

Chapter 3

The Impact of Notation Systems: From the Practical Knowledge of Surveyors to Babylonian Geometry

Peter Damerow

3.1 Introduction

Knowledge about the early development of human cognition can be gained from two sources:

1. Archaeologically excavated artifacts can be interpreted by elaborating the minimal cognitive preconditions that made the invention and use of these artifacts possible.
2. If archaeological findings suggest that social and economic settings in an early human community are similar to those known from indigenous peoples untouched by modernization processes, it can be concluded that their members possessed similar cognitive abilities required to perform similar activities.

For the conditions under which prehistoric humans lived, abilities reconstructed by methods of this type are, as a rule, perfectly sufficient to master the problems to which they were exposed in their environment. Compared to later historic developments, however, such abilities show characteristic differences.

Max Wertheimer showed that indigenous cultures do not share our context-independent concept of number, but solve problems involving the judgement of quantities with context-specific mental constructions he called *Zahlgebilde*.¹ Counting sequences, if there are any, remain rudimentary. They are often linked to the specific classes of objects to which they are applied. No elaborated instruments for measuring sizes are used. Consequently, the languages of non-literate peoples lack words designating abstract concepts of number and size.

The situation is similar in the case of the concepts of time and space. Norbert Elias has argued convincingly that the metric concept of time is a product of modern economy and technology and neither required nor functional in premodern societies.² The classical concept of space, and more so the concept of space as an aspect of the integrated concept of spacetime in contemporary non-classical physics, are also dependent on conditions provided by modern science and technology. By contrast, the previous chapter has shown how tasks involving abilities of spatial orientation were solved among non-literate peoples. The development of spatial knowledge in these communities is linked closely to external knowledge representations, in particular to coordinated actions, landmarks for orientation, and lexical and grammatical properties of spoken language that were used to express spatial

¹Wertheimer 1925.

²Elias 1984.

relations. Such abilities are not based on standardized measures representing spatial relationships quantitatively. No symbol systems exist which would allow for calculation techniques like those required, for instance, to determine the sizes of areas. The abilities remain context-specific and cannot be applied universally to spatial magnitudes and relations.

This raises the interesting question about the historical circumstances under which the nature of the conceptualization of space changed and what brought about these changes. Obviously, this development is somehow related to the invention of writing. Fortunately, in the case of what is possibly the earliest writing system, that is, cuneiform writing invented in Mesopotamia, an abundance of preserved sources from the time of its invention well into the time of Classical Antiquity provides information about the development of conceptual patterns and corresponding technologies and methods dealing with spatial relations.

The focus of the following analysis on this development will be concentrated on its early phase. The earliest sources that indicate innovations related to the conceptualization of spatial relations, in comparison to what is known from non-literate cultures, are administrative tools, in particular proto-cuneiform administrative tablets written between 3200 and 3000 BCE, which triggered the invention of writing. The analysis will cover the following time period of about 1500 years into the middle of the second millennium, when the so-called Babylonian mathematics was fully developed. It will be shown under which circumstances knowledge about spatial relations in this period developed from the basic forms known from non-literate societies into an esoteric art of formulating complex geometrical problems and solving them using sophisticated arithmetical tools applied to geometrical intuition. It will be shown that this development was not a direct consequence of the invention of writing. Rather, it has to be conceived as a coevolution of glottographic and non-glottographic symbol systems³ ending up with the dissociation into writing, arithmetic, and geometry.

This development originated in the context of the bureaucratic administration which necessarily evolved together with the concentration of economic and military power in the early Mesopotamian cities and the emergence of socially stratified, centralized states in the time period from the late fourth to the late third millennium BCE. Measurement technologies had to be developed and externally represented in written or other symbolic form in order to control the acquisition and redistribution of unprecedented amounts of resources. The quantification of spatial relations by length, area, and volume measures, in particular, was based primarily on the growing knowledge of surveyors and the reflection on their means and practices. The resulting mental constructions remained implicit, but can partly be reconstructed from the arithmetical operations documented by administrative records and by Babylonian mathematics. One of the surprising results of the analysis will be that the outcome of this development differs from the geometry we know from the Euclidean tradition. The kind of geometry that is part of Babylonian mathematics shows ‘non-Euclidean’ peculiarities such as the neglect of the role of angles, which result from the practices of the surveyors they reflect. It will become clear that the Babylonian case demonstrates that there is no canonical path to a single universal geometry.⁴

³The term *glottographic* denotes the dependence on spoken language. For a detailed discussion of the classification of early writing systems and their relation to spoken language (phonology, linearity, etc.), see Hyman 2006.

⁴The present paper is heavily based on my inaugural lecture 1994 at the University of Konstanz and its extended publication (Damerow 2001).

3.2 The origin of notation systems in Mesopotamia in the third millennium BCE

Rural communities as we know them from indigenous cultures apply simple techniques of quantifying such as using the breadth of fingers or the length of the forearm to control productive activities of knitting, weaving, or braiding. Long distances may be distinguished by counting the days of travel. However such quantifications remain embedded in the context in which they were used and do not give rise to their integration into a comprehensive concept of length. Accordingly, the assessment of areas or capacities by some kind of estimation or tool-based quantification does not cause any mental construction of a three-dimensional space that integrates lengths, areas, and volumes. It is reasonable to assume that the rural communities of early Mesopotamia applied similar techniques of rudimentary quantification to those known from extant indigenous cultures.

In the fourth millennium at the latest the nomadic tribes and rural communities in Mesopotamia were complemented by larger settlements and even fortified cities.⁵ This development was connected with unprecedented cognitive constructions. The first and foremost type of such constructions in the early phase of the development of new cognitive abilities triggered by administrative challenges was the integration of context-specific measures into comprehensive and standardized systems of measures represented by corresponding systems of symbolic notation.⁶

What is the evidence for such a cognitive development? Archeological finds from this period show that the new challenges of the situation required an extensive use of existing traditional methods to control quantities. This is indicated especially by the expansive use of certain geometrically shaped clay tokens whose functions were long unclear. Sealed spherical envelopes made of clay and containing combinations of such tokens finally made evident that the tokens were used as counters to document quantities of resources that were the subject of administrative transactions.⁷ Some of these envelopes bear impressed markings on their surfaces that – in view of the type of impression – can be identified as precursors of later numerical notations. At about the same time, such markings were pressed into the surface of sealed clay tablets, known as numerical tablets, the appearance of which immediately predates the invention of proto-cuneiform writing with its developed numerical notation systems.⁸

A close analysis of these archaeological findings and the results of deciphering the earliest proto-cuneiform administrative tablets strongly suggest that the invention of writing was in fact coupled with the emergence of completely new cognitive constructions, among them incipient forms of numbers.

Many prehistoric numerical notations preceding proto-cuneiform and cuneiform numerical notations consist of a few vertically or obliquely impressed marks similar in shape and size. They are comparable to one or more series of notches or dots as known from much older artifacts such as bones, tools, or cave paintings. It is likely that such repetitions of signs represent the quantity of the objects represented by the individual signs. Other notations on the prehistoric numerical tablets show a more complex structure. They often consist of sev-

⁵For an example, see the excavation of Habuba Kabira (Strommenger 1980).

⁶Nissen, Damerow, and Englund 1993.

⁷Schmandt-Besserat 1992.

⁸For an example see the numerical tablets of Jebel Aruda written in the second half of the fourth millennium BCE (Driel 1982).



Figure 3.1: Prehistoric sealed numerical tablet from Jebel Aruda with four series of impressions, repeated according to the quantities they represent

eral series of impressions that differ in size and shape. Generally, the symbols are arranged symmetrically and placed as if they would represent a hierarchy of quantities or numerical units.

Without knowing the context which might explain the function of such repeated symbols, it is impossible to decide definitively whether they represent units in the framework of an already existing number concept, or still represent objects such as animals, containers, or pieces of cheese. While tablets that display more than one type of repeated signs show them arranged in a way that resembles the structure of later numerical notations, they nonetheless share properties with simple types of numerical impressions, suggesting that they still represent objects and not units of a general metrological or numerical system.⁹

The most conspicuous properties indicating a non-numerical meaning of the individual signs are:

1. The *low degree of standardization of their shapes*. These shapes differ from tablet to tablet. Since we do not know the context of their usage it is for the most part impossible to decide whether signs on different tablets are variants of the same sign or denote different units with different values. Their meaning seems to be determined by the context of their use rather than by internal relations within a formal system of numeration, which would it make possible to identify their meaning independent of any knowledge about the context of their use.
2. Their *unlimited repeatability*. The numerical signs are frequently repeated more than ten times, even if they are placed in the middle of a complex numerical notation. The numerical notations of the prehistoric numerical tablets obviously lack the typi-

⁹This is true, for instance, for several of the above mentioned numerical tablets from Jebel Aruda, published by Driel 1982.

