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Soft behavior of a closed massless state in superstring and universality in the soft behavior of the dilaton

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Abstract

We consider the tree-level scattering amplitudes in the NS-NS (Neveu-Schwarz) massless sector of closed superstrings in the case where one external state becomes soft. We compute the amplitudes generically for any number of dimensions and any number and kind of the massless closed states through the subsubleading order in the soft expansion. We show that, when the soft state is a graviton or a dilaton, the full result can be expressed as a soft theorem factorizing the amplitude in a soft and a hard part. This behavior is similar to what has previously been observed in field theory and in the bosonic string. Differently from the bosonic string, the supersymmetric soft theorem for the graviton has no string corrections at subsubleading order. The dilaton soft theorem, on the other hand, is found to be universally free of string corrections in any string theory.

Contents

1 Introduction and Results

In this work we consider the tree-level scattering amplitudes of massless states in the NS-NS (Neveu-Schwarz) sector of closed superstrings, and analyze the behavior of the amplitude when one of the external states, either a graviton, dilaton or a Kalb-Ramond field, becomes soft with respect to the other (hard) states. This is a direct continuation of our earlier works in Ref. [1, 2], where we dealt with the same problem, but in the bosonic string only. In Ref. [1] the soft behavior of a massless closed string was computed through subsubleading order in the soft momentum, when the hard states were all tachyons, and through subleading order, when the hard states were any other massless closed state in the bosonic string. These soft behaviors were shown for the graviton and dilaton to generically admit soft theorems with the factorizing soft part being equal to the known leading and subleading soft theorems of a graviton in pure field theory [3], and of a dilaton in string theory [4]. The subsubleading soft behavior, when scattering on hard tachyons was found instead to have an additional factorizing piece, only relevant for a soft dilaton, as compared to the recently discovered subsubleading soft theorem for the graviton in field theory [5, 6, 7] (for complementary discussions, see also Ref.[8]).

As shown in [7], the field theory soft theorem results for the graviton all follow by just imposing gauge invariance of the scattering amplitudes. This same analysis can be extended to also cover the dilaton collectively with the graviton, and as shown in Ref. [9], one indeed recovers the additional piece at subsubleading order, found explicitly in Ref. [1], when scattering on hard tachyons, signalling universality of the soft theorem. In Ref. [2] we extended our analysis in the bosonic string by computing the subsubleading soft behavior of a massless closed state scattering on other hard massless closed states. The soft theorems for the graviton and dilaton were again uncovered, and it was shown that an additional factorizing soft operator proportional to the string slope α' appears at subsubleading order, when the soft state is a graviton, while the dilaton soft theorem remains equal to the field theory result of Ref. [9]. By including the α' corrections in the three-point amplitude for massless states, this was shown to again follow from gauge invariance of the scattering amplitudes. While this shows that the graviton soft theorem at subsubleading order is not universal, but depends on higher-order operators in its effective action, it is intriguing to think that the soft behavior of the dilaton is universal in any theory through subsubleading order, signalling some underlying hidden symmetry. (For recent discussions on existing relations between broken symmetries of Lagrangians and soft theorems, see Refs. [10, 9].) Indeed, as shown in Ref. [9], the dilaton soft theorem does bare striking resemblance to the soft theorem of the Nambu-Goldstone boson of spontaneously broken conformal symmetry, which is universal through subsubleading order.

In this work we are in fact going to confirm that the dilaton in superstrings obeys the same soft theorem as in the bosonic string through subsubleading order, when scattering on other massless closed states of the NS-NS sector of closed superstrings. The soft theorem of the graviton in superstrings, on the other hand, does not have any string corrections, in contrast to the bosonic string. Furthermore, we will show by using gauge invariance that also in the heterotic string, the dilaton soft operator has no string corrections at subsubleading order. In conclusion, we find that the dilaton soft operator is tree-level universal through subsubleading order in all string theories, and in particular, does not contain string corrections, making it the same as in field theory. We leave out in this work any discussions on the soft behavior of the Kalb-Ramond field. We plan to discuss it in a future work, collecting also our previous results in the bosonic string.

The subject of this work has seen tremendous progress in recent years, both in field theory and in string theories. We restrict ourselves here to referring only to the more related string theory papers [11], while the interested reader is invited to look up our recent paper [2] for a brief count of the progress also on the field theory side, including the relevant references.

Let us summarize the results of this work, while at the same time introducing our notation: The n-point tree-level scattering amplitudes, M_n, of closed massless superstrings can generically be written as a convolution of a bosonic part, M_n^b, with a supersymmetric part, M_n^s, as follows:

$$M_n = M_n^b * M_n^s \,, \tag{1.1}$$

The expressions for bosonic and supersymmetric parts of the n-point supersymmetric string amplitude are defined by:

$$\begin{aligned} M_n^b = \frac{8\pi}{\alpha'}\left(\frac{\kappa_D}{2\pi}\right)^{n-2} \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}|z_1 - z_2|^2} \prod_{i=1}^{2} d\theta_i \theta_i \prod_{i=1}^{2} d\bar{\theta}_i \bar{\theta}_i \prod_{i=3}^{n} d\theta_i \prod_{i=1}^{n} d\varphi_i \prod_{i=3}^{n} d\bar{\theta}_i \prod_{i=1}^{n} d\bar{\varphi}_i \\ \times \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j} \exp\left[\frac{1}{2}\sum_{i\neq j} \frac{C_i \cdot C_j}{(z_i - z_j)^2} + \sqrt{\frac{\alpha'}{2}} \sum_{i\neq j} \frac{C_i \cdot k_j}{z_i - z_j} + \text{c.c.}\right] , \end{aligned} \tag{1.2}$$

$$M_n^s = \exp\left[-\frac{1}{2}\sum_{i\neq j}\frac{A_i\cdot A_j}{z_i-z_j} + \text{c.c.}\right]. \tag{1.3}$$

where κ_D is the D-dimensional Newton's constant, dV_{abc} is the volume of the Möbius group, z_i are the Koba-Nielsen variables, φ_i and θ_i are Grassmannian integration variables, and we have introduced the following *superkinematical* quantities:

$$A_i^\mu = \varphi_i\epsilon_i^\mu + \sqrt{\frac{\alpha'}{2}}\theta_i k_i^\mu \quad ; \quad C_i^\mu = \varphi_i\theta_i\epsilon_i^\mu \,, \tag{1.4}$$

where ϵ_i^μ and k_i^μ are respectively the holomorphic polarization vector and momentum of the state i, and α' is the string slope.

Apart from the integration measure, M_n^b is equivalent at the integrand level to the same amplitude in the bosonic string; the integrands, in fact, become equal if one makes the identification $\theta_i\epsilon_i \to \epsilon_i$ and remembers that, after this substitution, ϵ_i has become a Grassmann variable. The difference in the measure between M_n^b and the bosonic string amplitude is only the presence in M_n^b of the integrals over the Grassmann variables θ_i, $\bar{\theta}_i$, and the additional factor $\prod_{i=1}^2 \theta_i\bar{\theta}_i/|z_1-z_2|^2$ coming from the correlator of the superghosts. The latter factor in the measure effectively kills any term involving $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$, which readily follows from an expansion of the exponentials and an integration over those variables. We will not need to use this property, and thus leave the integrand as it is.

When considering an amplitude with an additional state, which is soft, it is useful to factorize the string amplitude at the integrand level into a soft part S and a hard part as follows:

$$M_{n+1} = M_n * S \tag{1.5}$$

where M_n is the full superstring amplitude of n closed massless states, and S is a function that when convoluted with the integral expression for M_n provides the additional soft state involved in the amplitude. The function S can be further decomposed into its bosonic part and supersymmetric part as follows:

$$S = S_b + S_s + \bar{S}_s \,, \tag{1.6}$$

where S_b is the bosonic part and $S_s + \bar{S}_s$ is the supersymmetric part, with $\bar{S}_s$ being the complex conjugate of S_s. This decomposition is useful, since the bosonic part can be related to the soft function in the bosonic string. In fact, after the identification $\theta_i\epsilon_i \to \epsilon_i$, the bosonic soft function S_b is the same as the one given for the bosonic string in Ref. [1, 2],[1] which was computed therein through $\mathcal{O}(q)$, where q is the momentum of the soft state. In this work we need therefore only to consider the additional contributions

[1]The Grassmanian variables φ_i are equivalent to those of Ref. [1, 2] denoted therein by θ_i.

from the supersymmetric states, described by $S_s + \bar{S}_s$. We have computed this function through $\mathcal{O}(q)$, and our result reads:

$$
\begin{aligned}
S_s + \bar{S}_s &= \kappa_D \epsilon_\mu \bar{\epsilon}_\nu \sum_{i \neq j} \Bigg\{ \frac{q_\rho}{(k_i \cdot q)} \frac{\bar{A}_i^{[\rho} \bar{A}_j^{\nu]} k_i^\mu}{(\bar{z}_i - \bar{z}_j)} + q_\rho \left(\frac{\alpha'}{2} \right)^{\frac{3}{2}} \frac{q \cdot k_j \bar{C}_i^{[\rho,} k_i^{\nu]}}{\bar{z}_i - \bar{z}_j} \left(\frac{k_i^\mu}{q \cdot k_i} - \frac{k_j^\mu}{q \cdot k_j} \right) \\
&+ q_\rho \sqrt{\frac{\alpha'}{2}} \frac{\bar{A}^\rho_{\{i,} \bar{A}^\nu_{j\}}}{\bar{z}_i - \bar{z}_j} \sum_{l \neq i} \left[\frac{q \cdot k_l}{q \cdot k_i} \left(\frac{C_i^\mu}{z_i - z_l} + \sqrt{\frac{\alpha'}{2}} k_i^\mu \log |z_i - z_l|^2 \right) + \left(\frac{C_l^\mu}{z_i - z_l} - \sqrt{\frac{\alpha'}{2}} k_l^\mu \log |z_i - z_l|^2 \right) \right] \\
&+ q_\rho q_\sigma \Bigg[\left(\frac{1}{2} A^\sigma_{\{i,} A^\mu_{j\}} - \sqrt{\frac{\alpha'}{2}} C^\sigma_{\{i,} k^\mu_{j\}} \right) \sum_{l \neq i} \frac{\bar{A}^\rho_{\{i,} \bar{A}^\nu_{l\}}}{q \cdot k_i (z_i - z_j)(\bar{z}_i - \bar{z}_l)} - \frac{\alpha'}{2} \frac{\bar{C}_i^{[\sigma,} k_i^{\nu]} \bar{C}_j^\rho}{(\bar{z}_i - \bar{z}_j)^2} \left(\frac{k_j^\mu}{q \cdot k_j} - \frac{k_i^\mu}{q \cdot k_i} \right) \\
&- \sqrt{\frac{\alpha'}{2}} \sum_{l \neq i,j} \frac{k_i^\mu \left(\bar{C}_j^\sigma \bar{A}^\rho_{\{i,} \bar{A}^\nu_{l\}} + \frac{1}{2} \bar{C}_i^\sigma \bar{A}^\rho_{\{j,} \bar{A}^\nu_{l\}} \right)}{q \cdot k_i (\bar{z}_i - \bar{z}_j)(\bar{z}_i - \bar{z}_l)} - \sum_{l \neq i} \frac{C^\sigma_{[i,} C^\mu_{j]} \bar{A}^\rho_{\{i,} \bar{A}^\nu_{l\}}}{q \cdot k_i (z_i - z_j)^2 (\bar{z}_i - \bar{z}_l)} \Bigg] \Bigg\} + \text{c.c.} + \mathcal{O}(q^2) \,,
\end{aligned}
\tag{1.7}
$$

where the brackets and curly-brackets in the indices denote commutation and anticommutation of the indices, e.g.:

$$
\begin{aligned}
C_i^{[\rho,} k_i^{\nu]} &\equiv C_i^\rho k_i^\nu - C_i^\nu k_i^\rho \\
A^\mu_{\{i} A^\nu_{j\}} &\equiv A_i^\mu A_j^\nu + A_j^\mu A_i^\nu
\end{aligned}
\tag{1.8}
$$

