in (0, 1)$ such that whenever $|v(t)| \le \tau_0$, we have that

$$|v(t)| \le 10\sqrt{\mathcal{H}^1(\gamma) - T}.$$

Proof. We let $A = \gamma(0) = (1, 0, 0)$, $B = \gamma(T) = (\cos T, \sin T, 0)$, and let $C = \gamma(t_0)$ be a point in γ such that

$$|v(t_0)| = \max\{|v(t)| : t \in [0, T]\}.$$

We let γ_i, $i = 1, 2$, be two curve such that $\gamma_1(0) = A$, $\gamma_1(1) = C$, $\gamma_2(0) = B$ and $\gamma_2(1) = C$, and let $s \in [0, 1]$ be the smallest number such that $\gamma_1(s) \notin \gamma_2$, and put $D = \gamma_1(s)$. Then, by setting $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ the arc AD, BD and CD respectively, we have that

$$\mathcal{H}^1(\gamma) \ge \mathcal{H}^1(\gamma_1 \cup \gamma_2) \ge \mathcal{H}^1(\mathfrak{C}_1) + \mathcal{H}^1(\mathfrak{C}_2) + \mathcal{H}^1(\mathfrak{C}_3).$$

We see that $\mathfrak{C}_1 \cup \mathfrak{C}_2$ is a simple Lipschitz curve joining A and B, and let $\gamma_3 : [0, \ell] \to \mathbb{S}^2$ giving by

$$\gamma_3(t) = \left(\sqrt{1 - w(t)^2} \cos \theta(t), \sqrt{1 - w(t)^2} \sin \theta(t), w(t) \right)$$

be its parametrization by length. We assume that $\gamma_3(t_1) = D$, then $w'(t) > 0$ on $(0, t_1)$, or $w'(t) < 0$ on $(0, t_1)$, thus $|w(t)| = \int_0^{t_1} |w'(t)| dt$.

We let our τ_0 to be the small number $\tau_1 < 1$ as in Lemma 7.8 in [2]. If $\mathcal{H}^1(\gamma) - T \le \tau_0$, then we have that

$$\int_0^\ell |w'(t)|^2 dt \le 14(\ell - T),$$

thus

$$|w(t_1)| = \int_0^{t_1} |w'(t)| dt \le \left(t_1 \int_0^{t_1} |w'(t)|^2 dt \right)^{1/2} \le \sqrt{14\ell(\ell - T)}.$$

We get so that

$$\begin{aligned} |v(t_0)| &\le \mathcal{H}^1(\mathfrak{C}_3) + |w(t_1)| \le (\mathcal{H}^1(\gamma) - \ell) + \sqrt{14\ell(\ell - T)} \\ &\le \sqrt{14\mathcal{H}^1(\gamma)(\mathcal{H}^1(\gamma) - T)} \le 10\sqrt{\mathcal{H}^1(\gamma) - T}. \end{aligned}$$

If $\mathcal{H}^1(\gamma) - T > \tau_0$, then $v(t) \le \tau_0 \le 10\sqrt{\tau_0} \le 10\sqrt{\mathcal{H}^1(\gamma) - T}$. $\square$

Lemma 3.6. *Let a and b be two points in $\Omega_0 \cap \partial B(0,1)$ satisfying*

$$\frac{\pi}{3} \leq \operatorname{dist}_{\mathbb{S}^2}(a,b) \leq \frac{2\pi}{3}.$$

Let γ be a simple rectifiable curve in $\Omega_0 \cap \partial B(0,1)$ which joins a and b, and satisfies

$$\operatorname{length}(\gamma) \leq \operatorname{dist}_{\mathbb{S}^2}(a,b) + \tau_0,$$

where $\tau_0 > 0$ is as in Lemma 3.5. Then there is a constant $C > 0$ such that, for any $\eta > 0$, we can find a simple curve γ_* in $\Omega_0 \cap \partial B(0,1)$ which is a Lipschitz graph with constant at most η joining a and b, and satisfies that

$$\mathcal{H}^1(\gamma_* \setminus \gamma) \leq \mathcal{H}^1(\gamma \setminus \gamma_*) \leq C\eta^{-2}(\operatorname{length}(\gamma) - \operatorname{dist}_{\mathbb{S}^2}(a,b)).$$

The proof will be the same as in [2, p.875-p.878], so we omit it.

3.3 Compare surfaces

Let Γ be a Lipschitz curve in $\mathbb{S}^2$. We assume for simplicity that its extremities a and b lie in the horizontal plane. Let us assume that $a = (1,0,0)$ and $b = (\cos T, \sin T, 0)$ for some $T \in [\pi/3, 2\pi/3]$. We also assume that Γ is a Lipschitz graph with constant at most η, i.e. there is a Lipschitz function $s : [0,T] \to \mathbb{R}$ with $s(0) = s(T) = 0$ and $\operatorname{Lip}(s) \leq \eta$, such that Γ is parametrized by

$$z(t) = (w(t)\cos t, w(t)\sin t, s(t)) \text{ for } t \in [0,T],$$

where $w(t) = (1 - |s(t)|^2)^{1/2}$.

We set

$$D_T = \{(r\cos t, r\sin t)| \mid 0 < r < 1, 0 < t < T\},$$

and consider the function $v : \overline{D}_T \to \mathbb{R}$ defined by

$$v(r\cos t, r\sin t) = \frac{rs(t)}{w(t)} \text{ for } 0 \leq r \leq 1 \text{ and } 0 \leq t \leq T.$$

For any function $f : \overline{D}_T \to \mathbb{R}$, we denote by Σ_f the graphs of f over $\overline{D}_T$.

Lemma 3.7. *There is a universal constant $\kappa > 0$ such that we can find a Lipschitz function u on $\overline{D}_T$ satisfying that*

$$\operatorname{Lip}(u) \leq C\eta,$$

$$u(r,0) = u(r\cos T, r\sin T) = 0, \text{ for } 0 \leq r \leq 1, 0 \leq t \leq T,$$

$$u(r\cos t, r\sin t) = v(r\cos t, r\sin t) \text{ for } 0 \leq r \leq 1, 0 \leq t \leq T,$$

$$u(r\cos t, r\sin t) = 0, \text{ for } 0 \leq r \leq 2\kappa, 0 \leq t \leq T$$

and

$$\mathcal{H}^2(\Sigma_v) - \mathcal{H}^2(\Sigma_u) \geq 10^{-4}(\mathcal{H}^1(\Gamma) - T).$$

The proof is the same as Lemma 8.8 in [2], we omit it.

3.4 Retractions

We let $\boldsymbol{\mu}_r : \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping defined by $\boldsymbol{\mu}_r(x) = rx$ for any $r > 0$, and let $\Pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$ be the projection defined by $\Pi(x) = x/|x|$. In this subsection, we will construct a neighborhood retraction of set E, which satisfying the following conditions: $E \subset \Omega_0$ is a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$, $0 \in E$, and E is locally (Ω_0, L_0, h) sliding almost minimal at 0, E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ at 0.

For convenient, we will denote

$$j(r) = \frac{1}{r}\mathcal{H}^1(E \cap \partial B(0,r)) - \mathcal{H}^1(X \cap \partial B(0,1)),$$

and denote by $\mathscr{R}_1$ the set $\{r \in \mathscr{R} : j(r) \leq \tau_0\}$, where τ_0 is the small number considered as in Lemma 3.5.

For any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_1$, we take $\mathcal{X}_r \subset E \cap B(0,r) \cap L$ as following: if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$, we let $\mathfrak{X}_{r,1}$ and $\mathfrak{X}_{r,2}$ be the same as in Lemma 3.3, and let $\mathcal{X}_r = \{\mathfrak{X}_{r,1}, \mathfrak{X}_{r,2}\}$; if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$, we let $\mathfrak{X}_{r,1}$, $\mathfrak{X}_{r,2}$ and $\mathfrak{X}_{r,3}$ be the same as in Lemma 3.4, and let $\mathcal{X}_r = \{\mathfrak{X}_{r,1}, \mathfrak{X}_{r,2}, \mathfrak{X}_{r,3}\}$.

We take y_r as in Lemma 3.3 and Lemma 3.4. For any $x \in \mathcal{X}_r$, we let γ^x be the curve which joins x and y_r as in Lemma 3.3 and Lemma 3.4, let D_{x,y_r} be the sector determined by points 0, y_r and x. We denote by P_{x,y_r} the plane that contains 0, x and y_r, let $\mathcal{R}_{x,y_r}$ be a rotation such that $\mathcal{R}_{x,y_r}(y_r) = (r,0,0)$ and $\mathcal{R}_x(y_r) = (r\cos T_x, r\sin T_x, 0)$, where $T_x \in [\pi/3, 2\pi/3]$.

For any $x \in \mathcal{X}_r$, γ^x is a simple rectifiable curve in $\Omega_0 \cap \partial B(0,r)$, thus the curve $\Gamma^x = \Pi(\gamma^x)$ is a simple rectifiable curve in $\Omega_0 \cap \partial B(0,1)$, let Γ^x_* be the corresponding curve with respect to Γ^x as in Lemma 3.6. Let $z(t) = (w(t)\cos t, w(t)\sin t, s(t))$ be a parametrization of $\mathcal{R}_{x,y_r}(\Gamma^x_*)$, where $w(t) = \sqrt{1 - s(t)^2}$. Let Σ^x_v and Σ^x_u be the same as in Lemma 3.7. We put $T = \sum_{x\in\mathcal{X}_r} T_x$, and put

$$X = \bigcup_{x\in\mathcal{X}_r} D_{x,y_r},\ \Gamma_* = \bigcup_{x\in\mathcal{X}_r} \Gamma^x_*,\ \mathcal{M} = \bigcup_{x\in\mathcal{X}_r} \Sigma^x_v,\ \text{and}\ \Sigma = \bigcup_{x\in\mathcal{X}_r} \Sigma^x_u. \tag{3.3}$$

By Lemma 3.7, we get that

$$\mathcal{H}^2(\mathcal{M}) - \mathcal{H}^2(\Sigma) \geq 10^{-4}\left(\mathcal{H}^1(\Gamma_*) - T\right). \tag{3.4}$$

Lemma 3.8. *Let $\delta, \varepsilon < 1/2$ be positive numbers. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be unit vectors such that $|\langle v_2, v_i\rangle| \leq \delta$ for $i = 1, 3$. Then we have that*

$$|\langle v, v_3\rangle - \langle v_1, v_3\rangle\langle v, v_2\rangle| \leq (\varepsilon + \delta)|v|$$

and

$$|\langle v, v_1\rangle| \geq (1 - \varepsilon - \delta)|v|$$

for any $v \in \mathbb{R}^3$ with $\langle v, v_2\rangle = 0$ and $\operatorname{dist}(v, \operatorname{span}\{v_1, v_2\}) \leq \varepsilon|v|$; when $\langle v_1, v_3\rangle < 1$ and $\delta < 10^{-2}(1 - \langle v_1, v_3\rangle)^2$, we have that

$$|w_1| + |w_2| \leq \sqrt{2/\left(1 - \langle v_1, v_3\rangle - 10\sqrt{\delta}\right)}|w_1 - w_2|$$

for any $w_1, w_3 \in \mathbb{R}^3$ with $\langle v_i, w_i\rangle \geq (1 - \delta)|w_i|$, $i = 1, 3$.

Proof. We write $v = v^\perp + \lambda_1 v_1 + \lambda_2 v_2$, $\lambda_i \in \mathbb{R}$, $\langle v^\perp, v_i\rangle = 0$.

Since $\langle v, v_2\rangle = 0$, we have that $\lambda_2 = -\lambda_1\langle v_1, v_2\rangle$, thus

$$\lambda_1 = \frac{\langle v, v_1\rangle}{1-\langle v_1, v_2\rangle^2},\ \ \lambda_2 = -\frac{\langle v, v_1\rangle\langle v_1, v_2\rangle}{1-\langle v_1, v_2\rangle^2},$$

we get so that

$$v = v^\perp + \frac{\langle v, v_1\rangle v_1 - \langle v, v_1\rangle\langle v_1, v_2\rangle v_2}{1-\langle v_1, v_2\rangle^2}, \tag{3.5}$$

and then

$$\langle v, v_3\rangle = \langle v^\perp, v_3\rangle + \frac{\langle v_1, v_3\rangle - \langle v_2, v_3\rangle\langle v_1, v_2\rangle}{1-\langle v_1 . v_2\rangle^2}\langle v, v_1\rangle,$$

thus

$$|\langle v, v_3\rangle - \langle v_1, v_3\rangle\langle v, v_1\rangle| \le \varepsilon|v| + \frac{\delta^2+\delta}{1-\delta^2}|v| \le (\varepsilon + 2\delta)|v|.$$

We get also, from (3.4), that

$$|v| \le |v^\perp| + \frac{1+|\langle v_1 . v_2\rangle|}{1-\langle v_1, v_2\rangle^2}|\langle v, v_1\rangle| \le \varepsilon|v| + \frac{1}{1-\delta}|\langle v, v_1\rangle|,$$

thus

$$|\langle v, v_1\rangle| \ge (1-\varepsilon)(1-\delta)|v| \ge (1-\varepsilon-\delta)|v|.$$

□

Lemma 3.9. *For any $r \in (0.\mathfrak{r}) \cap \mathscr{R}_1$, we let Σ be as in (3.4). Then there is a Lipschitz mapping $p : \Omega_0 \to \Sigma$ with $\mathrm{Lip}(p) \le 50$, such that $p(z) \in L$ for $z \in L$, and that $p(z) = z$ for $z \in \Sigma$.*

Proof. By definition, we have that

$$\Sigma \setminus B(0, 9/10) = \mathcal{M} \setminus B(0, 9/10),$$

and that

$$\Sigma \cap B(0, 2\kappa) = X \cap B(0, 2\kappa).$$

For any $z \in \Omega_0 \setminus \{0\}$, we denote by $\ell(z)$ the line which is through 0 and z. Then $\partial D_{x,y_r} = \ell(x) \cup \ell(y_r)$. We fix any $\sigma \in (0, 10^{-3})$, put

$$R^x = \{z \in \Omega_0 \mid \mathrm{dist}(z, D_{x,y_r}) \le \sigma\,\mathrm{dist}(z, \partial D_{x,y_r})\},$$
$$R_1^x = \{z \in \Omega_0 \mid \mathrm{dist}(z, D_{x,y_r}) \le \sigma\,\mathrm{dist}(z, \ell(y_r))\},$$

and

$$R = \bigcup_{x\in\mathcal{X}_r} R^x, R_1 = \bigcup_{x\in\mathcal{X}_r} R_1^x.$$

Then we see that $R^x \subset R_1^x$, and that both of them are cones,

$$R^{x_i} \cap R^{x_j} = R_1^{x_i} \cap R_1^{x_j} = \ell(y_r) \text{ for } x_i, x_j \in \mathcal{X}_r, x_i \ne x_j.$$

Since Σ_u^x is a small Lipschitz graph over D_{x,y_r} bounded by two half lines of $\partial D_{x,y_r}$ with constant at most η, there is a constant $\bar\eta$ such that

$$\Sigma_u^x \subset R^x,$$

when $0 < \eta < \bar{\eta}$.

We will construct a Lipschitz retraction $p_0 : \Omega_0 \to R_1$ such that $p_0(z) = z$ for $z \in R_1$, $p_0(z) \in L$ for $z \in L$, and $\mathrm{Lip}(p_0) \leq 3$. We now distinguish two cases, depending on cardinality of $\mathcal{X}_r$.

Case 1: $\mathrm{card}(\mathcal{X}_r) = 2$. We assume that $\mathcal{X}_r = \{x_1, x_2\}$. Then $|y_r| = |x_1| = |x_2| = r$, and

$$0 \leq \langle x_1, x_2 \rangle + r^2 \leq 2\varepsilon^2 r^2.$$

Since $|y_r - \mathfrak{z}_r| \leq \varepsilon r$, we have that $|\langle y_r, x \rangle| \leq \varepsilon r^2$ for any $x \in L \cap \partial B(0, r)$.

We now let e_1 and e_2 be two unit vectors in L such that $\langle x_1, e_1 \rangle = \langle x_2, e_1 \rangle \geq 0$ and $e_2 = -e_1$. Then

$$0 \leq \langle x_i, e_1 \rangle \leq \varepsilon r.$$

We let Ω_1' and Ω_2' be the two connected components of $\Omega_0 \setminus (\cup_i D_{x_i, y_r})$ such that $e_i \in \Omega_i'$. We put $\Omega_i = \Omega_i' \setminus R_1$. We claim that

$$|\langle z_1 - z_2, e_i \rangle| \leq 5(\sigma + \varepsilon)|z_1 - z_2|$$

whenever $z_1, z_2 \in \partial\Omega_i$, $z_1 \neq z_2$, $i \in \{1, 2\}$.

Without loss of generality, we assume $z_1, z_2 \in \partial\Omega_1$, because for another case we will use the same treatment. We see that

$$\mathrm{dist}(z_i, D_{x_j, y_r}) = \sigma\, \mathrm{dist}(z_i, \ell(y_r)).$$

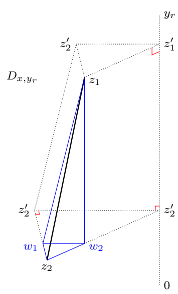

Figure 1: the angle between $z_1 - z_2$ and D_{x,y_r} is small.