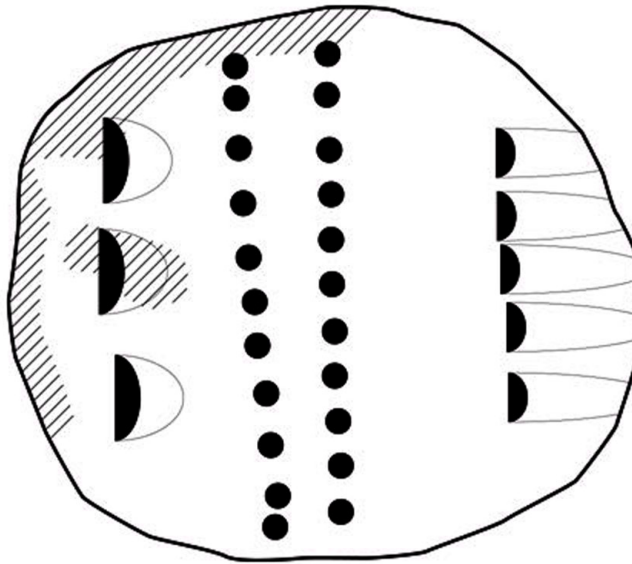


Figure 3.2: Numerical tablet from Jebel Aruda with a sign repetition exceeding any known relation between numerical or metrological units

cal bundling structure of all known notation systems of numbers. Even when repeated small units represent an amount greater than the next higher unit, they were nevertheless not converted. This makes it at the very least likely that these symbols represent real objects and not units of a numerical or metrological system.

These properties of prehistoric numerical notations strongly suggest that the numerical signs in fact still represent real units such as containers of different sizes, not values within a standardized system of context-independent measures. Their precise meaning was determined by the context in which they were used. This makes deciphering of the prehistoric numerical notations virtually impossible and, consequently, also the determination of the meaning of the combinations of clay tokens, which was transferred to the new medium of numerical tablets. In particular, it is impossible to find out whether prehistoric numerical notations existed which represented geometrical measures such as measures for length, area, and volume.

At the end of the fourth millennium, the situation changed radically. As a consequence of the invention of the proto-cuneiform writing system, the numerical notations were complemented with iconic graphs.¹⁰ The shapes of the signs became largely standardized. With few exceptions, repetitions of small units were systematically converted into higher units as soon as the repeated units exceeded their size. Several numerical notations qualified by

¹⁰Usually called ideograms. On the precise meaning of the term *iconic graph*, see Boltz 2006. Later, when the cuneiform writing system had developed into writing in the proper sense, that is, a system representing spoken language, such ideograms developed into a non-syllabic notation within a syllabic writing system. The same signs are then rightly called logograms or Sumerograms (i.e., adapted from Sumerian writing).

iconic graphs were now written on one and the same tablet. Sequences of such entries were frequently complemented by their total, using the conversion procedure determined by the size relationships of the applied notation system.

The internal consistency of the application of these rules to the numerical notations made it possible to decipher their meaning in spite of the systematic ambiguity which is a consequence of a persisting context-dependency of the meaning of the numerical signs.¹¹ In fact, the numerical signs had no fixed numerical values. They were arranged in different, but internally coherent systems of notations with changing numerical values depending on the system in which they were used. These systems were applied consistently in specific areas of application. In order to facilitate the identification of the system, the numerical signs were partly modified by additional strokes or impressions produced with the tip of a stylus. Generally, however, the context indicated by the iconic graphs accompanying the numerical notations, and the extremely different numerical values of the signs when they were used in different systems, were sufficient to identify the notation system used. Thus, the relation of the sign representing the unit in the counting system to the next higher unit in this system changes between one to six, one to ten, and one to eighteen, depending on the context of application. The numerical signs frequently changed even the order of their numerical values. The signs with the values of 60 and 3,600 units in the counting system applied to grain measures, for example, assume the values 180 and 60 respectively. Thus the first sign, which represented $\frac{1}{60}$ of the second sign, assumes a value 3 times higher than the second sign.¹²

It is obvious that – in spite of this still existing context dependency of the numerical signs' values on the context of their application – the consistent notation systems of the administrative proto-cuneiform tablets document a new kind of cognitive construction different from those of their prehistoric precursors. They represent the earliest well-documented notation systems of measures that are independent of the actual measuring tools, constituted by internal relations between units which determine their sizes. Among them, systems can be identified which implicitly define the dimensions of space: length, area, and volume.

The structures of the proto-cuneiform systems of numerical notations allow for some inferences about their origins. Most of them consist of a core group of measures with numerical relations between them that lack a systematic pattern. At the upper and the lower ends, these irregular patterns tend to become systematic, based on the sexagesimal pattern of one of the two counting series also documented by the administrative proto-cuneiform tablets. This distinction within the systems suggests that the core of a notation system represents inherited prehistoric measures which in a first phase of development had to be brought into standardized numerical relations taking into account the relative sizes determined by traditional measuring tools. In a second phase the core notation system was artificially expanded in order to cover the new challenges of a central administration of goods and resources. Since such an expansion was no longer determined by existing measures, this structure could be defined much more systematically than the structure of the core of the system.

In the case of the system of length measures, the core, which consisted of measures such as the finger and the cubit, is known only from later cuneiform sources. The reason

¹¹ For the results of deciphering of the proto-cuneiform numerical systems and an overview of the limited results of earlier attempts, see Damerow and Englund 1987. For a comprehensive account of the social context, see Nissen, Damerow, and Englund 1993 and the part written by R. Englund in Bauer, Englund, and Krebernik 1998, 15–233.

¹² See the overview of proto-cuneiform sign systems in Nissen, Damerow, and Englund 1993, 28–29.

may be that these measures were unimportant for the administrative activities recorded on proto-cuneiform tablets. It seems that in the context of these activities length measures were used predominantly for surveying.

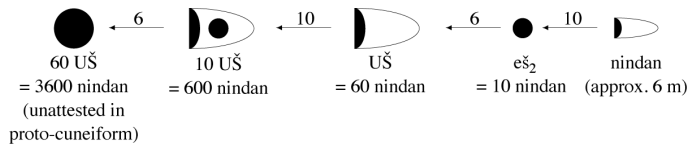


Figure 3.3: Relations between length measures on proto-cuneiform tablets

The length measures on the administrative proto-cuneiform tablets show a strict sexagesimal structure, suggesting that they were artificially created by officials of the central administration and played no role in prehistoric times. The basis of the system was the *nindan*, with a size of approximately six meters, probably derived from a measuring rod of the surveyors. This unit *nindan* was represented by the sign used in the sexagesimal counting system for the unit one. The higher units of the notation system for length measurements strictly follow the counting systems with signs for 10, 60, and 600, the last representing a length of about 3,600 meters. Contrary to what is known from the end of the third millennium BCE, such a distance still seems to extend beyond the field sizes at the beginning of the third millennium when the proto-cuneiform documents were written. At least, the size of 600 *nindan* is attested to on proto-cuneiform tablets only by so-called school texts.¹³

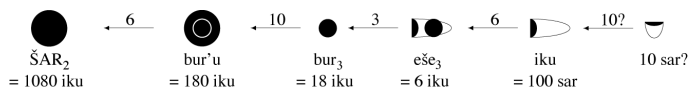


Figure 3.4: Relations between area measures on proto-cuneiform tablets

The situation is different in the case of area measures. The core notation system starts with a unit representing a size of about 0.36 hectares or 3,600 square meters, that is, 100 times of a unit later written with the cuneiform sign *sar*, designating a garden.¹⁴ This basic unit of the core system of 100 *sar* was again written with the sign used in the sexagesimal counting system for the unit 1. It was later designated as *iku*, a term representing a field of arable soil. The unit *iku* was followed by a unit six times greater, *eše*³, and a unit 18 times greater, *bur*³. This core group of area notations was obviously derived from prehistoric measures of fields, the sizes corresponding to what one would expect from rural communities where the sizes of fields are related to the work to be invested and the yield required to support a family or a clan. With the creation of the area notation system, the measures forming the core group must have been redefined by the numbers that most closely approximate their originally imprecise relations to each other, resulting in the unsystematically organized core

¹³The number of proto-cuneiform tablets documenting length measures is relatively small. They contain length measures of fields up to approximately seven kilometers. See, for example, the school text discussed in Nissen, Damerow, and Englund 1993, 50.

¹⁴There is one tablet containing a sign possibly representing ten *sar*; see Nissen, Damerow, and Englund 1993, 57.

group of units. This core group of area notations was expanded, again following the structure of sexagesimal counting. Accordingly, the proto-cuneiform administrative tablets contain signs for 10 and 60 *bur*³, which were artificial extensions of the core system to meet the requirements of controlling larger areas of arable land in a stratified society as compared to rural communities.

While the notation systems of length and area measures documented by proto-cuneiform administrative tablets were dependent on each other (as will be shown in the next section), notation systems of volume measures were still completely independent at that time. The reason is that they were used in completely different contexts, probably representing containers ranging from small ration bowls up to huge silos of about 260,000 liters.¹⁵ The main area of application of this notation system was to represent amounts of different types of grain or liquid grain products. Different types of grain and grain products such as grain, barley groats, or malt were indicated by modifying the numerical signs.

The core notation system for volume measures consists of five signs with values ranging from 4.8 to 4,320 liters. Taking into account that, as a staple food, grain had to be stored for a period from one harvest to the next even in small-scale rural communities, it is plausible that all six of these measures go back to containers already used in prehistoric times. This core notation system was extended to higher units again by artificially adding sexagesimally structured signs, documented for ten and 60 times of the highest unit of the core system. The system was extended to smaller units by using unit fractions from $\frac{1}{2}$ to $\frac{1}{6}$, and probably also $\frac{1}{10}$ of the smallest unit of the core system.

The creation of coherent notation systems for length, area and volume was a major step towards a context-independent concept of space. While it is true that all three systems were still closely related to certain areas of application, that is, surveying in the case of length and area measures, and storage facilities in the case of volume measures, they nevertheless changed the direction of the determination. The handling of real objects no longer determined the methods of their quantification, and thus the perception of the measures. Instead, the notation system determined how the objects had to be quantified, thereby turning them into realizations of amounts determined by the notation system as a tool of administrative control.