The above expression starts at $\mathcal{O}(q^0)$. This means that only the bosonic part contributes to the amplitude at $\mathcal{O}(q^{-1})$, and is thus responsible for the Weinberg graviton soft theorem. When the above expression is projected onto a soft state which is symmetric in its polarization, i.e. a graviton or a dilaton, we will show that the explicit results above can be reproduced by the following soft theorem:

$$
(M_{n+1})_S = (\hat{S}^{(-1)} + \hat{S}^{(0)} + \hat{S}^{(1)}) M_n + \mathcal{O}(q^2) \tag{1.9}
$$

where the subscript S denotes that the soft state must be symmetrically polarized, and

$$
\hat{S}^{(-1)} = \kappa_D \, \epsilon^S_{\mu\nu} \sum_{i=1} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} \,, \tag{1.10a}
$$

$$
\hat{S}^{(0)} = -i \kappa_D \epsilon^S_{\mu\nu} \sum_{i=1}^n \frac{q_\rho k_i^\nu J_i^{\mu\rho}}{k_i \cdot q} \,, \tag{1.10b}
$$

$$
\hat{S}^{(1)} = -\kappa_D \frac{\epsilon^S_{\mu\nu}}{2} \sum_{i=1}^n \left(\frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q} + \frac{q^\mu \eta^{\nu\rho} q^\sigma + q^\mu \eta^{\nu\sigma} q^\rho - \eta^{\mu\nu} q^\sigma q^\rho}{k_i \cdot q} \mathbf{A}_{i\rho\sigma} \right) \,, \tag{1.10c}
$$

and where $\epsilon^S_{\mu\nu} = \frac{1}{2}(\epsilon_\mu \bar{\epsilon}_\nu + \epsilon_\nu \bar{\epsilon}_\mu)$, J_i is the total angular momentum operator,

$$
J_i^{\mu\nu} = L_i^{\mu\nu} + S_i^{\mu\nu} + \bar{S}_i^{\mu\nu} \,, \tag{1.11}
$$

$$
L_i^{\mu\nu} = i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right) \,, \; S_i^{\mu\nu} = i \left(\epsilon_i^\mu \frac{\partial}{\partial \epsilon_{i\nu}} - \epsilon_i^\nu \frac{\partial}{\partial \epsilon_{i\mu}} \right) \,, \; \bar{S}_i^{\mu\nu} = i \left(\bar{\epsilon}_i^\mu \frac{\partial}{\partial \bar{\epsilon}_{i\nu}} - \bar{\epsilon}_i^\nu \frac{\partial}{\partial \bar{\epsilon}_{i\mu}} \right) \,,
$$

and $\mathbf{A}_i$ is an operator:

$$\mathbf{A}_{i\rho\sigma} = k_{i\rho}\frac{\partial}{\partial k_i^\sigma} + \Pi_{i\rho\sigma}\,, \quad \Pi_{i\rho\sigma} = \epsilon_{i\rho}\frac{\partial}{\partial \epsilon_i^\sigma} + \bar{\epsilon}_{i\rho}\frac{\partial}{\partial \bar{\epsilon}_i^\sigma} \tag{1.12}$$

that acts covariantly on the superkinematical variables, i.e.

$$\mathbf{A}_i^{\mu\rho} A_j^\sigma = \delta_{ij}\eta^{\sigma\rho} A_i^\mu\,, \quad \mathbf{A}_i^{\mu\rho} C_j^\sigma = \delta_{ij}\eta^{\sigma\rho} C_i^\mu\,. \tag{1.13}$$

It follows that $J_i^{\mu\nu}$ is also covariant acting on these variables, since

$$J_i^{\mu\nu} = i(\mathbf{A}_i^{\mu\nu} - \mathbf{A}_i^{\nu\mu})\,. \tag{1.14}$$

In contrast, we notice that the subsubleading soft operators obtained in the bosonic string [2]:

$$\hat{S}_{bos}^{(1)} = \hat{S}^{(1)} + \frac{\alpha'}{2}\kappa_D\,\epsilon_{\mu\nu}^S \sum_{i=1}^n \left(q^\sigma k_i^\nu \eta^{\rho\mu} + q^\rho k_i^\mu \eta^{\sigma\nu} - \eta^{\rho\mu}\eta^{\sigma\nu}(k_i\cdot q) - q^\rho q^\sigma \frac{k_i^\mu k_i^\nu}{q\cdot k_i}\right)\Pi_{i\rho\sigma}\,, \tag{1.15}$$

and, as we will show in this work, also in heterotic string:

$$\hat{S}_{het}^{(1)} = \hat{S}^{(1)} + \frac{\alpha'}{2}\kappa_D\,\epsilon_{\mu\nu}^S \sum_{i=1}^n \left(q^\sigma k_i^\nu \eta^{\rho\mu} + q^\rho k_i^\mu \eta^{\sigma\nu} - \eta^{\rho\mu}\eta^{\sigma\nu}(k_i\cdot q) - q^\rho q^\sigma \frac{k_i^\mu k_i^\nu}{q\cdot k_i}\right)\epsilon_{i\rho}\frac{\partial}{\partial \epsilon_i^\sigma}\,, \tag{1.16}$$

differ from Eq. (1.10c) by terms due to string corrections. These additional parts, proportional to α', do not act covariantly on the superkinematical variables, and thus are not supersymmetric operators. We have consistently found that they only appear in the bosonic and in the heterotic string. They furthermore vanish when projected onto the dilaton state. Therefore the subsubleading soft operator in Eq. (1.10c) is, nevertheless, universally valid for the dilaton in the bosonic string, in superstrings, in the heterotic string, and in field theory [9].

The paper is organized as follows: In Sec. 2 we review the superstring amplitude of $n+1$ massless closed states and rewrite it in a convenient form for computing its behavior in the limit where one of the external states becomes soft with respect to the momenta of the other n external states. Here we also introduce our notation. Then in Sec. 3 we show the calculational details of the soft part of the amplitude and provide our explicit results. In Sec. 4 we demonstrate that the explicit results for the graviton and the dilaton can be expressed equally as a soft theorem, where the soft part is provided by the action of an operator acting on the lower point amplitude involving only the n external hard states. We furthermore explicitly show how the supersymmetric part of the amplitude cancels the purely bosonic string corrections to the amplitude at the subsubleading order, found in Ref. [2]. In Sec. 5, using gauge invariance, we compute the string corrections in the heterotic string and we show that they do not contribute to the dilaton soft behavior. Finally, Sec. 6 offers our conclusions and remarks. An appendix is additionally provided for the details of the calculation in Sec. 4.

2 Amplitude of one soft and n massless closed superstrings

In this section, we review the closed superstring amplitude and rewrite it in a convenient form, when one particle is soft, which allows us to directly express the results using the calculations already done in the bosonic string in Ref. [1, 2].

The massless closed superstring vertex, in the $(-1,-1)$ and $(0,0)$ pictures, is given by the compact expression:

$$V^{(-p,-p)} = \frac{\kappa_D}{2\pi} \int d\theta \; \theta^p \; V^{(p)}(z,\theta;k) \int d\bar{\theta} \; \bar{\theta}^p \; \bar{V}^{(p)}(\bar{z},\bar{\theta};k)\,, \tag{2.1}$$

with

$$V^{(p)}(z,\theta;k) = e^{-p\phi(z)} \; \epsilon_\mu \; DX^\mu \; e^{i\sqrt{\frac{\alpha'}{2}}k\cdot X(z,\theta)}\,, \tag{2.2}$$

where θ and $\bar{\theta}$ are Grassmannian variables, $\epsilon_\mu \bar{\epsilon}_\nu = \epsilon_{\mu\nu}$ is the polarization of the massless state, and the superfield notation is given by

$$X^\mu(z,\theta) \equiv x^\mu(z) + \theta\psi^\mu(z)\,, \quad D \equiv \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}\,. \tag{2.3}$$

The relevant expectation values for massless amplitudes are:

$$\begin{aligned} \langle X^\mu(z_1,\theta_1)X^\nu(z_2,\theta_2)\rangle &= -\eta^{\mu\nu}\log(z_1 - z_2 - \theta_1\theta_2)\,, \\ \langle e^{-\phi(z_1)}\; e^{-\phi(z_2)}\rangle &= \frac{1}{z_1 - z_2}\,. \end{aligned} \tag{2.4}$$

The amplitude of $n+1$ massless states in closed superstring can be written as:

$$\begin{aligned} M_{n+1} =& \frac{8\pi}{\alpha'}\left(\frac{\kappa_D}{2\pi}\right)^{n-1} \int \frac{d^2z \prod_{i=1}^n d^2z_i d\theta d\bar{\theta}}{dV_{abc}|z_1 - z_2|^2}\left[\prod_{i=1}^2 d\theta_i\theta_i \prod_{i=3}^n d\theta_i\right]\left[\prod_{i=1}^2 d\bar{\theta}_i\bar{\theta}_i \prod_{i=3}^n d\bar{\theta}_i\right] \\ &\times \langle 0| \int d\varphi \; e^{i\left(\varphi\epsilon DX(z,\theta) + \sqrt{\frac{\alpha'}{2}}qX(z,\theta)\right)} \prod_{i=1}^n \left(\int d\varphi_i \; e^{i(\varphi_i\epsilon_i D_i X(z_i,\theta_i) + K_i X(z_i,\theta_i))}\right)|0\rangle \\ &\times \langle 0| \int d\bar{\varphi} \; e^{i\left(\bar{\varphi}\bar{\epsilon}\bar{D}X(\bar{z},\bar{\theta}) + \sqrt{\frac{\alpha'}{2}}qX(\bar{z},\bar{\theta})\right)} \prod_{i=1}^n \left(\int d\bar{\varphi}_i \; e^{i\left(\bar{\varphi}_i\bar{\epsilon}_i\bar{D}_i X(\bar{z}_i,\bar{\theta}_i) + K_i X(\bar{z}_i,\bar{\theta}_i)\right)}\right)|0\rangle\,, \end{aligned} \tag{2.5}$$

where new Grassmanian variables $(\varphi, \varphi_i, \bar{\varphi}, \bar{\varphi}_i)$ are introduced, and dV_{abc} is the volume of the Möbius group. The states with the indices 1 and 2 are in the $(-1,-1)$ picture, while the others are in the $(0,0)$ picture. This effectively means that, in the expressions for the integrands that follow, terms involving $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2$ can be equated to zero because of the overall integration measures $\int d\theta_i\theta_i$ and $\int d\bar{\theta}_i\bar{\theta}_i$ for $i = 1, 2$. Since this choice could have been made for any two of the n states, we will not explicitly impose these zero conditions in the expressions that follow.

The $n+1$ point amplitude, with the help of the correlation functions written in Eq. (2.4) and after having integrated over the variables θ and $\bar{\theta}$, reduces to an expression which can be factorized at the integrand level as follows:

$$M_{n+1} = M_n * S\,, \tag{2.6}$$

where by $*$ a convolution integral is understood, and the two parts M_n and S can be conveniently expressed in terms of the superkinematical quantities:

$$A_i^\mu = \varphi_i \epsilon_i^\mu + \sqrt{\frac{\alpha'}{2}}\theta_i k_i^\mu \quad ; \quad C_i^\mu = \varphi_i \theta_i \epsilon_i^\mu\,, \tag{2.7}$$

such that

$$\begin{aligned} M_n = \frac{8\pi}{\alpha'}\left(\frac{\kappa_D}{2\pi}\right)^{n-2} \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}|z_1 - z_2|^2} \prod_{i=1}^{2} d\theta_i \theta_i \prod_{i=1}^{2} d\bar{\theta}_i \bar{\theta}_i \prod_{i=3}^{n} d\theta_i \prod_{i=1}^{n} d\varphi_i \prod_{i=3}^{n} d\bar{\theta}_i \prod_{i=1}^{n} d\bar{\varphi}_i \\ \prod_{i<j} |z_i - z_j|^{\alpha' k_i k_j} \exp\left[\frac{1}{2}\sum_{i\neq j} \frac{C_i \cdot C_j}{(z_i - z_j)^2} + \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j} \frac{C_i \cdot k_j}{z_i - z_j} - \frac{1}{2} \sum_{i \neq j} \frac{A_i \cdot A_j}{z_i - z_j}\right] \\ \times \exp\left[\frac{1}{2}\sum_{i\neq j} \frac{\bar{C}_i \cdot \bar{C}_j}{(\bar{z}_i - \bar{z}_j)^2} + \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j} \frac{\bar{C}_i \cdot k_j}{\bar{z}_i - \bar{z}_j} - \frac{1}{2} \sum_{i \neq j} \frac{\bar{A}_i \cdot \bar{A}_j}{\bar{z}_i - \bar{z}_j}\right]\,, \end{aligned} \tag{2.8}$$

while for convenience we express S as a sum of three terms

$$S \equiv S_b + S_s + \bar{S}_s\,, \tag{2.9}$$

where S_b is the purely bosonic part, which is simply equal to the similar expression in the bosonic string after identifying $\theta_i \epsilon_i \to \epsilon_i$ (whereby ϵ_i becomes a Grassmann variable) and is given by: [2]

$$\begin{aligned} S_b = \frac{\kappa_D}{2\pi} \int d^2 z \prod_{l=1}^{n} |z - z_l|^{\alpha' q k_l} \prod_{l=1}^{n} \exp\left[-\sqrt{\frac{\alpha'}{2}} \frac{q \cdot C_l}{z - z_l} - \sqrt{\frac{\alpha'}{2}} \frac{q \cdot \bar{C}_l}{\bar{z} - \bar{z}_l}\right] \\ \times \left(\sum_{i=1}^{n} \frac{\epsilon \cdot C_i}{(z - z_i)^2} + \sum_{i=1}^{n} \sqrt{\frac{\alpha'}{2}} \frac{\epsilon \cdot k_i}{z - z_i}\right)\left(\sum_{j=1}^{n} \frac{\bar{\epsilon} \cdot \bar{C}_j}{(\bar{z} - \bar{z}_j)^2} + \sum_{j=1}^{n} \sqrt{\frac{\alpha'}{2}} \frac{\bar{\epsilon} \cdot k_j}{\bar{z} - \bar{z}_j}\right)\,, \end{aligned} \tag{2.10}$$

and S_s and $\bar{S}_s$ are the complex conjugates of each other and they provide the contributions

[2] For comparison with the expressions in Ref. [1, 2] we notice that the variables here denoted by φ_i are equivalent to the variables denoted by θ_i in those papers.