(1) In case $z_1, z_2 \in \partial R_1^{x_i} \cap \Omega_1$, without loss of generality, we assume that $z_1, z_2 \in \partial R_1^{x_1} \cap \Omega_1$. We let $\widetilde{z}_i \in D_{x_1, y_r}$ be such that

$$z_i - \widetilde{z}_i = \mathrm{dist}(z_i, D_{x_1, y_r}),\ i = 1, 2,$$

and let $z_i' \in \ell(y_r)$ be such that

$$|z_i - z_i'| = \mathrm{dist}(z_i, \ell(y_r)),$$

and put

$$w_1 = z_1 - \widetilde{z}_1 + \widetilde{z}_2,\ w_2 = z_1 - z_1' + z_2',$$

then we get that $z_1 - z_2 = (z_1 - w_2) + (w_2 - z_2)$. Moreover, we have that $z_1 - w_2$ is perpendicular to $w_2 - z_2$ and parallel to y_r. Thus $|w_2 - z_2| \leq |z_1 - z_2|$, $|z_1 - w_2| \leq |z_1 - z_2|$ and

$$\mathrm{dist}(w_2 - z_2, \mathrm{span}\{x_1, y_r\}) = \sigma |w_2 - z_2|.$$

We apply Lemma 3.8 to get that

$$|\langle z_1 - w_2, e_1 \rangle| \leq \varepsilon |z_1 - w_2|$$

and

$$|\langle w_2 - z_2, e_1 \rangle| \leq (\sigma + 3\varepsilon)\, |w_2 - z_2|,$$

thus

$$|\langle z_1 - z_2, e_1 \rangle| \leq |\langle z_1 - w_2, e_1 \rangle| + |\langle w_2 - z_2, e_1 \rangle| \leq (\sigma + 4\varepsilon)\, |z_1 - z_2|.$$

(2) In case $z_1 \in \partial R^{x_1} \cap \Omega_1$, $z_2 \in \partial R^{x_2} \cap \Omega_1$. We let $\widetilde{z}_i \in D_{x_i, y_r}$ be such that

$$|z_i - \widetilde{z}_i| = \mathrm{dist}(z_i, D_{x_i, y_r}),\ \ i = 1, 2,$$

and let $z_i' \in \ell(y_r)$ be such that

$$|z_i - z_i'| = \mathrm{dist}(z_i, \ell(y_r)),\ \ i = 1, 2.$$

Then by Lemma 3.8, we have that

$$\left\langle z_i - z_i', \frac{x_i}{|x_i|} \right\rangle \geq (1 - \sigma - \varepsilon)|z_i - z_i'|,\ \ i = 1, 2.$$

Since $z_1 - z_2 = (z_1 - z_1') + (z_2' - z_2) + (z_1' - z_2')$,

$$|\langle z_1' - z_2', e_1 \rangle| \leq \varepsilon |z_1' - z_2'| \leq \varepsilon |z_1 - z_2|$$

and

$$|\langle z_i - z_i', e_1 \rangle| \leq (\sigma + \varepsilon)\, |z_i - z_i'|,$$

we get that

$$\begin{aligned} |\langle z_1 - z_2, e_1 \rangle| &\leq |\langle z_1 - z_1', e_1 \rangle| + |\langle z_2' - z_2, e_1 \rangle| + |\langle z_1' - z_2', e \rangle| \\ &\leq 2 \cdot (\sigma + \varepsilon)\, (|z_1 - z_1'| + |z_2 - z_2'|) + \varepsilon |z_1 - z_2|. \end{aligned}$$

Since $z_1' - z_2'$ is perpendicular to $z_1 - z_1'$ and $z_2 - z_2'$, and

$$\left\langle z_i - z_i', \frac{x_i}{|x_i|} \right\rangle \geq (1 - \sigma - \varepsilon)|z_i - z_i'|,\ \ i = 1, 2,$$

and

$$\left\langle \frac{x_1}{|x_1|}, \frac{x_2}{|x_2|} \right\rangle \leq -1 + 2\varepsilon^2,$$

we get, by Lemma 3.8, that

$$|z_1 - z_1'| + |z_2 - z_2'| \leq \left(\frac{1}{1 - \varepsilon^2 - 5\sqrt{\sigma + \varepsilon}} \right)^{1/2} |(z_1 - z_1') - (z_2 - z_2')| \leq 2|z_1 - z_2|.$$

Thus

$$\langle z_1 - z_2, e \rangle \leq (4\sigma + 5\varepsilon)|z_1 - z_2|.$$

We now define $p_0 : \Omega_0 \to R_1$ as follows: for any $z \in \Omega_i$, we let $p_0(z)$ be the unique point in $\partial\Omega_i$ such that $p_0(z) - z$ parallels e; and for any $z \in R_1$, we let $p_0(z) = z$. Since $p_0(z) - z$ parallels e, we see that $p_0(L) \subset L$. We will check that

$$p_0 \text{ is Lipschitz with } \operatorname{Lip}(p_0) \leq \frac{2}{1 - 5(\sigma + \varepsilon)}.$$

Indeed, for any $z_1, z_2 \in \Omega_0$, we put

$$p_0(z_i) = z_i + t_i e,\ t_i \in \mathbb{R},$$

then

$$\begin{aligned} |t_1 - t_2| &= |\langle (t_1 - t_2)e, e\rangle| \\ &\leq |\langle p_0(z_1) - p_0(z_2), e\rangle| + |\langle z_1 - z_2, e\rangle| \\ &\leq 5(\sigma + \varepsilon)|p_0(z_1) - p_0(z_2)| + |z_1 - z_2|, \end{aligned}$$

and

$$|p_0(z_1) - p_0(z_2)| \leq |z_1 - z_2| + |t_1 - t_2| \leq 5(\sigma + \varepsilon)|p_0(z_1) - p_0(z_2)| + 2|z_1 - z_2|,$$

thus

$$|p_0(z_1) - p_0(z_2)| \leq \frac{2}{1 - 5(\sigma + \varepsilon)}|z_1 - z_2|.$$

Case 2: $\operatorname{card}(\mathcal{X}_r) = 3$. We assume that $\mathcal{X}_r = \{x_1, x_2, x_3\}$, then

$$|\langle x_i, y_r\rangle| \leq \varepsilon r^2, \left(-\sqrt{3}\varepsilon - \frac{1}{2}\right) r^2 \leq \langle x_i, x_j\rangle \leq \left(-\frac{1}{2} + 2\varepsilon\right) r^2.$$

We put

$$e_1 = \frac{x_2 + x_3}{|x_2 + x_3|}, e_2 = \frac{x_1 + x_3}{|x_1 + x_3|}, e_3 = \frac{x_2 + x_1}{|x_2 + x_1|},$$

and let Ω_1', Ω_2' and Ω_3' be the three connected components of $\Omega_0 \setminus (\cup_i D_{x_i,y_r})$ such that $e_i \in \Omega_i'$. By putting $\Omega_i = \Omega_i' \setminus R_1$, we claim that

$$\left(\frac{1}{2} - 5(\sigma + \varepsilon)\right) |z_1 - z_2| \leq |\langle z_1 - z_2, e_i\rangle| \leq \left(\frac{1}{2} + 5(\sigma + \varepsilon)\right) |z_1 - z_2|$$

whenever $z_1, z_2 \in \partial\Omega_i$, $z_1 \neq z_2$, $i \in \{1, 2, 3\}$.

Indeed, we only need to check the case $z_1, z_2 \in \partial\Omega_1$, and the other two cases will be the same. Since $-\sqrt{3}\varepsilon - 1/2 \leq \langle x_i, x_j\rangle \leq 1/2 + 2\varepsilon$, we have that $(1/2 - \varepsilon)r \leq \langle x_i, e_1\rangle \leq (1/2 + \varepsilon)r$ for $i = 2, 3$.

If $z_1, z_2 \in \partial R_1^{x_2} \cap \Omega_1$ or $z_1, z_2 \in \partial R_1^{x_3} \cap \Omega_1$, we assume that $z_1, z_2 \in \partial R_1^{x_2} \cap \Omega_1$, and let $\widetilde{z}_i \in D_{x_2,y_r}$ be such that

$$z_i - \widetilde{z}_i = \operatorname{dist}(z_i, D_{x_2,y_r}),\ i = 1, 2,$$

and let $z_i' \in \ell(y_r)$ be such that

$$|z_i - z_i'| = \operatorname{dist}(z_i, \ell(y_r)),$$

and put

$$w_1 = z_1 - \widetilde{z}_1 + \widetilde{z}_2,\ w_2 = z_1 - z_1' + z_2',$$

then we get that $z_1 - w_2$ is perpendicular to $w_2 - z_2$ and parallel to y_r. Since $z_1 - z_2 = (z_1 - w_2) + (w_2 - z_2)$, we have that $|w_2 - z_2| \le |z_1 - z_2|$, $|z_1 - w_2| \le |z_1 - z_2|$ and

$$\operatorname{dist}(w_2 - z_2, \operatorname{span}\{x_1, y_r\}) = \sigma |w_2 - z_2|.$$

We apply Lemma 3.8 to get that

$$|\langle z_1 - w_2, e_1\rangle| \le \varepsilon |z_1 - w_2|$$

and

$$|\langle w_2 - z_2, e_1\rangle| \le \left(\frac{1}{2} + \varepsilon + \sigma + \varepsilon\right) |w_2 - z_2|,$$

thus

$$|\langle z_1 - z_2, e_1\rangle| \le |\langle z_1 - w_2, e_1\rangle| + |\langle w_2 - z_2, e_1\rangle| \le \left(\frac{1}{2} + \sigma + 3\varepsilon\right) |z_1 - z_2|.$$

If $z_1 \in \partial R^{x_2} \cap \Omega_1$, $z_2 \in \partial R^{x_3} \cap \Omega_1$, we let $\widetilde{z}_i \in D_{x_i, y_r}$ be such that

$$|z_1 - \widetilde{z}_1| = \operatorname{dist}(z_1, D_{x_2, y_r}),\ \ |z_2 - \widetilde{z}_2| = \operatorname{dist}(z_2, D_{x_3, y_r})$$

and let $z_i' \in \ell(y_r)$ be such that

$$|z_i - z_i'| = \operatorname{dist}(z_i, \ell(y_r)),\ \ i = 1, 2.$$

Since $z_1 - z_2 = (z_1 - z_1') + (z_2' - z_2) + (z_1' - z_2')$,

$$|\langle z_1' - z_2', e_1\rangle| \le \varepsilon |z_1' - z_2'| \le \varepsilon |z_1 - z_2|$$

and

$$|\langle z_i - z_i', e_1\rangle| \le \left(\frac{1}{2} + \varepsilon + \sigma + \varepsilon\right) |z_i - z_i'|,$$

we get that

$$\begin{aligned} |\langle z_1 - z_2, e_1\rangle| &\le |\langle z_1 - z_1', e_1\rangle| + |\langle z_2' - z_2, e_1\rangle| + |\langle z_1' - z_2', e\rangle| \\ &\le \left(\frac{1}{2} + \sigma + 2\varepsilon\right) (|z_1 - z_1'| + |z_2 - z_2'|) + \varepsilon |z_1 - z_2|. \end{aligned} \tag{3.6}$$

By Lemma 3.8, we have that

$$\left\langle z_1 - z_1', \frac{x_2}{|x_2|} \right\rangle \ge (1 - \sigma - \varepsilon) |z_1 - z_1'|$$

and

$$\left\langle z_2 - z_2', \frac{x_3}{|x_3|} \right\rangle \ge (1 - \sigma - \varepsilon) |z_2 - z_2'|.$$

Applying Lemma 3.8 with $\langle x_2/|x_2|, x_3/|x_3|\rangle \le -1/2 + 2\varepsilon$, we get that

$$\begin{aligned} |z_1 - z_1'| + |z_2 - z_2'| &\le \left(\frac{2}{1 + 1/2 - 2\varepsilon - 10\sqrt{\sigma + \varepsilon}}\right)^{1/2} |(z_1 - z_1') - (z_2 - z_2')| \\ &\le \frac{2}{\sqrt{3}} \left(1 - \frac{2\varepsilon + 10\sqrt{\sigma + \varepsilon}}{3}\right) |z_1 - z_2|. \end{aligned}$$

We get, from (3.4), that

$$|\langle z_1 - z_2, e_1\rangle| \le \frac{2}{3}|z_1 - z_2|.$$

For any $z \in \Omega_i$, we now let $p_0(z)$ be the unique point in $\partial\Omega_i$ such that $p_0(z) - z$ parallels e; and for $z \in R_1$, we let $p_0(z) = z$. Then $p_0(L) \subset L$. We will check that

$$p_0 \text{ is Lipschitz with } \operatorname{Lip}(p_0) \le 6.$$

For any $z_1, z_2 \in \Omega_i$, we put

$$p_0(z_j) = z_j + t_j e_i,\ t_i \in \mathbb{R},\ j = 1, 2,$$

then

$$\begin{aligned}|t_1 - t_2| &= |\langle (t_1 - t_2)e_i, e_i\rangle| \\ &\le |\langle p_0(z_1) - p_0(z_2), e_i\rangle| + |\langle z_1 - z_2, e_i\rangle| \\ &\le \frac{2}{3}|p_0(z_1) - p_0(z_2)| + |z_1 - z_2|,\end{aligned}$$

and

$$|p_0(z_1) - p_0(z_2)| \le |z_1 - z_2| + |t_1 - t_2| \le \frac{2}{3}|p_0(z_1) - p_0(z_2)| + 2|z_1 - z_2|,$$

thus

$$|p_0(z_1) - p_0(z_2)| \le 6|z_1 - z_2|.$$

By the definition of R^x and R_1^x, we have that

$$R^x = \{z \in R_1^x \mid \operatorname{dist}(z, D_{x,y_r}) \le \sigma \operatorname{dist}(z, \ell(x))\}.$$

Similar as above, we can that, for any $z_1, z_2 \in R_1^x \cap \partial R^x$ with $[z_1, z_2] \cap D_{x,y_r} = \emptyset$, if $\operatorname{card}(\mathcal{X}_r) = 2$ then

$$|\langle z_1 - z_2, e_i\rangle| \le 5(\sigma + \varepsilon)|z_1 - z_2|;$$

if $\operatorname{card}(\mathcal{X}_r) = 3$ then

$$|\langle z_1 - z_2, e_i\rangle| \le \left(\frac{1}{2} + \sigma + 3\varepsilon\right)|z_1 - z_2|,$$

where e_i is the vector in 3.4 such that $z_1, z_2 \in \Omega_i$.

We now consider the mapping $p_1 : R_1 \to R$ defined by

$$p_1(z) = \begin{cases} z, & \text{for } z \in R, \\ z - te_i \in \partial R \cap \Omega_i, & \text{for } z \in \Omega_i. \end{cases}$$

By the same reason as above, we get that

$$\operatorname{Lip}(p_1) \le \frac{2}{1 - 1/2 - \sigma - 3\varepsilon} \le 5.$$

We define a mapping $p_2 : R \cap \overline{B(0,1)} \to \Sigma$ as follows: we know Σ_u^x is the graph of u over D_{x,y_r}, thus for any $z \in R^x$, there is only one point in the intersection of Σ_u^x and the line which is perpendicular to D_{x,y_r} and through z, we define $p_2(z)$ to be the unique intersection point. That is, $p_2(z)$ is the unique point in Σ_u^x such that $p_2(z)-z$ is perpendicular to D_{x,y_r}. We will show that p_2 is Lipschitz and $\mathrm{Lip}(p_2)\leq 1+10^4\eta$. Indeed, for any points $z_1,z_2\in R^x$, we let $\widetilde{z}_i$, $i=1,2$, be the points in D_{x,y_r} such that $z_i-\widetilde{z}_i$ is perpendicular to D_{x,y_r}, then

$$|(p_2(z_1)-z_1)-(p_2(z_2)-z_2)|=|u(\widetilde{z}_1)-u(\widetilde{z}_2)|\leq \mathrm{Lip}(u)|\widetilde{z}_1-\widetilde{z}_2|\leq \mathrm{Lip}(u)|z_1-z_2|,$$

thus

$$|p_2(z_1)-p_2(z_2)|\leq (1+\mathrm{Lip}(u))|z_1-z_2|\leq (1+10^4\eta)|z_1-z_2|.$$

Let $p_3:\mathbb{R}^3\to\mathbb{R}^3$ be the mapping defined by

$$p_3(x)=\begin{cases} x, & |x|\leq 1\\ \frac{x}{|x|}, & |x|>1.\end{cases}$$

Then $p=p_3\circ p_2\circ p_3\circ p_1\circ p_0$ is our desire mapping. □

Lemma 3.10. *For any $r\in(0.\mathfrak{r})\cap\mathscr{R}_1$, we let Σ be as in (3.4), and let Σ_r be given by $\boldsymbol{\mu}_r(\Sigma)$. Then we have that*

$$\mathcal{H}^2(E\cap B(0,r))\leq \mathcal{H}^2(\Sigma_r)+2550\int_{E\cap\partial B(0,r)}\mathrm{dist}(z,\Sigma_r)d\mathcal{H}^1(z)+(2r)^2h(2r).$$

Proof. For any $\xi>0$, we consider the function $\psi_\xi:[0,\infty)\to\mathbb{R}$ defined by

$$\psi_\xi(t)=\begin{cases} 1, & 0\leq t\leq 1-\xi\\ -\frac{t-1}{\xi}, & 1-\xi<t\leq 1\\ 0, & t>1,\end{cases}$$

and the mapping $\phi_\xi:\Omega_0\to\Omega_0$ defined by

$$\phi_\xi(z)=\psi_\xi(|z|)p(z)+(1-\psi_\xi(|z|))z.$$

Then we get that $\phi_\xi(L)\subset L$. For any $t\in[0,1]$, we put

$$\varphi_t(z)=tr\phi_\xi\left(z/r\right)+(1-t)z,\ \text{for } z\in\Omega_0.$$

Then $\{\varphi_t\}_{0\leq t\leq 1}$ is a sliding deformation, and we get that

$$\mathcal{H}^2(E\cap\overline{B(0,r)})\leq\mathcal{H}^2(\varphi_1(E)\cap\overline{B(0,r)})+(2r)^2h(2r).$$

Since $\psi_\xi(t)=1$ for $t\in[0,1-\xi]$, we get that

$$\varphi_1(E\cap B(0,(1-\xi)r))=p(E\cap B(0,(1-\xi)r))\subset\Sigma_r.$$

We set $A_\xi=B(0,r)\setminus B(0,(1-\xi)r)$. By Theorem 3.2.22 in [7], we get that

$$\mathcal{H}^2(\varphi_1(E\cap A_\xi))\leq\int_{E\cap A_\xi}\mathrm{ap}\,J_2(\varphi_1|_E)(z)d\mathcal{H}^2(z). \tag{3.7}$$

For any $z \in A_\xi$ and $v \in \mathbb{R}^3$, we have, by setting $z' = z/r$, that

$$D\varphi_1(z)v = \psi_\xi(|z'|)Dp(z')v + (1 - \psi_\xi(|z'|))v + \psi'_\xi(|z'|)\langle z/|z|, v\rangle(rp(z') - z).$$

