

Properties of an affine transport equation and its holonomy

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Abstract An affine transport equation was used recently to study properties of angular momentum and gravitational-wave memory effects in general relativity. In this paper, we investigate local properties of this transport equation in greater detail. Associated with this transport equation is a map between the tangent spaces at two points on a curve. This map consists of a homogeneous (linear) part given by the parallel transport map along the curve plus an inhomogeneous part, which is related to the development of a curve in a manifold into an affine tangent space. For closed curves, the affine transport equation defines a “generalized holonomy” that takes the form of an affine map on the tangent space. We explore the local properties of this generalized holonomy by using covariant bitensor methods to compute the generalized holonomy around geodesic polygon loops. We focus on triangles and “parallelogramoids” with sides formed from geodesic segments. For small loops, we recover the well-known result for the leading-order linear holonomy ($\sim \text{Riemann} \times \text{area}$), and we derive the leading-order inhomogeneous part of the generalized holonomy ($\sim \text{Riemann} \times \text{area}^{3/2}$). Our bitensor methods let us naturally compute higher-order corrections to these leading results. These corrections reveal the form of the finite-size effects that enter into the holonomy for larger loops; they could also provide quantitative errors on the leading-order results for finite loops.

Keywords Bitensors · Holonomies · Transport Equations

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1 Introduction

In general relativity, vectors and tensors are most often transported between tangent spaces at different spacetime points using the Levi-Civita connection on the tangent bundle (the unique connection that is both metric-compatible and torsion-free). Within specific contexts in general relativity, there are often physical reasons to transport specific vectors by specialized transport equations. For example, along an accelerating worldline, it is often useful to carry vectors using Fermi-Walker transport (see, e.g., [1]). Similarly, for a spinning point particle, its 4-momentum and angular-momentum tensor are jointly transported through the coupled Mathisson-Papapetrou equations [2, 3] (the dual of these equations, the Killing transport equations, also transport a vector and antisymmetric tensor in a related way—see, e.g., [4]). In [5], one of the authors and a collaborator introduced an affine transport equation for vectors that proved useful for measuring physical effects related to the gravitational-wave memory and for transporting a type of special-relativistic linear and angular momentum in general relativity. We review the definition of this transport equation and describe some of its properties in the next subsection.

1.1 Affine transport equations of [5]

The aim of [5] was to define an operational method by which observers in asymptotically flat spacetimes could measure the linear and angular momentum of the spacetime geometry from the spacetime curvature and its derivatives in the vicinity of an observer’s timelike worldline. The observers could then compare their measured values of linear and angular momentum by using a specific transport equation (which has the form of an affine map between the tangent spaces along a curve that connects two points along the two different worldlines). In the case of spacetimes that are stationary, followed by a burst of gravitational waves with memory, and then stationary again, there was observer dependence in the measured angular momentum that was a consequence of the gravitational waves’ memory. Since the memory is related to the supertranslation degree of freedom in the Bondi-Metzner-Sachs (BMS) group [6, 7], the measurement and transport procedure was able to probe aspects of the angular-momentum ambiguity in general relativity.

The covariant method for transporting angular momentum and measuring gravitational-wave memory effects was based on a system of differential equations called “affine transport” in [5]. Given a vector ξ^a along a curve $x(\lambda)$, the affine transport of ξ^a was defined by

$$\dot{x}^b \nabla_b \xi^a = \alpha \dot{x}^a, \quad (1)$$

where $\dot{x}^a = dx^a/d\lambda$ is the tangent to the curve. For simplicity, we will assume $\alpha = 1$ throughout the remainder of this paper. The solution ξ^a at $x = x(\lambda)$, given the vector $\xi^{a'}$ at an initial point $x' = x(0)$, can be written as a sum of

homogeneous and inhomogeneous parts,

$$\xi^a = \Lambda^a_{a'} \xi^{a'} + \Delta \xi^a. \quad (2)$$

Here $\Lambda^a_{a'}(\lambda)$ is the parallel transport map along the curve from x' to x , which satisfies

$$\dot{x}^b \nabla_b \Lambda^a_{a'} = 0, \quad \Lambda^a_{a'}(0) = \delta^a_{a'}, \quad (3)$$

and $\Delta \xi^a(\lambda)$ is the inhomogeneous part of the solution, satisfying

$$\dot{x}^b \nabla_b \Delta \xi^a = \dot{x}^a, \quad \Delta \xi^a(0) = 0. \quad (4)$$

The solution takes the form of an affine map between the two tangent spaces at x and x' , which was the motivation for the name affine transport.