from the additional supersymmetric states. They are given by

$$\begin{aligned}\bar{S}_s = \frac{\kappa_D}{2\pi} \int d^2z \prod_{l=1}^{n} |z-z_l|^{\alpha' q k_l} \prod_{l=1}^{n} \exp\left[-\sqrt{\frac{\alpha'}{2}}\frac{q\cdot C_l}{z-z_l} - \sqrt{\frac{\alpha'}{2}}\frac{q\cdot \bar{C}_l}{\bar{z}-\bar{z}_l}\right] \\ \times \left[\frac{1}{2}\sum_{i=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{q\cdot A_i}{z-z_i}\sum_{j=1}^{n}\frac{\epsilon\cdot A_j}{z-z_j}\sum_{l=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{q\cdot \bar{A}_l}{\bar{z}-\bar{z}_l}\sum_{m=1}^{n}\frac{\bar{\epsilon}\cdot \bar{A}_m}{\bar{z}-\bar{z}_m}\right. \\ \left. + \left(\sum_{i=1}^{n}\frac{\epsilon\cdot C_i}{(z-z_i)^2} + \sum_{i=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{\epsilon\cdot k_i}{z-z_i}\right)\sum_{j=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{q\cdot \bar{A}_j}{\bar{z}-\bar{z}_j}\sum_{l=1}^{n}\frac{\bar{\epsilon}\cdot \bar{A}_l}{\bar{z}-\bar{z}_l}\right] ,\end{aligned} \tag{2.11}$$

and S_s is given by the complex conjugate of this expression, where complex conjugation sends $z_i \to \bar{z}_i$, $\epsilon_i^\mu \to \bar{\epsilon}_i^\mu$, $\theta_i \to \bar{\theta}_i$, and $\varphi_i \to \bar{\varphi}_i$, while the momenta k_i are left invariant. The superkinematical quantities A_i^μ and C_i^μ are respectively anticommuting and commuting kinematic factors. Furthermore, since $\varphi_i^2 = \theta_i^2 = 0$, they obey the following useful identities:

$$A_i^\mu A_i^\nu = \sqrt{\frac{\alpha'}{2}} C_i^{[\mu,} k_i^{\nu]} \, , \quad C_i^\mu C_i^\nu = A_i^\mu C_i^\nu = 0 \, , \tag{2.12}$$

where we have used the notation $C_i^{[\mu,} k_i^{\nu]} \equiv C_i^\mu k_i^\nu - C_i^\nu k_i^\mu$. This antisymmetrizing notation will be used throughout this paper. Furthermore an equivalent notation will be used with curly brackets for denoting symmetrization.

Let us remark that M_n can be decomposed in a bosonic and a supersymmetric part as well, as follows:

$$M_n = M_n^b * M_n^s \, , \tag{2.13}$$

where the first part yields the complete bosonic case and is given by

$$\begin{aligned}M_n^b = \frac{8\pi}{\alpha'}\left(\frac{\kappa_D}{2\pi}\right)^{n-2} \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}|z_1-z_2|^2} \prod_{i=1}^{2} d\theta_i \theta_i \prod_{i=3}^{n} d\theta_i \prod_{i=1}^{n} d\varphi_i \prod_{i=1}^{2} d\bar{\theta}_i\bar{\theta}_i \prod_{i=3}^{n} d\bar{\theta}_i \prod_{i=1}^{n} d\bar{\varphi}_i \\ \prod_{i<j} |z_i - z_j|^{\alpha' k_i k_j} \exp\left[\frac{1}{2}\sum_{i\neq j}\frac{C_i\cdot C_j}{(z_i-z_j)^2} + \sqrt{\frac{\alpha'}{2}}\sum_{i\neq j}\frac{C_i\cdot k_j}{z_i-z_j} + \text{c.c.}\right] ,\end{aligned} \tag{2.14}$$

and the second part gives the supplement of the additional superstring states and reads

$$M_n^s = \exp\left[-\frac{1}{2}\sum_{i\neq j}\frac{A_i\cdot A_j}{z_i-z_j} + \text{c.c.}\right] . \tag{2.15}$$

3 Soft expansion through subsubleading order

The integral S_b has been computed through subsubleading order in q, that is through $\mathcal{O}(q)$, in Refs. [1, 2]. Thus for this work we only need to consider the other parts of S, i.e. S_s and $\bar{S}_s$, where the latter can be conveniently written in the following compact form:

$$\begin{aligned}\bar{S}_s = \sqrt{\frac{\alpha'}{2}}\kappa_D\epsilon_\mu\bar{\epsilon}_\nu\Bigg\{ q_\rho \sum_{i,j,l=1}\bar{A}_j^\rho\bar{A}_l^\nu\left(C_i^\mu I_{ii}^{jl}+\sqrt{\frac{\alpha'}{2}}k_i^\mu I_i^{jl}\right)\\
+\sqrt{\frac{\alpha'}{2}}\,q_\rho q_\sigma\sum_{i,j,l,m=1}\bar{A}_l^\sigma\bar{A}_m^\nu\Bigg[\left(\frac{1}{2}A_i^\rho A_j^\mu-\sqrt{\frac{\alpha'}{2}}C_i^\rho k_j^\mu\right)I_{ij}^{lm}\\
-C_i^\rho C_j^\mu I_{ijj}^{lm}-C_i^\mu\bar{C}_j^\rho I_{ii}^{jlm}-\sqrt{\frac{\alpha'}{2}}k_i^\mu\bar{C}_j^\rho I_i^{jlm}\Bigg]\Bigg\}\,,\end{aligned}\tag{3.1}$$

where all the integrals involved in the calculus of the amplitude are represented as:

$$I_{i_1 i_2\dots}^{j_1 j_2\dots}=\int\frac{d^2z}{2\pi}\frac{\prod_{l=1}^n|z-z_l|^{\alpha' q k_l}}{(z-z_{i_1})(z-z_{i_2})\cdots(\bar{z}-\bar{z}_{j_1})(\bar{z}-\bar{z}_{j_2})\cdots}\,.\tag{3.2}$$

Notice that according to Eq. (2.12) the term involving $C_i^\rho C_j^\mu$ vanishes for $i=j$, and that the terms involving $\bar{C}_j^\rho$ vanish for $j=l,m$. It turns out that all integrals involved in the calculation have already been computed in Ref. [2], and they are all obtained from two master integrals, I_i^i and I_i^j, through an iteratively use of the identities:

$$I_{ii}^j=\frac{1}{1-\frac{\alpha'}{2}(qk_i)}\partial_{z_i}I_i^j\tag{3.3}$$

valid even for $i=j$ and

$$I_{i_1 i_2\dots}^{j_1 j_2\dots}=\frac{I_{i_1\dots}^{j_1 j_2\dots}-I_{i_2\dots}^{j_1 j_2\dots}}{z_{i_1}-z_{i_2}}=\frac{I_{i_1\dots}^{j_1\dots}-I_{i_1\dots}^{j_2\dots}-I_{i_2\dots}^{j_1\dots}+I_{i_2\dots}^{j_2\dots}}{(z_{i_1}-z_{i_2})(\bar{z}_{j_1}-\bar{z}_{j_2})}=\dots\tag{3.4}$$

The explicit expressions of the master integrals are [1, 2]:

$$\begin{aligned}I_i^i=&\frac{2}{\alpha'(k_iq)}\left(1+\alpha'\sum_{j\neq i}(k_jq)\log|z_i-z_j|+\frac{(\alpha')^2}{2}\sum_{j\neq i}\sum_{k\neq i}(k_jq)(k_kq)\log|z_i-z_j|\log|z_i-z_k|\right)\\
&+(\alpha')^2\sum_{j\neq i}(k_jq)\log^2|z_i-z_j|+\log\Lambda^2+\mathcal{O}(q^2)\,,\end{aligned}\tag{3.5}$$

$$\begin{aligned}I_i^j=&\sum_{m\neq i,j}\frac{\alpha'(qk_m)}{2}\left(\mathrm{Li}_2\left(\frac{\bar{z}_i-\bar{z}_m}{\bar{z}_i-\bar{z}_j}\right)-\mathrm{Li}_2\left(\frac{z_i-z_m}{z_i-z_j}\right)-2\log\frac{\bar{z}_m-\bar{z}_j}{\bar{z}_i-\bar{z}_j}\log\frac{|z_i-z_j|}{|z_i-z_m|}\right)\\
&-\log|z_i-z_j|^2+\log\Lambda^2+\mathcal{O}(q^2)\,,\end{aligned}\tag{3.6}$$

with Λ a cut off that cancels in the final expression of the amplitude. The notation of two momenta in a round bracket is hereafter used to denote $(k_j q) \equiv k_j \cdot q$. It is worthwhile to notice that only I_i^i shows a pole in the soft momentum and therefore the integrals $I_{i_1 i_2 \ldots}^{j_1 j_2 \ldots}$ can yield a term of $\mathcal{O}(q^{-1})$ only if one of its lower indices is equal to one of the upper ones.

To derive $\bar{S}_s$ through subsubleading order, let us first notice that with the integrand explicitly containing a factor of q, the leading part can only be of $\mathcal{O}(q^0)$, and therefore the entire $\mathcal{O}(q^{-1})$ terms are produced by the bosonic part only. Next, to obtain the terms of order q^0 and q, we notice by inspection of Eq. (3.1) that the integrals I_{ii}^{jl} and I_i^{jl} must be equated through the $\mathcal{O}(q^0)$, while for all other integrals only the leading q^{-1} order is relevant. The integral I_{ii}^{jlm}, only relevant at $\mathcal{O}(q^{-1})$, does not contribute, since by having two lower indices equal it cannot be divergent in the soft momentum. For the same reason, all the other integrals which are only relevant at $\mathcal{O}(q^{-1})$ contribute only when one of the indices l or m is equal to i or j.

The complete expression through $\mathcal{O}(q)$ of $\bar{S}_s$ can be explicitly given in the following form, where each integral is now unique and we discard integrals that do not give any relevant contribution:

$$\begin{aligned}
\bar{S}_s = \sqrt{\frac{\alpha'}{2}} \kappa_D \epsilon_\mu \bar{\epsilon}_\nu q_\rho \Bigg\{ & \frac{\alpha'}{2} \sum_{i=1} \bar{C}_i^{[\rho} k_i^{\nu]} \Big(k_i^\mu I_i^{ii} + \sum_{j \neq i} k_j^\mu I_j^{ii} \Big) + \sum_{i \neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} \Big(C_i^\mu I_{ii}^{ji} + \sqrt{\frac{\alpha'}{2}} k_i^\mu I_i^{ij} \Big) \\
& + \sum_{i \neq j \neq l} \bar{A}_j^\rho \bar{A}_l^\nu \Big(C_i^\mu I_{ii}^{jl} + \sqrt{\frac{\alpha'}{2}} k_i^\mu I_i^{jl} \Big) + \sqrt{\frac{\alpha'}{2}} q_\sigma \Bigg[\sum_{i \neq j} \sum_{i \neq l} \frac{1}{2} \bar{A}^\sigma_{\{i} \bar{A}^\nu_{l\}} A^\rho_{\{i} A^\mu_{j\}} I_{ij}^{li} - \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j} \bar{A}^\sigma_{\{i} \bar{A}^\nu_{j\}} C_j^\rho k_i^\mu I_{ij}^{ij} \\
& - \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j \neq l} \bar{A}^\sigma_{\{i} \bar{A}^\nu_{l\}} C^\rho_{\{i} k^\mu_{j\}} I_{ij}^{il} - \sum_{i \neq j} \bar{A}^\sigma_{\{i} \bar{A}^\nu_{j\}} C_j^\rho C_i^\mu I_{jii}^{ij} - \sum_{i \neq j \neq l} C_j^\rho C_i^\mu \Big(\bar{A}^\sigma_{\{i} \bar{A}^\nu_{l\}} I_{jii}^{il} + \bar{A}^\sigma_{\{j} \bar{A}^\nu_{l\}} I_{jii}^{jl} \Big) \\
& - \frac{\alpha'}{2} \sum_{i \neq j} \bar{C}_i^{[\sigma} k_i^{\nu]} k_i^\mu \bar{C}_j^\rho I_i^{iij} - \frac{\alpha'}{2} \sum_{i \neq j} \bar{C}_j^{[\sigma} k_j^{\nu]} k_i^\mu \bar{C}_i^\rho I_i^{jji} - \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j \neq l} k_i^\mu \Big(\bar{A}^\sigma_{\{i} \bar{A}^\nu_{l\}} \bar{C}_j^\rho I_i^{jli} + \bar{A}_l^\sigma \bar{A}_j^\nu \bar{C}_i^\rho I_i^{lji} \Big) \Bigg] \Bigg\} \\
& + \mathcal{O}(q^2)
\end{aligned} \tag{3.7}$$

where we made explicit use of Eq. (2.12) and particularly of the identity $\bar{A}_i^\rho \bar{A}_i^\nu = \sqrt{\frac{\alpha'}{2}} C_i^{[\rho,} k_i^{\nu]}$. We recall for convenience the notations:

$$\begin{aligned}
C_i^{[\rho,} k_i^{\nu]} &\equiv C_i^\rho k_i^\nu - C_i^\nu k_i^\rho \\
A^\mu_{\{i} A^\nu_{j\}} &\equiv A_i^\mu A_j^\nu + A_j^\mu A_i^\nu = A_i^{[\mu} A_j^{\nu]}
\end{aligned} \tag{3.8}$$

where the latter equality is due to the Grassmannian nature of the A_i.