For any $z \in A_\xi \cap E$, we let $v_1, v_2 \in T_z E$ be such that

$$|v_1| = |v_2| = 1,\ v_1 \perp z \text{ and } v_2 \perp v_1,$$

then we have that $\langle z/|z|, v\rangle = \cos\theta(z)$, and that

$$|\psi_\xi(|z'|)Dp(z')v_i + (1 - \psi_\xi(|z'|))v_i| \leq |Dp(z')v_i| \leq \mathrm{Lip}(p),$$

thus

$$\begin{aligned} \mathrm{ap}\, J_2(\varphi_1|_E)(z) &= |D\varphi_1(z)v_1 \wedge D\varphi_1(z)v_2| \\ &\leq \mathrm{Lip}(p)^2 + \frac{1}{\xi}\mathrm{Lip}(p)\cos\theta(z)|rp(z') - z|. \end{aligned} \tag{3.8}$$

Since $p(\widetilde{z}) = \widetilde{z}$ for any $\widetilde{z} \in \Sigma$, we have that

$$|p(z') - z'| = |p(z') - p(\widetilde{z}) + \widetilde{z} - z'| \leq (\mathrm{Lip}(p) + 1)|\widetilde{z} - z'|,$$

then we get that

$$|p(z') - z'| \leq (\mathrm{Lip}(p) + 1)\,\mathrm{dist}(z, \Sigma).$$

We now get, from (3.4), that

$$\mathrm{ap}\, J_2(\varphi_1|_E)(z) \leq \mathrm{Lip}(p)^2 + \frac{1}{\xi}\mathrm{Lip}(p)(\mathrm{Lip}(p) + 1)\,\mathrm{dist}(z, \Sigma_r)\cos\theta(z),$$

plug that into (3.4) to get that

$$\begin{aligned} \mathcal{H}^2(\varphi_1(E \cap A_\xi)) &\leq 2500\mathcal{H}^2(E \cap A_\xi) + \frac{2550}{\xi}\int_{E \cap A_\xi}\mathrm{dist}(z, \Sigma_r)\cos\theta(z)d\mathcal{H}^2(z) \\ &\leq 2500\mathcal{H}^2(E \cap A_\xi) + \frac{2550}{\xi}\int_{(1-\xi)r}^{r}\int_{E \cap \partial B(0,t)}\mathrm{dist}(z, \Sigma_r)d\mathcal{H}^1(z)dt, \end{aligned}$$

we let $\xi \to 0+$, then we get that, for almost every r,

$$\lim_{\xi \to 0+}\mathcal{H}^2(\varphi_1(E \cap A_\xi)) \leq 2550r\int_{E \cap \partial B(0,r)}\mathrm{dist}(z, \Sigma_r)d\mathcal{H}^1(z),$$

for such r, we have that

$$\mathcal{H}^2(E \cap B(0,r)) \leq \mathcal{H}^2(\Sigma_r) + 2550r\int_{E \cap \partial B(0,r)}\mathrm{dist}(z, \Sigma_r)d\mathcal{H}^1(z) + (2r)^2 h(2r).$$

□

3.5 The main comparison statement

For any $x, y \in \Omega_0 \cap \partial B(0,1)$, if $|x - y| < 2$, we denote by $g_{x,y}$ the geodesic on $\Omega_0 \cap \partial B(0,1)$ which join x and y.

Lemma 3.11. *Let $\tau \in (0, 10^{-4})$ be a given. Then there is a constant $\vartheta > 0$ such that the following hold. Let $a \in \partial B(0,1)$ and $b, c \in L \cap \partial B(0,1)$ be such that $\operatorname{dist}(a,(0,0,1)) \le \tau$, $\operatorname{dist}(b,(1,0,0)) \le \tau$ and $\operatorname{dist}(c,(-1,0,0)) \le \tau$. Let X be the cone over $g_{a,b} \cup g_{a,c}$. Then there is a Lipschitz mapping $\varphi : \Omega_0 \to \Omega_0$ with $\varphi(E \cap L) \subset L$, $|\varphi(z)| \le 1$ when $|z| \le 1$, and $\varphi(z) = z$ when $|z| > 1$, such that*

$$\mathcal{H}^2(\varphi(X) \cap \overline{B(0,1)}) \le (1-\vartheta)\mathcal{H}^2(X) + \frac{\vartheta\pi}{2}.$$

Proof. The proof will be similar to the proof of Lemma 14.4 in [2]. □

Lemma 3.12. *Let $\tau \in (0, 10^{-4})$ be a given. Then there is a constant $\vartheta > 0$ such that the following hold. Let $a \in \partial B(0,1)$ and $b, c, d \in L \cap \partial B(0,1)$ be such that $\operatorname{dist}(a,(0,0,1)) \le \tau$, $\operatorname{dist}(b,(-1/2,\sqrt{3}/2,0)) \le \tau$, $\operatorname{dist}(c,(-1/2,-\sqrt{3}/2,0)) \le \tau$ and $\operatorname{dist}(d,(1,0,0)) \le \tau$. Let X be the cone over $g_{a,b} \cup g_{a,c} \cup g_{a,d}$. Then there is a Lipschitz mapping $\varphi : \Omega_0 \to \Omega_0$ with $\varphi(E \cap L) \subset L$, $|\varphi(z)| \le 1$ when $|z| \le 1$, and $\varphi(z) = z$ when $|z| > 1$, such that*

$$\mathcal{H}^2(\varphi(X) \cap \overline{B(0,1)}) \le (1-\vartheta)\mathcal{H}^2(X) + \frac{3\vartheta\pi}{4}.$$

Proof. The proof will be similar to the proof of Lemma 14.6 in [2]. □

Let $E \subset \Omega_0$ be a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$ and $0 \in E$. Suppose that E is locally (Ω_0, L_0, h) sliding almost minimal at 0, and that E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ at 0.

We will denote by $\mathscr{R}_2$ the set

$$\left\{ r \in \mathscr{R}_1 : \varepsilon(r) + 10 j(r)^{1/2} \le 10200^{-1}(1 - 2\cdot 10^{-4}) \right\},$$

and denote by B_t the open ball $B(0,t)$ sometimes for short for any $t > 0$.

Lemma 3.13. *For any $r \in (0, \mathfrak{r}) \cap \mathscr{R}_2$, we have that*

$$\begin{aligned}\mathcal{H}^2(E \cap B_r) \le (1 - 2\cdot 10^{-4})\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + (2\cdot 10^{-4} - \vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1)\\ + \vartheta\kappa^2 r^2 \Theta(0) + (2r)^2 h(2r).\end{aligned}$$

Proof. Without loss of generality, we assume that $x = 0$. Let Σ, Σ_r, ξ, ψ_ξ, ϕ_ξ and $\{\varphi_t\}_{0\le t\le 1}$ be the same as in the proof of Lemma 3.10.

We see that

$$\varphi_1(E \cap B(0,(1-\xi)r)) = p(E \cap B(0,(1-\xi)r)) \subset \Sigma_r,$$

and that $\Sigma \cap B(0, 2\kappa) = X \cap B(0, 2\kappa)$, where X is a cone defined in (3.4). We see that if $\Theta(0) = \pi/2$, then X satisfies the conditions in Lemma 3.11; if $\Theta(0) = 3\pi/4$, then X satisfies the conditions in Lemma 3.12. Thus we can find a Lipschitz mapping $\Omega_0 \to \Omega_0$ with $\varphi(E \cap L) \subset L$, $|\varphi(z)| \le 1$ when $|z| \le 1$, and $\varphi(z) = z$ when $|z| > 1$, such that

$$\mathcal{H}^2(\varphi(X) \cap \overline{B(0,1)}) \le (1-\vartheta)\mathcal{H}^2(X \cap B(0,1)) + \vartheta\Theta(x).$$

Let $\widetilde{\varphi}:\Omega_0\to\Omega_0$ be the mapping defined by $\widetilde{\varphi}(x)=r\varphi(x/r)$, then

$$\begin{aligned}\mathcal{H}^2(E\cap B(0,r))&\le \mathcal{H}^2(\widetilde{\varphi}\circ\varphi_1(E)\cap\overline{B(0,r)})+(2r)^2h(2r)\\ &\le \mathcal{H}^2(\widetilde{\varphi}\circ\varphi_1(E\cap B(0,(1-\xi)r)))+\mathcal{H}^2(\varphi_1(E\cap A_\xi))\\ &\le \mathcal{H}^2(\Sigma_r\setminus\overline{B(0,\kappa r)})+(1-\vartheta)(\kappa r)^2\mathcal{H}^2(X\cap B(0,1))\\ &\quad+\vartheta\cdot(\kappa r)^2\Theta(0)+\mathcal{H}^2(\varphi_1(E\cap A_\xi)).\end{aligned}$$

But we see that $\Sigma_r=\{rx:x\in\Sigma\}$, $\Sigma\cap B(0,2\kappa)=X\cap B(0,2\kappa)$, and

$$\lim_{\xi\to 0+}\mathcal{H}^2(\varphi_1(E\cap A_\xi))\le 2550\int_{E\cap\partial B(0,r)}\mathrm{dist}(z,\Sigma_r)d\mathcal{H}^1(z),$$

we have that

$$\mathcal{H}^2(\Sigma_r\setminus\overline{B(0,\kappa r)})=r^2\left(\mathcal{H}^2(\Sigma)-\mathcal{H}^2(X\cap B(0,\kappa))\right),$$

and that

$$\begin{aligned}\mathcal{H}^2(E\cap B(0,r))\le r^2\mathcal{H}^2(\Sigma)&-(\kappa r)^2\mathcal{H}^2(X\cap B(0,1))\\ &+(1-\vartheta)(\kappa r)^2\mathcal{H}^2(X\cap B(0,1))+(\kappa r)^2\vartheta\cdot\Theta(0)\\ &+2550\int_{E\cap\partial B(0,r)}\mathrm{dist}(z,\Sigma_r)d\mathcal{H}^1(z)+(2r)^2h(2r).\end{aligned}$$

By (3.4), we get that

$$\begin{aligned}\mathcal{H}^2(\Sigma)&\le\mathcal{H}^2(\mathcal{M})-10^{-4}(\mathcal{H}^1(\Gamma_*)-T)\\ &=(1/2-10^{-4})\mathcal{H}^1(\Gamma_*)+10^{-4}\mathcal{H}^1(X\cap\partial B(0,1)),\end{aligned}$$

and then

$$\begin{aligned}\mathcal{H}^2(E\cap B_r)\le(1/2-10^{-4})r^2\mathcal{H}^1(\Gamma_*)&+(10^{-4}-\vartheta\kappa^2/2)r^2\mathcal{H}^1(X\cap\partial B_1)\\ &+\vartheta\kappa^2r^2\Theta(0)+2550\int_{E\cap\partial B_r}\mathrm{dist}(z,\Sigma_r)d\mathcal{H}^1(z)+(2r)^2h(2r).\end{aligned}$$

For any $\varepsilon>\varepsilon(r)$, there exists cone Z of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ such that

$$d_{0,r}(E,Z)\le\varepsilon,$$

by the construction of X and $\mathcal{M}$, we see that

$$d_{0,r}(X,Z)\le\varepsilon,$$

thus

$$d_{0,r}(E,X)\le 2\varepsilon.$$

By Lemma ??, we have that

$$d_{0,1}(\mathcal{M},X)\le 10j(r)^{1/2}.$$

We get that for any $z\in E\cap\partial B(0,r)$,

$$\mathrm{dist}(\boldsymbol{\mu}_{1/r}(z),\mathcal{M})\le 2\varepsilon(r)+10j(r)^{1/2},$$

thus

$$\operatorname{dist}(z,\Sigma_r)=r\operatorname{dist}(\boldsymbol{\mu}_{1/r}(z),\Sigma)=r\operatorname{dist}(\boldsymbol{\mu}_{1/r}(z),\mathcal{M})\le 2r\varepsilon(r)+20rj(r)^{1/2},$$

because $\Sigma\setminus B(0,9/10)=\mathcal{M}\setminus B(0,9/10)$. We get that

$$\begin{aligned}\int_{E\cap\partial B(0,r)}\operatorname{dist}(z,\Sigma_r)d\mathcal{H}^1(z)&\le 2r(\varepsilon(r)+10j(r)^{1/2})\mathcal{H}^1(E\cap\partial B(0,r)\setminus\Sigma_r)\\&\le 2r(\varepsilon(r)+20j(r)^{1/2})(\mathcal{H}^1(E\cap\partial B_r)-r\mathcal{H}^1(\Gamma_*)).\end{aligned}$$

By Lemma 3.6, we have that

$$\mathcal{H}^1(\Gamma_*\setminus\Gamma)\le\mathcal{H}^1(\Gamma\setminus\Gamma_*)\le C\eta^2(\mathcal{H}^1(\Gamma)-\mathcal{H}^1(X\cap\partial B(0,1))),$$

so that

$$\mathcal{H}^1(X\cap\partial B(0,1))\le\mathcal{H}^1(\Gamma_*)\le\mathcal{H}^1(\Gamma)\le\mathcal{H}^1(\boldsymbol{\mu}_{1/r}(E\cap\partial B_r)),$$

thus

$$\begin{aligned}\mathcal{H}^2(E\cap B_r)\le{}&(1/2-10^{-4})r^2\mathcal{H}^1(\Gamma_*)+(10^{-4}-\vartheta\kappa^2/2)r^2\mathcal{H}^1(X\cap\partial B_1)\\&+5100(\varepsilon(r)+10j(r)^{1/2})r(\mathcal{H}^1(E\cap\partial B_r)-r\mathcal{H}^2(\Gamma_*))\\&+\vartheta\kappa^2r^2\Theta(0)+(2r)^2h(2r).\end{aligned}$$

Since $r\in(0,\mathfrak{r})\cap\mathscr{R}_2$, we have that

$$5100\left(\varepsilon(r)+10j(r)^{1/2}\right)\le\frac{1}{2}(1-2\cdot10^{-4})$$

thus

$$\begin{aligned}\mathcal{H}^2(E\cap B_r)\le{}&(1-2\cdot10^{-4})\frac{r}{2}\mathcal{H}^1(E\cap\partial B_r)+(2\cdot10^{-4}-\vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X\cap\partial B_1)\\&+\vartheta\kappa^2r^2\Theta(0)+(2r)^2h(2r).\end{aligned}$$

□

Theorem 3.14. *There exist $\lambda,\mu\in(0,10^{-3})$ and $\mathfrak{r}_1>0$ such that, for any $0<r<\mathfrak{r}_1$,*

$$\mathcal{H}^2(E\cap B_r)\le(1-\mu-\lambda)\frac{r}{2}\mathcal{H}^1(E\cap\partial B_r)+\mu\frac{r^2}{2}\mathcal{H}^1(X\cap\partial B_1)+\lambda\Theta(0)r^2+4r^2h(2r).$$

Proof. We put $\tau_1=\min\{\tau_0,10^{-12}(1-\vartheta\kappa^2)^2\}$, and take δ such that

$$\kappa<\delta<\kappa+(8\vartheta)^{-1}(1-2\cdot10^{-4})\Theta(0)\tau_1. \tag{3.9}$$

We see that $\varepsilon(r)\to0$ as $r\to0+$, there exist $\mathfrak{r}_1\in(0,\mathfrak{r})$ such that, for any $r\in(0,\mathfrak{r}_1)$,

$$\varepsilon(r)\le10^{-1}\min\{\tau_1,\vartheta(\delta^2-\kappa^2)\}. \tag{3.10}$$

If $r\in(0,\mathfrak{r}_1)$ and $j(r)\le\tau_1$, then $r\in\mathscr{R}_2$, then by Lemma 3.13, we have that

$$\begin{aligned}\mathcal{H}^2(E\cap B_r)\le{}&(1-2\cdot10^{-4})\frac{r}{2}\mathcal{H}^1(E\cap\partial B_r)+(2\cdot10^{-4}-\vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X\cap\partial B_1)\\&+\vartheta\kappa^2r^2\Theta(0)+(2r)^2h(2r).\end{aligned}$$

We only need to consider the case $r \in (0, \mathfrak{r}_1)$, $j(r) > \tau_1$ and $\mathcal{H}^1(E \cap \partial B_r) < +\infty$, thus

$$\mathcal{H}^1(X \cap \partial B_1) + \tau_1 \le \frac{1}{r}\mathcal{H}^1(E \cap B(0,r)). \tag{3.11}$$

By the construction of X, we see that $X \cap B(0,1)$ is Lipschitz neighborhood retract, let U be a neighborhood of $X \cap B(0,1)$ and $\varphi_0 : U \to X \cap B(0,1)$ be a retraction such that $|\varphi_0(x) - x| \le r/2$. We put $U_1 = \boldsymbol{\mu}_{8r/9}(U)$, $\varphi_1 = \boldsymbol{\mu}_{8r/9} \circ \varphi_0 \circ \boldsymbol{\mu}_{9/(8r)}$, and let $s : [0,\infty) \to [0,1]$ be a function given by

$$s(t) = \begin{cases} 1, & 0 \le t \le 3r/4, \\ -(8/r)(t - 7r/8), & 3r/4 < t \le 7r/8, \\ 0, & t > 7r/8. \end{cases}$$

We see that there exist sliding minimal cone Z such that $d_{0,1}(X,Z) \le \varepsilon(r)$, thus $d_{0,r}(E,X) \le 2\varepsilon(r)$, then for any $x \in E \cap B(0,r) \setminus B(0,3r/4)$,

$$\operatorname{dist}(x,X) \le 2\varepsilon(r)r \le \frac{8\varepsilon(r)}{3}|x|.$$

We consider the mapping $\psi : \Omega_0 \to \Omega_0$ defined by

$$\psi(x) = s(|x|)\varphi_1(x) + (1 - s(|x|))x,$$

then $\psi(L) = L$ and $\psi(x) = x$ for $|x| \ge 8r/9$.

We take $\mathfrak{r}_1 > 0$ such that, for any $r \in (0, \mathfrak{r}_1)$,

$$\{x \in \Omega_0 \cap B(0,1) : \operatorname{dist}(x,X) \le 3\varepsilon(r)\} \subset U.$$

Then we get that $\psi(x) \in X$ for any $x \in E \cap B(0,3r/4)$;

$$\operatorname{dist}(\psi(x),X) \le 3\varepsilon(r)|x| \text{ for any } x \in E \cap B(0,r) \setminus B(0,3r/4);$$

and $\Psi(E \cap B_r) \cap B(0,r/4) = X \cap B(0,r/4)$.