3.3 The problem of multiplication and of the calculation of areas of fields

In addition to the creation of coherent notation systems there was another, even more influential innovation made possible by the creation of these systems. Most of these systems were constructed independently, based only on the different quantities of the same type of measure. However, in the case of length and area measures it seems that they depended on length measures from the very beginning. This connection must have influenced the construction of the notation system for areas and made it dependent on the system of length measures. This dependency probably reflects the fact that the sizes of fields were determined by surveyors on the basis of length measurements.

How precisely did surveyors assign values of sizes to the fields? While most of the entries of proto-cuneiform tablets concerning fields give their sizes without any information

¹⁵On the determination of the absolute sizes of the signs for capacity measures, see Damerow and Englund 1987, 153–154.

on how the surveyors determined them, a small group of extant tablets contains measurements of the lengths and widths of fields together with the resulting areas. These examples show that the areas were determined implicitly by the relation that an area unit has the size of an area of 1 unit length and 1 unit width. We know from the later tradition that 1 *sar* was the size of an area of 1 *nindan* length and 1 *nindan* width. Accordingly, 1 *iku* was the size of an area of 10 *nindan* length and 10 *nindan* width. Although these relations are not directly attested to in proto-cuneiform surveying documents, the relations between higher units makes evident that the area notations in these documents were already based on these relations. In accordance with the basic relation between *nindan* and *iku* known from later documents, proto-cuneiform documents show, for instance, that the area of 2 *bur*³, which is 360 *iku*, was the size of a field where length and width both measure 60 *nindan*.

How were the sizes of fields calculated on the basis of these relations? From a modern point of view and assuming that the fields were rectangular their areas have to be calculated by multiplying length and width. For this purpose, the measures of length and width would have to be converted into the same unit, preferably into the smallest unit *nindan*. The numbers of these units would have to be multiplied by some kind of algorithm, followed by a conversion into the standard area measures. But the evidence on the extant tablets precludes this interpretation.

1. The calculation of areas by multiplying the number of units of the length by the number of units of the width presupposes that the rectangular area is conceived as being covered with unit squares the number of which is defined as the size of the area. But the idea to determine the size of an area by covering it with squares was an unfamiliar idea to the whole Babylonian mathematical tradition. This is because the surveyors who coined the basic geometrical concepts created what Solomon Gandz described as a *geometry of lines*,¹⁶ that is, a geometry without any concept of an *angle*. The size of an area was conceived as being determined by the length and width of the area independent of its shape. Any area with equal length and width was considered to have the same size. It will become clear in the following that this kind of geometry was perfectly adapted to carry out the task of the surveyors to determine the sizes of fields making use of only length measurements.
2. In proto-cuneiform documents the length measures were recorded with a precision down to 1 *nindan*. The area measures, however, were never recorded with a precision down to 1 square *nindan*, that is, to the level of 1 *sar*. While the lowest unit in area notations is usually the *iku*, areas are not usually recorded at this level of precision, but use *iku* as the smallest unit. It is unlikely that the units of *sar* which would automatically result from a multiplication of numbers of *nindan* were simply suppressed.
3. The two next higher units, the *eše*³ with 600 *sar* and the *bur*³ with 1800 *sar*, can no longer be expressed as squares of a length recorded with the applied notation system. But in any case they were too big to justify the precision of 1 *nindan* used by the surveyors of that time as the basic unit of length measurements. One must acknowledge that the sizes of areas were not perceived as the result of the multiplication of sexagesimal numbers to yield the number of square *nindan*, followed by a conversion into the irregularly structured system of area measures. Obviously, the notations representing

¹⁶Gandz 1929.

highly differentiated metrologies did not encourage the development of methods to perform abstract multiplications.

The neglect of algorithms for performing the multiplication of arbitrary sexagesimal numbers is characteristic until the end of the third millennium. The third millennium administrative tablets document instead three different kinds of implicit multiplication already evident in proto-cuneiform administrative documents.

1. Similar to the multiplication designated by the English term *times*, quantitative entries could be multiplied by small whole numbers, a technique which can easily be performed as repeated addition. From a modern point of view this arithmetical technique is the application of a dimensionless numeric operator to a qualified magnitude.
2. Linear relations between two sets of quantities such as the numbers of workers and corresponding amounts of rations to feed them can easily be extended to tables of corresponding pairs of values by applying the first type of multiplication to both of the corresponding values. In this case the factor by which the amounts of rations could be calculated from the numbers of workers, usually a fraction, remains completely implicit. From a modern point of view this arithmetical technique is the application of a dimensionless numeric operator to a linear function.
3. Areas of fields are, as we have seen, somehow calculated from length measures. From a modern point of view this arithmetical technique determines values by a bilinear function from one type of a specific kind of qualified magnitude, represented by length measures, into another type of qualified magnitude, represented by area measures.

The last type of multiplication is the subject of the discussion here. It is the only type where it is not obvious what kind of operations were performed in order to obtain the result. However, an atypical early Dynastic tablet excavated at Fara provides some critical information for the reconstruction of this procedure since it bears on its reverse some scribbled numerical notations that seem to be intermediate results of the calculation procedure.¹⁷

In the following entries on the obverse of the tablet the measures of a field and its area are registered:¹⁸

1. The *second entry* contains its length, measuring 1 UŠ 1 eš² 2 nindan.¹⁹
2. The *third entry* contains its width, measuring 1 UŠ 2 eš² 2 nindan.²⁰

¹⁷See the interpretation of this tablet in Damerow 2001, 260–261. A rough drawing of the tablet was published by Jestin 1937, TSS 51. According to a photo and an unpublished copy of the tablet by Kazuya Maekawa discussed by Jöran Friberg at the *Third Workshop on Concept Development in Babylonian Mathematics* in December 1985 in Berlin, the drawing Jestin published contains serious errors. In particular, this drawing does not distinguish between a numerical sign written at the left edge of the tablet and the scribbled notation on the reverse nearby. The reconstruction of the numerical notations presented here is based on the photo of the tablet and Maekawa's copy.

¹⁸Since the first and the last entry on the obverse and the first line on the reverse are unrelated to the calculations discussed in the following, they are not taken into account here. The first entry on the obverse which is nearly illegible, probably due to an erasure by the ancient scribe, seems to contain an area notation of 1 eše³ 3 iku. This area equals the notation in the clearly legible first line on the reverse. The relation to the rest of the tablet is unclear. The colophon in the last entry of the obverse seems to contain a geographical designation and a title.

¹⁹The value seems to have been corrected by the ancient scribe from 1 UŠ 2 eš² 2 nindan to 1 UŠ 1 eš² 2 nindan.

²⁰In addition to the metrological notation of the width and the sign for the width of a field, the entry contains three oblique strokes which might be interpreted as representing three units of smaller measure. Since such a small length measure does not occur on any other tablet related to surveying from the Fara period, the meaning of the three strokes remains obscure. Smaller units known from later times have various sizes. 1 nindan equals 4 ni-kaš, 6 kuš-numun, or 12 kuš, see for instance the tables of calculated areas on the tablets OIP 14, 70 Edzard 1969 and

3. The *fourth entry* contains an area notation of $3 \text{ bur}^3 5 \text{ iku}$. Assuming that the readings of the measures of length and width are correct, the area of the field was $3 \text{ bur}^3 5 \text{ iku} 4 \text{ sar}$ which is extremely close to the area of $3 \text{ bur}^3 5 \text{ iku}$ registered in the *fourth entry*.

How was this area calculated? The answer is provided by four of the five scribbled numerical notations on the reverse of the tablet:

1. The *second line* contains the notation $2 \text{ bur}^3 1 \text{ eše}^3 1 \text{ iku} 20 \text{ sar}$, which is precisely the area of a field with a length of $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$ and the width of 1 UŠ .
2. The *third line* contains the notation $2 \text{ eše}^3 2 \text{ iku}$ which is close to the area of a field with the same length of $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$ and the width of 2 eš^2 .²¹
3. The *fourth line* contains the notation $3 \text{ bur}^3 3 \text{ iku}$, which is the sum of the second and the third line, thus representing the area of a field with a length of $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$ and a width of $1 \text{ UŠ} 2 \text{ eš}^2$. Again, the precise value of $3 \text{ bur}^3 3 \text{ iku} 60 \text{ sar}$ was rounded by dropping the 60 sar.
4. Finally the *fifth line* contains the area of a field of $1 \frac{1}{2} \text{ iku} (?)$ which is the rounded value of a field with the same length of $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$ and the width of 2 nindan .²²

Thus three of the scribbled numerical notations on the reverse of the tablet represent the results of the calculation of partial areas of the field with a length of $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$, and the three parts of the width 1 UŠ , 2 eš^2 , and 2 nindan . Their sum, $3 \text{ bur}^3 4 \frac{1}{2} \text{ iku}$, is the approximate area of a field with a length measuring $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$ and a width measuring $1 \text{ UŠ} 1 \text{ eš}^2 2 \text{ nindan}$ as registered in the second and third entry on the obverse of the tablet. Rounded up to $3 \text{ bur}^3 5 \text{ iku}$, the calculated area equals the fourth entry on the obverse of the tablet.

This suggests that the ancient scribes used a calculation procedure based on the knowledge of the area for each pair of units of length and width measures:

If the length equals 1 nindan and the width 1 nindan then the area equals 1 sar.

If the length equals 1 nindan and the width 1 eš^2 then the area equals 10 sar.

If the length equals 1 nindan and the width 1 UŠ then the area equals 60 sar.