Only after [5] was written did it come to the attention of the authors of [5] that the transport equations (1) and their solution (2) are related to other aspects of general relativity and differential geometry. The vector $\Delta \xi^a$ also appears as the development of a curve on a manifold into the affine tangent space at the curve's starting point. This is sometimes equivalently described as rolling the manifold along the initial tangent space without slipping or twisting [8, 9, 10]. More specifically, a vector $\Delta \xi^{a'}$ at x' is equivalent to the displacement vector in the initial affine tangent space that points between the initial and final values of the rolling (or developing) curve, and $\Delta \xi^a = \Lambda^a_{a'} \Delta \xi^{a'}$ is its parallel transport along the curve in the manifold from x' to x . In flat spacetime, $\Delta \xi^a$ is the net displacement vector from x' to x , and in curved spacetime, $\Delta \xi^a$ provides a curve-dependent notion of a displacement vector between the two points.

In addition, there is another construction in which the transport equation (1) appears, as we now describe. As a slight generalization of the linear bundle of orthonormal frames, one can consider the affine frame bundle, in which the frame field is defined in an affine space and consequently an additional vector defining the origin of this affine tangent space is also required. A connection on this affine frame bundle prescribes that the vector is transported via (1) (see, e.g., [11] for a discussion of this in the relativity literature and for its use in understanding spacetimes with conical deficits). It has long been known [8] that there is a one-to-one mapping between connections on the affine frame bundle and connections with torsion on the linear frame bundle. In fact, for a connection on the affine frame bundle, the curvature of this connection contains both the usual Riemann curvature of the linear frame bundle and the torsion as the curvature associated with the part of the connection that determines the transport of the additional vector (see, e.g., [11]). Thus, the holonomy for the affine frame bundle has a “translational” part related to the torsion and a “rotational” part associated with the Riemann curvature.

We will work exclusively with the metric-compatible, torsion-free derivative ∇_a in this paper, however. Within this context, it is not immediately clear how the holonomy found from solving the affine transport equation around a closed curve (which was called a “generalized holonomy” in [5]) will behave in the limit of an infinitesimal loop. For large loops the inhomogeneous part

of the solution is known to be nonvanishing because of nonlocal effects of spacetime curvature [12, 11, 5]. In these three references, the inhomogeneous part of the solution has direct physical relevance, because it can be used to find spacetimes that contain “torsion without torsion,” understand certain spinning cosmic-string spacetimes, and measure gravitational-wave memory effects and observer dependence in angular momentum, respectively.

Our focus in this paper, therefore, will be to investigate the local geometrical properties of the affine transport equation (1) and its holonomy around infinitesimal loops for torsion-free connections. Our aim will be to find the relevant physical information that can be extracted from these local holonomies. When we compute this generalized holonomy around small (contractible) loops in a generic (smooth) pseudo-Riemannian manifold, we find that the inhomogeneous solution is a higher-order effect in the size of the loop for a torsion-free connection. In addition, the inhomogeneous part depends on just the Riemann tensor so that it contains the same physical data as the linear holonomy associated with parallel transport. Furthermore, by carefully defining the loop and using the methods of covariant bitensor calculus (see, e.g., [13, 14, 15]), we can compute higher-order corrections to the generalized holonomy in the size of the loop. It remains the case that the homogeneous and inhomogeneous solutions contain similar information about the gradient of the Riemann tensor, but the inhomogeneous solution scales more rapidly with the loop’s size. We next provide an overview of the methods, results, and organization of this paper in the next subsection.

1.2 Summary of the results of this paper

By specializing the solution in (2) to a closed curve beginning and ending at a point x , we note that the solution to the affine transport equation (1) around the loop defines an affine map on the tangent space at x . It takes an initial vector ξ_0^a at x and returns a final vector ξ^a at x :

$$\dot{x}^b \nabla_b \xi^a = \dot{x}^a \quad \Rightarrow \quad \xi^a = \Lambda^a_b \xi_0^b + \Delta \xi^a. \quad (5)$$

The linear map Λ^a_b is the linear holonomy associated with the metric-compatible, torsion-free derivative operator ∇_a , and the vector $\Delta \xi^a$ is the inhomogeneous contribution to generalized holonomy, $(\Lambda^a_b, \Delta \xi^a)$. We compute the generalized holonomy along the curves in Fig. 1, and we list the results of our calculation in equations (6)–(9).