We now consider the mapping $\Pi_1 : \Omega_0 \to \Omega_0$ defined by

$$\Pi_1(x) = s(4|x|)x + (1 - s(4|x|))\Pi(x),$$

and the mapping $\psi_1 : \Omega_0 \to \Omega_0$ defined by

$$\psi_1(x) = \begin{cases} \Pi_1 \circ \psi(x), & |x| \le r, \\ x, & |x| \ge r. \end{cases}$$

We have that ψ_1 is Lipschitz, $\psi_1(L) = L$ and $\psi_1(B(0,r)) \subset \overline{B(0,r)}$,

$$\psi_1(E \cap B(0,r)) \subset X \cap B(0,r) \cup \{x \in \partial B_r : \operatorname{dist}(x,X) \le 3r\varepsilon(r)\}.$$

Let φ be the same as in Lemma 3.11 and Lemma 3.12, and let $\psi_2 = \boldsymbol{\mu}_\delta \circ \varphi \circ \boldsymbol{\mu}_{1/\delta} \circ \psi_1$. Then we have that

$$\begin{aligned}\mathcal{H}^2(E \cap \overline{B(0,r)}) &\le \mathcal{H}^2(\psi_2(E \cap \overline{B(0,r)})) + (2r)^2 h(2r) \\ &\le (1 - \vartheta\delta^2)\mathcal{H}^2(X \cap B(0,r)) + \vartheta\delta^2\Theta(0)r^2 + 4r^2h(2r) \\ &\quad + \mathcal{H}^2(\{x \in \partial B_r : \operatorname{dist}(x,X) \le 3r\varepsilon(r)\}) \\ &\le (1 - \vartheta\delta^2)\mathcal{H}^2(X \cap B(0,r)) + \vartheta\delta^2\Theta(0)r^2 \\ &\quad + 4r\varepsilon(r)\mathcal{H}^1(X \cap \partial B_r) + 4r^2h(2r) \\ &\le (1 - \vartheta\delta^2 + 8\varepsilon(r))\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \vartheta\delta^2\Theta(0)r^2 + 4r^2h(2r)\end{aligned} \tag{3.12}$$

We take $\mu = 2 \cdot 10^{-4} - \vartheta\kappa^2$ and $\lambda = \vartheta\kappa^2$, then by (3.5) and (3.5), we have that

$$8\varepsilon(r) < \vartheta(\delta^2 - \kappa^2)$$

and

$$\vartheta(\delta^2 - \kappa^2)\Theta(0) \leq (1 - 2 \cdot 10^{-4})\frac{\tau_1}{2}.$$

We get from (3.5) and (3.5) that

$$\begin{aligned}
\mathcal{H}^2(E \cap \overline{B_r}) &\leq (1 - 2 \cdot 10^{-4})\frac{r^2}{2}(\mathcal{H}^1(X \cap \partial B_1) + \tau_1) - (1 - 2 \cdot 10^{-4})\frac{\tau_1 r^2}{2} \\
&\quad + \mu\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \vartheta\kappa^2\Theta(0)r^2 + 4r^2 h(2r) \\
&\quad + (8\varepsilon(r) - \vartheta\delta^2 + \vartheta\kappa^2)\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + (\vartheta\delta^2 - \vartheta\kappa^2)\Theta(0)r^2 \\
&\leq (1 - \lambda - \mu)\frac{r}{2}\mathcal{H}^1(E \cap \partial B_r) + \mu\frac{r^2}{2}\mathcal{H}^1(X \cap \partial B_1) + \lambda\Theta(0)r^2 + 4r^2 h(2r).
\end{aligned}$$

□

For convenient, we put $\lambda_0 = \lambda/(1-\lambda)$, $f(r) = \Theta(0,r) - \Theta(0)$ and $u(r) = \mathcal{H}^1(E \cap B(0,r))$ for $r > 0$. Since $f(r) = r^{-2}u(r) - \Theta(0)$ and u is a nondecreasing function, we have that, for any $\lambda_1 \in \mathbb{R}$ and $0 < r \leq R < +\infty$,

$$R^{\lambda_1} f(R) - r^{\lambda_1} f(r) \geq \int_r^R \left(t^{\lambda_1} f(t)\right)' dt,$$

thus

$$f(r) \leq r^{-\lambda_1} R^{\lambda_1} f(R) + r^{-\lambda_1} \int_r^R \left(t^{\lambda_1} f(t)\right)' dt. \tag{3.13}$$

Corollary 3.15. *If the gauge function h satisfy*

$$h(t) \leq C_h t^{\alpha},\ 0 < t \leq \mathfrak{r}_1 \text{ for some } C_h > 0,\ \alpha > 0,$$

then for any $0 < \beta < \min\{\alpha, 2\lambda_0\}$, there is a constant $C = C(\lambda_0, \alpha, \beta, \mathfrak{r}_1, C_h) > 0$ such that

$$|\Theta(0,\rho) - \Theta(0)| \leq C\rho^{\beta} \tag{3.14}$$

for any $0 < \rho \leq \mathfrak{r}_1$.

Proof. For any $r > 0$, we put $u(r) = \mathcal{H}^2(E \cap B(0,r))$. Then u is differentiable for $\mathcal{H}^1$-a.e. $r \in (0, \infty)$.

Recall that $\mathscr{R}$ is the set $r \in (0,\infty)$ such that $\mathcal{H}^1(E \cap B(0,r)) < \infty$ and u is differentiable at r, and we have that $\mathcal{H}^1((0,\infty) \setminus \mathscr{R}) = 0$.

By Theorem 3.14 and Lemma 2.1, we have that for any $r \in (0, \mathfrak{r}_1) \cap \mathscr{R}$,

$$\begin{aligned}
u(r) &\leq (1-\lambda)\frac{r}{2}\mathcal{H}^1(E \cap \partial B(0,r)) + \lambda\Theta(0)r^2 + 4r^2 h(2r) \\
&\leq (1-\lambda)\frac{r}{2}u'(r) + \lambda\Theta(0)r^2 + 4r^2 h(2r),
\end{aligned}$$

thus

$$rf'(r) \geq \frac{2\lambda}{1-\lambda}f(r) - \frac{8}{1-\lambda}h(2r) = 2\lambda_0 f(r) - 8(1+\lambda_0)h(2r),$$

and

$$\left(r^{-2\lambda_0}f(r)\right)' = r^{-1-2\lambda_0}\left(rf'(r)-2\lambda_0\right) \geq -8(1+\lambda_0)r^{-1-2\lambda_0}h(2r).$$

We get, from (3.5), so that, for any $0<r<R\leq \mathfrak{r}_1$,

$$f(r) \leq r^{2\lambda_0}R^{-2\lambda_0}f(R) + 8(1+\lambda_0)r^{2\lambda_0}\int_r^R t^{-1-2\lambda_0}h(2t)dt. \tag{3.15}$$

Since $h(t)\leq C_h t^\alpha$, we have that

$$f(r) \leq (r/R)^{-2\lambda_0}f(R) + 2^{3+\alpha}(1+\lambda_0)C_h r^{2\lambda_0}\int_r^R t^{\alpha-2\lambda_0-1}dt.$$

If $\alpha > 2\lambda_0$, then

$$f(r) \leq \left(f(R) + 2^{3+\alpha}(1+\lambda_0)(1+\lambda_0)(\alpha-2\lambda_0)^{-1}C_h R^\alpha\right)(r/R)^{2\lambda_0}; \tag{3.16}$$

if $\alpha = 2\lambda_0$, then

$$f(r) \leq f(R)(r/R)^\alpha + 2^{\alpha+3}(1+\lambda_0)C_h r^\alpha \ln(R/r),$$

thus, for any $\beta\in(0,\alpha)$,

$$\begin{aligned} f(r) &\leq f(R)r^\alpha + 2^{\alpha+3}(1+\lambda_0)C_h r^\beta R^{\alpha-\beta}\frac{\ln(R/r)}{(R/r)^{\alpha-\beta}} \\ &\leq \left(f(R) + 2^{\alpha+3}(1+\lambda_0)C_h(\alpha-\beta)^{-1}e^{-1}R^\alpha\right)(r/R)^\beta; \end{aligned} \tag{3.17}$$

if $\alpha < 2\lambda_0$, then

$$\begin{aligned} f(r) &\leq f(R)(r/R)^{2\lambda_0} + 2^{\alpha+3}(1-\lambda_0)C_h r^{2\lambda_0}\cdot(2\lambda_0-\alpha)^{-1}\left(r^{\alpha-2\lambda_0}-R^{\alpha-2\lambda_0}\right) \\ &\leq \left((r/R)^{2\lambda_0-\alpha}f(R) + 2^{\alpha+3}(1-\lambda_0)C_h(2\lambda_0-\alpha)^{-1}R^\alpha\right)(r/R)^\alpha. \end{aligned} \tag{3.18}$$

Hence (3.15) follows from (3.5), (3.5), (3.5) and Theorem 2.3. Indeed, there is a constant $C_1(\alpha,\beta,\lambda_0)>0$ such that

$$r^{2\lambda_0}\int_r^R t^{\alpha-2\lambda_0-1}dt \leq C_1(\alpha,\beta,\lambda_0)R^\alpha\cdot(r/R)^\beta, \tag{3.19}$$

and there is a constant $C_2(\alpha,\beta,\lambda_0)>0$ such that

$$f(r) \leq \left(f(R) + C_2(\alpha,\beta,\lambda_0)C_h\cdot R^\alpha\right)(r/R)^\alpha.$$

□

Remark 3.16. *If the gauge function h satisfy that*

$$h(t) \leq C\left(\ln\left(\frac{A}{t}\right)\right)^{-b}$$

for some $A,b,C>0$, then (3.5) implies that there exist $R>0$ and constant $C(R,\lambda,b)$ such that

$$f(r) \leq C(R,\lambda,b)\left(\ln\left(\frac{A}{r}\right)\right)^{-b} \text{ for } 0<r\leq R.$$

4 Approximation of E by cones

In this section, we also assume that $E \subset \Omega_0$ is a 2-rectifiable set with $\mathcal{H}^2(E) < \infty$ and $0 \in E$, and that E is locally (Ω_0, L_0, h) sliding almost minimal at 0. Suppose in addition that E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$ or $\mathbb{Y}_+$ at 0. We let $\varepsilon(r) = \varepsilon_P(r)$ if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{P}_+$; and let $\varepsilon(r) = \varepsilon_Y(r)$ if E is locally C^0-equivalent to a sliding minimal cone of type $\mathbb{Y}_+$.

For any $r > 0$, we put

$$f(r) = \Theta(0,r) - \Theta(0),\ \ F(r) = f(r) + 8h_1(r),\ \ F_1(r) = F(r) + 8h_1(r),$$

and put

$$\Xi(r) = rf'(r) + 2f(r) + 16h(2r) + 32h_1(r),$$

if $r \in \mathscr{R}$.

We denote by $X(r)$ and $\Gamma(r)$, respectively, the cone X and the set Γ which are defined in (3.4), and by $\gamma(r)$ the set $\boldsymbol{\mu}_r(\Gamma(r))$. For any $r_2 > r_1 > 0$, we put

$$A(r_1, r_2) = \{x \in \mathbb{R}^3 : r_1 \leq |x| \leq r_2\}.$$

Lemma 4.1. *For any $0 < r < R < \infty$ with $\mathcal{H}^2(E \cap \partial B_r) = \mathcal{H}^2(E \cap \partial B_R) = 0$, we have that*

$$\int_{E\cap A(r,R)} \frac{1 - \cos\theta(x)}{|x|^2} d\mathcal{H}^2(x) \leq F(R) - F(r), \tag{4.1}$$

and

$$\mathcal{H}^2\left(\Pi(E \cap A(r,R))\right) \leq \int_{E\cap A(r,R)} \frac{\sin\theta(x)}{|x|^2} d\mathcal{H}^2(x). \tag{4.2}$$

Proof. We see that for $\mathcal{H}^2$-a.e. $x \in E$, the tangent plane $\mathrm{Tan}^2(E, x)$ exists, we will denote by $\theta(x)$, the angle between the line $[0, x]$ and the plane $\mathrm{Tan}^2(E, x)$. For any $t > 0$, we put $u(t) = \mathcal{H}^2(E \cap B(0,t))$, then $u : (0, \infty) \to [0, \infty]$ is a nondecreasing function. By Lemma 2.2, we have that

$$u(t) \leq \frac{t}{2}\mathcal{H}^1(E \cap \partial B(0,t)) + 4t^2 h(2t),$$

for $\mathcal{H}^1$-a.e. $t \in (0, \infty)$. Considering the mapping $\phi : \mathbb{R}^3 \to [0, \infty)$ given by $\phi(x) = |x|$, we have, by (2), that

$$\mathrm{ap}\, J_1(\phi|_E)(x) = \cos\theta(x)$$

for $\mathcal{H}^2$-a.e. $x \in E$.

Apply Theorem 3.2.22 in [7], we get that

$$\begin{aligned}
&\int_{E\cap A(r,R)} \frac{1}{|x|^2}\cos\theta(x)d\mathcal{H}^2(x) = \int_r^R \frac{1}{t^2}\mathcal{H}^1(E\cap\partial B(0,t)ddt\\
&\geq 2\int_r^R \frac{u(t)}{t^3}dt - 8\int_r^R \frac{h(2t)}{t}dt\\
&= 2\int_r^R \frac{1}{t^3}\int_{E\cap B(0,t)} d\mathcal{H}^2(x)dt - 8(h_1(R)-h_1(r))\\
&= 2\int_{E\cap B(0,R)}\int_{\max\{r,|x|\}}^R \frac{1}{t^3}dtd\mathcal{H}^2(x) - 8(h_1(R)-h_1(r))\\
&= \int_{E\cap A(r,R)} \frac{1}{|x|^2}d\mathcal{H}^2(x) + r^{-2}u(r) - R^{-2}u(R) - 8(h_1(R)-h_1(r)),
\end{aligned}$$

thus (4.1) holds.

By a simple computation, we get that

$$\operatorname{ap} J_2\Pi(x) = \frac{\sin\theta(x)}{|x|^2},$$

we now apply Theorem 3.2.22 in [7] to get (4.1). □

We get from above Lemma that

$$\mathcal{H}^2(\Pi(E\cap A(r,R))) \leq \frac{r_2}{r_1}\left(2\Theta(0,R)\right)^{1/2}\left(F(R)-F(r)\right)^{1/2}$$

Lemma 4.2. *For any $r\in(0,\mathfrak{r}_1)\cap\mathscr{R}$, if $\Xi(r)\leq\mu\tau_0$, then*

$$d_H(\Gamma(r), X(r)\cap\partial B(0,1)) \leq \mu^{-1/2}\Xi(r)^{1/2}.$$

Proof. By lemma 2.1, we get that

$$\frac{1}{r}\mathcal{H}^1(E\cap\partial B(0,r)) \leq 2\Theta(0) + rf'(r) + 2f(r),$$

By Theorem 3.14, we get that

$$\begin{aligned}
r^2\Theta(0,r) &\leq (1-\lambda-\mu)\frac{r}{2}\mathcal{H}^1(E\cap\partial B_r) + \mu\frac{r^2}{2}\mathcal{H}^1(X\cap\partial B_1) + \lambda\Theta(0)r^2 + 4r^2h(2r)\\
&\leq \frac{1}{2}(1-\lambda-\mu)r^2(2\Theta(0)+rf'(r)+2f(r)) + \mu\frac{r^2}{2}\mathcal{H}^1(X\cap\partial B_1)\\
&\quad + \lambda\Theta(0)r^2 + 4r^2h(2r),
\end{aligned}$$

thus

$$\mathcal{H}^1(X\cap\partial B_1) \geq 2\Theta(0) + \frac{2(\lambda+\mu)}{\mu}f(r) - \frac{1-\lambda-\mu}{\mu}rf'(r) - \frac{\mu}{8}h(2r).$$

Hence

$$\begin{aligned}
j(r) &= \frac{1}{r}\mathcal{H}^1(E\cap B_r) - \mathcal{H}^1(X\cap\partial B_1)\\
&\leq \frac{1-\lambda}{\mu}rf'(r) - \frac{2\lambda}{\mu}f(r) + \frac{8}{\mu}h(2r)\\
&\leq \frac{1}{\mu}(rf'(r) + 16h_1(r) + 16h(2r)).
\end{aligned}$$

Since

$$\mathcal{H}^1(X \cap \partial B_1) \leq \mathcal{H}^1(\Gamma_*(r)) \leq \mathcal{H}^1(\Gamma(r)) \leq \mathcal{H}^1(\boldsymbol{\mu}_{1/r}(E \cap \partial B_r)),$$

we have that

$$0 \leq \mathcal{H}^1(\Gamma(r)) - \mathcal{H}^1(X \cap B_1) \leq j(r) \leq \frac{1}{\mu}\Xi(r),$$

by Lemma 3.5, we get that for any $z \in \Gamma(r)$,

$$\operatorname{dist}(z, X \cap \partial B(0,1)) \leq \left(\frac{\Xi(r)}{\mu}\right)^{1/2}.$$

□

Lemma 4.3. *For any $0 < r_1 < r_2 < (1-\tau)\mathfrak{r}$, if P is a plane such that $\mathcal{H}^1(E \cap P \cap B_{\mathfrak{r}}) < \infty$ and $P \cap \mathcal{X}_r = \emptyset$ for any $r \in [r_1, r_2]$, then there is a compact path connected set*

$$\mathcal{C}_{P,r_1,r_2} \subset E \cap P \cap A(r_2, r_1)$$

such that

$$\mathcal{C}_{P,r_1,r_2} \cap \gamma(t) \neq \emptyset \text{ for } r_1 \leq t \leq r_2.$$

Proof. We let ϱ be the same as in 3. Since $\|\Phi - \mathrm{id}\|_\infty \leq \tau\varrho$, we get that

$$\Phi^{-1}\left(E \cap \overline{B(0,r_2)}\right) \subset Z_{0,\varrho} \cap \overline{B(0, r_2 + \tau\varrho)}.$$

We put

$$\mathbb{X} = Z_{0,\varrho} \cap \overline{B(0, r_2 + \tau\varrho)},$$
$$F = \mathbb{X} \cap \Phi^{-1}(E \cap P_z).$$

We take $x_1, x_2 \in \mathcal{X}_r$, $x_2 \neq x_1$, such that $\Phi^{-1}(x_1)$ and $\Phi^{-1}(x_2)$ are contained in two different connected components of $\mathbb{X} \backslash F$. By Lemma 3.2, there is a connected closed subset F_0 of F such that $\Phi^{-1}(x)$ and $\Phi^{-1}(x_2)$ are still contained in two different connected components of $\mathbb{X} \backslash F_0$. Then $F_0 \cap \phi^{-1}(\gamma(t)) \neq \emptyset$ for $0 < t \leq r_2$; otherwise, if $F_0 \cap \phi^{-1}(\gamma(t_0)) = \emptyset$, then x_1 and x_2 are in the same connected component of $\Phi(\mathbb{X}) \backslash \Phi(F_0)$, thus $\Phi^{-1}(x_1)$ and $\Phi^{-1}(x_2)$ are in the same connected component of $\mathbb{X} \setminus F_0$, absurd!