If the length equals 1 eš^2 and the width 1 eš^2 then the area equals 1 iku.

If the length equals 1 eš^2 and the width 1 UŠ then the area equals 1 eše^3 .

If the length equals 1 UŠ and the width 1 UŠ then the area equals 2 bur^3 .

A field with a length or a width that is a multiple of a length unit has an area which is the same multiple of the corresponding *unit area*, that is, the area grows linearly with its length or width. Thus, partial areas such as those represented by the scribbled notations on the reverse of the tablet discussed above could easily be calculated from the *unit areas* by using the well-documented techniques to operate with linear relations.

This procedure was necessarily specific for the calculation of areas and was never adapted to other problems that, from a modern point of view, involve multiplication. It was based only on the arithmetical techniques described above. Furthermore, it was implicitly assumed that the sum of the sizes of partial areas equals the size of the total area.

CUNES 50-08-001 Friberg 2007a, 419–425, 499–500. For all three units the additional area (between 18 and 54 sar) the additional area would not substantially change the result but bring it closer to the total area recorded in the fourth entry of the obverse.

²¹The precise value would have been $2 \text{ eše}^3 2 \text{ iku} 40 \text{ sar}$. The 40 sar have obviously been dropped.

²²The precise value would have been 1 iku 42 sar.

Besides these conditions the procedure did not presuppose any arithmetical techniques that are unattested by the extant sources. In particular, the sizes of areas could be determined without making a detour through an intermediate sexagesimal result.

The precondition for effective application of such a procedure was, however, the knowledge of the sizes of *unit areas* for all combinations of length and width units. Tablets dating back to the first half of the third millennium, in fact, document that learning the sizes of such *unit areas* was an important part of the training of the scribes. From the Early Dynastic II period around 2700 BCE onward, school tablets survived containing tables or problems which could be calculated or solved by applying simple linear operations to the *unit areas*. Tables of this kind usually do not contain the *unit areas* directly but rather areas calculated from single-digit sizes of length and width, that is, sizes represented by multiples of a single sign.

The earliest tablet containing such a table, dates back to the Fara period.²³ Three columns list, first, the length of a field, followed by its width (equal to the length) and the calculated area. The lengths and widths go down from 10 to 1 UŠ, which equals 6 eš², continuing from 5 to 1 eš², and ending up with 5 nindan, which is half of 1 eš².

Such tables do not necessarily contain lengths and width with equal sizes. A recently published tablet²⁴ is quite similar to the tablet from the Early Dynastic II period, but starts with a length of 5 nindan and a width of 5 UŠ, continuing by doubling both sizes to 1 eš² and 10 UŠ, respectively, subsequently adding 1 eš² and 10 UŠ to length and width, respectively. This operation goes up to 5 eš² and 50 UŠ.²⁵

Of special interest concerning the learning of the sizes of *unit areas* as a precondition for calculating areas is an Old Akkadian tablet some one hundred years younger, containing a problem and its solution.²⁶ The problem concerns small Old Akkadian length and area measures.

The length measures used are:

2 zipaḥ	= 1 GIŠ.BAD (= a cubit of approx. 50 cm)
2 GIŠ.BAD	= kuš ³ -numun
6 kuš ³ -numun	= 1 nindan
10 nindan	= 1 eš ²

The area measures used are:

60 gin ² -tur	= 1 gin ² (= approx. 0.6 square meter)
60 gin ²	= 1 sar
100 sar	= 1 iku

²³The tablet VAT 12593 was originally published by Deimel 1923 and has been discussed several times in the literature, see e.g. Powell 1976, 429–431, Nissen, Damerow, and Englund 1993, 136–139, Damerow 2001, 262–263, Friberg 2007a, 149–150.

²⁴Tablet MS 3047 of the Schøyen Collection, published by Friberg 2007a, 150–153, 484.

²⁵This is followed by an entry with huge sizes that are difficult to interpret.

²⁶Limet 1973, no. 36; see Powell 1976, 426–427.

The tablet contains the problem to calculate the area of a quadrilateral with equal sides the lengths of which are the sum of the involved length units. Thus, each side has the length:

$$1 \text{ eš}^2 \text{ 1 nindan 1 kuš}^3\text{-numun 1 GIŠ.BAD 1 zipah}$$

If, in fact, the calculation method was based on the knowledge of the *unit areas*, this seemingly strange problem finds a simple explanation: The solution of the problem requires adding up the 25 *unit areas*, each combining two length units. With this problem the teacher therefore was able to check whether his pupils knew these *unit areas* or were able to calculate them.

	ēš ₂	nindan	kuš ₃ -numun	GIŠ.BAD	zipah	
eš ₂	1 iku	10 sar	1 sar 40 gin ₂	50 gin ₂	25 gin ₂	Factor 10
nindan	10 sar	1 sar	10 gin ₂	5 gin ₂	2 gin ₂ 30 gin ₂ -tur	Factor 6
kuš ₃ -numun	1 sar 40 gin ₂	10 gin ₂	1 gin ₂ 40 gin ₂ -tur	50 gin ₂ -tur	25 gin ₂ -tur	Factor 2
GIŠ.BAD	50 gin ₂	5 gin ₂	50 gin ₂ -tur	25 gin ₂ -tur	12 gin ₂ -tur 30/60	Factor 2
zipah	25 gin ₂	2 gin ₂ 30 gin ₂ -tur	25 gin ₂ -tur	12 gin ₂ -tur 30/60	6 gin ₂ -tur 15/60	

Figure 3.5: *Unit areas* that had to be known to solve the problem on an Old Akkadian school text (Limet 36)

This procedure remained difficult to perform, however. In the third millennium, tablets containing length and area measures often show errors in the calculation of the areas, and partial areas including small measures were often treated negligently. Thus, in this case as well, the given solution is not correct.

The tablet contains the solution:

$$1 \frac{1}{4} \text{ (iku) gan}_2 \text{ 2} \frac{1}{2} \text{ šar 6 gin}_2 \text{ 15 gin}_2\text{-tur}$$

The correct solution would have been:

$$1 \frac{1}{4} \text{ (iku) gan}_2 \text{ 2} \frac{1}{2} \text{ šar 6 gin}_2\text{-tur } \frac{15}{60} \text{ gin}_2\text{-tur}$$

The cause of the error is obvious.²⁷ The smallest *unit area* has the size:

$$6 \text{ gin}_2\text{-tur } \frac{15}{60} \text{ gin}_2\text{-tur}$$

²⁷Powell 1976, 427 has argued that the error proves that, in spite of the lack of any direct textual evidence, the principles of the sexagesimal positional system were already used in the early third millennium. Friberg 2005a, § 4.3 argued convincingly that other interpretations of the cause of the error are possible. He gives an geometrical reason different from the interpretation favored here.

It was erroneously interpreted as:

6 gin₂ 15 gin₂-tur

Given that no designation is known for $\frac{1}{60}$ of the small unit gin₂-tur (about 1 square decimeter) the error may well have been caused by the scribe's unfamiliarity with such small units, which, in fact, played no role in the contemporary practice of surveying.

3.4 The invention of the *surveyors' formula*

The determination of the size of an area from its measured length and width met only some of the challenges with which the surveyors of the administration of the early states were confronted. The fields they had to measure were rarely regular enough that their sizes could be determined on the basis of just measuring a length and a width. They usually were quadrilaterals with four unequal sides.

In order to solve the problem to calculate the size of such fields, the surveyors of the third millennium Mesopotamia already used what is called the *surveyors' formula*. This formula is well-known, in particular from later Greek and Roman surveying technology. The formula determines the area of an irregularly shaped polygon with four edges as the result of the multiplication of the means of opposite sides.

Determining the mean of two measures was a simple problem given the notation of measures used by the ancient scribes of Mesopotamia. As a rule, the opposite sides of a field did not differ too much so that the number of higher units of the notation of their sizes was equal. These units could simply be neglected. Only the differing parts of lower units had to be added for each pair, and then half the result had to be appended to the strings of higher units in order to get an area with equal opposite sides.

This procedure is well documented by sources from the second half of the third millennium. However, a favorable circumstance allows us to trace the roots of this formula back to the end of the fourth millennium, that is, to the origins of cuneiform writing. Among the earliest proto-cuneiform tablets from the Uruk IV period there is a group of school tablets containing exercises in area calculations without giving the result. One of these tablets turned out to provide crucial information about the early history of the *surveyors' formula*. The tablet is heavily damaged, but the geometry of the two surviving fragments made it possible to completely reconstruct its content. Each side contains two columns with two entries each. On both sides, the entries in the first column are qualified by an archaic sign for the length of a field, known from other proto-cuneiform tablets containing results of surveying. Accordingly, the entries in the second columns of both sides are qualified by an archaic sign for the width of a field. Obviously, each side contains the measures of an irregularly shaped field. The tablet does not give any information on the areas of the fields, but the application of the *surveyors' formula* results in both cases in the same 'round number' of 600 bur₃ or 10 šar₂.

There are several indications that the tablet contains a school text:

1. The text contains an exercise, giving the primary data, but not the solution.
2. The field area to be calculated is unrealistically large, corresponding to an area of a field with a length of about 7 km and a width of about 5 km. At the beginning of the third millennium, such a huge field would not fit into what is known about sizes

of individual fields at that time. Numerical area notations of this order of magnitude occur only as totals of several fields.

3. The solution would have to be written with a sign for $10 \text{ } \check{s}ar_2$ which is not well attested by other documents of the time. Usually, this sign is the same sign as the sign for $10 \text{ } bur_3$, excluding misreadings only by its position in front of the sign $\check{s}ar_2$. It seems that the exercise was composed with the aim to demonstrate the difficulty of denoting large areas with the proto-cuneiform means of the time.