We first compute the generalized holonomy around a small geodesic triangle (on the left in Fig. 1) defined by three points x , x' and x'' . We assume that these points are within a convex normal neighborhood of one another so that there exist unique geodesic segments connecting them. The solution can be expressed in terms of the two vectors u^a and v^a at x that yield the points x' and x'' under the exponential map. For the loop followed counterclockwise ($x \rightarrow x' \rightarrow x'' \rightarrow x$)—the “ u, v triangle” or $\Delta_{u,v}$, for short—we show that the

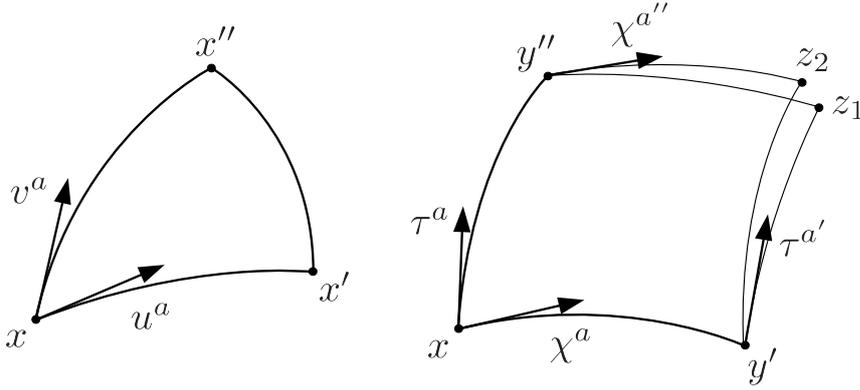


Fig. 1 *Left:* What we call the “ u, v triangle”, denoted by $\Delta_{u,v}$, which is traversed counterclockwise (the $x \rightarrow x' \rightarrow x'' \rightarrow x$ direction). The generalized holonomy of this loop is given by (6) and (7). The points x' and x'' are the images of the exponential maps of u^a and v^a at x . *Right:* There are two possible parallelogramoid loops that can be defined by a pair of vectors χ^a and τ^a at x , which we label by $x \rightarrow y' \rightarrow z_1 \rightarrow y'' \rightarrow x$ and $x \rightarrow y' \rightarrow z_2 \rightarrow y'' \rightarrow x$. The points y' and y'' are obtained through the exponential maps of χ^a and τ^a at x , respectively. The vector $\tau^{a'}$ at y' is the parallel transport (along the $x \rightarrow y'$ geodesic) of τ^a at x , and the point z_1 comes from the exponential map of $\tau^{a'}$ at y' ; there is then a unique geodesic linking z_1 to y'' and completing the first parallelogramoid. The vector $\chi^{a''}$ at y'' is the parallel transport (along the $x \rightarrow y''$ geodesic) of χ^a at x , and the point z_2 is the result of the exponential map of $\chi^{a''}$ at y'' ; there is then a unique geodesic that links y' to z_2 , thereby closing the second parallelogramoid. Through third order in distance, the generalized holonomy is the same around either loop, and (8) and (9) give the holonomy of what we call the χ, τ parallelogramoid loop, $\diamond_{\chi, \tau}$.

linear holonomy associated with parallel transport is

$$\Lambda^a_b(\Delta_{u,v}) = \delta^a_b + \frac{1}{2} R^a_{bcd} v^c u^d + \frac{1}{6} R^a_{bcd;e} v^c u^d (v^e + u^e) + O(4), \quad (6)$$

and the inhomogeneous solution is

$$\Delta \xi^a(\Delta_{u,v}) = \frac{1}{6} R^a_{bcd} (v^b + u^b) v^c u^d + O(4). \quad (7)$$

Here $O(n)$ stands for terms with n or more factors of the vectors u^a and v^a (i.e., n powers of distance). The result (7) shows that the leading-order inhomogeneous part of the generalized holonomy scales as the area of the triangle to the three-halves power (three powers of distance) times the Riemann tensor. Because we computed (7) using the torsion-free, metric-compatible derivative ∇_a , the solution depends just upon the Riemann tensor and is a higher-order effect. We also give an exact series solution to all orders in distance (written in terms of usual two-point coincidence limits of the parallel propagator), and we present explicit results through fourth-order in distance in Appendix A.