The $\mathcal{O}(q^0)$ part of $\bar{S}_s$ is obtained from the term involving I_i^{jl} only, since I_{ii}^{jl} does not have a $\mathcal{O}(q^{-1})$ term, and the only nonzero part reads:

$$\bar{S}_s = \kappa_D \epsilon_\mu \bar{\epsilon}_\nu \frac{\alpha'}{2} q_\rho \sum_{i\neq j} \left(\bar{A}_j^\rho \bar{A}_i^\nu + \bar{A}_i^\rho \bar{A}_j^\nu \right) k_i^\mu I_i^{ij} + \mathcal{O}(q)$$
$$= \kappa_D \epsilon_\mu \bar{\epsilon}_\nu \sum_{i\neq j} \frac{q_\rho \bar{A}_i^{[\rho} \bar{A}_j^{\nu]} k_i^\mu}{(k_i \cdot q)(\bar{z}_i - \bar{z}_j)} + \mathcal{O}(q)\,. \tag{3.9}$$

It is worth noticing that this expression does not involve any overall α'-factor.

Finally we express explicitly the terms of $\mathcal{O}(q)$, which after some simplifications read:

$$\begin{aligned}
\bar{S}_s\Big|_{\mathcal{O}(q)} =\; & \kappa_D \epsilon_\mu \bar{\epsilon}_\nu \sum_{i\neq j} \Bigg\{ q_\rho \left(\frac{\alpha'}{2}\right)^{\frac{3}{2}} \frac{q k_j \bar{C}_i^{[\rho,} k_i^{\nu]}}{\bar{z}_i - \bar{z}_j} \left(\frac{k_i^\mu}{q k_i} - \frac{k_j^\mu}{q k_j} \right) \\
& + q_\rho \sqrt{\frac{\alpha'}{2}} \frac{\bar{A}^\rho_{\{i,} \bar{A}^\nu_{j\}}}{\bar{z}_i - \bar{z}_j} \sum_{l\neq i} \left[\frac{q k_l}{q k_i} \left(\frac{C_i^\mu}{z_i - z_l} + \sqrt{\frac{\alpha'}{2}} k_i^\mu \log|z_i - z_l|^2 \right) + \left(\frac{C_l^\mu}{z_i - z_l} - \sqrt{\frac{\alpha'}{2}} k_l^\mu \log|z_i - z_l|^2 \right) \right] \\
& + q_\rho q_\sigma \Bigg[\left(\frac{1}{2} A^\sigma_{\{i,} A^\mu_{j\}} - \sqrt{\frac{\alpha'}{2}} C^\sigma_{\{i,} k^\mu_{j\}} \right) \sum_{l\neq i} \frac{\bar{A}^\rho_{\{i,} \bar{A}^\nu_{l\}}}{q k_i (z_i - z_j)(\bar{z}_i - \bar{z}_l)} - \frac{\alpha'}{2} \frac{\bar{C}_i^{[\sigma,} k_i^{\nu]} \bar{C}_j^\rho}{(\bar{z}_i - \bar{z}_j)^2} \left(\frac{k_j^\mu}{q k_j} - \frac{k_i^\mu}{q k_i} \right) \\
& - \sqrt{\frac{\alpha'}{2}} \sum_{l\neq i,j} \frac{k_i^\mu \left(\bar{C}_j^\sigma \bar{A}^\rho_{\{i,} \bar{A}^\nu_{l\}} + \frac{1}{2} \bar{C}_i^\sigma \bar{A}^\rho_{\{j,} \bar{A}^\nu_{l\}} \right)}{q k_i (\bar{z}_i - \bar{z}_j)(\bar{z}_i - \bar{z}_l)} - \sum_{l\neq i} \frac{C^\sigma_{[i,} C^\mu_{j]} \bar{A}^\rho_{\{i,} \bar{A}^\nu_{l\}}}{q k_i (z_i - z_j)^2 (\bar{z}_i - \bar{z}_l)} \Bigg] \Bigg\}\,.
\end{aligned} \tag{3.10}$$

4 Soft action on the lower-point amplitude

In Sec. 3 we have seen that the n-point string amplitudes with all massless external legs can be written as the convolution integral of M_n^b with M_n^s. The dependence of M_n^b on the momenta and polarizations is the same as for the amplitude of n massless particles in the bosonic string, which is in turn already known to obey a soft theorem through subsubleading order when the soft particle is a graviton or dilaton [1, 2], i.e.

$$M_{n+1}^b = M_n * S_b = \left(\hat{S}^{(-1)}_{\text{bos}} + \hat{S}^{(0)}_{\text{bos}} + \hat{S}^{(1)}_{\text{bos}} \right) M_n^b + \mathcal{O}(q^2)\,, \tag{4.1}$$

where

$$\hat{S}^{(-1)}_{\rm bos} = \kappa_D \, \epsilon^S_{\mu\nu} \sum_{i=1} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} \, , \tag{4.2a}$$

$$\hat{S}^{(0)}_{\rm bos} = - \, i\kappa_D \epsilon^S_{\mu\nu} \sum_{i=1}^n \frac{q_\rho k_i^\nu J_i^{\mu\rho}}{k_i \cdot q} \, , \tag{4.2b}$$

$$\begin{aligned}\hat{S}^{(1)}_{\rm bos} = & - \kappa_D \frac{\epsilon^S_{\mu\nu}}{2} \sum_{i=1}^n \Bigg[\frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q} + \frac{q^\mu \eta^{\nu\rho} q^\sigma + q^\mu \eta^{\nu\sigma} q^\rho - \eta^{\mu\nu} q^\sigma q^\rho}{q k_i} \mathbf{A}_{i\rho\sigma} \\ & - \alpha' \left(q^\sigma k_i^\nu \eta^{\rho\mu} + q^\rho k_i^\mu \eta^{\sigma\nu} - \eta^{\rho\mu} \eta^{\sigma\nu} (k_i \cdot q) - q^\rho q^\sigma \frac{k_i^\mu k_i^\nu}{k_i \cdot q} \right) \Pi_{i\rho\sigma} \Bigg] \, , \end{aligned} \tag{4.2c}$$

where the different quantities and operators were defined in the introduction, Eq. (1.11)-(1.12).

In this section we will establish a soft theorem for gravitons and dilatons in superstring amplitudes. By using the above results for M_n^b, we will do this by showing that also M_n^s satisfies similar soft identities. In this way we will crucially see how the supersymmetric part cancels the α'-terms in the soft theorem of the bosonic string, Eq. (4.2c), leaving a superstring soft theorem free of any α'-correction through subsubleading order. Let us first notice the trivial leading order result,

$$\begin{aligned} M_{n+1} = M_n * (S_b + S_s + \bar{S}_s) = M_n * S_b + \mathcal{O}(q^0) = (M_n^b * M_n^s) * S_b + \mathcal{O}(q^0) \\ = M^b_{n+1} * M_n^s + \mathcal{O}(q^0) = \hat{S}^{(-1)}_{\rm bos} \, M_n + \mathcal{O}(q^0) \end{aligned} \tag{4.3}$$

thus at leading order we can trivially identify $\hat{S}^{(-1)} = \hat{S}^{(-1)}_{\rm bos}$.

In order to identify the superstring soft operator at subleading order, it is useful, in analogy with the bosonic calculation [1], to make the holomorphic and antiholomorphic sectors completely independent. This is achieved by replacing, in the antiholomorphic sector, the momentum k of the hard particles with a spurious quantity $\bar{k}$. By doing this, the integrand of a closed string amplitude completely factorizes, at the cost of $M_n \equiv M_n(k_i, \epsilon_i, \bar{k}_i, \bar{\epsilon}_i)$ only becoming a physical amplitude after identifying $\bar{k}$ with k. This, however, leads us to introduce holomorphic angular momentum operators,

$$L_i^{\mu\rho} = i \left(k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}} \right) \, , \quad S_i^{\mu\rho} = i \left(\epsilon_i^\mu \frac{\partial}{\partial \epsilon_{i\rho}} - \epsilon_i^\rho \frac{\partial}{\partial \epsilon_{i\mu}} \right) \, , \tag{4.4}$$

with similar expressions for the antiholomorphic quantities. The action of these operators on the superkinematical variables, defined in Eq. (2.7), gives:

$$\begin{aligned} (L_i + S_i)^{\mu\rho} A_j^\sigma = i\delta_{ij} \left(\eta^{\sigma\rho} A_i^\mu - \eta^{\sigma\mu} A_i^\rho \right) \, , \quad (\bar{L}_i + \bar{S}_i)^{\mu\rho}_i \bar{A}_j^\sigma = i\delta_{ij} \left(\eta^{\sigma\rho} \bar{A}_i^\mu - \eta^{\sigma\mu} \bar{A}_i^\rho \right) \, , \\ (L_i + S_i)^{\mu\rho} C_j^\sigma = i\delta_{ij} \left(\eta^{\sigma\rho} C_i^\mu - \eta^{\sigma\mu} C_i^\rho \right) \, , \quad (\bar{L}_i + \bar{S}_i)^{\mu\rho} \bar{C}_j^\sigma = i\delta_{ij} \left(\eta^{\sigma\rho} \bar{C}_i^\mu - \eta^{\sigma\mu} \bar{C}_i^\rho \right) \, . \end{aligned} \tag{4.5}$$

From these identities it is straightforward to show a pseudo-soft theorem at subleading order for any soft state (graviton, dilaton, Kalb-Ramond) in the following form:

$$M_{n+1} = \quad -i\kappa_D \epsilon_\mu \bar{\epsilon}_\nu \sum_{i=1}^{n} \left[\frac{q_\rho k_i^\nu (L_i + S_i)^{\mu\rho}}{qk_i} + \frac{q_\rho k_i^\mu (\bar{L}_i + \bar{S}_i)^{\nu\rho}}{qk_i} \right] M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}} + \mathcal{O}(q) \tag{4.6}$$

This is easiest to see by noting that in the bosonic string the same expression holds for the $\mathcal{O}(q^0)$ part, as shown in Ref. [1], and therefore also for M_n^b as defined in this work, and since the operator above is linear on $M_n = M_n^b * M_n^s$, it needs only to be checked that the operation above on M_n^s reproduces $S_s + \bar{S}_s$ at $\mathcal{O}(q^0)$, given explicitly in Eq. (3.9).

By taking the symmetric, respectively antisymmetric combinations of the above expression in the polarization of the soft state, it is possible to turn the above pseudo-soft theorem into a physical soft theorem. We postpone the full antisymmetric analysis to a future work, and here focus on the symmetric part, which reads:

$$(M_{n+1})_S = -i\kappa_D \epsilon^S_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho k_i^\nu}{qk_i} \left(L_i + \bar{L}_i + S_i + \bar{S}_i \right)^{\mu\rho} M_n(k_i, \epsilon_i; \bar{k}_i, \bar{\epsilon}_i) \Big|_{k=\bar{k}} + \mathcal{O}(q) \tag{4.7}$$

where the sub/superscript S is for symmetric and where $\epsilon^S_{\mu\nu} = \frac{1}{2}(\epsilon_\mu \bar{\epsilon}_\nu + \epsilon_\nu \bar{\epsilon}_\mu)$. Now using the equivalence $(L_i + \bar{L}_i)^{\mu\rho} M_n(k_i; \bar{k}_i)|_{k=\bar{k}} \equiv L_i^{\mu\rho} M_n^s(k_i)$, we can readily set $\bar{k} = k$ and thus get:

$$\begin{aligned}
(M_{n+1})_S &= -i\kappa_D \epsilon^S_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho k_i^\nu}{qk_i} \left(L_i + S_i + \bar{S}_i \right)^{\mu\rho} M_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q) \\
&= -i\kappa_D \epsilon^S_{\mu\nu} \sum_{i=1}^{n} \frac{q_\rho k_i^\nu J_i^{\mu\rho}}{qk_i} M_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q) \\
&\equiv \hat{S}^{(0)} M_n(k_i, \epsilon_i, \bar{\epsilon}_i) + \mathcal{O}(q)
\end{aligned} \tag{4.8}$$

where we identified the total angular momentum operator $J_i^{\mu\rho} = L_i^{\mu\rho} + S_i^{\mu\rho} + \bar{S}_i^{\mu\rho}$, and in the last line we defined the subleading operator $\hat{S}^{(0)}$. This result is the well-known subleading soft theorem for the graviton. Here we have shown, however, that it also applies to the dilaton, by taking its proper polarization tensor, and furthermore that in superstring theory there are no string corrections to the soft operator through this order. It follows that to the subleading order, the soft theorem for the graviton and dilaton in superstring theory is exactly the same as in bosonic string theory. Since $\hat{S}^{(0)} = \hat{S}^{(0)}_{\text{bos}}$, we could equally well have shown this from the computation:

$$\begin{aligned}
\hat{S}^{(0)} M_n = \hat{S}^{(0)} (M_n^b * M_n^s) &= (\hat{S}^{(0)} M_n^b) * M_n^s + M_n^b * (\hat{S}^{(0)} M_n^s) \\
&= \left[M_n * S_b + M_n * (S_s + \bar{S}_s) \right]_{\mathcal{O}(q^0)}
\end{aligned} \tag{4.9}$$

and checking that $\hat{S}^{(0)} M_n^s$ reproduces $S_s + \bar{S}_s$ at $\mathcal{O}(q^0)$.