In addition to the attestation of the earliest known use of the *surveyors' formula*, the tablet provides further important information. Starting from a predetermined result, the exercises are constructed accordingly on both sides of the tablet to produce the desired result. In the present case, the desired result is $10 \text{ } \check{s}ar_2$. One of the possibilities to arrive at this solution is to choose a field of 1,200 nindan length (written as two times the sign for 600 nindan) and 900 nindan width (written with one sign for 600 nindan and five signs for 60 nindan). From this basic data the problem on the obverse of the tablet is constructed by adding 30 nindan to the first width and subtracting 30 nindan from the second width. The problem on the reverse of the tablet is constructed by subtracting 220 nindan from the first length and adding 220 nindan to the second length, and furthermore adding 380 nindan to the first width and subtracting 380 nindan from the second width. Calculating the means of opposite sides then results in the simple exercise to calculate the area of a field of 1200 nindan length and 900 nindan width.

This exercise tablet thus throws some light on the origins of the *surveyors' formula*. The ancient surveyors apparently assumed that the area of a field remains equal if they subtract some part from one side of a field and add a part of the same size to the opposite side.

From a modern point of view, the *surveyors' formula* yields only an approximate value of the 'real' area, that is, the procedure approximates the area as it is defined in the tradition of Euclidean geometry. But this cannot be the way the ancient surveyors of the third and second millennium BCE perceived the values they calculated. As long as the surveyed fields do not deviate too much from a rectangle, the *surveyors' formula* does in fact yield a very close approximation to the Euclidean area. Even if the surveyors had had any idea of an Euclidean area concept, it would have been difficult for them to find any empirical evidence of a difference from what they calculated using the *surveyors' formula*.

Moreover, for the time before the first millennium BCE there are no indications of any alternative concept of area that might have taken into account not only length measures but also the angles between the sides of an area. For the ancient surveyors an area must have been implicitly defined as the result of what they had calculated. There is not even a term for 'area' distinct from the numerical notations representing specific quantities. Only the term *iku* (sign form GANA₂) developed from a specific area unit into a general term for fields, used as a semantic marker of area notations. Furthermore, the term *a-ša₃* designating a field, also written with the iconic graph GANA₂, ultimately developed into a general term for a calculated area in the Babylonian mathematics of the second millennium BCE. But, as will be shown in the following, the term was always identified with the result of the *surveyors' formula*.

3.5 Sophisticated surveying techniques in the Ur III period

The second half of the third millennium BCE was the era of the rise of big empires in Mesopotamia, culminating in the empire of the *Third Dynasty of Ur* (known as the Ur III period) which integrated southern and northern Mesopotamia and even parts of southwestern Iran for somewhat less than 100 years.²⁸ This integration was accompanied by reforms in the techniques of social control initiated by the powerful emperor Šulgi and, in particular, in the bookkeeping techniques used by the centralized administration.

One of these innovations concerned the administration of arable land. Šulgi created a new category of royal domains that were distributed to high-ranking officials in exchange for their services. Fields such as those of the royal domains could take on considerable size and an irregular shape.

This may be the reason for the emergence of a new type of cuneiform tablets documenting the work of surveyors. These documents do not simply give the measures of the surveyed fields but also drawings of their shape. The fields are drawn as polygons approximating the shapes of the real fields. These polygons were divided into triangles and quadrilaterals. More precisely, mostly one or more quadrilaterals were used to roughly approximate the field. The area of this group of quadrilaterals was designated by the Sumerian term *temen*. The term is usually translated as ‘fundament’ since it was primarily used to designate the foundation of a temple or a similar building, indicated by pegs nailed into the ground before the building was erected. The sketched *temen* was then further adapted to the shape of the field by adding small triangles and quadrilaterals to the edges where they did not reach the real border of the field and by subtracting such small areas where the edges went beyond them.

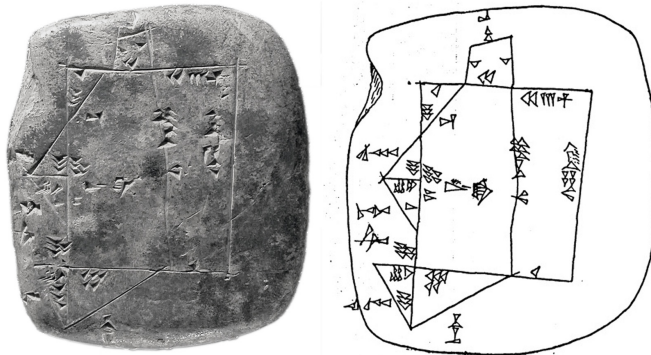


Figure 3.6: Field map of the Ur III period (MVN 10, 214)

Triangles were considered to be half of a quadrilateral with the longest side as the common side of both halves. Accordingly, the size of their area was calculated by halving the product of their two smaller sides. The size of the area of quadrilaterals was calculated using the *surveyors' formula*. In this way, first the size of the *temen* was calculated by

²⁸Gibson and Biggs 1991.

adding up the partial areas in the case of complex shapes of the *temen*. The measuring and calculation of the sizes of the small appended or cut-off areas was often simplified by measuring only three of the four sides of a quadrilateral, leaving out the measurement of the outer border. The results of the calculation of the sizes of the appended or cut-off areas were then added to, or subtracted from, the size of the area of the *temen* in order to obtain the size of the total area of the irregularly shaped field.

It is characteristic of the field maps of the Ur III period that they were never drawn to scale. The shapes of the drawn fields always turn out to be heavily distorted when compared with the real fields that were surveyed, as far as their shapes can be approximately reconstructed from the given measures.²⁹ This characteristic of the field maps can easily be explained by their function. For the administrators who needed to know the sizes of the fields because they had to decide on the amount of required seed grain, the required work force, or the expected harvest, and consequently for the surveyors who prepared the necessary data for them, the geometry of the fields was completely irrelevant. Using the iconic representation of the topology of the irregularly shaped fields, the drawings rather served to determine how the measurements had to be processed in order to arrive at their correct total sizes.

3.6 From context-dependent to abstract notations of quantities

Besides the use of maps to document complex shapes of fields and to organize the calculation of their areas, the Ur III period with its centralized administration provided the setting for what can be considered the most important mathematical invention of the ancient Near East, that is, the invention of the sexagesimal positional system of numerical notation.

Before the Ur III period numerical notations were based without exception on signs or sign combinations representing the absolute value of metrological units or, in the case of counting, on specific signs for each base unit, that is, specific signs for 1, 10, 60, 600, 3600 etc. The innovation consisted in repeating the signs for 1 and 10 for the higher units, that is, using 1, 10, 1, 10, 1 etc. for the same sequence of units, distinguishing them only by their position. Eleanor Robson characterized this innovation by arguing that the sexagesimal place value system

... changed the status of numbers from properties of real-world objects to independent entities that could be manipulated without regard to absolute value or metrological system.³⁰

²⁹According to the figures given on the field map MVN 10, 214, presented here as an example, the large quadrilateral in the center of the tablet has a length of 170 nindan (= 2 UŠ 50 nindan, written along the left side), an upper width of 53 ½ nindan (= 10 + 20 + 23 ½ nindan, written along the upper side), and a lower width of 60 nindan (= 50 + 10 nindan, written along the lower side). If the map were drawn to scale it should thus be approximately 3 times as long as it is wide. But the drawing on the tablet is close to a square. The reverse of the tablet (not depicted here) lists the essential figures required by the administration in three lines. The first line contains the sum of the small areas to be added to the large quadrilateral. Accordingly, the third line contains the sum of the areas to be subtracted. For some reason, the area to be subtracted consists of the whole right part of the large quadrilateral (damaged soil?) and the small triangle at its upper left corner. The second line contains the remaining area of the large quadrilateral, designated as *core temen*. The fourth line contains the name of the field, a fifth line the name of the official who authorized the document. The interpretation of the figures in the first three lines differs slightly from the figures given on the map, but comes close enough to them not to exceed the typical lack of accuracy frequently found in surveying documents. For details on the tablet discussed here, see Fuÿe 1915; Dunham 1986.

³⁰Robson 2008, 78.

Although the missing zero and the missing separation between whole units and fractions (comparable to the decimal comma in the modern decimal positional system) caused difficulties in applying the new system, the purpose of the innovation is clear and its inherent potential is obvious. Its invention was a precondition for the development of efficient calculation methods, in particular of universally applicable algorithms for multiplication and division.

Unfortunately, the origin of the new system is difficult to trace. For a long time the question was simply neglected, assuming that dealing with quantities is in any case a universal human faculty and the symbolic representation of numbers merely a circumstantial realization of a common human numeracy. The fact that all known sources documenting the sexagesimal positional system come from the first dynasty of Isin, the short-lived dynasty of Larsa, and the first dynasty of Babylon (the so-called Old Babylonian period), or later periods, that is the time from approximately 2000 BCE onwards, did not provoke any investigation of the origin of this system.

This situation changed over the last decades, along with a growing interest in texts of the third millennium that somehow indicate mathematical knowledge, but do not belong to the corpus of Babylonian mathematics in the narrow sense as it was understood by the pioneers of the decipherment of mathematical cuneiform texts, in particular Otto Neugebauer and Françoise Thureau-Dangin.³¹ In this context the search for the earliest attestations of numerical notations written in the Ur III period, that is in the last century of the third millennium, became an issue for research.