Since $\mathcal{H}^1(\Phi(F_0)) \leq \mathcal{H}^1(E \cap P_z \cap B_\varrho) < \infty$, we get that $\Phi(F_0)$ is path connected. We take $z_1 \in \Phi(F_0) \cap \gamma(r_1)$ and $z_2 \in \Phi(F_0) \cap \gamma(r_2)$, and let $g : [0,1] \to \Phi(F_0)$ be a path such that $g(0) = z_1$ and $g(1) = z_2$. We take $t_1 = \sup\{t \in [0,1] : |g(t)| \leq r_1\}$ and $t_2 = \inf\{t \in [t_1, 1] : |g(t)| \geq r_2\}$. Then $\mathcal{C}_{z,r_1,r_2} = g([t_1, t_2])$ is our desire set. □

Lemma 4.4. *Let $T \in [\pi/4, 3\pi/4]$ and $\varepsilon \in (0, 1/2)$ be given. Suppose that F a 2-rectifiable set satisfying*

$$F \subset \partial B(0,1) \cap \{(t\cos\theta, t\sin\theta, x_3) \in \mathbb{R}^3 \mid t \geq 0, |\theta| \leq T/2, |x_3| \leq \varepsilon\}.$$

Then we have, by putting $\mathcal{P}_\theta = \{(t\cos\theta, t\sin\theta, x_3) \mid t \geq 0, x_3 \in \mathbb{R}\}$, that

$$\int_{-T/2}^{T/2} \mathcal{H}^1(F \cap \mathcal{P}_\theta) d\theta \leq (1+\varepsilon)\mathcal{H}^2(F)$$

Proof. For any $x = (x_1, x_2, x_3) \in F$, we have that $x_1^2 + x_2^2 + x_3^2 = 1$ and $|x_3| \leq \varepsilon$, thus $x_1^2 + x_2^2 \geq 1 - \varepsilon^2$. Since $|\theta| \leq T/2 \leq 3\pi/8$, we get that the mapping $\phi : F \to \mathbb{R}$ given by

$$\phi(x_1, x_2, x_3) = \arctan \frac{x_2}{x_1}$$

is well defined and Lipschitz. Moreover, we have that

$$\operatorname{ap} J_1 \phi(x) = (x_1^2 + x_2^2)^{-1/2} \leq (1 - \varepsilon^2)^{-1/2} \leq 1 + \varepsilon.$$

Hence

$$\int_{-T/2}^{T/2} \mathcal{H}^1(F \cap \mathcal{P}_\theta) d\theta = \int_F \operatorname{ap} J_1 \phi(x) d\mathcal{H}^2(x) \leq (1 + \varepsilon)\mathcal{H}^2(F).$$

□

For any $0 < t_1 \leq t_2$, we put

$$E_{t_1,t_2} = \Pi \left(\{x \in E : t_1 \leq |x| \leq t_2\} \right).$$

For any $t > 0$, we put

$$\bar{\varepsilon}(t) = \sup\{\varepsilon(r) : r \leq t\}.$$

Lemma 4.5. *If $r_2 > r_1 > 0$ satisfy that $8(1 + r_2/r_1)\bar{\varepsilon}(r_2) < 1/2$, then we have that*

$$\int_{X(t) \cap \partial B(0,1)} \mathcal{H}^1 \left(P_z \cap E_{r_1,r_2} \right) d\mathcal{H}^1(z) \leq 2\mathcal{H}^2 \left(E_{r_1,r_2} \right), \ \forall r_1 \leq t \leq r_2.$$

Proof. For any $\delta > 0$, we can find sliding minimal cone Z in Ω_0 with sliding boundary L such that

$$d_{0,r}(E, Z) \leq 2\varepsilon(r),$$

thus

$$d_{0,r}(X(r), Z) \leq 2\varepsilon(r)$$

and

$$d_{0,r}(E, X(r)) \leq 4\varepsilon(r).$$

We get that

$$\begin{aligned} d_{0,t}(X(t), X(r_2)) &\leq d_{0,t}(E, X(t)) + d_{0,t}(E, X(r_2)) \\ &\leq 4\bar{\varepsilon}(r_2) + 4\frac{r_2}{t}\bar{\varepsilon}(r_2). \end{aligned}$$

But

$$\operatorname{dist}(x, X(r_2)) \leq 4r_2\varepsilon(r_2), \text{ for any } x \in E \cap B(0, r_2),$$

and

$$\operatorname{dist}(\Pi(x), X(r_2)) \leq \frac{4r_2\varepsilon(r_2)}{|x|}, \text{ for any } x \in E \cap A(r_1, r_2),$$

we get so that

$$\operatorname{dist}(\Pi(x), X(t)) \leq (8r_2/r_1 + 4)\bar{\varepsilon}(r_2) < \frac{1}{2}.$$

We now apply Lemma 4.4 to get the result. □

Lemma 4.6. *Let $\varepsilon \in (0,1/2)$ be given. Let $A \subset \partial B(0,1)$ be an arc of a great circle such that $0 < \mathcal{H}^1(A) \leq \pi$ and*

$$\operatorname{dist}(x, L) \leq \varepsilon, \forall x \in A.$$

Then there is a constant $C > 0$ such that

$$\operatorname{dist}(x, L) \leq \frac{\pi^2}{2\mathcal{H}^1(A)^2} \int_A \operatorname{dist}(x, L) d\mathcal{H}^1(x), \ \forall x \in A.$$

Proof. We let P be the plane such that $A \subset P$, let $v_0 \in P \cap L \cap \partial B(0,1)$ and $v_2 \in P \cap \partial B(0,1)$ be two vectors such that v_0 is perpendicular to v_1. Then A can be parametrized as $\gamma : [\theta_1, \theta_2] \to A$ given by

$$\gamma(t) = v_0 \cos t + v_1 \sin t,$$

where $\theta_2 - \theta_1 = \mathcal{H}^1(A)$. We write $v_1 = w + w^\perp$ with $w \in L$ and $w^\perp$ perpendicular to L. Since $\operatorname{ap} J_1 \gamma(t) = 1$ for any $t \in [\theta_1, \theta_2]$, by Theorem 3.2.22 in [7], we have that

$$\begin{aligned}\int_A \operatorname{dist}(x, L)\mathcal{H}^1(x) &= \int_{\theta_1}^{\theta_2} \operatorname{dist}(\gamma(t), L) dt = \int_{\theta_1}^{\theta_2} |w^\perp \sin t| dt \\ &\geq 2|w^\perp| \left(1 - \cos \frac{\theta_2 - \theta_1}{2}\right) \geq \frac{2(\theta_2 - \theta_1)^2}{\pi^2} |w^\perp|,\end{aligned}$$

and that

$$\operatorname{dist}(x, L) \leq |w^\perp| \leq \frac{\pi^2}{2\mathcal{H}^1(A)^2} \int_A \operatorname{dist}(x, L) d\mathcal{H}^1(x).$$

□

Lemma 4.7. *Let r_1 and r_2 be the same as in Lemma 4.3. If $\Xi(r_i) \leq \mu\tau_0$, $(1 + r_2/r_1)\bar{\varepsilon}(r_2) \leq 1/10$, then we have that*

$$d_{0,1}(X(r_1), X(r_2)) \leq \frac{30 r_2}{r_1} \Theta(0, r_2)^{1/2} \cdot F(r_2)^{1/2} + 2\pi\mu^{-1/2} \cdot \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right)$$

Proof. For $z \in X(r_2) \cap \partial B_1$, if $z \notin \{y_r\} \cup \mathcal{X}_r$, we will denote by P_z the plane which is through 0 and z and perpendicular to $\operatorname{Tan}(X(r_2) \cap \partial B_1, z)$. By Lemma 4.2, we have that

$$|z - a| \leq \mu^{-1/2} \Xi(r_1)^{1/2}, \forall a \in \Gamma(r_2) \cap P_z.$$

Since $\mathcal{C}_{P_z, r_1, r_2} \cap \gamma(r_i) \neq \emptyset$, $i = 1, 2$, we take $b_i \in \mathcal{C}_{P_z, r_1, r_2} \cap \gamma(r_i)$, then

$$|\Pi(b_1) - \Pi(b_2)| \leq \mathcal{H}^1(\Pi(\mathcal{C}_{P_z, r_1, r_2})) \leq \mathcal{H}^1(P_z \cap E_{r_1, r_2}),$$

thus

$$\begin{aligned}\operatorname{dist}(z, X(r_1) \cap \partial B_1) &\leq |z - \Pi(b_2)| + |\Pi(b_2) - \Pi(b_1)| + \operatorname{dist}(\Pi(b_1), X(r_1) \cap \partial B_1) \\ &\leq \mathcal{H}^1(P_z \cap E_{r_1, r_2}) + \mu^{-1/2} \left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right).\end{aligned}$$

For any $x \in \mathcal{X}_r$, we let A_x be the arc in $\partial B(0,1)$ which join $\Pi(x)$ and $\Pi(y_r)$, We see that $X(r_2) \cap \partial B(0,1) = \cup_{x\in\mathcal{X}_r} A_x$, and $\mathcal{H}^1(A_x) \geq (1/2 - \bar{\varepsilon}(r_2))\pi \geq \pi/4$. Suppose $z \in A_x$, then

$$\begin{aligned}
\operatorname{dist}(z, X(r_1)) &\leq \frac{\pi^2}{2\mathcal{H}^1(A_x)^2} \int_{A_x} \operatorname{dist}(z, X(r_1)) d\mathcal{H}^1(x) \\
&\leq \frac{2\pi}{\mathcal{H}^1(A_x)} \int_{A_x} \mathcal{H}^1(P_z \cap E_{r_1,r_2}) d\mathcal{H}^1(x) + 2\pi\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right) \\
&\leq 16\mathcal{H}^2(E_{r_1,r_2}) + 2\pi\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right) \\
&\leq \frac{16r_2}{r_1}\left(2\Theta(0,r_2)\right)^{1/2} F(r_2)^{1/2} + 2\pi\mu^{-1/2}\left(\Xi(r_1)^{1/2} + \Xi(r_2)^{1/2}\right)
\end{aligned}$$

□

Remark 4.8. *For any cones X_1 and X_2, we see that*

$$d_H(X_1 \cap \partial B(0,1), X_2 \cap \partial B(0,1)) \leq 2d_{0,1}(X_1, X_2).$$

Since $\Xi(r) = [rF_1(r)]'$ for any $r \in \mathscr{R}$, we get that

$$\int_{r_1}^{r_2} \Xi(t)dt \leq r_2F_1(r_2) - r_1F_1(r_1),$$

For any $\zeta > 2$, if $r_1 \leq r_2 \leq r$, then by Chebyshev's inequality, we get that,

$$\mathcal{H}^1\left(\left\{t \in [r_1, r_2] \;\middle|\; \Xi(t) \leq \zeta F_1(r)^{2/3}\right\}\right) \geq r_2 - r_1 - \frac{1}{\zeta} rF_1(r)^{1/3},$$

thus $\left\{t \in [r_1, r_2] \;\middle|\; \Xi(t) \leq \zeta F_1(r)^{2/3}\right\} \neq \emptyset$ when $r_2 - r_1 > (1/\zeta)rF_1(r)^{1/3}$.

Lemma 4.9. *Let $R_0 < (1-\tau)\mathfrak{r}$ be a positive number such that $F(R_0) \leq \mu\tau_0/4$ and $\bar{\varepsilon}(R_0) \leq 10^{-4}$. For any $r \in \mathscr{R} \cap (0, R_0)$, if $\Xi(r) \leq \mu\tau_0$, then there is a constant $C = C(\mu, \Theta(0))$ such that*

$$\operatorname{dist}(x, E) \leq Cr\left(F_1(r)^{1/3} + \Xi(r)^{1/2}\right),\ x \in X(r) \cap B_r.$$

Proof. For any $k \geq 0$, we take $r_k = 2^{-k}r$. Then there exists $t_k \in [r_k, r_{k-1}]$ such that

$$\Xi(t_k) \leq \frac{\int_{r_k}^{r_{k-1}} \Xi(t)dt}{r_{k-1} - r_k} \leq \frac{r_{k-1}F_1(r_{k-1})}{r_{k-1}/2} = 2F_1(r_{k-1}).$$

We let $X_k = X(t_k)$, then for any $j > i \geq 1$, we have that

$$\begin{aligned}
d_{0,1}(X_i, X_j) &\leq \sum_{k=i}^{j-1} d_{0,1}(X_k, X_{k+1}) \\
&\leq 60\left(\Theta(0) + \mu\tau_0/4\right)^{1/2} \sum_{k=i}^{j-1} F_1(t_k)^{1/2} + 2\pi\mu^{-1/2} \sum_{k=i}^{j-1}\left(\Xi(t_k)^{1/2} + \Xi(t_{k+1})^{1/2}\right) \\
&\leq \left(60\left(\Theta(0) + \mu\tau_0/4\right)^{1/2} + 4\pi\mu^{-1/2}\right) \sum_{k=i}^{j-1} 2F_1(t_k)^{1/2} + F_1(t_{k-1})^{1/2} \\
&\leq C_1(\mu, \Theta(0))(j-i)F_1(r_{i-1})^{1/2} = C_1(\mu, \Theta(0))F_1(r_{i-1})^{1/2}\log_2(r_i/r_j),
\end{aligned} \tag{4.3}$$

where $C_1(\mu,\Theta(0))=3\left(60\left(\Theta(0)+\mu\tau_0/4\right)^{1/2}+4\pi\mu^{-1/2}\right)$.

For any $x\in X(r)\cap B_r$ with $\Xi(|x|)\leq\mu\tau_0$, we assume that $t_{k+1}\leq|x|<t_k$, then

$$\begin{aligned}\mathrm{dist}(x,E)&\leq d_H(X(r)\cap B_{|x|},X(|x|)\cap B_{|x|})+d_H(X(|x|)\cap B_{|x|},\gamma(|x|))\\&\leq 2|x|d_{0,1}(X(r),X(|x|))+\mu^{-1/2}|x|\Xi(|x|)^{1/2}\\&\leq 2|x|(d_{0,1}(X(|x|),X_k)+d_{0,1}(X_k,X_1)+d_{0,1}(X_1,X(r)))+\mu^{-1/2}|x|\Xi(|x|)^{1/2}\\&\leq(4\pi+1)\mu^{-1/2}|x|\left(\Xi(|x|)^{1/2}+\Xi(r)^{1/2}\right)+C_2(\mu,\Theta(0))|x|F_1(r)^{1/2}\log_2(r/|x|)\\&\leq(4\pi+1)\mu^{-1/2}|x|\Xi(|x|)^{1/2}+C_3(\mu,\Theta(0))r\left(\Xi(r)^{1/2}+F_1(r)^{1/2}\right)\end{aligned}$$

For any $0\leq a\leq b\leq r$, we put

$$I(a,b)=\left\{t\in[a,b]\;\middle|\;\Xi(t)\leq F_1(r)^{2/3}\right\},$$

then $I(a,b)\neq\emptyset$ when $b-a>rF_1(r)^{1/3}$. If $|x|\in I(0,r)$, then

$$\mathrm{dist}(x,E)\leq C_4(\mu,\Theta(0))r\left(F_1(r)^{1/3}+\Xi(r)^{1/2}\right).$$

We let $\{s_i\}_{i=0}^{m+1}\subset[0,r]$ be a sequence such that

$$0=s_0<s_1<\cdots<s_m<s_{m+1}=r,\;\; s_i\in I(0,r),$$

and

$$s_{i+1}-s_i\leq 2rF_1(r)^{1/3}.$$

For any $x\in X(r)\cap B_r$, if $s_i\leq|x|<s_{i+1}$ for some $0\leq i\leq m$, we have that

$$\begin{aligned}\mathrm{dist}(x,E)&\leq\left|x-\frac{s_i}{|x|}x\right|+\mathrm{dist}\left(\frac{s_i}{|x|}x,E\right)\\&\leq(s_{i+1}-s_i)+C_4(\mu,\Theta(0))r\left(F_1(r)^{1/3}+\Xi(r)^{1/2}\right)\\&\leq(C_4(\mu,\Theta(0))+2)r\left(F_1(r)^{1/3}+\Xi(r)^{1/2}\right).\end{aligned}$$

□

Definition 4.10. Let $U\subset\mathbb{R}^3$ be an open set, $E\subset\mathbb{R}^3$ be a set of Hausdorff dimension 2. E is called Ahlfors-regular in U if there is a $\delta>0$ and $\xi_0\geq 1$ such that, for any $x\in E\cap U$, if $0<r<\delta$ and $B(x,r)\subset U$, we have that

$$\xi_0^{-1}r^2\leq\mathcal{H}^2(E\cap B(x,r))\leq\xi_0 r^2.$$

Lemma 4.11. *Let R_0 be the same as in Lemma 4.9. If E is Ahlfors-regular, and $r\in\mathscr{R}\cap(0,R_0)$ satisfies $\Xi(r)\leq\mu\tau_0$, then there is a constant $C=C(\mu,\xi_0,\Theta(0))$ such that*

$$\mathrm{dist}(x,X(r))\leq Cr\left(F_1(r)^{1/4}+\Xi(r)^{1/2}\right),\;\; x\in E\cap B(0,9r/10).$$

Proof. Let $\{X_k\}_{k\geq 1}$ be the same as in (4). For any $t \in \mathscr{R}$ with $t_{k+1} \leq t < t_k$, $\Xi(t) \leq \mu\tau_0$ and $x \in \gamma(t)$, we have that

$$\begin{aligned}\operatorname{dist}(x, X(r)) &\leq d_H(\gamma(t), X(|x|) \cap B_{|x|}) + d_H(X(|x|) \cap B_{|x|}, X(r)) \\ &\leq (4\pi + 1)\mu^{-1/2}|x|\Xi(|x|)^{1/2} + C_3(\mu, \Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/2}\right)\end{aligned}$$

We put

$$J(0, r) = \{t \in [0, r] : \Xi(t) > F_1(r)^{1/2}\}.$$

For any $x \in \gamma(t)$ with $t \in (0, r) \setminus J(0, r)$, we have that

$$\operatorname{dist}(x, X(r)) \leq C_5(\mu, \Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/4}\right).$$

We put

$$E_1 = \bigcup_{t\in J(0,r)} (E \cap \partial B_t),\ E_2 = \bigcup_{t\in(0,r)\setminus J(0,r)} (E \cap B_t \setminus \gamma(t)),$$

and

$$E_3 = E \cap B_r \setminus (E_1 \cup E_2) = \bigcup_{t\in(0,r)\setminus J(0,r)} \gamma(t).$$

Then

$$\begin{aligned}\mathcal{H}^2(E_1 \cup E_2) &= \int_{E\cap B_r} d\mathcal{H}^2(x) - \int_{E_3} d\mathcal{H}^2(x) \\ &\leq \int_{E\cap B_r} d\mathcal{H}^2(x) - \int_{E_3} \cos\theta(x) d\mathcal{H}^2(x) \\ &= \int_{E\cap B_r} (1 - \cos\theta(x)) d\mathcal{H}^2(x) + \int_{E_1\cup E_2} \cos\theta(x) d\mathcal{H}^2(x) \\ &\leq r^2 F(r) + \int_0^r \mathcal{H}^1(E_1 \cap \partial B_t)dt + \int_0^r \mathcal{H}^1(E_2 \cap \partial B_t)dt \\ &\leq r^2 F(r) + \int_{J(0,r)} (2\Theta(0) + tf'(t) + 2f(t))tdt + \mu^{-1}\int_0^r t\Xi(t)dt \\ &\leq (2 + \mu^{-1})r^2 F_1(r) + 2\Theta(0)\int_{\{t\in[0,r]:\Xi(t)>F_1(r)^{1/2}\}} tdt \\ &\leq (2 + \mu^{-1})r^2 F_1(r) + \frac{2\Theta(0)}{F_1(r)^{1/2}}\int_0^r t\Xi(t)dt \\ &\leq C_6(\mu, \Theta(0))r^2 F_1(r)^{1/2},\end{aligned}$$

where $C_6(\mu, \Theta(0)) = (2 + \mu^{-1})(\mu\tau_0/4)^{1/2} + 2\Theta(0)$.