At the subsubleading order we proceed by considering the recently established soft theorem in the bosonic string Eq. (4.2c). Let us also recall that the α'-terms in Eq. (4.2c) arise as a consequence of gauge invariance together with the fact that the three-point amplitude in the bosonic string has terms with higher powers in α'. In superstring these latter terms are missing in the three-point amplitude of massless closed states. We thus do not expect that the subsubleading soft operator for the superstring contains the part proportional to α'. We therefore would like to check, as an ansatz, whether the action

$$-\kappa_D \frac{\epsilon^S_{\mu\nu}}{2} \sum_{i=1}^{n} \left[\frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{qk_i} + \frac{q^\mu \eta^{\nu\rho} q^\sigma + q^\mu \eta^{\nu\sigma} q^\rho - \eta^{\mu\nu} q^\sigma q^\rho}{qk_i} \mathbf{A}_{i\rho\sigma} \right] M_n \equiv \hat{S}^{(1)} M_n \,, \tag{4.10}$$

reproduces the explicit results derived in the previous section. Let us first notice that the term involving $J_i^{\mu\rho} J_i^{\nu\sigma}$ is a nonlinear operator. Therefore the above action, decomposed on the M_n^b and M_n^s parts, gives:

$$\begin{aligned} \hat{S}^{(1)} M_n &= \hat{S}^{(1)} (M_n^b * M_n^s) \\ &= (\hat{S}^{(1)} M_n^b) * M_n^s + M_n^b * (\hat{S}^{(1)} M_n^s) - \kappa_D\, \epsilon^S_{\mu\nu}\, q_\rho q_\sigma \sum_{i=1}^{n} \frac{(J_i^{\mu\rho} M_n^b) * (J_i^{\nu\sigma} M_n^s)}{qk_i} \,, \end{aligned} \tag{4.11}$$

and we would like to check whether this reproduces the explicit expressions given for $M_n * (S_b + S_s + \bar{S}_s)$. Since $\hat{S}^{(1)} M_n^b$ does not reproduce fully the complete subsubleading soft behavior of $M_n^b * S_b$, it is useful to know explicitly the remaining part, which is simply derived from the action of the α'-terms in Eq. (4.2c), reading:

$$\begin{aligned} &(M_n^b * S_b)\Big|_{\mathcal{O}(q)} - (\hat{S}^{(1)} M_n^b) \\ &= \kappa_D \epsilon^S_{\mu\nu} \frac{\alpha'}{2} \sum_{i=1}^{n} \left(q^\sigma k_i^\nu \eta^{\rho\mu} + q^\rho k_i^\mu \eta^{\sigma\nu} - \eta^{\rho\mu}\eta^{\sigma\nu} (k_i \cdot q) - q^\rho q^\sigma \frac{k_i^\mu k_i^\nu}{qk_i} \right) \Pi_{i\rho\sigma} M_n^b \\ &= M_n^b * \left[\kappa_D \epsilon^S_{\mu\nu} \frac{\alpha'}{2} \sum_{i=1}^{n} \sum_{j\neq i} \frac{q_\rho q_\sigma}{qk_i} C_i^{[\mu} k_i^{\rho]} \left(\frac{C_j^{[\sigma} k_i^{\nu]}}{(z_i - z_j)^2} + \sqrt{\frac{\alpha'}{2}} \frac{k_j^{[\sigma} k_i^{\nu]}}{z_i - z_j} \right) + \text{c.c} \right] . \end{aligned} \tag{4.12}$$

We will explicitly show that this part of S_b is exactly cancelled by the additional supersymmetric contributions coming from $S_s + \bar{S}_s$. Having the above expression at hand and the result from Ref. [1, 2], we will not need to compute the first term in Eq. (4.11) involving $\hat{S}^{(1)} M_n^b$. We need only to consider the action of the last two operators of Eq. (4.11). The derivation is straightforward but tedious, and we therefore leave it in the appendix.

The result is:

$$
\begin{aligned}
&M_n^b *(\hat{S}^{(1)} M_n^s) - \kappa_D\, \epsilon^S_{\mu\nu}\, q_\rho q_\sigma \sum_{i=1}^n \frac{(J_i^{\mu\rho} M_n^b) * (J_i^{\nu\sigma} M_n^s)}{q k_i} \\
&\quad = (M_n^b * M_n^s) * \kappa_D \epsilon^S_{\mu\nu} \sqrt{\frac{\alpha'}{2}} \Bigg\{ \\
&\quad q_\rho \Bigg[\sum_{i\neq j\neq l} \bar{A}_i^\rho \bar{A}_j^\nu \bigg(C_l^\mu I^{(q^0)}{}^{ij}_{ll} + \sqrt{\frac{\alpha'}{2}} k_l^\mu I^{(q^0)}{}^{ij}_{l} \bigg) + \sum_{i\neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} \bigg(C_j^\mu I^{(q^0)}{}^{ij}_{jj} + \sqrt{\frac{\alpha'}{2}} k_i^\mu I^{(q^0)}{}^{ij}_{i} \bigg) \Bigg] \\
&\quad + \sqrt{\frac{\alpha'}{2}} q_\rho q_\sigma \Bigg[\sum_{i\neq j} \sum_{l\neq i} \frac{1}{2} \bar{A}^\sigma_{\{l} \bar{A}^\nu_{i\}} A^\rho_{\{j} A^\mu_{i\}} I^{(q^{-1})}{}^{il}_{ij} - \sqrt{\frac{\alpha'}{2}} \sum_{i\neq j\neq l} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} k^\mu_{\{i} C^\sigma_{l\}} I^{(q^{-1})}{}^{ij}_{il} \\
&\quad - \sqrt{\frac{\alpha'}{2}} \sum_{i\neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} k_i^\mu C_j^\sigma I^{(q^{-1})}{}^{ij}_{ij} - \sum_{i\neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} C_i^\mu C_j^\sigma I^{(q^{-1})}{}^{ij}_{iij} \\
&\quad - \sum_{i\neq j\neq l} C_i^\mu C_l^\sigma \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} I^{(q^{-1})}{}^{ij}_{iil} - \sum_{i\neq j\neq l} C_i^\mu C_l^\sigma \bar{A}^\rho_{\{l} \bar{A}^\nu_{j\}} I^{(q^{-1})}{}^{lj}_{iil} - \sqrt{\frac{\alpha'}{2}} \sum_{i\neq j\neq l} \bar{A}^\rho_{\{j} \bar{A}^\nu_{i\}} k_i^\mu \bar{C}_l^\sigma I^{(q^{-1})}{}^{ilj}_{i} \\
&\quad - \sqrt{\frac{\alpha'}{2}} \sum_{i\neq l\neq j} \bar{A}_j^\rho \bar{A}_l^\nu k_i^\mu \bar{C}_i^\sigma I^{(q^{-1})}{}^{ijl}_{i} + \sum_{i\neq j} \frac{C_j^{[\rho} k_j^{\nu]} C_i^{[\mu} k_i^{\sigma]}}{q k_i (z_i - z_j)^2} \bigg) \Bigg] \Bigg\} + \text{c.c.}
\end{aligned}
\tag{4.13}
$$

The derivation in the Appendix involves first computing the action of the operators and then rewriting everything in terms of the expressions for the integrals $I^{j_1 j_2 \dots}_{i_1 i_2 \dots}$, up to the relevant order. Therefore we have introduced the superscripts (q^a), $a = -1, 0$, on the $I^{j_1 j_2 \dots}_{i_1 i_2 \dots}$, denoting the relevant order in q to which the integrals $I^{j_1 j_2 \dots}_{i_1 i_2 \dots}$ have been identified. In this way we can directly compare this expression with the explicit expression in Eq. (3.7) for S_s through $\mathcal{O}(q)$. The last term, which has not been expressed in terms of $I^{j_1 j_2 \dots}_{i_1 i_2 \dots}$, is the 'left-over' term from this identification procedure. All the other terms can be matched one-by-one with similar terms in Eq. (3.7). In Eq. (3.7) only the terms not involving A_i's remain unmatched. Specifically we have:

$$
\begin{aligned}
&\Big[\hat{S}^{(1)} M_n - (\hat{S}^{(1)} M_n^b) * M_n^s \Big] - \big[M_n * (S_s + \bar{S}_s) \big]_S \Big|_{\mathcal{O}(q)} \\
&= \kappa_D \epsilon^S_{\mu\nu} \Bigg[\frac{\alpha'}{2} q_\rho q_\sigma \sum_{i\neq j} \frac{C_j^{[\rho} k_j^{\nu]} C_i^{[\mu} k_i^{\sigma]}}{q k_i (z_i - z_j)^2} - \left(\frac{\alpha'}{2}\right)^{3/2} q_\rho \sum_{i=1} C_i^{[\rho} k_i^{\nu]} \Big(k_i^\mu I^i_{ii} + \sum_{j\neq i} k_j^\mu I^j_{ii} \Big) \\
&\qquad + \left(\frac{\alpha'}{2}\right)^2 q_\rho q_\sigma \sum_{i\neq j} C_i^{[\sigma} k_i^{\nu]} k_i^\mu C_j^\rho I^i_{iij} + \left(\frac{\alpha'}{2}\right)^2 q_\rho q_\sigma \sum_{i\neq j} C_j^{[\sigma} k_j^{\nu]} k_i^\mu C_i^\rho I^i_{jji} \Bigg] + \text{c.c.} \\
&= \kappa_D \epsilon^S_{\mu\nu} \left[\frac{\alpha'}{2} q_\rho q_\sigma \sum_{i\neq j} \frac{C_i^{[\mu} k_i^{\rho]} C_j^{[\sigma} k_i^{\nu]}}{q k_i (z_i - z_j)^2} - \left(\frac{\alpha'}{2}\right)^{\frac{3}{2}} \sum_{i=1}^n \sum_{j\neq i} q_\rho q_\sigma \frac{C_i^{[\rho} k_i^{\nu]} k_i^{[\mu} k_j^{\sigma]}}{q k_i (z_i - z_j)} \right] + \text{c.c}\,,
\end{aligned}
\tag{4.14}
$$

where the first part of the left-hand side was identified with Eq. (4.13) using Eq. (4.11). To arrive to the final equality we made use of:

$$I^i_{ii} = \sum_{j\neq i} \frac{qk_j}{qk_i(z_i - z_j)} + \mathcal{O}(q) \quad ; \quad I^j_{ii} = -\frac{1}{z_i - z_j} + \mathcal{O}(q) \tag{4.15}$$

$$I^i_{iij} = -\frac{2}{\alpha' qk_i(z_i - z_j)^2} + \mathcal{O}(q^0) \quad ; \quad I^j_{iij} = \frac{2}{\alpha' qk_j(z_i - z_j)^2} + \mathcal{O}(q^0) \tag{4.16}$$

The right-hand side of Eq. (4.14) is exactly equal to the α'-correction in the bosonic string, given in Eq. (4.12). Since the soft-behavior of the bosonic part M_n^b is given by Eq. (4.1), which is exactly $\hat{S}^{(1)}$ plus the above α' corrections, we arrive at the conclusion that:

$$\hat{S}^{(1)} M_n = M_n * (S_b + S_s + \bar{S}_s)_S\big|_{\mathcal{O}(q)} = (M_{n+1})_S\big|_{\mathcal{O}(q)} \tag{4.17}$$

This is a subsubleading soft theorem for the graviton and dilaton in the supersymmetric string, with $\hat{S}^{(1)}$ defined in Eq. (4.10), and it is simply equal to the field theory result derived in Refs. [7, 9], without further string corrections. To be specific, what we have just observed is that the α' corrections appearing in S_b are exactly cancelled by the additional supersymmetry parts $S_s + \bar{S}_s$.

5 String corrections in heterotic string from gauge invariance

Both in Ref. [2] for the bosonic string and in this paper for the superstring we have computed the soft behavior through subsubleading order by explicitly performing the string integrals. On the other hand, in Ref. [2] we have also determined the soft behavior, including the string corrections, by imposing gauge invariance and the fact that the three-point amplitude involving massless particles already has string corrections. In this section, we extend this second procedure to the heterotic string fixing also in this case the string corrections at subsubleading order for a soft graviton or dilaton. It turns out that, as in the bosonic string, the soft graviton behavior includes string corrections that are, however, absent for a soft dilaton. This implies that the soft behavior of the dilaton is uniquely encoded in an operator universally applicable to all string theories and field theory.

The basic ingredient is the three-point amplitude involving gravitons, dilatons and Kalb-Ramond fields that in the heterotic string is equal to

$$2\kappa_D \big[\eta^{\mu\mu_i} q_i^\alpha - \eta^{\mu\alpha_i} q^{\mu_i} + \eta^{\mu_i\alpha} k_i^\mu - \frac{\alpha'}{2} k_i^\mu q^{\mu_i} q_i^\alpha\big] \big[\eta^{\nu\nu_i} q_i^\beta - \eta^{\nu\beta_i} q^{\nu_i} + \eta^{\nu_i\beta_i} k_i^\nu\big] \,, \tag{5.1}$$

where the three particles have the following momenta and polarizations: (q, μ, ν), (k_i, μ_i, ν_i) and $((-q - k_i), \alpha_i, \beta_i)$. In writing the previous equation we have used momentum conservation and we have eliminated terms that are zero when we saturate it with the three polarization vectors.