We next consider the holonomies around small “Levi-Civita parallelogramoids” [16], the quadrilaterals formed from geodesic segments that are the closest approximation in curved space to a flat-space parallelogram. As described in

Fig. 1 and Sec. 4, we can use a pair of vectors χ^a and τ^a at a point x to define two distinct parallelogramoid loops starting and ending at x . However, both loops in Fig. 1, when traversed in the counterclockwise direction from x , have the same generalized holonomy through third order in distance. It is given by

$$\Lambda^a_b(\diamond_{\chi,\tau}) = \delta^a_b + R^a_{bcd}\tau^c\chi^d + \frac{1}{2}R^a_{bcd;e}\tau^c\chi^d(\tau^e + \chi^e) + O(4), \quad (8)$$

and

$$\Delta\xi^a(\diamond_{\chi,\tau}) = \frac{1}{2}R^a_{bcd}(\tau^b + \chi^b)\tau^c\chi^d + O(4). \quad (9)$$

We show how the parallelogramoid solution can be obtained from a composition of the solutions for two triangles (which, through this order, is additive).

We now outline how we arrive at these results. In Sec. 2, we describe some basic mathematical results that will be needed to derive the generalized holonomy around a triangle and parallelogramoid. Specifically, in Sec. 2.1, we give the solution to the affine transport equation along a geodesic segment in terms of fundamental bitensors (the parallel propagator and derivatives of Synge's world function). Section 2.2 covers the mathematical framework for defining the geodesic triangles. The framework is similar to that used in [17, 13] to derive the curvature corrections to the law of cosines (a result which we reproduce below). Section 3.1 contains a derivation of the linear holonomy around the triangle and discussion about how this method relates to other procedures for computing the holonomy (e.g., using a path-ordered integral). Section 3.2 gives the inhomogeneous contribution to the generalized holonomy. We treat the generalized holonomy of the parallelogramoid in Sec. 4, and in Sec. 5 we discuss the implications of these calculations for the program described in [5]. We conclude in Sec. 6. Appendix A contains fourth-order terms for the generalized holonomy of the geodesic triangle.

2 Mathematical preliminaries

Because the formalism of covariant bitensors is carefully explained in the review paper [15], we will refer the reader to that resource for more background and detail on bitensor calculus. We generally adopt the notation of [15], except that we use Latin rather than Greek tensor indices, and we often interchange the role of the primed and unprimed indices (corresponding to tensor indices in the tangent spaces of different spacetime points) relative to [15]. Thus, for two spacetime points x and x' connected by a geodesic, we will use $\sigma(x, x')$ to denote Synge's world function (half the squared proper distance along the geodesic between the points) and $g^{a'}_a$ to denote the parallel propagator (which was written as $\Lambda^{a'}_a$ in the introduction). We will use semicolons preceding indices to denote the covariant derivative operator (e.g., $\sigma_{;a}$ or $\sigma_{;a'}$) and square brackets around quantities to denote coincidence limits (e.g., $[g^{a'}_b] = \delta^a_b$).

2.1 Affine transport along a geodesic segment

Consider an affinely parametrized geodesic $x'(\lambda)$ with tangent $u^{a'}(\lambda)$,

$$u^{a'} = \frac{dx^{a'}}{d\lambda}, \quad \frac{Du^{a'}}{D\lambda} = u^{b'} \nabla_{b'} u^{a'} = 0, \quad (10)$$

and let $x = x'(0)$ be a fixed initial point on the geodesic, where the tangent is u^a . The affine transport equation,

$$u^{b'} \nabla_{b'} \xi^{a'} = u^{a'}, \quad (11)$$

has a formal solution along the (assumed unique) geodesic connecting x to $x'(\lambda)$, which in the language of bitensor calculus [13, 14, 15] is given by

$$\xi^{a'} = g^{a'}{}_a(x, x') \xi^a + \sigma^{;a'}(x, x'). \quad (12)$$

Here $\xi^{a'}$ is the solution at x' , ξ^a is the initial value at x , $g^{a'}{}_a(x, x')$ is the parallel propagator, and $\sigma^{;a'}(x, x')$ is the covariant derivative at x' of Synge's world function $\sigma(x, x')$.