We see that, for any $x \in E_3$,

$$\operatorname{dist}(x, X(r)) \leq C_5(\mu, \Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/4}\right).$$

If $x \in E \cap B(0, 9r/10)$ with

$$\operatorname{dist}(x, X(r)) > C_5(\mu, \Theta(0))r\left(\Xi(r)^{1/2} + F_1(r)^{1/4}\right) + s$$

for some $s \in (0, r/10)$, then $E \cap B(x,s) \subset E_1 \cup E_2$, thus

$$\mathcal{H}^2(E \cap B(x,s)) \le C_6(\mu, \Theta(0)) r^2 F_1(r)^{1/2}.$$

But on the other hand, by Ahlfors-regular property of E, we have that

$$\mathcal{H}^2(E \cap B(x,s)) \ge \xi_0^{-1} s^2.$$

We get so that

$$s \le C_6(\mu, \Theta(0))^{1/2} \cdot \xi_0^{1/2} \cdot r F_1(r)^{1/4}.$$

Therefore, for $x \in E \cap B(0, 9r/10)$,

$$\operatorname{dist}(x, X(r)) \le \left(C_6(\mu, \Theta(0))^{1/2} \cdot \xi_0^{1/2} + C_5(\mu, \Theta(0)) \right) \left(\Xi(r)^{1/2} + F_1(r)^{1/4} \right).$$

□

For any $k \ge 0$, we take $R_k = 2^{-k} R_0$ and $s_k \in [R_{k+1}, R_k]$ such that

$$\Xi(s_k) \le \frac{\int_{R_{k+1}}^{R_k} \Xi(t) dt}{R_k - R_{k+1}} \le 2F_1(R_k).$$

We put $X_k = X(s_k)$. Then for any $j \ge i \ge 2$, we have that

$$\begin{aligned} d_{0,1}(X_i, X_j) &\le \frac{C_1(\mu, \Theta(0))}{3} \sum_{k=i}^{j-1} \left(2F_1(s_k)^{1/2} + F_1(s_{k-1})^{1/2} \right) \\ &\le C_1(\mu, \Theta(0)) \sum_{k=i-1}^{j-1} F_1(R_k)^{1/2} \\ &\le \frac{C_1(\mu, \Theta(0))}{\ln 2} \sum_{k=i-1}^{j-1} \int_{R_k}^{R_{k-1}} \frac{F_1(t)^{1/2}}{t} dt \\ &= \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_{R_{i-2}}^{R_{j-1}} \frac{F_1(t)^{1/2}}{t} dt. \end{aligned}$$

If the gauge function h satisfy that $h(r) \le C(\ln(A/r))^{-b}$, $0 < r \le R_0$, for some $A > R_0$, $C > 0$ and $b > 3$, then

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \le \frac{C}{b-1} \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1},$$

and then Remark 3.16 implies that

$$F(r) \le C_1 \left(\ln \left(\frac{A}{r} \right) \right)^{-b} + \frac{C}{b-1} \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1} \le C_2 \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1},$$

thus

$$\int_0^{R_0} \frac{F_1(t)^{1/2}}{t} dt < +\infty. \tag{4.4}$$

In case (4) holds, X_k converges to a cone $X(0)$, and

$$d_{0,1}(X(0), X_k) \le \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_0^{R_{k-2}} \frac{F_1(t)^{1/2}}{t} dt.$$

Lemma 4.12. *If* (4) *holds, then* $X(0)$ *is a minimal cone.*

Proof. We see that for any $r \in (0, \mathfrak{r}) \cap \mathscr{R}$, there exist sliding minimal cone $Z(r)$ such that $d_{0,1}(X(r), Z(r)) \leq 20\varepsilon(r)^{1/2}$. But $\varepsilon(r) \to 0$ as $r \to 0+$, we get that

$$d_{0,1}(Z(s_k), X(0)) \to 0.$$

Since $Z(s_k)$ is sliding minimal for any k, we get that $X(0)$ is also sliding minimal. □

For any $r \in \mathscr{R} \cap (0, R_0)$ with $\Xi(r) \leq \mu\tau_0$, we assume $R_{k+1} \leq r < R_k$, by Lemma 4.7, we have that

$$\begin{aligned} d_{0,1}(X(0), X(r)) &\leq d_{0,1}(X(0), X_{k+3}) + d_{0,1}(X_{k+3}, X(r)) \\ &\leq \frac{C_1(\mu, \Theta(0))}{\ln 2} \int_0^{R_{k+1}} \frac{F_1(t)^{1/2}}{t} dt \\ &\quad + \frac{30r}{s_{k+3}} \Theta(0, r)^{1/2} F_1(r)^{1/2} + 2\pi\mu^{-1/2} \left(\Xi(s_{k+3})^{1/2} + \Xi(r)^{1/2} \right) \\ &\leq 10 C_1(\mu, \Theta(0)) \left(\Xi(r)^{1/2} + F_1(r)^{1/2} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right) \end{aligned} \tag{4.5}$$

Theorem 4.13. *If* (4) *holds, and* E *is AR, then* E *has unique tangent cone* $X(0)$ *at* 0*, and there is a constant* $C = C_{10}(\mu, \Theta, \xi_0)$ *such that*

$$d_{0,9r/10}(E, X(0)) \leq C \left(F_1(r)^{1/4} + \int_0^r \frac{F(t)^{1/2}}{t} dt \right) \tag{4.6}$$

In particular,

- *if* $h(r) \leq C_h(\ln(A/r))^{-b}$ *for some* $A, C_h > 0$, $b > 3$ *and* $0 < r \leq R_0 < A$, *then*

  $$d_{0,r}(E, X(0)) \leq C'(\ln(A_1/r))^{-(b-3)/4},\ 0 < r \leq 9R_0/10,\ A_1 \leq 10A/9;$$

- *if* $h(r) \leq C_h r^{\alpha_1}$ *for some* $C_h, \alpha_1 > 0$, *and* $0 < r \leq r_0$, $0 < r_0 \leq \min\{1, R_0\}$, *then*

  $$d_{0,r}(E, X(0)) \leq C(r/r_0)^{\beta},\ 0 < r \leq 9r_0/10,\ 0 < \beta < \alpha_1,$$

  where

  $$C \leq C_{11}(\mu, \lambda_0, \alpha_1, \beta, C_h, \xi_0, \Theta(0)) \left(F(r_0)^{1/4} + r_0^{\alpha_1/4} \right).$$

Proof. From (4) and Lemma 4.9, we get that, for any $x \in X(0) \cap B_r$ where $r \in \mathscr{R} \cap (0, R_0)$ such that $\Xi(r) \leq \mu\tau_0$,

$$\operatorname{dist}(x, E) \leq C_7(\mu, \xi_0, \Theta(0)) r \left(\Xi(r)^{1/2} + F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right).$$

Similarly to the proof of Lemma 4.9, we still consider

$$I(a, b) = \left\{ t \in [a, b] \,\middle|\, \Xi(t) \leq F_1(r)^{2/3} \right\},\ 0 \leq a \leq b \leq r,$$

we have that $I(a,b) \neq \emptyset$ whenever $b-a > rF_1(r)^{1/3}$. We let $\{s_i\}_0^{m+1} \subset [0,r]$ be a sequence such that

$$0 = s_0 < s_1 < \cdots < s_m < s_{m+1} = r, \;\; s_i \in I(0,r),$$

and

$$s_{i+1} - s_i \le 2rF_1(r)^{1/3}.$$

For any $r \in (0, R_0)$, we assume that $s_i \le r < s_{i+1}$, $x \in X(0) \cap \partial B_r$.

$$\begin{aligned} \operatorname{dist}(x, E) &\le \left| x - \frac{s_i}{|x|} x \right| + \operatorname{dist}\left(\frac{s_i}{|x|} x, E \right) \\ &\le C_8(\mu, \xi_0, \Theta(0)) r \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right) \end{aligned} \tag{4.7}$$

From (4) and Lemma 4.11, we have that, for any $x \in X(0) \cap B(0, 9r/10)$ where $r \in \mathscr{R} \cap (0, R_0)$ such that $\Xi(r) \le \mu\tau_0$,

$$\operatorname{dist}(x, X(0)) \le C_9(\mu, \xi_0, \Theta(0)) \left(\Xi(r)^{1/2} + F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right).$$

Similarly to the proof of Lemma 4.11, we can get that

$$\operatorname{dist}(x, X(0)) \le C_{10}(\mu, \xi_0, \Theta(0)) \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right). \tag{4.8}$$

We get, from (4) and (4), that (4.13) holds.

If $h(r) \le C_h(\ln(A/r))^{-b}$ for some $A, C_h > 0$ and $b > 3$ and $0 < r \le R_0 < A$, then

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \le \frac{C_h}{b-1} \left(\ln \left(\frac{A}{r} \right) \right)^{-b+1},$$

and by Remark 3.16 we have that

$$F(r) \le C'' \left(\ln \frac{A}{r} \right)^{-b+1}$$

where

$$C'' \le C(R_0, \lambda, b) \left(\ln \frac{A}{r} \right)^{-1} + \frac{C_1}{b-1} \le C(R_0, \lambda, b) \left(\ln \frac{A}{R_0} \right)^{-1} + \frac{C_1}{b-1}$$

is bounded, thus

$$\int_0^r \frac{F_1(t)^{1/2}}{t} dt \le C''' \left(\ln \frac{A}{r} \right)^{(-b+3)/2}$$

Hence we get that

$$\begin{aligned} d_{0,9r/10}(E, X(0)) &\le C_{10}(\mu, \xi_0, \Theta(0)) \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right) \\ &\le C' \left(\ln \frac{A}{r} \right)^{-(b-3)/4}. \end{aligned}$$

If $h(r) \leq C_h r^{\alpha_1}$ for some $C_h, \alpha_1 > 0$ and $0 < r \leq r_0$, then

$$h_1(r) = \int_0^r \frac{h(2t)}{t} dt \leq \frac{C_h}{\alpha_1}(2r)^{\alpha_1}.$$

We see, from the proof of Corollary 3.15, that

$$f(r) \leq (f(r_0) + C_2(\alpha_1, \beta, \lambda_0) C_h r_0^{\alpha_1})\,(r/r_0)^{\beta},\ \forall 0 < \beta < \alpha_1,$$

thus

$$F_1(r) = f(r) + 16 h_1(r) \leq (f(r_0) + C_2'(\alpha_1, \beta, \lambda_0) C_h r_0^{\alpha_1})(r/r_0)^{\beta}.$$

Then

$$\begin{aligned} d_{0,9r/10}(E, X(0)) &\leq C_{10}(\mu, \xi_0, \Theta(0)) \left(F_1(r)^{1/4} + \int_0^r \frac{F_1(t)^{1/2}}{t} dt \right) \\ &\leq C(r/r_0)^{\beta/4}, \end{aligned}$$

where

$$C \leq C_{10}'(\mu, \xi_0, \Theta(0))(F(r_0)^{1/4} + C_2''(\alpha_1, \beta, \lambda_0, C_h) r_0^{1/4}).$$

□

Remark 4.14. *If the gauge function h satisfies that*

$$\int_0^{R_0} \frac{1}{r} \left(\int_0^r \frac{h(2t)}{t} dt \right)^{1/2} dr < +\infty \tag{4.9}$$

and

$$\int_0^{R_0} r^{-1+\frac{\lambda}{1-\lambda}} \left(\int_r^{R_0} t^{-1-\frac{2\lambda}{1-\lambda}} h(2t) dt \right)^{1/2} dr < +\infty, \tag{4.10}$$

then by (3.5), we have that (4) holds.

5 Parameterization of sliding almost minimal sets

Let n, $d \leq n$ and k be nonnegative integers, $\alpha \in (0,1)$. By a d-dimensional submanifold of class $C^{k,\alpha}$ of $\mathbb{R}^n$ we mean a subset M of $\mathbb{R}^n$ satisfying that for each $x \in M$ there exist s neighborhood U of x in $\mathbb{R}^n$, a mapping $\Phi : U \to \mathbb{R}^n$ which is a diffeomorphism of class $C^{k,\alpha}$ between its domain and image, and a d dimensional vector subspace Z of $\mathbb{R}^n$ such that

$$\Phi(M \cap U) = Z \cap \Phi(U).$$

In this section, we assume that $\Omega \subset \mathbb{R}^3$ is a closed domain such that the boundary $\partial\Omega$ is a 2-dimensional submanifold of class $C^{1,\alpha}$ for some $\alpha > 0$. Let $E \subset \Omega$ be a closed set such that $E \in SAM(\Omega, \partial\Omega, h)$ and $\partial\Omega \subset E$, $x_0 \in \partial\Omega$. We always assume that the gauge function h satisfies (4.14) and (4.14). We put $\lambda_0 = \lambda/(1-\lambda)$, and

$$h_2(\rho) = \int_0^{\rho} \frac{1}{r} \left(\int_0^r \frac{h(2t)}{t} dt \right)^{1/2} dt$$

and

$$h_3(\rho) = \int_0^\rho r^{-1+\lambda_0} \left(\int_r^{R_0} t^{-1-2\lambda_0} h(2t) dt \right)^{1/2} dt$$

We see, from Proposition 4.1 in [3], that E is Ahlfors-regular in $B(x_0, R_0)$, i.e. there exist $\delta_1 > 0$ and $\xi_1 \geq 1$ such that for any $x \in E \cap B(x_0, R_0)$, if $0 < r < \delta_1$ and $B(x, r) \subset B(x_0, R_0)$, we have that

$$\xi_1^{-1} r^2 \leq \mathcal{H}^2(E \cap B(x, r)) \leq \xi_1 r^2.$$

We see from Theorem 3.10 in [6] that there only there kinds of possibility for the blow-up limits of E at x_0, they are the plane $\mathrm{Tan}(\partial\Omega, x_0)$, cones of type $\mathbb{P}_+$ union $\mathrm{Tan}(\partial\Omega, x_0)$, and cones of type $\mathbb{Y}_+$ union $\mathrm{Tan}(\partial\Omega, x_0)$. By Proposition 29.53 in [3], we get so that

$$\Theta_E(x_0) = \pi, \ \frac{3\pi}{2}, \ \text{or} \ \frac{7\pi}{4}.$$

If $\Theta_E(x_0) = \pi$, then there is a neighborhood U_0 of x_0 in $\mathbb{R}^3$ such that $E \cap U_0 = \partial\Omega \cap U_0$. In the next content of this section, we put ourself in the case $\Theta_E(x_0) = 3\pi/2$ or $7\pi/4$.

Lemma 5.1. *There exist $r_0 = r_0(x_0) > 0$ and a mapping $\Psi = \Psi_{x_0} : B(0, r_0) \to \mathbb{R}^3$, which is a diffeomorphism of class $C^{1,\alpha}$ from $B(0, r_0)$ to $\Psi(B(0, r_0))$, such that*

$$\Psi(0) = x_0, \Psi(\Omega_0 \cap B_{r_0}) \subset \Omega \cap B(x_0, R_0), \Psi(L_0 \cap B_{r_0}) \subset \partial\Omega \cap B(x_0, R_0),$$

and that $D\Psi(0)$ is a rotation satisfying that

$$D\Psi(0)(\Omega_0) = \mathrm{Tan}(\Omega, x_0) \ \textit{and} \ D\Psi(0)(L_0) = \mathrm{Tan}(\partial\Omega, x_0).$$

Proof. By definition, there are an open set $U, V \subset \mathbb{R}^3$ and a diffeomorphism $\Phi : U \to V$ of class $C^{1,\alpha}$ such that $x_0 \in U$, $0 = \Phi(x_0) \in V$ and

$$\Phi(U \cap \partial\Omega) = Z \cap V,$$

where Z is a plane through 0. Indeed, we have that

$$Z = D\Phi(x_0) \, \mathrm{Tan}(\partial\Omega, x_0)$$

and

$$\Phi(U \cap \Omega) = V \cap D\Phi(x_0) \, \mathrm{Tan}(\Omega, x_0).$$

We will denote by A the linear mapping given by $A(v) = D\Phi(x_0)^{-1} v$, and assume that $A(V) = B(0, r)$ is a ball. Let Φ_1 be a rotation such that $\Phi_1(\mathrm{Tan}(\partial\Omega, x_0)) = L_0$ and $\Phi_1(\mathrm{Tan}(\Omega, x_0)) = \Omega_0$. Then we get that $\Phi_1 \circ A \circ \Phi$ is also $C^{1,\alpha}$ mapping which is a diffeomorphism between U and $B(0, r)$,

$$D(\Phi_1 \circ A \circ \Phi)(x_0) \, \mathrm{Tan}(\Omega, x_0) = \Phi_1(\mathrm{Tan}(\Omega, x_0)) = \Omega_0,$$
$$D(\Phi_1 \circ A \circ \Phi)(x_0) \, \mathrm{Tan}(\partial\Omega, x_0) = \Phi_1(\mathrm{Tan}(\partial\Omega, x_0)) = L_0,$$

and

$$\Phi_1 \circ A \circ \Phi(U \cap \partial\Omega) = \Phi_1 \circ A(Z \cap V) = L_0 \cap B(0, r),$$
$$\Phi_1 \circ A \circ \Phi(U \cap \partial\Omega) = \Phi_1 \circ A(V \cap D\Phi(x_0) \, \mathrm{Tan}(\Omega, x_0)) = \Omega_0 \cap B(0, r).$$

We now take $r_0 = r$ and $\Psi = (\Phi_1 \circ A \circ \Phi)^{-1}|_{B(0,r)}$ to get the result.