The leading term of the scattering amplitude of $(n+1)$ massless particles, when one of them becomes soft, is given by the diagram where the soft particle is attached to the other hard external particles. Since we are only interested in the term corresponding to string corrections and of order q in the momentum of the soft particle, this pole term is given by,

$$M_{n+1}^{\mu\nu}(k_1 \dots k_n,\, q) \sim -\alpha' \kappa_D \sum_{i=1}^{n} \epsilon^i_{\mu_i} \bar{\epsilon}^i_{\nu_i} k_i^\mu k_i^\nu q^{\mu_i} q^{\alpha_i} \eta^{\nu_i \beta_i} \frac{\eta_{\alpha_i r_i} \eta_{\beta_i s_i}}{2 k_i q} M_n^{r_i s_i}(k_i + q)\,. \tag{5.2}$$

In order to get a gauge invariant expression we have to add also a term that is regular in the soft limit ($q \sim 0$):

$$M_{n+1}^{\mu\nu}(k_1 \dots k_n,\, q)\, |_{\alpha'} = -\alpha' \kappa_D \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{2 k_i q} q^\rho q^\sigma T_{i\,\rho\sigma}\; M_n(k_i + q) + N^{\mu\nu}(q, k_i)\,, \tag{5.3}$$

where we also used

$$T_{i\,\rho\sigma} = \epsilon_{i\,\rho} \frac{\partial}{\partial \epsilon_i^\sigma} \quad ; \quad M_n(k_i + q) \equiv \epsilon_i^{r_i} \bar{\epsilon}_i^{s_i} M^{r_i s_i}(k_i + q)\,, \tag{5.4}$$

and we have omitted to strip off the polarization vectors for the other, $j \neq i$, $n-1$ states. Since the pole term is symmetric under the exchange of the indices μ and ν, gauge invariance can only determine the symmetric part of $N^{\mu\nu}$. This is consistent with the fact that gauge invariance does not fix the term of order q in the soft limit of the Kalb-Ramond field.

Gauge invariance implies:

$$q_\mu M_{n+1}^{\mu\nu}\, |_{\alpha'} = -\frac{\alpha' \kappa_D}{2} \sum_{i=1}^{n} k_i^\nu q^\rho q^\sigma T_{i\,\rho\sigma} M_n(k_i + q) + q_\mu N^{\mu\nu}(q, k_i) = 0\,. \tag{5.5}$$

Expanding for small q we get $N^{\mu\nu}(q = 0; k_i) = 0$ and

$$\frac{\partial}{\partial q_\rho} N^{\mu\nu} + \frac{\partial}{\partial q_\mu} N^{\rho\nu} = \frac{\alpha' \kappa_D}{2} \sum_{i=1}^{n} k_i^\nu \left(T_i^{\mu\rho} + T_i^{\rho\mu}\right) M_n(k_i)\,. \tag{5.6}$$

Inserting it in Eq. (5.3) we get

$$\begin{aligned} M_{n+1}^{\mu\nu}\, |_{\alpha'} = \; & -\frac{\alpha' \kappa_D}{2} \sum_{i=1}^{n} \frac{k_i^\nu k_i^\mu}{k_i q} q^\rho q^\sigma T_{i\,\rho\sigma} M_n(k_i) \\ & + \frac{\alpha' \kappa_D}{8} \sum_{i=1}^{n} q_\rho \left[k_i^\nu \left(T_i^{\mu\rho} + T_i^{\rho\mu}\right) + k_i^\mu \left(T_i^{\nu\rho} + T_i^{\rho\nu}\right)\right] M_n \\ & + \frac{1}{4} q_\rho \left[\frac{\partial}{\partial q_\rho} N^{\mu\nu} - \frac{\partial}{\partial q_\mu} N^{\rho\nu} + \frac{\partial}{\partial q_\rho} N^{\nu\mu} - \frac{\partial}{\partial q_\nu} N^{\rho\mu}\right]\,, \end{aligned} \tag{5.7}$$

where we have symmetrized under the exchange of ν and μ because, as already observed, the amplitude has such a symmetry. Imposing gauge invariance on the index ν; i.e. $q_\nu M_{n+1}^{\mu\nu} = 0$, we get the following condition:

$$q_\nu q_\rho \left[\frac{\partial}{\partial q_\rho} N^{\mu\nu} - \frac{\partial}{\partial q_\mu} N^{\rho\nu} \right] = \alpha' \kappa_D q_\nu q_\rho \sum_{i=1}^{n} \left[k_i^\mu T_i^{\nu\rho} - \frac{1}{2} k_i^\nu \left(T_i^{\mu\rho} + T_i^{\rho\mu} \right) \right] M_n(k_i) \tag{5.8}$$

that implies

$$\begin{aligned} &\frac{1}{2} \left[\left(\frac{\partial}{\partial q_\rho} N^{\mu\nu} - \frac{\partial}{\partial q_\mu} N^{\rho\nu} \right) + \left(\frac{\partial}{\partial q_\nu} N^{\mu\rho} - \frac{\partial}{\partial q_\mu} N^{\nu\rho} \right) \right] \\ &= \alpha' \kappa_D \sum_{i=1}^{n} \left[\frac{1}{2} k_i^\mu (T_i^{\nu\rho} + T_i^{\rho\nu}) - \frac{1}{4} k_i^\nu (T_i^{\mu\rho} + T_i^{\rho\mu}) - \frac{1}{4} k_i^\rho (T_i^{\mu\nu} + T_i^{\nu\mu}) \right] M_n(k_i) \,. \end{aligned} \tag{5.9}$$

From the previous relation we can extract the part that is symmetric under the exchange of μ and ν obtaining

$$\begin{aligned} &\frac{1}{4} \left[\left(\frac{\partial}{\partial q_\rho} N^{\mu\nu} - \frac{\partial}{\partial q_\mu} N^{\rho\nu} \right) + \left(\frac{\partial}{\partial q_\rho} N^{\mu\nu} - \frac{\partial}{\partial q_\nu} N^{\mu\rho} \right) \right] \\ &= \alpha' \kappa_D \sum_{i=1}^{n} \left[\frac{1}{8} k_i^\mu (T_i^{\nu\rho} + T_i^{\rho\nu}) + \frac{1}{8} k_i^\nu (T_i^{\mu\rho} + T_i^{\rho\mu}) - \frac{1}{4} k_i^\rho (T_i^{\mu\nu} + T_i^{\nu\mu}) \right] M_n(k_i) \,, \end{aligned} \tag{5.10}$$

which fixes the last part of Eq. (5.7). An alternative way of deriving the previous expression is by noticing that Eq. (5.6), together with the symmetric and antisymmetric parts of Eq. (5.9), under the exchange of μ and ν, actually allow to determine the derivative of $N^{\mu\nu}$:

$$\frac{\partial}{\partial q_\rho} N^{\mu\nu} = \frac{\alpha' \kappa_D}{4} \sum_{i=1}^{n} \left[k_i^\mu (T^{\nu\rho} + T^{\nu\rho}) + k_i^\nu (T^{\mu\rho} + T^{\mu\rho}) - k_i^\rho (T^{\mu\nu} + T^{\nu\mu}) \right] M_n(k_i) \,. \tag{5.11}$$

One can then use this to fix the last part of Eq. (5.7), equivalent to Eq. (5.10).

Inserting Eq. (5.10) in Eq. (5.7) we finally get the completely fixed string corrections in the case of the heterotic string:

$$M_{n+1}^{\mu\nu} \mid_{\alpha'} = - \frac{\alpha' \kappa_D}{4} \sum_{i=1}^{n} \left[\frac{k_i^\mu k_i^\nu}{k_i q} q^\rho q^\sigma - q^\rho k_i^\nu \eta^{\mu\sigma} - q^\rho k_i^\mu \eta^{\nu\sigma} + (k_i q) \eta^{\mu\sigma} \eta^{\nu\rho} \right] \left(T_i^{\rho\sigma} + T_i^{\sigma\rho} \right) M_n(k_i) \,, \tag{5.12}$$

By saturating it with the dilaton polarization $\epsilon_{\mu\nu}^{(D)} = \eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu$ we get

$$\epsilon_{\mu\nu}^{(D)} M_{n+1}^{\mu\nu} \mid_{\alpha'} = - \frac{\alpha' \kappa_D}{2} \sum_{i=1}^{n} \left[-2 q^\rho k_i^\sigma + (k_i q) \eta^{\rho\sigma} \right] \frac{1}{2} \left(T_i^{\rho\sigma} + T_i^{\sigma\rho} \right) M_n(k_i) = 0 \,, \tag{5.13}$$

which vanishes because of transversality, $(k_i \epsilon_i) = 0$, gauge invariance, $k_i^\sigma \frac{\partial}{\partial \epsilon_i^\sigma} M_n = 0$, and momentum conservation, $\sum_{i=1}^{n} k_i = -q$.

In conclusion, as in the bosonic string and in superstring, also in the heterotic string the soft theorem of the dilaton has no α' corrections.

6 Conclusions and remarks

In this paper we have computed superstring amplitudes with an arbitrary number of massless external states in the kinematic region where one of the massless states carries low momentum, be it a graviton, dilaton or a Kalb-Ramond field. The soft behaviour of the amplitude has been determined through the subsubleading order. When the soft external state is a graviton or a dilaton we have further been able to identify soft operators that, when acting on the amplitude involving only the hard states, reproduce our results, thus demonstrating a soft theorem for these states.

The calculation is an extension of the one done in Ref. [2] for the bosonic string and despite the much more complicate expressions of the amplitudes it requires exactly the same ingredients and techniques developed for the bosonic theory.

In the case of the graviton, we have found that the soft operators coincide up to subsubleading order with the ones already identified in the literature without any string correction. More specifically, we have shown that the string corrections appearing in the bosonic string are exactly cancelled by the additional supersymmetric contributions to the amplitude. This result confirms the validity of the procedures developed in Ref. [7, 2] where the soft behaviour is determined via gauge invariance from the interaction vertices with three massless closed string states. The absence of string corrections in the soft theorem is a consequence of the absence of such corrections in the three-point amplitude of massless states in superstring theory.

In the case of the dilaton we have found a universal soft behavior; i.e. it is the same in superstring, as well as in heterotic and bosonic string. The universality is a consequence of the vanishing of the string corrections to the soft theorem in all models. It thus also coincide with the field theory result. The dilaton soft operator contains the generators of scale transformations at subleading order, and the special conformal transformations at subsubleading order, as shown in Refs. [9, 2]. Curiously this property is similar to the soft theorem, derived recently also in Ref. [9], of another scalar known as a dilaton; i.e. the Nambu-Goldstone boson of spontaneously broken conformal symmetry. Both dilatons couple to the trace of the energy momentum tensor, but they obey slightly different soft theorems through the subsubleading order. Understanding this difference, as well as understanding the physical origin of the string dilaton soft behavior, are indeed problems that deserve further studies.

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A Explicit action of the subsubleading soft operator

In this appendix we compute the action of the subsubleading soft operator given in Eq. (4.10) on the n-point amplitude with only hard particles. We denote this operator by $\hat{S}^{(1)}$, i.e.:

$$\hat{S}^{(1)} = -\kappa_D \frac{\epsilon^S_{\mu\nu}}{2} \sum_{i=1}^{n} \left[\frac{q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{qk_i} + \frac{q^\mu \eta^{\nu\rho} q^\sigma + q^\mu \eta^{\nu\sigma} q^\rho - \eta^{\mu\nu} q^\sigma q^\rho}{qk_i} \mathbf{A}_{i\rho\sigma} \right], \tag{A.1}$$

with J_i the total angular momentum operator and $\mathbf{A}_i$ given in Eq. (1.12).

We observed in Sec. 2 that the superstring amplitudes with generically n-massless states can be decomposed at the integrand level into two parts; i.e. $M_n = M_n^b * M_n^s$, where one part is related to to the bosonic string, and the other part is a pure superstring contribution. We can therefore write the action of $\hat{S}^{(1)}$ on M_n as follows:

$$\begin{aligned} \hat{S}^{(1)} M_n &= \hat{S}^{(1)}(M_n^b * M_n^s) \\ &= (\hat{S}^{(1)} M_n^b) * M_n^s + M_n^b * (\hat{S}^{(1)} M_n^s) - \kappa_D\, \epsilon^S_{\mu\nu}\, q_\rho q_\sigma \sum_{i=1}^{n} \frac{(J_i^{\mu\rho} M_n^b) * (J_i^{\nu\sigma} M_n^s)}{qk_i} \end{aligned} \tag{A.2}$$

The first term, where the soft operator acts on the bosonic string amplitude M_n^b, has already been determined in Ref. [2], and given in Eq. (4.1), for $\alpha' = 0$. Here we analyze the remaining action of $\hat{S}^{(1)}$ on the full superstring amplitude.