In contrast to the high number of cuneiform tablets documenting the use of the sexagesimal positional system in the following periods, attestations of the use of this system in the Ur III period are extremely scarce.³² There is, at least, evidence that the system was developed enough to establish tables of reciprocals which became the main tools for performing division.³³ However, neither is it possible to decide on the basis of current research results whether the invention of the new system was a unique creation ‘out of the box’ or the result of a gradual extension and improvement of some basic ideas, nor is it possible to determine the location of its origin, the paths of its dissemination, or by whom and to what extent it was used in this period.

However, the inherent potential of the sexagesimal positional system of numerical notations is obvious. The new system provided a basis for the development of efficient algorithms for performing multiplication and division independent of the context of application. In particular, the system made it possible to merge the different forms of multiplication into one unique numerical operation. The cumbersome method the surveyors used to determine the areas of fields, for example, could be replaced by such a multiplication algorithm. The precondition was, of course, that the traditional numerical notations be converted into the new system.

The abundance of extant early second millennium tablets dedicated to the use of the new system shows unambiguously that this potential was in fact realized shortly after its invention. The majority of these tablets contain tables which were probably produced in the context of scribal schools to teach them to perform the operations of the new system. A brief

³¹Neugebauer 1935/1937; Thureau-Dangin 1938; Neugebauer and Sachs 1945.

³²Robson 2008, 75–84.

³³See the tables documented by Robson 2008, 81–83 and by Oelsner in his contribution to Høyrup and Damerow 2001, 53–59.

survey of the most frequent types of such tables elucidates how systematically the potentials of the new system were explored and transformed into techniques to handle quantitative data:³⁴

1. *Metrological tables* containing lists of traditional metrological notations, in particular notations of capacity measures, weight measures, length measures, and area measures, ordered according to the sizes of the values they represent in the first column, and corresponding notations in the new sexagesimal positional system.
2. *Multiplication tables* containing the multiples 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 30, 40, 50, and 60 of certain fixed numbers, for example 6: 1 times 6, 2 times 6, etc.
3. *Tables of reciprocals* containing, as a rule, the reciprocals of 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, 50, 54, 60, 1 4 (= 60 + 4 = 64), and 1 21 (= 60 + 21 = 81).
4. *Tables of coefficients* used for converting values such as, in the context of architecture, to convert a volume into the number of bricks required to fill it; in the context of metal work to convert an area to be gold-plated into the required amount of gold; in the context of geometry to convert the length of the side of a square into its diagonal.³⁵

This enormous corpus of hundreds of tablets from the short period of approximately 500 years after the invention of the sexagesimal positional system represents an unprecedented type of tablets with an arithmetical content. While these tablets demonstrate a radical break with a tradition of 1000 years of dealing with quantitative data, they also exhibit a strong continuity, which explains the rapid development of the new system.

1. The *Metrological tables* document the basic continuity between the traditional metrological systems and the sexagesimal positional system. The underlying technique of transformation assured the smooth conversion of traditional data into the new format and, conversely, of sexagesimal positional data into the traditional form.
2. The *Multiplication tables* were created using the method familiar from treating linear relations between two sets of quantities by applying repeated additions to both sets.
3. The *Tables of reciprocals* were created by using and further developing a technique that had been developed in the context of the strictly sexagesimally structured weight measures to replace unit fractions such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., with a combination of multiples of smaller units, that is, 30, 20, 15, 12, etc. units.³⁶
4. The *Tables of coefficients* make the implicate factors of linear relations explicite. These tables show that the invention of the sexagesimal positional system had the immediate consequence of merging the three former types of multiplications (n times x , linear

³⁴Comprehensive lists and editions of such tables are supplied by Hilprecht 1906, 57–70 and Plate 1–30, Neugebauer 1935/1937, Vol. 1, 4–82, Neugebauer and Sachs 1945, 11–36, Robson 1999, 193–207, and Proust 2008, 15–73.

³⁵There are further types of tables which are, however, less common. Such tables contain, for example, squares and square roots, cubes and cube roots, exponentials and logarithms, and, to mention a unique finding, columns of figures based on Pythagorean numbers, that is, whole numbers a , b , and c that satisfy the condition $a^2 + b^2 = c^2$. On the various, partly incompatible interpretations of this extraordinary tablet (Plimpton 322) see Neugebauer and Sachs 1945, 38–41, Schmidt 1980, Friberg 1981, Damerow 2001, 231–232 and 294, and Robson 2001.

³⁶Since this method fails if the denominator has prime factors other than two, three, and five, which are the only ones contained in the base sixty of the system, such values are simply omitted in the tables of reciprocals. Furthermore, the calculation of reciprocals of notations with several digits is much more complicated. See Sachs 1947.

relations with an implicit factor, and area calculations) into one context-independent operation of multiplication.

The extent to which the sexagesimal positional system was used for bureaucratic everyday tasks is an open question, as is the extent to which such tables were used by administrators as reference tables and not merely as exercises in scribal schools. But, whoever used them must have adopted a concept of numerical notations that was based on cognitive constructions which were independent of specific contexts to a greater degree than those underlying the traditional metrological systems. This is particularly obvious in the case of length, area, and capacity measures. When represented by sexagesimal positional notations, they all turned out to be merely specific kinds of quantities, so that the traditional methods of calculation could be replaced by unified methods of addition, subtraction, multiplication and division. Only the last of these operations, namely division, remained problematic since it was not universally applicable.

There are further differences from the modern decimal positional system. *First*, the sexagesimal positional system lacked any indication of the absolute size. For example, the vertical wedge represents 1, but likewise also $\frac{1}{3600}$, $\frac{1}{60}$, 60, 60^2 , 60^3 , etc. Thus, additions and subtractions required an independent control of the absolute sizes. *Second*, the system lacked a sign for zero so that notations that required an inner zero became ambiguous.

Division was performed by multiplying by the reciprocal of the divisor, but the calculation of reciprocals as they were listed in the *tables of reciprocals* remained difficult and, as mentioned above, could not be performed if the prime factorization of the divisor contained prime factors other than two, three and five, namely the prime factors of sixty.³⁷

In spite of the integration of metrological notations into a closed system of abstract numerical operations, this system evidently differs from the modern number concept. It is, in fact, difficult to imagine how, long before the creation of modern mathematical formalism, a numerical system without an inherent absolute value of its entities could be conceived. In fact, neither in the Sumerian, nor in the Akkadian language did a word exist that can be convincingly interpreted as a designation of such an entity. In particular, a word for *number* was missing in these languages.³⁸

The conceptual change from metrological to sexagesimal positional notation had its most dramatic consequences for cognitive constructions related to spatial relations. While in the context of traditional metrological notations length measures, field areas, and capacities of containers were associated with only certain external domains of human activities, and connected only through these activities, they now became entities internally linked within the system of the operations that constituted the sexagesimal positional system. The following section will show the consequences of this epistemic shift, resulting in the unparalleled knowledge system of Babylonian geometry.

³⁷Two, three, and five have the reciprocals 30, 20, and 12 respectively. The prime factorization of any number containing only these prime factors has a reciprocal which is the product of the corresponding reciprocals. No other prime number has a finite representation by units of the sexagesimal system. Thus, any number containing a prime factor other than two, three, and five also has no finite representation in this system.

³⁸Frequently the Sumerian word 'šid' with the Akkadian translation 'minūtu' or the Akkadian word 'mānu' are considered as candidates for representing the term 'number'. This attribution is, however, an anachronistic projection of modern concepts, neglecting the contexts in which the terms were used, as they usually related to operations and not to entities.

3.7 The heritage of the surveyors in Babylonian mathematics

The types of knowledge discussed so far were more or less forms of the knowledge of practitioners. It is characteristic of such knowledge that it is determined by the goals of its practical use. These goals limit the range of applications and at the same time the potential for further developments. However, social institutions, in which such knowledge played a different role, developed along with the accumulation and specialization of the knowledge of practitioners. This was the case with institutions of education and training, which fulfilled the function of providing the qualifications the practitioners needed. While it is likely that basic qualifications of practitioners were transmitted to apprentices by participation and imitation, as is known from non-literate indigenous cultures,³⁹ the development of writing was surely accompanied by a growing institutionalization of the transmission of knowledge.⁴⁰

This new type of knowledge transmission changed the character of the knowledge itself. Knowledge transmitted by institutionalized education was no longer structured and limited by the actual goals of the practitioners but rather, to a certain extent, determined by the aim of teaching how to use the available tools and elaborating their inherent potentials. Consequently, together with the establishment of institutionalized teaching, new types of texts arose, which usually are designated as *school texts*, showing one or more of the following characteristics:

1. The texts were *neither signed by a responsible official nor dated*.
2. The *scope of the exercises was unrealistically extended* beyond any practical needs.
3. The *values of exercises were systematically varied* and often documented in the form of tables representing relations between conditions and results, for instance, tables of length measures together with the calculated areas.
4. *Conditions and results of administrative procedures were exchanged* in order to produce new types of exercises: Results of typical operations of practitioners such as a calculated area together with, for instance, the length of a field were given from which the width had to be calculated.

While the operations with traditional metrological notations were not substantially influenced by school traditions, the impact on operations with the sexagesimal positional system was outstanding. Beyond the simplification and integration of arithmetical operations, the arithmetization of geometrical entities such as length or area measures made unrealistic operations such as the addition of lengths and area measures possible. Following the habit of teachers to construct exercises systematically by modifying the conditions and by exchanging the conditions and results of administrative procedures, completely new problems were generated. A typical example consisting of twenty-four problems with more or less systematically varied conditions is provided by a frequently discussed Old Babylonian tablet written in the first half of the second millennium BCE.⁴¹ In an anachronistic modern translation, the first four problems and their solutions recorded on this tablet read as follows:

1. The area and the side of a square I have added: 45 is it.
2. The side of a square I have subtracted from its area: 14 30 is it.