□

Let $U \subset \mathbb{R}^n$ be an open set. For any mapping $\Psi : U \to \mathbb{R}^n$ of class $C^{1,\alpha}$, we will denote by C_Ψ the constant defined by

$$C_\Psi = \sup\left\{ \frac{\|D\Psi(x) - D\psi(y)\|}{|x-y|^\alpha} : x, y \in U, x \neq y \right\}. \tag{5.1}$$

Then

$$\Psi(x) - \Psi(y) = \int_0^1 D\Psi(y + t(x-y))dt \cdot (x-y)$$

and

$$\begin{aligned} |\Psi(x) - \Psi(y) - D\Psi(y)(x-y)| &\leq \int_0^1 C_\Psi (t|x-y|)^\alpha dt \cdot |x-y| \\ &\leq \frac{C_\Psi}{\alpha+1} |x-y|^{1+\alpha}. \end{aligned}$$

For any $0 < \rho \leq r_0$, we set $U_\rho = \Phi(B_\rho)$, $M_\rho = \Psi^{-1}(E \cap U_\rho)$ and

$$\Lambda(\rho) = \max\left\{ \operatorname{Lip}\left(\Psi_{B_\rho}\right), \operatorname{Lip}\left(\Psi_{U_\rho}^{-1}\right) \right\}.$$

Then $\Lambda(\rho) \leq 1/(1 - C_\Psi \rho^\alpha)$ when $C_\Psi \rho^\alpha < 1$.

Lemma 5.2. *For any $1 < \rho \leq r_0$, M_ρ is local almost minimal in B_ρ at 0 with gauge function*

$$H(t) \leq 4\Lambda(\rho)^2 h(\Lambda(\rho)t) + Ct^\alpha, \ 0 < t < \rho,$$

where $C = C(\rho)$ is a constant such that $0 < C \leq \xi_1 \Lambda(\rho) C_\Psi (4 + C_\Psi \rho^\alpha)$. Moreover, we have that

$$M_\rho \in GSAM\left(B_\rho, \Lambda(\rho)^4, 2\rho, \Lambda(\rho)^4 h\left(2\rho\Lambda(\rho)\right)\right)$$

Proof. We see that

$$\operatorname{diam}(U_\rho) \leq 2\rho \operatorname{Lip}\left(\Psi|_{B_\rho}\right) \leq 2\rho\Lambda(\rho)$$

and

$$E \cap U_\rho \in GSAM(U_\rho, 1, \operatorname{diam}(U_\rho), h(2\operatorname{diam}(U_\rho))),$$

By Proposition 2.8 in [3], we have that

$$M_\rho \in GSAM\left(B_\rho, \Lambda(\rho)^4, 2\rho, \Lambda(\rho)^4 h\left(2\rho\Lambda(\rho)\right)\right)$$

By Proposition 4.1 in [3], we get that M_ρ is Ahlfors-regular in B_ρ. Indeed, we can get a little more, that is, for any $x \in M_\rho$ with $0 < r\Lambda(\rho) < \delta_1$ and $B(x,r) \subset B(0,\rho)$, we have that

$$\left(\xi_1 \Lambda(\rho)\right)^{-1} r^2 \leq \mathcal{H}^2(M_\rho \cap B(x,r)) \leq \left(\xi_1 \Lambda(\rho)\right) r^2.$$

Let $\{\varphi_t\}_{0 \leq t \leq 1}$ be a sliding deformation of M_r in B_ρ. Then

$$\left\{\Psi \circ \varphi_t \circ \Psi^{-1}\right\}_{0 \leq t \leq 1}$$

is a sliding deformation of E in U_ρ. Hence we get that

$$\mathcal{H}^2(E \cap U_\rho) \leq \mathcal{H}^2(\Psi \circ \varphi_t \circ \Psi^{-1}(E \cap U_\rho)) + h(2\operatorname{diam}(U_\rho))^2 \operatorname{diam}(U_\rho)^2 \tag{5.2}$$

For any 2-rectifiable set $A \subset B_r$, by Theorem 3.2.22 in [7], we have that

$$\operatorname{ap} J_2(\Psi|_A)(x) = \left\| \wedge_2 \left(D\Psi(x)|_{\operatorname{Tan}(A,x)} \right) \right\|$$

and

$$\mathcal{H}^2(\Psi(A \cap B_\rho)) = \int_{A \cap B_\rho} \operatorname{ap} J_2(\Psi|_A)(x) d\mathcal{H}^2(x)$$

By (5), we get that

$$\int_{A \cap B_\rho} (1 - C_\Psi |x|^\alpha)^2 d\mathcal{H}^2 \le \mathcal{H}^2(\Psi(A \cap B_\rho)) \le \int_{A \cap B_\rho} (1 + C_\Psi |x|^\alpha)^2 d\mathcal{H}^2. \quad (5.3)$$

Thus

$$\begin{aligned} \mathcal{H}^2(\Psi(M_\rho)) &\ge (1 - C_\Psi \rho^\alpha)^2 \mathcal{H}^2(M_\rho) \ge \mathcal{H}^2(M_\rho) - 2C_\Psi \rho^\alpha \mathcal{H}^2(M_\rho) \\ &\ge \mathcal{H}^2(M_\rho) - 2\xi_1 C_\Psi \Lambda(\rho) \rho^{2+\alpha}, \end{aligned}$$

and

$$\mathcal{H}^2(\Psi(\varphi_1(M_\rho))) \le \mathcal{H}^2(\varphi_1(M_\rho)) + \xi_1 \Lambda(\rho) C_\Psi \left(2 + C_\Psi \rho^\alpha\right) \rho^{2+\alpha}.$$

Combine these two equations with (5), we get that

$$\begin{aligned} \mathcal{H}^2(M_\rho) \le \mathcal{H}^2(\varphi_1(M_\rho)) &+ \xi_1 \Lambda(\rho) C_\Psi \left(4 + C_\Psi \rho^\alpha\right) \rho^{2+\alpha} \\ &+ 4 \left(\Lambda(\rho)\right)^2 h\left(2\rho\Lambda(\rho)\right) \rho^2. \end{aligned}$$

□

Lemma 5.3. *Let $E_1 \subset \Omega_0$ be a 2-rectifiable set, $x \in E_1$, X a cone centered at 0, $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism of class $C^{1,\alpha}$. Then there exist $C > 0$ such that, for any $r > 0$ and $\rho > 0$ with $B(\Phi(x), \rho) \subset \Phi(B(x, r))$,*

$$d_{\Phi(x),\rho}\left(\Phi(E_1), \Phi(x) + D\Phi(x)X\right) \le \left(Cr^\alpha + \|D\Phi(x)\| d_{x,r}(E_1, x + X)\right) \frac{r}{\rho}.$$

Proof. Since Φ is of class $C^{1,\alpha}$, we have that

$$|\Phi(y) - \Phi(x) - D\Phi(x)(y - x)| \le \frac{C_\Phi}{\alpha + 1} |x - y|^{1+\alpha},$$

by putting $C_1 = C_\Phi/(\alpha + 1)$, we get that

$$\operatorname{dist}(\Phi(y), \Phi(x) + D\Phi(x)X) \le C_1 |y - x|^{1+\alpha} \text{ for } y \in x + X.$$

For any $z \in E_1 \cap B_r$ and $y \in x + X$, we have that

$$\begin{aligned} |\Phi(z) - \Phi(y)| &\le |\Phi(z) - \Phi(y) - D\Phi(x)(z - y)| + \|D\Phi(x)\| \cdot |z - y| \\ &\le \|D\Phi(x)\| \cdot |z - y| + C_1 |z - x|^{1+\alpha} + C_1 |y - x|^{1+\alpha}, \end{aligned}$$

thus

$$\operatorname{dist}(\Phi(z), \Phi(x + X)) \le \|D\Phi(x)\| r d_{x,r}(E_1, x + X) + 2C_1 r^{1+\alpha},$$

hence

$$\operatorname{dist}(\Phi(z), \Phi(x) + D\Phi(x)X) \le \|D\Phi(x)\| r d_{x,r}(E_1, x + X) + 3C_1 r^{1+\alpha}. \quad (5.4)$$

For any $z \in X \cap B_r$, $\Phi(x) + D\Phi(x)z \in \Phi(x) + D\Phi(x)X$, and

$$\begin{aligned}\operatorname{dist}(\Phi(x) + D\Phi(x)z, \Phi(E_1)) &= \inf\{|\Phi(y) - \Phi(x) - D\Phi(x)z| : y \in E_1\} \\ &\leq \inf\{C_1 r^{1+\alpha} + \|D\Phi(x)\| \cdot |y - x - z| : y \in E_1\} \\ &\leq \|D\Phi(x)\| r d_{x,r}(x + X, E_1) + C_1 r^{1+\alpha}.\end{aligned} \tag{5.5}$$

We get from (5) and (5) that

$$d_{\Phi(x),\rho}(\Phi(E_1), \Phi(x) + D\Phi(x)X) \leq \frac{r}{\rho}\left(3C_1 r^\alpha + \|D\Phi(x)\| \cdot d_{x,r}(E_1, x + X)\right)$$

□

Theorem 5.4. *Let Ω, $E \subset \Omega$, $x_0 \in \partial\Omega$ and h be the same as in the beginning of this section. Then there is a unique tantent cone X of E at x_0; moreover, if the gauge function h satisfy that*

$$h(t) \leq C_h t^{\alpha_1} \text{ for some } C_h > 0, \alpha_1 > 0 \text{ and } 0 < t < t_0, \tag{5.6}$$

then there exists $\rho_0 > 0$ such that, for any $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}$,

$$d_{x_0,\rho}(E, x_0 + X) \leq C(\rho/\rho_0)^{\beta/4},\ 0 < \rho \leq 9\rho_0/20,$$

where C is a constant satisfying that

$$C \leq C_{20}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1)(F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})^{1/4},$$

and $F_E(x_0, r) = r^{-2}\mathcal{H}^2(E \cap B(x_0, r)) - \Theta_E(x_0) + 16h_1(r)$.

Proof. Let $r \in (0, r_0)$ be such that $C_\Psi r^\alpha \leq 1/2$ and $2r \leq R_0$. Then $\Lambda(r) \leq 2$. By Lemma 5.2, we have that M_r is loacal almost minimal at 0 with gauge function H satisfying that

$$H(t) \leq 16h(2t) + C_r t^\alpha,\ 0 < t < r, \tag{5.7}$$

where $C_r \leq \xi_1 \Lambda(r) C_\Psi (4 + C_\Psi r^\alpha) \leq 9\xi_1 C_\Psi$ is a constant.

We put $f_{M_r}(\rho) = \Theta_{M_r}(0, \rho) - \Theta_{M_r}(0)$. Then we get, from (3.5) and (3.5), that

$$\begin{aligned}f_{M_r}(\rho) &\leq \left(r^{-2\lambda_0} f_{M_r}(r)\right)\rho^{2\lambda_0} + 8(1 + \lambda_0)\rho^{2\lambda_0}\int_\rho^r t^{-1-2\lambda_0} H(2t)dt \\ &\leq \left(r^{-2\lambda_0} f_{M_r}(r)\right)\rho^{2\lambda_0} + 2^{7+2\lambda_0}(1 + \lambda_0)\rho^{2\lambda_0}\int_{2\rho}^{2r} \frac{h(2t)}{t^{1+2\lambda_0}}dt \\ &\quad + 2^{\alpha+3}(1 + \lambda_0) C_r \cdot C_1(\alpha, \beta, \lambda_0) r^\alpha \cdot (\rho/r)^\beta,\end{aligned}$$

where $C_1(\alpha, \beta, \lambda_0)$ is the constant in (3.5).

We get from (5) that

$$H_1(\rho) = \int_0^\rho \frac{H(2s)}{s} ds \leq 16h_1(2\rho) + \frac{C_r}{\alpha}(2\rho)^\alpha,$$

by setting $F_1(\rho) = f_{M_r}(\rho) + 16H_1(\rho)$, we have that

$$\begin{aligned}F_1(\rho) &\leq C_{12}(\lambda_0, \alpha, \beta, r)(\rho/r)^\beta + 2^8 h_1(2\rho) + 2^{4+\alpha} C_r \alpha^{-1}\rho^\alpha \\ &\quad + 2^{7+2\lambda_0}(1 + \lambda_0)\rho^{2\lambda_0}\int_{2\rho}^{2r} \frac{h(2t)}{t^{1+2\lambda_0}}dt,\end{aligned}$$

where

$$C_{12}(\lambda_0,\alpha,\beta,r)\le f_{M_r}(r)+2^{\alpha+3}(1+\lambda_0)C_rC_1(\alpha,\beta,\lambda_0)r^{\alpha}.$$

Hence

$$\int_0^t \frac{F_1(\rho)^{1/2}}{\rho}d\rho\le C_{12}(\lambda_0,\alpha,\beta,r)^{1/2}(2/\beta)(t/r)^{\beta}+16h_2(2t)+C_{13}(\alpha,r)t^{\alpha/2}$$
$$+2^{4+\lambda_0}(1+\lambda_0)^{1/2}\int_0^t\rho^{-1+\lambda_0}\left(\int_{2\rho}^{2r}\frac{h(2s)}{s^{1+2\lambda_0}}ds\right)^{1/2}d\rho,$$

where $C_{13}(\alpha,r)\le 2^{3+\alpha/2}\alpha^{-3/2}C_r^{1/2}$, thus

$$\int_0^t \frac{F_1(\rho)^{1/2}}{\rho}d\rho<+\infty,\ \text{for } 0<t\le r.$$

We now apply Theorem 4.13, there is a unique tangent cone T of M_r at 0, thus there is a unique tangent cone X of E at x_0.

For any $R\in(0,R_0)$, we put

$$f_E(x_0,R)=R^{-2}\mathcal{H}^2(E\cap B(x_0,R))-\Theta_E(x_0)$$

and

$$F_E(x_0,R)=f_E(x_0,R)+16h_1(R).$$

We see, from (5) and $B(x_0,\rho/\Lambda(\rho))\subset U_\rho\subset B(x_0,\rho\Lambda(\rho))$, that

$$(1-C_\Psi\rho^{\alpha})^2(f_{M_r}(\rho)+\Theta_E(x_0))\le\rho^{-2}\mathcal{H}^2(E\cap U_\rho)\le(1+C_\Psi\rho^{\alpha})^2(f_{M_r}(\rho)+\Theta_E(x_0)),$$

so that

$$f_{M_r}(\rho)\le(1-C_\Psi\rho^{\alpha})^{-4}f_E(x_0,\rho\Lambda(\rho))+4\Theta_E(x_0)C_\Psi\rho^{\alpha},$$

and

$$f_{M_r}(\rho)\ge(1-C_\Psi^2\rho^{2\alpha})^2f_E(x_0,\rho/\Lambda(\rho))+2\Theta_E(x_0)C_\Psi^2\rho^{2\alpha}.$$

Thus we get that

$$C_{12}(\lambda_0,\alpha,\beta,r)\le 16f_E(x_0,2r)+(9\xi_1\cdot 2^{\alpha+3}(1+\lambda_0)C_1(\alpha,\beta,\lambda_0)+4\Theta_E(0))C_\Psi r^{\alpha}.$$

If h satisfy (5.6), we take $0<\rho_0\le\min\{r,t_0\}$, then

$$h_1(\rho)\le\frac{C_h}{\alpha_1}(2\rho)^{\alpha_1},\ H_1(\rho)\le\frac{2^{4+2\alpha_1}C_h}{\alpha_1}\rho^{\alpha_1}+\frac{2^{\alpha}C_r}{\alpha}\rho^{\alpha},\ 0<\rho\le\rho_0,$$

and

$$F_1(\rho)\le C_{13}(\lambda_0,\alpha,\beta,\rho_0,C_h)(\rho/\rho_0)^{\beta}+2^{8+\alpha_1}\alpha_1^{-1}C_h\rho^{\alpha_1}+C_{14}(\alpha,\xi_1,C_\Psi)\rho^{\alpha},\tag{5.8}$$

where $C_{13}(\lambda_0,\alpha_1,\beta,\rho_0,C_h)$ and $C_{14}(\alpha,\xi_1,C_\Psi)$ are constant satisfying that

$$C_{13}(\lambda_0,\alpha_1,\beta,\rho_0,C_h)\le C_{12}(\lambda_0,\alpha,\rho_0)+2^{7+4\alpha_1}(1+\lambda_0)C_1(\alpha_1,\beta,\lambda_0)C_h\rho_0^{\alpha_1}$$

and

$$C_{14}(\alpha,\xi_1,C_\Psi)\le 2^{8+\alpha}\alpha^{-1}\xi_1C_\Psi.$$

We get so that (5) can be rewrite as

$$F_1(\rho) \le C_{15}(\lambda_0, \alpha, \alpha_1, \beta, \xi_1)(F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})(\rho/\rho_0)^{\beta/4}.$$

By Theorem 4.13, we have that

$$\begin{aligned} d_{0,9\rho/10}(M_r, T) &\le C_{16}(\mu, \xi_0)\left(F_1(\rho)^{1/4} + \int_0^\rho \frac{F_1(t)^{1/2}}{t} dt\right) \\ &\le C_{17}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1) G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4}, \end{aligned}$$

where

$$G_E(x_0, \rho_0) = (F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})^{1/4}.$$