The action of the angular momentum operator on M_n^b and M_n^s, given respectively in Eq. (2.14) and (2.15), is easily computed and reads:

$$J_i^{\mu\rho} M_n^b = i M_n^b * \sum_{j\neq i=1}^{n} \left[\frac{\alpha'}{2} k_i^{[\mu} k_j^{\rho]} \log|z_i - z_j|^2 + \left(\sqrt{\frac{\alpha'}{2}} \frac{C^\mu_{\{i,} k^\rho_{j\}} - C^\rho_{\{i,} k^\mu_{j\}}}{z_i - z_j} + \frac{C_i^{[\mu,} C_j^{\rho]}}{(z_i - z_j)^2} + \text{c.c} \right) \right] \tag{A.3}$$

and

$$J_i^{\mu\rho} M_n^s = i M_n^s * \sum_{j\neq i=1}^{n} \left[\frac{A^\rho_{\{i} A^\mu_{j\}}}{z_i - z_j} + \text{c.c.} \right] \tag{A.4}$$

where the antisymmetric and symmetric combinations of the indices are denoted with $k_i^{[\mu} k_j^{\nu]} = k_i^\mu k_j^\nu - k_i^\nu k_j^\mu$ and $k_i^{\{\mu} k_j^{\nu\}} = k_i^\mu k_j^\nu + k_i^\nu k_j^\mu$.

Let us consider in Eq. (A.2) the 'mixing' part, which by the above formulas can be written as

$$\begin{aligned} &- \sum_{i=1}^{n} \frac{q_\rho q_\sigma}{qk_i} (J_i^{\mu\rho} M_n^b) * (J_i^{\nu\sigma} M_n^s) = M_n * \sum_{i=1}^{n} \frac{q_\rho q_\sigma}{qk_i} \sum_{j\neq i} \frac{\bar{A}^\rho_{\{i,} \bar{A}^\nu_{j\}}}{\bar{z}_i - \bar{z}_j} \sum_{l\neq i} \left[\frac{\alpha'}{2} k_i^{[\mu,} k_l^{\sigma]} \log|z_i - z_l|^2 \right. \\ &\left. + \sqrt{\frac{\alpha'}{2}} \frac{C^\mu_{\{i,} k^\sigma_{l\}}}{z_i - z_l} - \sqrt{\frac{\alpha'}{2}} \frac{C^\sigma_{\{i,} k^\mu_{l\}}}{z_i - z_l} + \frac{C_i^{[\mu,} C_l^{\sigma]}}{(z_i - z_l)^2} + \sqrt{\frac{\alpha'}{2}} \frac{\bar{C}_l^{[\mu} k_i^{\sigma]}}{\bar{z}_i - \bar{z}_l} + \right] + \text{c.c.}, \end{aligned} \tag{A.5}$$

where we made use of the Grassmannian identity $\bar{A}_i^\alpha \bar{C}_i^\beta = 0$, cf. Eq. (2.12), to cancel some terms.

The idea is now to rewrite every term in terms of the integrals $I_{i_1 i_2 \ldots}^{j_1 j_2 \ldots}$ to be able to directly compare with the expression in Eq. (3.7). All the identities involving the integrals that we give in this appendix, are obtained starting from Eqs. (3.3), (3.4) and the explicit expression of the master integrals.

Let us consider the terms one by one:

- The terms containing the logarithm can be equivalently written as:

$$\begin{aligned}
&\sum_{i\neq j}\sum_{l\neq i}\frac{\alpha'}{2}q_\rho\left(\frac{qk_l}{qk_i}k_i^\mu - k_l^\mu\right)\left(\frac{A^\rho_{\{i}A^\nu_{j\}}}{z_i - z_j} + \text{c.c}\right)\log|z_i - z_l|^2 \\
&\quad = -\sum_{i\neq j\neq l}\frac{\alpha'}{2}k_l^\mu q_\rho \frac{A^\rho_{\{i}A^\nu_{j\}}}{z_i - z_j}\log|z_i - z_l|^2 \\
&\qquad + \sum_{i\neq j}\frac{\alpha'}{2}k_i^\mu q_\rho \frac{A^\rho_{\{i}A^\nu_{j\}}}{z_i - z_j}\left(\log|z_i - z_j|^2 + \sum_{i\neq l}\frac{qk_l}{qk_i}\log|z_i - z_l|^2\right) + \text{c.c} \\
&\quad = \sum_{i\neq l\neq j}\frac{\alpha'}{2}k_l^\mu q_\rho A_i^\rho A_j^\nu {I^{(q^0)}}^l_{ij} + \sum_{i\neq j}\frac{\alpha'}{2}k_i^\mu q_\rho A^\rho_{\{i}A^\nu_{j\}} {I^{(q^0)}}^i_{ij} + \text{c.c}
\end{aligned} \tag{A.6}$$

  where we have used Eqs. (3.4), (3.5) and (3.6) to identify:

$${I^{(q^0)}}^l_{ij} = \frac{\log\frac{|z_j - z_l|^2}{|z_i - z_l|^2}}{z_i - z_j} \quad ; \quad {I^{(q^0)}}^i_{ij} = \frac{\log|z_i - z_j|^2}{z_i - z_j} + \sum_{i\neq l}\frac{qk_l}{qk_i}\frac{\log|z_i - z_l|^2}{z_i - z_j} \tag{A.7}$$

  Here $I^{(q^0)}$ denotes soft expansion of the integral I through $\mathcal{O}(q^0)$.

- The term involving $C^\mu_{\{i}k^\sigma_{l\}}$ in Eq. (A.5) can be written as:

$$\begin{aligned}
&\sum_{i=1}^n \sqrt{\frac{\alpha'}{2}}\frac{q_\rho q_\sigma}{qk_i}\sum_{l;j\neq i}\frac{\bar{A}^\rho_{\{i,}\bar{A}^\nu_{j\}}}{\bar{z}_i - \bar{z}_j}\frac{C^\mu_{\{i}k^\sigma_{l\}}}{z_i - z_l} \\
&\quad = \sum_{i\neq j\neq l}\sqrt{\frac{\alpha'}{2}}q_\rho C_l^\mu \bar{A}_i^\rho \bar{A}_j^\nu I^{ji}_{ll} + \sum_{i\neq j}\sqrt{\frac{\alpha'}{2}}q_\rho C_i^\mu \bar{A}^\rho_{\{j}\bar{A}^\nu_{i\}} I^{ij}_{ii} + \mathcal{O}(q^2)
\end{aligned} \tag{A.8}$$

  where we have used Eqs. (3.3), (3.4) and the master integrals to get:

$${I^{(q^0)}}^{ji}_{ll} = \frac{1}{\bar{z}_i - \bar{z}_j}\left(\frac{1}{z_i - z_l} - \frac{1}{z_j - z_l}\right) \; ; \; {I^{(q^0)}}^{ij}_{ii} = \frac{1}{\bar{z}_i - \bar{z}_j}\left(\sum_{l\neq i}\frac{qk_l}{qk_i(z_i - z_l)} + \frac{1}{z_i - z_j}\right) \tag{A.9}$$

- In the same way the term in Eq. (A.5) involving $C^\sigma_{\{i}k^\mu_{l\}}$ becomes:

$$
\begin{aligned}
&-\sum_{i=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{q_\rho q_\sigma}{qk_i}\sum_{l;j\neq i}\frac{\bar{A}^\rho_{\{i,}\bar{A}^\nu_{j\}}}{\bar{z}_i-\bar{z}_j}\frac{C^\sigma_{\{i}k^\mu_{l\}}}{z_i-z_l} \\
&= -\left(\frac{\alpha'}{2}\right)^{\frac{3}{2}} q_\sigma q_\rho\left(\sum_{i\neq j\neq l} k^\mu_i C^\sigma_l\left(\bar{A}^\rho_{\{i}\bar{A}^\nu_{j\}}I^{ij}_{il}+\bar{A}^\rho_{\{l}\bar{A}^\nu_{j\}}I^{lj}_{il}\right)+\sum_{i\neq j} k^\mu_i C^\sigma_j \bar{A}^\rho_{\{i}\bar{A}^\nu_{j\}}I^{ij}_{ij}\right)+\mathcal{O}(q^2)
\end{aligned}
\tag{A.10}
$$

where

$$
I^{ij}_{il}=\frac{2}{\alpha' qk_i(\bar{z}_i-\bar{z}_j)(z_i-z_l)}+O(q^0)\ ;\quad I^{lj}_{il}=-\frac{2}{\alpha' qk_l(\bar{z}_l-\bar{z}_j)(z_i-z_l)}+O(q^0) \tag{A.11}
$$

- The term in Eq. (A.5) involving $C^{[\mu}_i C^{\sigma]}_l$ is rewritten in the form:

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{qk_i}\sum_{l;j\neq i}\frac{\bar{A}^\rho_{\{i,}\bar{A}^\nu_{j\}}}{\bar{z}_i-\bar{z}_j}\frac{C^{[\mu}_i C^{\sigma]}_l}{(z_i-z_l)^2} \\
&= -\frac{\alpha'}{2}q_\sigma q_\rho\left(\sum_{i\neq i\neq l} C^\mu_i C^\sigma_l \bar{A}^\rho_{\{i}\bar{A}^\nu_{j\}}I^{ij}_{iil}+\sum_{i\neq i\neq l} C^\mu_i C^\sigma_l \bar{A}^\rho_{\{l}\bar{A}^\nu_{j\}}I^{lj}_{iil}+\sum_{i\neq j}\bar{A}^\rho_{\{i}\bar{A}^\nu_{j\}}C^\mu_i C^\sigma_j I^{ij}_{iij}\right)+\mathcal{O}(q^2)
\end{aligned}
\tag{A.12}
$$

where we have used the identities:

$$
I^{ij}_{iil}=-\frac{2}{\alpha' qk_i(z_i-z_l)^2(\bar{z}_i-\bar{z}_j)}+\mathcal{O}(q^0)\quad ;\quad I^{lj}_{iil}=\frac{2}{\alpha' qk_l(z_i-z_l)^2(\bar{z}_l-\bar{z}_j)}+\mathcal{O}(q^0)
$$

$$
I^{ij}_{iij}=-\frac{2}{\alpha'(\bar{z}_i-\bar{z}_j)(z_i-z_j)^2}\left(\frac{1}{qk_i}+\frac{1}{qk_j}\right)+\mathcal{O}(q^0) \tag{A.13}
$$

- Finally, the term in Eq. (A.5) involving $\bar{C}^{[\mu}_l k^{\sigma]}_i$ can be written as:

$$
\begin{aligned}
&\sum_{i=1}^{n}\sqrt{\frac{\alpha'}{2}}\frac{q_\rho q_\sigma}{qk_i}\sum_{l;j\neq i}\frac{\bar{A}^\rho_{\{i,}\bar{A}^\nu_{j\}}}{\bar{z}_i-\bar{z}_j}\frac{\bar{C}^{[\mu}_l k^{\sigma]}_i}{\bar{z}_i-\bar{z}_l} \\
&= -\sum_{i\neq j\neq l}\sqrt{\frac{\alpha'}{2}}q_\rho\frac{\bar{C}^\mu_i\bar{A}^\rho_j\bar{A}^\nu_l}{(\bar{z}_i-\bar{z}_l)(\bar{z}_i-\bar{z}_j)}-\sum_{i\neq j\neq l}\left(\frac{\alpha'}{2}\right)^{\frac{3}{2}} q_\rho q_\sigma k^\mu_i\bar{C}^\sigma_l\bar{A}^\rho_{\{j}\bar{A}^\nu_{i\}}I^{ilj}_i+\mathcal{O}(q^2)
\end{aligned}
\tag{A.14}
$$

where the following identity was used:

$$
I^{ilj}_i=\frac{2}{\alpha' qk_i(\bar{z}_i-\bar{z}_l)(\bar{z}_i-\bar{z}_j)}+\mathcal{O}(q^0)\,. \tag{A.15}
$$

Next we consider the 'pure' supersymmetric part of Eq. (A.2) and analyze the term:

$$
\begin{aligned}
&-\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{2qk_i}M_n^b * (J_i^{\mu\rho}J_i^{\nu\sigma}M_n^s)\\
&=-\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{2qk_i}M_n^b * J_i^{\mu\rho}\left[i\left(\sum_{j\neq i}\frac{A^\sigma_{\{i,}A^\nu_{j\}}}{z_i-z_j}+\text{c.c}\right)M_n^s\right]\\
&=\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{2qk_i}M_n^b * \left[\left(\sum_{j\neq i}\frac{A_i^\mu A_j^\nu\eta^{\sigma\rho}+A_i^\rho A_j^\sigma\eta^{\mu\nu}-\eta^{\nu\rho}A_i^\mu A_j^\sigma-\eta^{\sigma\mu}A_i^\rho A_j^\nu}{z_i-z_j}+\text{c.c}\right)\right.\\
&\left.\quad+\sum_{j,l\neq i}\left(\frac{(A^\rho_{\{i,}A^\mu_{j\}})(A^\sigma_{\{i,}A^\nu_{l\}})}{(z_i-z_j)(z_i-z_l)}+\frac{(A^\rho_{\{i,}A^\mu_{j\}})(\bar{A}^\sigma_{\{i,}\bar{A}^\nu_{l\}})}{(z_i-z_j)(\bar{z}_i-\bar{z}_l)}+\text{c.c.}\right)\right]M_n^s
\end{aligned}
\tag{A.16}
$$

The first term after the second equality involving $\eta^{\sigma\rho}$ vanishes since $q^2=0$, while all the other terms under the same parenthesis can be rewritten in terms of the following differential operator acting on the M_n^s:

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{2qk_i}M_n^b * \sum_{j\neq i}\left(\frac{A_i^\rho A_j^\sigma\eta^{\mu\nu}-\eta^{\nu\rho}A_i^\mu A_j^\sigma-\eta^{\sigma\mu}A_i^\rho A_j^\nu}{z_i-z_j}+\text{c.c}\right)M_n^s\\
&=-M_n^b * \sum_{i=1}^{n}\left(\frac{q^\sigma q^\rho\eta^{\nu\mu}-q^\rho q^\mu\eta^{\nu\sigma}-q^\sigma q^\mu\eta^{\rho\mu}}{2k_iq}\right)\mathbf{A}_{i\,\rho\sigma}M_n^s
\end{aligned}
\tag{A.17}
$$

This is nothing but the second part of $\hat{S}^{(1)}$ with opposite sign, as given in Eq. (A.1). Thus the two cancel.