³⁹Alt 1956.

⁴⁰Sjöberg 1973; Sjöberg 1975, Robson 2008, 31–33, 40–53, 97–106, 115–136, and 192–198.

⁴¹BM 13901, see Neugebauer 1935/1937, Vol. 3, 1–14.

3. The third of the area of a square I have subtracted. The third of its side I have added: 20 is it.
4. The third of the area of a square I have subtracted. The side I have added: 4 46 40 is it.

In all four problems, the task is, of course, to determine the length of the side. Keeping in mind that the notations in the sexagesimal positional system are determined except for a power of sixty as a factor, the construction of these problems is straightforward and simple using the multiplication tables of the sexagesimal positional system. In all cases the side of the square to be determined is 30 (times 60^n). The given areas are calculated as follows:

1. 30 times 30 is 15 (times 60). 15 plus 30 is 45 (the given area).
2. 30 times 30 is 15 (times 60). 15 (times 60) minus 30 is 14 30 (the given area).
3. 30 times 30 is 15 (times 60). The third of 15 is 5. 15, the area, minus 5 is 10. The third of 30, the side, is 10. 10 plus 10 are 20 (the given area).
4. 20 times 20 is 6 40 (6 times 60 plus 40). The third of 6 40 is 2 13 20 (times $\frac{1}{60}$). 6 40, the area, minus 2 13 20 (times $\frac{1}{60}$) is 4 26 40 (times $\frac{1}{60}$). 4 26 40 (times $\frac{1}{60}$) plus 20, the side, is 4 46 40 (times $\frac{1}{60}$, the given area).

While the arithmetical operations of multiplication and division involved are quite simple and their results independent of the order of magnitude with regard to the power of sixty, there is obviously a difficult problem if additions and subtractions are involved. Adding or subtracting sexagesimal positional notations produces different results if the relative position of the notations is different with regard to the power of sixty. The ambiguity of the notation system thus causes an ambiguity in the operations to be performed. It was therefore always common sense that the ancient scribes must have kept track of the absolute values of their notations without, however, ever attempting to find a notation to express these values.

This common-sense interpretation is supported to a certain degree by recent philological work.⁴² It turns out that the concepts represented by the terms used are partly arithmetical and partly geometrical, representing the traditional distinction between different kinds of metrologically determined operations, in particular different kinds of multiplication.⁴³ Thus, in the present example the term for multiplication is identical with the term for the area of a field, the terms for addition and subtraction seem to have an arithmetical origin, even though the values to which they are applied, the sizes of lengths and areas, also have a clear geometrical meaning. It is therefore plausible, that the ancient scribes kept track of the absolute values of the arithmetical notations by keeping in mind their metrological meanings.

In contrast to the simplicity of problems based on the operations of the practitioners, reverse operations led to more sophisticated problem solutions. Albeit recorded on many of the tablets, they are still poorly understood.⁴⁴ In modern terms, the examples given above represent second-degree equations:

⁴²Høyrup 1990; Høyrup 2002.

⁴³Høyrup 1990, 45–69, Høyrup 2002, 18–49.

⁴⁴Several or even the majority of historians of science will probably not agree with this statement. Translated in terms of modern mathematics most of the problems are in fact understandable. The solutions appear to be either correct or somehow flawed, allegedly due to an insufficient qualification of the ancient scribes. From the viewpoint of historical epistemology, however, such errors shed some light on the inherent mental operations of Babylonian mathematics. In spite of the progress made in the interpretation of Babylonian mathematics after the *geometric turn*, the present understanding of this ‘non-Euclidean’ kind of mathematics is deficient since there is still

$$x^2 + x = a \quad (3.1)$$

$$x^2 - x = a \quad (3.2)$$

$$x^2 - \alpha x^2 + \beta x = a \quad (3.3)$$

$$x^2 - \alpha x^2 + x = a \quad (3.4)$$

Solutions to such problems, if featured on the tablets, describe only the operations to be performed with the given sexagesimal positional notations. Any reflection on the reasons for the operations performed is usually missing. Accordingly, the first step to understand the performed operations was to rewrite them in generalized form using modern algebraic notations with the result that the operations follow roughly the algebraic solution formula of second-degree equations. Despite the fact that the origins of the solution procedure remained obscure, Babylonian mathematics seemed to be some kind of Babylonian algebra.

The new geometric interpretation of key terms of Babylonian mathematics undermined this understanding. For the first time, some Babylonian problem solutions, at least, became plausible on the basis of hypothetical geometrical figures, which might have existed on scrap pads or as memorized mental images.

As a consequence of this *geometric turn* in the interpretation of problems posed in Babylonian mathematics, there is a tendency to supplement them systematically with Euclidean-style figures. Given that there is a common basis of practitioners' knowledge in Babylonian and Euclidean geometry, it does not come as a surprise that the Euclidean-style figures offer some explanation for the solution procedures. However, this approach disguises the close connection with the surveyors' tradition. Babylonian mathematics still did not account for angles and thus for the similarity of shapes.⁴⁵ The incompatibility of this characteristic, which was inherited from the surveying of fields based exclusively on length measurements, is particularly obvious in cases where tablets themselves illustrate problems with drawings that contradict a Euclidean interpretation. This is the case, for instance, when triangles were drawn which cannot be right-angled, but the interpretation has to treat them as if they had a right angle.⁴⁶

The geometrical problems of Babylonian mathematics were determined by

1. the length of straight lines, which may approximate curved lines,
2. the assumption that the size of a figure which consists of partial areas equals the sum of these partial areas,

no comprehensive reconstruction of the mental models and operations that provide a coherent image of the partly successful and partly erroneous solutions of the problems documented on the tablets of Babylonian mathematics. For a critical reassessment of some interpretations of tablets allegedly dealing with the Pythagorean theorem see Damerow 2001.

⁴⁵Høyrup 2002, 228 discussing the role of angles and of similarity justifies the use of Euclidean-style figures:

Exactly *how* it [i.e. the right angle] was understood we do not know – the texts speak too little about it to allow us to understand – but it is beyond doubt that a (probably intuitive) concept of similarity or 'same shape' was at hand.

⁴⁶See for instance the tablet Strssbg. 364 Neugebauer 1935/1937, volume 1, 248–356 and volume 3, plate 11. See also the overview of similar problems provided by Friberg 2007b, 244–268. Friberg 2005b, 46–50 has convincingly shown that Egyptian mathematics dealt with problems such as those on Strssbg. 364 in the same way.

3. the assumption that the area of a quadrilateral is determined by the *surveyors' formula*.

With the Euclidean concept in mind, these assumptions could lead to counterintuitive consequences. On one of the Old Babylonian mathematical tablets,⁴⁷ for instance, a quadrilateral is cut into two equal pieces. The scribe asked for the length of the dividing line.

From a Euclidean perspective and its concept of the size of an area this question is pure nonsense, because the length of the dividing line depends on the angle of the cut. From the perspective of the Babylonian concept of the size of an area, however, the problem makes perfect sense. The condition that the size of the whole area calculated according to the *surveyors' formula* equals the sum of the sizes of the cut parts, also calculated according to the *surveyors' formula*, is a strong condition contradicting the Euclidean concept of the size of an area. Under this strong condition, the length of the dividing line is, in fact, mathematically determined by a simple relation between the lengths of the two opposite sides of the quadrilateral, which are not cut by the dividing line, and the length of the dividing line itself. If a and b are the lengths of the two opposite sides and c is the length of the dividing line, then the Babylonian assumptions imply the relation:⁴⁸

$$a^2 + b^2 = 2c^2 \quad (3.5)$$

This example is not an exceptional one but rather the simplest kind on a type of tablets that deal with the division of irregular fields according to specific criteria. In all likelihood, this type of problem goes back to the problems of the surveyors, even though such problems are not documented by administrative tablets. It is known, however, that officials were allotted arable land as compensation for services and deliveries of agricultural products. It is quite possible that the allotments had to have a specific size so that, contrary to what is known from surveyors' administrative documents, length measures had to be determined under certain conditions for given area measures. At least, school tablets from the second half of the third millennium BCE remain, which ask for length measures to be calculated from a given area which was a difficult problem until the sexagesimal positional system was invented. In any case, the majority of tablets dealing with the division of fields belong to the mathematical tablets of the Old Babylonian period.

From the origin of writing at the beginning of the third millennium up to at least the first half of the first millennium BCE, all mathematical tablets dealing with areas were based on the *surveyors' formula*. From the viewpoint of modern mathematics, all calculations on these tablets are merely approximations of results which can be developed on the basis of Euclidean geometry.

But is it possible that this was also the perspective of the Babylonian scribes? There is, in fact, no evidence whatsoever that they considered their calculation of areas to be some kind of approximation of a different understanding of 'true' areas for which they had neither a theoretical nor even a conceptual basis. While they probably were aware of the fact that their calculated areas were not precise in any absolute sense, they surely did not see any difference between the dependence of area measures on external circumstances and the limited precision of measures such as the measures of lengths, weights, volumes, or economic

⁴⁷YBC 4675, published by Neugebauer and Sachs 1945, 44–48.