Apply Lemma 5.3, and by setting $X = D\Psi(0)T$, we get that, for any $\rho \in (0, 9\rho_0/10)$,

$$\begin{aligned} d_{x_0,\rho/2}(E, x_0 + X) &\le d_{x_0,\rho/\Lambda(\rho)}(E, x_0 + D\Psi(0)T) \\ &\le 6C_\Psi \rho^\alpha + 2d_{x,\rho}(M_r, T) \\ &\le 6C_\Psi \rho^\alpha + C_{18}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1) G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4} \\ &\le C_{19}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1) G_E(x_0, \rho_0)(\rho/\rho_0)^{\beta/4}. \end{aligned}$$

The radius ρ_0 is chosen to be such that

$$0 < \rho_0 \le \min\left\{1, t_0, r_0(x_0), R_0/2, (2C_\Psi)^{-1/\alpha}\right\}$$

and $R_0 > 0$ is chosen to be such that

$$F_{M_r}(R_0) \le \mu\tau_0/4,\ \bar{\varepsilon}(R_0) \le 10^{-4},\ R_0 < (1-\tau)\mathfrak{r}.$$

□

Lemma 5.5. *For any $\tau > 0$ small enough, there exists $\varepsilon_2 = \varepsilon_2(\tau) > 0$ such that the following hold: E is an sliding almost minimal set in Ω with sliding boundary $\partial\Omega$ and gauge function h, $x_0 \in E \cap \partial\Omega$, Ψ is a mapping as in Lemma 5.1 and C_Ψ is the constant as in (5), if $r_0 > 0$ satisfy that $C_\Psi r_0^\alpha \le \varepsilon_2$, $h(2r_0) \le \varepsilon_2$ and $F_E(x_0, r_0) \le \varepsilon_2$, then for any $r \in (0, 9r_0/10)$, we can find sliding minimal cone $Z_{x_0,r}$ in $\mathrm{Tan}(\Omega, x_0)$ with sliding boundary $\mathrm{Tan}(\partial\Omega, x_0)$ such that*

$$\begin{aligned} \mathrm{dist}(x, Z_{x_0,r}) \le \tau r,\ x \in E \cap B(x_0, (1-\tau)r) \\ \mathrm{dist}(x, E) \le \tau r,\ x \in Z_{x_0,r} \cap B(x_0, (1-\tau)r), \end{aligned}$$

and for any ball $B(x, t) \subset B(x_0, (1-\tau)r)$,

$$|\mathcal{H}^2(Z_{x_0,r} \cap B(x, t)) - \mathcal{H}^2(E \cap B(x, t))| \le \tau r^2.$$

Moreover, if $E \supset \partial\Omega$, then $Z_{x_0,r} \supset \mathrm{Tan}(\partial\Omega, x_0)$.

Proof. It is a consequence of Proposition 30.19 in [3]. □

Corollary 5.6. *Let Ω, $E \subset \Omega$, $x_0 \in \partial\Omega$, h and F_E be the same as in Theorem 5.4. Suppose that the gauge function h satisfying*

$$h(t) \le C_h t^{\alpha_1} \text{ for some } C_h > 0, \alpha_1 > 0 \text{ and } 0 < t < t_0. \tag{5.9}$$

Then there exists $\delta > 0$ and constant $C = C_{20}(\mu, \lambda_0, \alpha, \alpha_1, \beta, \xi_1) > 0$ for $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}$ such that, whenever $0 < \rho_0 \leq \min\{1, t_0, r_0(x_0)\}$ satisfying

$$F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1} \leq \delta,$$

we have that, for $0 < \rho \leq 9\rho_0/20$,

$$d_{x_0,\rho}(E, x_0 + \mathrm{Tan}(E, x_0)) \leq C(F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1})^{1/4} (\rho/\rho_0)^{\beta/4}.$$

Proof. By Theorem 5.4, there exist $\rho_0 > 0$ such that

$$d_{x_0,\rho}(E, x_0 + \mathrm{Tan}(E, x_0)) \leq C(\rho/\rho_0)^{\beta/4},\ 0 < \rho \leq 9\rho_0/20,$$

where $\rho_0 > 0$ is chosen to be such that

$$0 < \rho_0 \leq \min\left\{1, t_0, r_0(x_0), R_0/2, (2C_\Psi)^{-1/\alpha}\right\} \tag{5.10}$$

and $R_0 > 0$ is chosen to be such that

$$F_{M_r}(R_0) \leq \mu\tau_0/4,\ \bar{\varepsilon}(R_0) \leq 10^{-4},\ R_0 < (1-\tau)\mathfrak{r}.$$

By Lemma 5.5, there exists $\delta > 0$ such that if $F_E(x_0, 2\rho_0) + C_\Psi \rho_0^\alpha + C_h \rho_0^{\alpha_1} \leq \delta$, then (5) holds, and we get the result. □

Lemma 5.7. *Let Ω, E, x_0 and h be the same as in Theorem 5.4. Suppose that $\Theta_E(x_0) = 3\pi/2$. Then there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in B(x_0, r) \cap E$ and $0 < \rho < 2r$, we can find cone $Z_{x,r}$, which is a half plane in $\mathrm{Tan}(\Omega, x_0)$ union $\mathrm{Tan}(\partial\Omega, x_0)$ when $x \in \partial\Omega$ and a plane in $\mathbb{R}^3$ when $x \notin \partial\Omega$, satisfying that*

$$d_{x,\rho}(E, Z_{x,\rho}) \leq C\rho^\beta.$$

Proof. By Corollary 5.6, there exist $\delta > 0$ and $C > 0$ such that whenever $0 < \rho_0 \leq \min\{1, t_0, r_0(x_0)\}$ satisfying

$$F_E(x_0, 2\rho_0) + C_{\Psi_{x_0}} \rho_0^\alpha + C_h \rho_0^{\alpha_1} \leq \delta,$$

we have that, for $0 < \rho \leq 9\rho_0/20$,

$$d_{x_0,\rho}(E, x_0 + \mathrm{Tan}(E, x_0)) \leq C\delta^{1/4}(\rho/\rho_0)^\beta,$$

where $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}/4$. We take $\rho_1 \in (0, \rho_0)$ such that

$$F_E(x_0, 2\rho) + C_{\Psi_{x_0}} \rho^\alpha + C_h \rho^{\alpha_1} \leq \min\{\delta/2, \varepsilon_2(\tau)\}, \forall 0 < \rho \leq \rho_1.$$

If $x \in \partial\Omega \cap B(x_0, \rho_1/10)$, we take $t = \rho_1/2$, then apply Lemma 5.5 with $r = |x - x_0| + t$ to get that

$$\mathcal{H}^2(E \cap B(x, t)) \leq \mathcal{H}^2(Z_{x,r} \cap B(x, t)) + \tau r^2,$$

thus

$$\Theta_E(x, t) \leq \frac{1}{t^2}\mathcal{H}^2(Z_{x,r} \cap B(x, t)) + 4\tau \leq \frac{\pi}{2} + C_{\Psi_{x_0}} r^\alpha + 4\tau,$$

and

$$F_E(x,t) \le C_{\Psi_{x_0}} r^\alpha + 4\tau + 16h_1(t).$$

We get that $F_E(x, 2\rho) + C_{\Psi_x}\rho^\alpha + C_h\rho^{\alpha_1} \le \delta$ for $0 < \rho \le t/2$. Thus

$$d_{x,r}(E, x + \mathrm{Tan}(E,x)) \le C\delta^{1/4}(r/t)^\beta,\ 0 < r < 9t/20. \tag{5.11}$$

If $x \in \Omega \cap B(x_0, \rho_1/10) \setminus \partial\Omega$, we take $t = t(x) = \mathrm{dist}(x, \partial\Omega)$ then apply Lemma 5.5 with $r = |x - x_0| + t$ to get that

$$\mathcal{H}^2(E \cap B(x,t)) \le \mathcal{H}^2(Z_{x,r} \cap B(x,t)) + \tau r^2,$$

thus

$$\Theta_E(x,t) \le \frac{1}{t^2}\mathcal{H}^2(Z_{x,r} \cap B(x,t)) + 4\tau \le \pi + 4\tau.$$

It is follow from Section 12 in [1] that there is a constent $C_1 > 0$ such that

$$d_{x,r}(E, Z_{x,r}) \le C_1(r/t)^\beta,\ 0 < r < 9t/20 \tag{5.12}$$

for some plane $Z_{x,r}$.

There exists a constant $C_2 > 0$ such that for any $x \in B(x_0, \rho_1/10)$, there exists $x_1 \in B(x_0, \rho_1/5) \cap \partial\Omega$ with $x_1 \in \overline{E \setminus \Omega}$ such that

$$|x - x_1| \le C_2 \,\mathrm{dist}(x, \partial\Omega).$$

We take $0 < a < \beta/(1+\beta)$. For any $x \in B(x_0, \rho_1/5) \cap \partial\Omega$, if $r \le C_3 t^{1/(1-a)}$, then we get from (5) that

$$d_{x,r}(E, Z_{x,r}) \le C_1 C_3^{\beta(a-1)} r^{a\beta};$$

if $C_3 t^{1/(1-a)} < r < \rho_1/5$, then by (5), we have that

$$\begin{aligned}
d_{x,r}(E, x_1 + \mathrm{Tan}(E,x_1)) &\le \frac{|x - x_1| + r}{r} d_{x_1, |x-x_1|+r}(E, x_1 + \mathrm{Tan}(E, x_1)) \\
&\le C_4\left(1 + \frac{C_1 t}{r}\right)\left(\frac{r + C_1 t}{\rho_1/2}\right)^\beta \\
&\le C_5(1 + C_6 r^{-a})^{\beta+1} r^\beta \le C_7 r^{\beta - a\beta - a}.
\end{aligned}$$

We get so that there is a minimal cone $Z_{x,r}$ such that

$$d_{x,r}(E, Z_{x,r}) \le C_8 r^{\beta_1}$$

for $\beta_1 = \min\{a\beta, \beta - a\beta - a\}$ and any $0 < r < \rho_1/5$. $\square$

Lemma 5.8. *Let Ω, E, x_0 and h be the same as in Theorem 5.4. Suppose that $\Theta_E(x_0) = 7\pi/4$. Then there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in B(x_0, r) \cap E$ and $0 < \rho < 2r$, we can find cone $Z_{x,r}$, which is a cone of type $\mathbb{Y}_+$ in $\mathrm{Tan}(\Omega, x_0)$ union $\mathrm{Tan}(\partial\Omega, x_0)$ when $x \in \partial\Omega$ and a cone of type $\mathbb{Y}$ in $\mathbb{R}^3$ when $x \notin \partial\Omega$, satisfying that*

$$d_{x,\rho}(E, Z_{x,\rho}) \le C\rho^\beta.$$

Proof. By Corollary 5.6, there exist $\delta > 0$ and $C > 0$ such that whenever $0 < \rho_0 \le \min\{1, t_0, r_0(x_0)\}$ satisfying

$$F_E(x_0, 2\rho_0) + C_{\Psi_{x_0}}\rho_0^\alpha + C_h\rho_0^{\alpha_1} \le \delta,$$

we have that, for $0 < \rho \le 9\rho_0/20$,

$$d_{x_0,\rho}(E, x_0 + \mathrm{Tan}(E, x_0)) \le C\delta^{1/4}(\rho/\rho_0)^\beta,$$

where $0 < \beta < \min\{\alpha, \alpha_1, 2\lambda_0\}/4$. We take $\rho_1 \in (0, \rho_0)$ such that

$$F_E(x_0, 2\rho) + C_{\Psi_{x_0}}\rho^\alpha + C_h\rho^{\alpha_1} \le \min\{\delta/2, \varepsilon_2(\tau)\}, \forall 0 < \rho \le \rho_1.$$

If $x \in \partial\Omega \cap B(x_0, \rho_1/10)$, we take $t = |x - x_0|/2$, then apply Lemma 5.5 with $r = |x - x_0| + t$ to get that

$$\mathcal{H}^2(E \cap B(x, t)) \le \mathcal{H}^2(Z_{x,r} \cap B(x, t)) + \tau r^2,$$

thus

$$\Theta_E(x, t) \le \frac{1}{t^2}\mathcal{H}^2(Z_{x,r} \cap B(x, t)) + 9\tau \le \frac{\pi}{2} + C_{\Psi_{x_0}}r^\alpha + 9\tau,$$

and

$$F_E(x, t) \le C_{\Psi_{x_0}}r^\alpha + 9\tau + 16h_1(t).$$

We get that $F_E(x, 2\rho) + C_{\Psi_x}\rho^\alpha + C_h\rho^{\alpha_1} \le \delta$ for $0 < \rho \le t/2$. Thus

$$d_{x,r}(E, x + \mathrm{Tan}(E, x)) \le C\delta^{1/4}(r/t)^\beta, \ 0 < r < 9t/20.$$

If $x \in \partial\Omega \cap B(x_0, \rho_1/10) \setminus \partial\Omega$, we put $E_Y = \{x \in E : \Theta_E(x) = 3\pi/2\}$ and take

$$t = t(x) = \min\{\mathrm{dist}(x, \partial\Omega), \mathrm{dist}(x, E_Y)\},$$

then apply Lemma 5.5 with $r = |x - x_0| + t$ to get that

$$\mathcal{H}^2(E \cap B(x, t)) \le \mathcal{H}^2(Z_{x,r} \cap B(x, t)) + \tau r^2,$$

thus

$$\Theta_E(x, t) \le \frac{1}{t^2}\mathcal{H}^2(Z_{x,r} \cap B(x, t)) + 4\tau \le \frac{3\pi}{2} + 4\tau.$$

It is follow from Section 12 in [1] that

$$d_{x,r}(E, x + \mathrm{Tan}(E, x)) \le C(r/t)^\beta, \ 0 < r < 9t/20.$$

A Similar argument as above, we will get the result. □

Corollary 5.9. *Let Ω, E, x_0 and h be the same as in Theorem 5.4. Then there exist a radius $r > 0$, a number $\beta > 0$ and a constant $C > 0$ such that, for any $x \in B(x_0, r) \cap E$ and $0 < \rho < 2r$, we can find cone $Z_{x,r}$, which is a sliding minimal cone when $x \in \partial\Omega$ and a minimal cone in $\mathbb{R}^3$ when $x \notin \partial\Omega$, satisfying that*

$$d_{x,\rho}(E, Z_{x,\rho}) \le C\rho^\beta. \tag{5.13}$$

Proof. It is follow from Lemma 5.7 and Lemma 5.8. □

Proof of Theorem 1.2. It is follow from Corollary 5.9 and the generalization in [1] of Reifenberg's topological disk. More precisely, Section 10 in [1] gives a C^1 estimates, but in our case, we can get a bit little more that equation (10.22) in [1] and (5.9) give a $C^{0,\beta}$-Hölder estimates of its differential. □

6 Existence of the Plateau problem with sliding boundary conditions

Let $\Omega \subset \mathbb{R}^3$ be a closed domain such that the boundary $\partial\Omega$ is a 2-dimensional manifold of class $C^{1,\alpha}$ for some $\alpha > 0$. Let $E_0 \subset \Omega$ be a closed set with $E_0 \supset \partial\Omega$. We denote by $\mathscr{C}(E_0)$ be the collection of all competitors of E_0.

Theorem 6.1. *There exists $E \in \mathscr{C}(E_0)$ such that*

$$\mathcal{H}^2(E \setminus \partial\Omega) = \inf\{\mathcal{H}^2(S \setminus \partial\Omega) : S \in \mathscr{C}(E_0)\}$$

Proof. We put

$$m_0 = \inf\{\mathcal{H}^2(S \setminus \partial\Omega) : S \in \mathscr{C}(E_0)\}.$$

If $m_0 = +\infty$, we have nothing to do; if $m_0 = 0$. We now assume that $0 < m_0 < +\infty$.

Let $\{S_i\} \subset \mathscr{C}_0$ be a sequence of competitors bounded by $B(0,R)$ such that

$$\lim_{i\to\infty} \mathcal{H}^2(S_i \setminus \partial\Omega) = m_0.$$

Apply Lemme 5.2.6 in [8], we can fined a sequence of open sets $\{U_i\}$ and a sequence of competitors $\{E_i\} \subset \mathscr{C}(E_0)$ of E_0 bounded by $B(0,R+1)$ such that

- $U_i \subset U_{i+1}$, $\cup_{i\geq 1} U_i = B(0,R+2) \setminus \partial\Omega$;
- $E_i \cap U_i \in QM(U_i, M, \mathrm{diam}(U_i))$ for constant $M > 0$;
- $\mathcal{H}^2(E_i) \leq \mathcal{H}^2(S_i) + 2^{-i}$.

We assume that E_i converge locally to E in $B(0,R+2)$, pass to subsequence if necessary, then by Corollary 21.15 in [3], we get that E is sliding minimal.

We get, from Theorem 1.2 and Theorem 1.15 in [2], that E is a Lipschitz neighborhood retract. But we see that E_i converges to E, we get so that E is a competitor. □

References

[1] G. David, T. De Pauw, and T. Toro, *A generalization of reifenberg's theorem in $\mathbb{R}^3$*, Geometric and Functional Analysis **18** (2008), 1168–1235.

[2] Guy David, *$C^{1+\alpha}$-regularity for two-dimensional almost-minimal sets in $\mathbb{R}^n$*, J. Geom. Anal. **20** (2010), no. 4, 837–954. MR2683770

[3] ______, *Local regularity properties of almost and quasiminimal sets with a sliding boundary condition* (2014), Preprint, available at http://arxiv.org/abs/1401.1179.

[4] ______, *Should we solve Plateau's problem again?*, Advances in analysis: the legacy of Elias M. Stein, 2014, pp. 108–145. MR3329849

[5] K. J. Falconer, *The geometry of fractal sets*, Vol. 85, Cambridge university press, 1986.

[6] Yangqin Fang, *Hölder regularity at the boundary of two-dimensional sliding almost minimal sets*, Adv. Calc. Var. (accepted) (2016).

[7] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.

[8] V. Feuvrier, *Un résultat d'existence pour les ensembles minimaux par optimisation sur des grilles polyédrales*, Ph.D. Thesis, 2008.

[9] C. Kuratowski, *Topologie*, Vol. 2, Éditions Jacques Gabay, Sceaux, 1992.

[10] J. E. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math **103** (1976), 489–539.

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