The term in Eq. (A.16) involving four unbarred A_i's gives:

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{q_\rho q_\sigma}{2qk_i}M_n^b * \sum_{j;l\neq i}\left(\frac{A_i^\rho A_j^\mu A_i^\sigma A_l^\nu+A_i^\rho A_j^\mu A_l^\sigma A_i^\nu+A_j^\rho A_i^\mu A_i^\sigma A_l^\nu+A_j^\rho A_i^\mu A_l^\sigma A_i^\nu}{(z_i-z_j)(z_i-z_l)}+\text{c.c.}\right)M_n^s\\
&=M_n * \sqrt{\frac{\alpha'}{2}}q_\rho q_\sigma\left(\sum_{i\neq l\neq j}\frac{A_j^\rho A_l^\nu C_i^{[\mu}k_i^{\sigma]}}{2qk_i(z_i-z_j)(z_i-z_l)}+\sqrt{\frac{\alpha'}{2}}\sum_{i\neq j}\frac{C_j^{[\rho}k_j^{\nu]}C_i^{[\mu}k_i^{\sigma]}}{2qk_i(z_i-z_j)^2}+[\mu\leftrightarrow\nu]\right)+\text{c.c.}\\
&=M_n * \sqrt{\frac{\alpha'}{2}}q_\rho\left(\sum_{i\neq l\neq j}\frac{A_j^\rho A_l^\nu C_i^\mu}{2(z_i-z_j)(z_i-z_l)}-\frac{\alpha'}{4}q_\sigma A_j^\rho A_l^\nu k_i^\mu C_i^\sigma I^i_{ijl}\right.\\
&\qquad\left.+\sqrt{\frac{\alpha'}{2}}q_\sigma\sum_{i\neq j}\frac{C_j^{[\rho}k_j^{\nu]}C_i^{[\mu}k_i^{\sigma]}}{2qk_i(z_i-z_j)^2}+[\mu\leftrightarrow\nu]\right)+\text{c.c.}+\mathcal{O}(q^2)
\end{aligned}
\tag{A.18}
$$

where we have used Eq. (2.12) and the identities:

$$I^{i}_{ijl} = \frac{2}{\alpha' k_i q (z_i - z_j)(z_i - z_l)} + \mathcal{O}(q^0) \quad ; \quad q_\rho q_\sigma A^\rho_i A^\sigma_i = 0 \quad ; \quad \sum_{i \neq j \neq l} \frac{A^\mu_i A^\nu_i (qA_j)(qA_l)}{qk_i(z_i - z_l)(z_i - z_j)} = 0 \tag{A.19}$$

The last identity comes out due to the different parity of the numerator and denominator in the exchange of the indices l and j. We observe that the term involving $A^\rho_j A^\nu_l C^\mu_i$ in Eq. (A.18) will cancel the similar term coming from Eq. (A.14).

The last term in Eq. (A.16) can be equivalently written in the form:

$$q_\rho q_\sigma \sum_{i\neq j} \sum_{i \neq l} \frac{A^\mu_{\{i} A^\rho_{j\}} \bar{A}^\nu_{\{i} \bar{A}^\sigma_{l\}}}{2qk_i(z_i - z_j)(\bar{z}_i - \bar{z}_l)} = \frac{\alpha'}{2} q_\rho q_\sigma \sum_{i\neq j} \sum_{i \neq l} \frac{1}{2} A^\mu_{\{i} A^\rho_{j\}} \bar{A}^\nu_{\{i} \bar{A}^\sigma_{l\}} I^{il}_{ij} + \mathcal{O}(q^2) \tag{A.20}$$

where we have used the identities

$$I^{il}_{ij} = \frac{2}{\alpha' k_i q (z_i - z_j)(\bar{z}_i - \bar{z}_l)} + \mathcal{O}(q^0) \quad ; \quad I^{ij}_{ij} = \frac{2}{\alpha' |z_i - z_j|^2} \left[\frac{1}{k_i q} + \frac{1}{k_j q} \right] + \mathcal{O}(q^0)$$

which follow from Eqs. (3.4) and (3.5).

We can now summarize the result of Eq. (A.2). We are only interested in the second and third part in that expression, since we know already the result of $\hat{S}^{(1)} M^b_n$ from Ref. [1, 2]. In other words, cf. Eq. (A.2), we have found that

$$\begin{aligned}
&\hat{S}^{(1)} \, (M^b_n * M^s_n) - (\hat{S}^{(1)} M^b_n) * M^s_n = (M^b_n * M^s_n) * \epsilon^S_{\mu\nu} \sqrt{\frac{\alpha'}{2}} \Bigg\{ \\
&\quad q_\rho \Bigg[\sum_{i\neq j \neq l} \bar{A}^\rho_i \bar{A}^\nu_j \left(C^\mu_l I^{(q^0)}{}^{ij}_{ll} + \sqrt{\frac{\alpha'}{2}} k^\mu_l I^{(q^0)}{}^{ij}_{l} \right) + \sum_{i \neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} \left(C^\mu_j I^{(q^0)}{}^{ij}_{jj} + \sqrt{\frac{\alpha'}{2}} k^\mu_i I^{(q^0)}{}^{ij}_{i} \right) \Bigg] \\
&\quad + \sqrt{\frac{\alpha'}{2}} q_\rho q_\sigma \Bigg[\sum_{i\neq j} \sum_{l \neq i} \frac{1}{2} \bar{A}^\sigma_{\{l} \bar{A}^\nu_{i\}} A^\rho_{\{j} A^\mu_{i\}} I^{(q^{-1})}{}^{il}_{ij} - \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j \neq l} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} k^\mu_{\{i} C^\sigma_{l\}} I^{(q^{-1})}{}^{ij}_{il} \\
&\quad - \sqrt{\frac{\alpha'}{2}} \sum_{i \neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} k^\mu_i C^\sigma_j I^{(q^{-1})}{}^{ij}_{ij} - \sum_{i \neq j} \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} C^\mu_i C^\sigma_j I^{(q^{-1})}{}^{ij}_{iij} \\
&\quad - \sum_{i\neq j \neq l} C^\mu_i C^\sigma_l \bar{A}^\rho_{\{i} \bar{A}^\nu_{j\}} I^{(q^{-1})}{}^{ij}_{iil} - \sum_{i\neq j \neq l} C^\mu_i C^\sigma_l \bar{A}^\rho_{\{l} \bar{A}^\nu_{j\}} I^{(q^{-1})}{}^{lj}_{iil} - \sqrt{\frac{\alpha'}{2}} \sum_{i\neq j \neq l} \bar{A}^\rho_{\{j} \bar{A}^\nu_{i\}} k^\mu_i \bar{C}^\sigma_l I^{(q^{-1})}{}^{ilj}_{i} \\
&\quad - \sqrt{\frac{\alpha'}{2}} \sum_{i\neq l \neq j} \bar{A}^\rho_j \bar{A}^\nu_l k^\mu_i \bar{C}^\sigma_i I^{(q^{-1})}{}^{ijl}_{i} + \sum_{i \neq j} \frac{C^{[\rho}_j k^{\nu]}_j C^{[\mu}_i k^{\sigma]}_i}{qk_i(z_i - z_j)^2} \Bigg) \Bigg] \Bigg\} + \text{c.c.}
\end{aligned} \tag{A.21}$$

References

[1] P. Di Vecchia, R. Marotta and M. Mojaza, *JHEP* **1505**, 137 (2015) `arXiv:1502.05258`.

[2] P. Di Vecchia, R. Marotta and M. Mojaza, *JHEP* **1606**, 054 (2016), `arXiv:1604.03355`.

[3] S. Weinberg, *Phys. Rev.* **135**, B1049 (1964);
S. Weinberg, *Phys. Rev.* **140**, B516 (1965);
D. J. Gross and R. Jackiw, *Phys. Rev.* **166**, 1287 (1968);
R. Jackiw, *Phys. Rev.* **168**, 1623 (1968).

[4] M. Ademollo, A. D'Adda, R. D'Auria, F. Gliozzi, E. Napolitano, S. Sciuto and P. Di Vecchia, *Nucl. Phys.* **B94**, (1975) 221;
J. Shapiro, *Phys. Rev.* **D11** (1975) 2937.

[5] F. Cachazo and A. Strominger, `arXiv:1404.4091`.

[6] J. Broedel, M. de Leeuw, J. Plefka, M. Rosso, *Phys.Rev.* **D90** no. 6, (2014) 065024, `arXiv:1406.6574`.

[7] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, Phys. Rev. D **90**, no. 8, 084035 (2014), `arXiv:1406.6987`.

[8] A. Strominger, *JHEP* **1407**, (2014) 152,`arXiv:1312.2229`;
T. He, V. Lysov, P. Mitra and A. Strominger,*JHEP* **1505**, (2015) 151, `arXiv:1401.7026`;
B. U. W. Schwab and A. Volovich, *Phys. Rev. Lett.* **113** no. 10, (2014) 101601, `arXiv:1404.7749`;
Z. Bern, S. Davies, and J. Nohle, *Phys. Rev.* **D90** no. 8, (2014) 085015, `arXiv:1405.1015`;
S. He, Y.-t. Huang, and C. Wen, *JHEP* **12** (2014) 115, `arXiv:1405.1410`;
A. J. Larkoski, *Phys. Rev.* **D90** no. 8, (2014) 087701, `arXiv:1405.2346`;
N. Afkhami-Jeddi, `arXiv:1405.3533`;
C. D. White, *Phys. Lett.* **B737** (2014) 216–222, `arXiv:1406.7184`;
M. Zlotnikov, *JHEP* **10** (2014) 148, `arXiv:1407.5936`;
C. Kalousios and F. Rojas, *JHEP* **01** (2015) 107, `arXiv:1407.5982`;
Y.-J. Du, B. Feng, C.-H. Fu, and Y. Wang, *JHEP* **11** (2014) 090, `arXiv:1408.4179`;
D. Bonocore, E. Laenen, L. Magnea, L. Vernazza, and C. D. White, *Phys. Lett.* **B742** (2015) 375–382, `arXiv:1410.6406`;
A. Sabio Vera and M. A. Vazquez-Mozo, *JHEP* **03** (2015) 070, `arXiv:1412.3699`;
A. E. Lipstein, *JHEP* **06** (2015) 166, `arXiv:1504.01364`;
S. D. Alston, D. C. Dunbar, and W. B. Perkins, *Phys. Rev.* **D92** no. 6, (2015) 065024,

arXiv:1507.08882;
J. Rao and B. Feng, arXiv:1604.00650.

[9] P. Di Vecchia, R. Marotta, M. Mojaza and J. Nohle, *Phys. Rev. D* **93**, no. 8, 085015 (2016), arXiv:1512.03316.

[10] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *JHEP* **1009** (2010) 016, arXiv:0808.1446;
C. Cheung, K. Kampf, J. Novotny and J. Trnka, *Phys. Rev. Lett.* **114**, no. 22, 221602 (2015) arXiv:1412.4095;
R. Boels and W. Wormsbecher, arXiv:1507.08162;
Y. Huang, C. Wen, *JHEP* **1512** (2015) 143, arXiv:1509.07840;
I. Low, *Phys.Rev.* **D93** no.4 (2016) 045032 , arXiv:1512.01232;
H. Luo, C. Wen, *JHEP* **1603** (2016) 088, arXiv:1512.06801;
M. Bianchi, A. L. Guerrieri, Y. Huang, C. Lee, C. Wen, arXiv:1605.08697.

[11] B. U. W. Schwab, *JHEP* **08** (2014) 062, arXiv:1406.4172;
M. Bianchi, S. He, Y. Huang, C. Wen, *Phys.Rev.* **D92** (2015) no.6, 065022, hrefhttp://arxiv.org/abs/1406.5155 arXiv:1406.5155 ;
B. U. W. Schwab, *JHEP* **03** (2015) 140, arXiv:1411.6661;
M. Bianchi and A. L. Guerrieri, *JHEP* **09** (2015) 164, arXiv:1505.05854;
A. L. Guerrieri, IFAE 2015, Rome, Italy, April 8-10, 2015, arXiv:1507.08829;
P. Di Vecchia, R. Marotta, and M. Mojaza, *Fortschr. Phys.* **64** (2016) 389, arXiv:1511.04921.