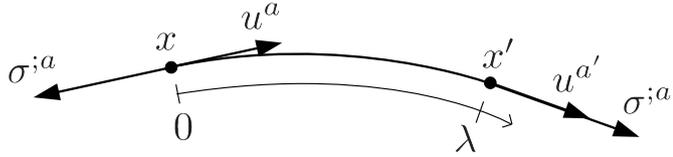


Fig. 2 The affinely parametrized geodesic $x'(\lambda)$ with initial point $x = x'(0)$. The tangent is parallel transported from u^a at x to $u^{a'}$ at x' . The derivatives of the world function both point outward from the geodesic segment: $\sigma^{;a} = -\lambda u^a$ and $\sigma^{;a'} = \lambda u^{a'}$.

That (12) satisfies (11) follows from the following properties: The world function is related to the tangents and the affine parameter interval by

$$\sigma(x, x') = \frac{1}{2} \lambda^2 u^2, \quad \sigma^{;a} = -\lambda u^a, \quad \sigma^{;a'} = \lambda u^{a'}, \quad (13)$$

and it satisfies

$$\sigma^{;b} \sigma^{;a}{}_b = \sigma^{;a}, \quad \sigma^{;b'} \sigma^{;a'}{}_{b'} = \sigma^{;a'}. \quad (14)$$

Dividing the second equation of (14) by λ and using the last equation of (13) shows that the second term in (12) is the inhomogeneous (particular) solution to (11). That the first term of (12) is the homogeneous solution follows from the second of the identities

$$\sigma^{;b} g^{a'}{}_{a;b} = 0, \quad \sigma^{;b'} g^{a'}{}_{a;b'} = 0, \quad (15)$$

and the condition $g^{a'}_a \rightarrow \delta^{a'}_a$ as $x' \rightarrow x$ that defines the parallel propagator. Also note that while the tangent is parallel transported, the world function derivatives are minus the parallel transports of each other:

$$u^{a'} = g^{a'}_a u^a, \quad \sigma^{;a'} = -g^{a'}_a \sigma^{;a}. \quad (16)$$

The above properties will be used often throughout the remainder of the paper.

2.2 Geodesic triangles

We now describe our framework for defining geodesic triangles (see Fig. 3). The somewhat lengthy and detailed definition of the triangle is necessary to compute $\Delta\xi^a$ without ambiguity (and also to obtain any higher-order corrections to both Λ^a_b and $\Delta\xi^a$).

We start at a fixed base point x with two vectors u^a and v^a , and we then follow the geodesics with initial tangents u^a and v^a for affine parameter intervals λ and ε to reach the points x' and x'' , respectively. As in (13), the tangents are related to the world-function derivatives by

$$\lambda u^a = -\sigma^{;a}(x, x'), \quad \varepsilon v^a = -\sigma^{;a}(x, x''). \quad (17)$$

This defines affinely parametrized geodesics $x'(\lambda)$ and $x''(\varepsilon)$ emanating from x . The tangents to these geodesics at x' and x'' are given by

$$\lambda u^{a'} = \sigma^{;a'}(x, x'), \quad \varepsilon v^{a''} = \sigma^{;a''}(x, x''), \quad (18)$$

which are parallel transports of (17) [cf. (16)]. We assume there is then a unique geodesic segment connecting x' to x'' . Its tangents are denoted by $w^{a'}$ at x' and $w^{a''}$ at x'' , and are assumed to be normalized so that the affine parameter interval from x' to x'' is 1. In terms of derivatives of the world function, they are given by

$$w^{a'} = -\sigma^{;a'}(x', x''), \quad w^{a''} = \sigma^{;a''}(x', x''), \quad (19)$$

and they are related to each other by parallel transport along the geodesic connecting x' and x'' .