⁴⁸For a simple proof see Bruins 1955, 46 and Vogel 1959, 70.

exchange rates. But there were, of course, practitioners' *rules of thumb* for achieving reproducible results, such as checking the reliability of a balance in equilibrium by exchanging the sides of the load and the balance weight, or applying the *surveyors' formula* only to shapes which differ not too much from rectangles.

The scribes who composed Babylonian mathematics inherited this understanding of area measures from the surveyors and made the *surveyors' formula* a core operation of posing problems and establishing ways to solve them. This insight challenges the common mainstream interpretation of Babylonian mathematics altogether. While in many cases taking into account the non-Euclidean concept of area and disregarding the angles of their shapes may not lead to substantially different interpretations, an understanding of these characteristics of Babylonian mathematics turns out to be crucial as soon as the division of areas and cut-and-paste procedures are involved, and when the aim is to reconstruct the epistemic basis and the origins of the solutions to the sophisticated problems of this esoteric art of the scribes.

3.8 Conclusion

This chapter investigated the development of abilities to deal with spatial relations in ancient Mesopotamia.⁴⁹ It covers the time period from pre-history in the fourth millennium BCE to the first half of the second millennium BCE when Babylonian mathematics reached its first climax. At the beginning of the chapter the questions were posed as to the historical circumstances under which the nature of the conceptualization of space changed from spatial orientation techniques in pre-historic times to Babylonian geometry and what brought about these changes.

Summarizing the results of the chapter, two major innovations can be identified that triggered fundamental changes related to spatial cognition. Both innovations concern the representation of spatial relations by notation systems. The first innovation was closely related to the transition from rural communities in the alluvial plane of Mesopotamia to the centralized and stratified societies of the first city states. This transition was associated with the development of administrative tools such as seals, tokens, numerical impressions into clay, and finally the invention of the proto-cuneiform writing system, a non-glottographic system of operations with symbols. These operations represented administrative activities and were used to control them. They were based on the standardization of measures and their integration into systems. The second innovation that triggered fundamental changes was the invention of the sexagesimal positional system of numerical notation.

The overall development of knowledge which had its origin in the practitioners' knowledge of surveyors and its climax in the sophisticated geometry of Babylonian mathematics can be characterized as a fundamental shift in the socially shared conceptual structures used to understand spatial relationships. This development shows how content-specific mental models were transformed into an abstract mathematical model of space, which however, differs fundamentally from Euclidean geometry which later became the basis of mathematical developments in the European mathematical tradition.

⁴⁹Peter Damerow passed away before he could finish this concluding section, which he had headed "Mental models of spatial relations in the third millennium BCE." We have not attempted to complete the section. – MS.

Bibliography

- Alt, Robert (1956). *Vorlesungen über die Erziehung auf frühen Stufen der Menschheitsentwicklung*. Berlin: Volk und Wissen.
- Bauer, Josef, Robert K. Englund, and Manfred Krebernik (1998). *Mesopotamien: Späturuk-Zeit und Frühdynastische Zeit*. Annäherungen 1. Göttingen: Vandenhoeck & Ruprecht.
- Boltz, William G. (2006). Pictographic Myths. In: *Sprache und Denken in China und Japan*. Ed. by Wolfgang Behr and Heiner Roetz. 30. Bochum Yearbook of East Asian Studies (BJOAF). Bochum: Faculty of East Asian Studies at Ruhr-Universität Bochum, 39–54.
- Bruins, E. M. (1955). On the System of Babylonian Geometry. *Sumer* 11(1):44–49.
- Damerow, Peter (2001). Kannten die Babylonier den Satz des Pythagoras? Epistemologische Anmerkungen zur Natur der Babylonischen Mathematik. In: *Changing Views on Ancient Near Eastern Mathematics*. Ed. by Jens Høyrup and Peter Damerow. Berlin: Reimer, 219–310.
- Damerow, Peter and Robert K. Englund (1987). Die Zahlzeichensysteme der Archaischen Texte aus Uruk. In: *Zeichenliste der Archaischen Texte aus Uruk (ATU 2)*. Ed. by Margret W. Green and Hans J. Nissen. 2. Archaische Texte aus Uruk. Berlin: Gebr. Mann, 117–166.
- Deimel, Anton (1923). *Schultexte aus Fara*. Die Inschriften von Fara. Leipzig: Hinrichs.
- Driel, G. van (1982). Tablets from Jebel Aruda. In: *Zikir Sumin*. Ed. by G. Van Driel. Leiden: Brill, 12–25.
- Dunham, Sally (1986). Sumerian Words for Foundation. *Revue d'Assyriologie* 80:31–63.
- Edzard, Dietz O. (1969). Eine altsumerische Rechentafel. In: *Lisan mithurti: Festschrift Wolfram Freiherr von Soden*. Ed. by Wolfgang Röllig. Kevelaer: Butzon und Bercker.
- Elias, Norbert (1984). *Über die Zeit*. Frankfurt a. M.: Suhrkamp.
- Friberg, Jöran (1981). Methods and Traditions of Babylonian Mathematics: Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations. *Historia Mathematica* 8:277–318.
- (2005a). On the Alleged Counting with Sexagesimal Place Value Numbers in Mathematical Cuneiform Texts from the Third Millennium BC. *Cuneiform Digital Library Journal* 2005:2.
- (2005b). *Unexpected Links between Egyptian and Babylonian Mathematics*. New Jersey: World Scientific.
- (2007a). *A Remarkable Collection of Babylonian Mathematical Texts*. New York: Springer.
- (2007b). *Amazing Traces of a Babylonian Origin in Greek Mathematics*. New Jersey: World Scientific.
- Fuße, Allotte de la (1915). Un cadastre de Djokha. *Revue d'Assyriologie* 12(1):47–54.
- Gandz, Solomon (1929). The Origin of Angle-Geometry. *ISIS* 12(1929):452–481.
- Gibson, McGuire and Robert D. Biggs (1991). *The Organization of Power: Aspects of Bureaucracy in the Ancient Near East*. 2nd ed. Chicago: The Oriental Institute of the University of Chicago.
- Hilprecht, H. V. (1906). *Mathematical, Metrological and Chronological Tablets from the Temple Library of Nippur*. Philadelphia: Department of Archaeology, University of Pennsylvania.

- Høyrup, Jens (1990). Algebra and Naive Geometry. *Altorientalische Forschungen* 17:27–69, 262–354.
- (2002). *Length, Width, Surfaces: A Portrait of Old Babylonian Mathematics and Its Kins*. New York: Springer.
- Høyrup, Jens and Peter Damerow, eds. (2001). *Changing Views on Ancient Near Eastern Mathematics*. Berliner Beiträge zum vorderen Orient. Berlin: Reimer.
- Hyman, Malcom D. (2006). Of Glyphs and Glottography. *Language and Communication* 26(3-4):231–249.
- Jestin, R. (1937). *Tablettes sumérienne de Shuruppak conservées au Musée se Stamboul*. Paris: Boccard.
- Limet, Henri (1973). *Étude de documents de la periode d'Agade appartenant à l'Université de Liège*. Bibliothèque de la Faculté de Philosophie et Lettres de l'Université de Liège. Paris: Société d'Éditions « Les Belles Lettres ».
- Neugebauer, Otto (1935/1937). *Mathematische Keilschrifttexte*. Berlin: Springer.
- Neugebauer, Otto and Abraham Sachs (1945). *Mathematical Cuneiform Texts*. New Haven: American Oriental Society.
- Nissen, Hans Jörg, Peter Damerow, and Robert K. Englund (1993). *Archaic Bookkeeping: Early Writing and Techniques of Economic Administration in the Ancient Near East*. Chicago: Chicago University Press.
- Powell, Marvin A. (1976). The Antecedents of Old Babylonian Place Notation and the Early History of Babylonian Mathematics. *Historia Mathematica* 3:417–439.
- Proust, Christine (2008). *Tablettes mathématiques de la collection Hilprecht*. Wiesbaden: Harrassowitz.
- Robson, Eleanor (1999). *Mesopotamian Mathematics, 2100-1600 BC. Technical Constants in Bureaucracy and Education*. Oxford: Oxford University Press.
- (2001). Neither Sherlock Holmes nor Babylon. A Reassessment of Plimpton 322. *Historia Mathematica* 28:167–206.
- (2008). *Mathematics in Ancient Iraq. A Social History*. Princeton: Princeton University Press.
- Sachs, Abraham (1947). Babylonian Mathematical Texts I. Reciprocals of Regular Sexagesimal Numbers. *Journal of Cuneiform Studies* 1(3):219–240.
- Schmandt-Besserat, Denise (1992). *Before Writing*. Austin: University of Texas Press.
- Schmidt, Olaf (1980). On Plimpton 322. Pythagorean Numbers in Babylonian Mathematics. *Centaurus* 24:4–13.
- Sjöberg, Ake W. (1973). Der Vater und sein missratener Sohn. *Journal of Cuneiform Studies* 25:105–169.
- (1975). Der Examenstext A. *Zeitschrift für Assyriologie* 64:137–177.
- Strommenger, Eva (1980). *Habuba Kabira, eine Stadt vor 5000 Jahren*. Mainz: Zabern.
- Thureau-Dangin, Françoise (1938). *Textes mathématiques babyloniens*. Leiden: Brill.
- Vogel, Kurt (1959). *Vorgriechische Mathematik, Teil II: Die Mathematik der Babylonier*. Hannover: Schroedel.
- Wertheimer, Max (1925). Über das Denken der Naturvölker: Zahlen und Zahlgebilde. In: *Drei Abhandlungen zur Gestalttheorie*. Ed. by Max Wertheimer. Erlangen: Verlag der philosophischen Akademie, 106–163.