In our calculations below, we will fix the base point x and the vectors u^a and v^a at x , and we will vary the affine parameters λ and ε . From this perspective, the points x' and x'' vary along the fixed geodesics determined by u^a and v^a at x . Quantities expressible as functions of x' and x'' , such as $w^{a'} = -\sigma^{;a'}(x', x'')$ or $g^{a''}_{a'}(x', x'')$, can then be expressed as functions of λ and ε , and can be differentiated according to

$$\frac{D}{D\lambda} = u^{a'} \nabla_{a'}, \quad \frac{D}{D\varepsilon} = v^{a''} \nabla_{a''}. \quad (20)$$

Note that these two derivatives commute, because $\nabla_{a'}$ and $\nabla_{a''}$ commute, and because $u^{a'}$ is independent of ε and $v^{a''}$ is independent of λ . Also note that the quantities $u^{a'}$ and $g^{a'}_a(x, x')$ depend only on λ ; $v^{a''}$ and $g^a_{a''}(x, x'')$ depend only on ε ; and $w^{a'}$, $w^{a''}$, and $g^{a''}_{a'}(x', x'')$ depend on both λ and ε . These are all of the quantities necessary to define the generalized holonomy around the triangle.

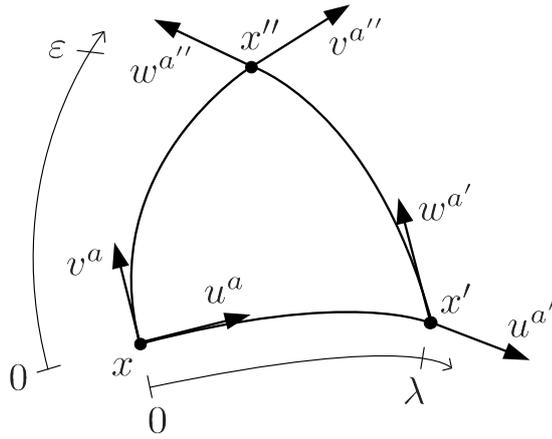


Fig. 3 The geodesic triangle associated with the three points x , x' , and x'' . We find it convenient to treat the triangle as a function of a fixed point x and fixed initial tangents u^a and v^a at x with two affine parameters λ and ε that parameterize two of the geodesic legs. The tangents are parallel transported along the legs: u^a at x to $u^{a'}$ at x' , v^a at x to $v^{a''}$ at x'' , and $w^{a'}$ at x' to $w^{a''}$ at x'' .

3 Generalized holonomy for geodesic triangles

3.1 Linear part of the holonomy

We now turn to the calculation of the holonomy of parallel transport around the geodesic triangle loop of the previous section. When following the loop counterclockwise ($x \rightarrow x' \rightarrow x'' \rightarrow x$), the holonomy of parallel transport is given by

$$A^a_b(\Delta_{u,v}) = g^a_{a'} g^{a''}_{b'} g^{b'}_b = \sum_{m,n=0}^{\infty} \frac{\lambda^m \varepsilon^n}{m!n!} A^a_{b(m,n)}. \quad (21)$$

We leave out the arguments of the parallel propagators, because they can be understood from their indices. Assuming x , u^a , and v^a fixed, while λ and ε vary to change the locations of x' and x'' , we write the holonomy tensor as a covariant Taylor series in λ and ε (the second equality). The coefficients $A^a_{b(m,n)}$ are constant tensors at x that will depend on u^a , v^a , and the local geometry at x .

We briefly digress to discuss the series solution for the holonomy in (21). There is also a different formal solution for the holonomy in terms of path-ordered integrals of the frame components of the Riemann tensor along a closed curve (see, e.g., [18]). Unlike the solution in (21), the path-ordered-integral solution is typically given for an arbitrary curve (not necessarily a triangle formed from geodesics) and it is expressed in terms of integrals over all the points along the curve (not necessarily just the initial point x). If the path-ordered-integral solution were specialized to a curve that traces out

a small geodesic triangle, and the components of the Riemann tensor along the curve were expanded in terms of covariant bitensors around the point x , then the path-ordered-integral solution should reduce to (21). Because the series solution in (21) generalizes quite straightforwardly to computing the inhomogeneous solution, we choose to use this method rather than the path-ordered integral hereafter.

We can calculate the coefficients $\Lambda^a_{b(m,n)}$ by repeatedly differentiating (21), using the operators $\frac{D}{D\lambda} = u^{a'} \nabla_{a'}$ and $\frac{D}{D\varepsilon} = v^{a''} \nabla_{a''}$ of (20). When doing so, we will frequently use the identities

$$u^{a'}_{;b'} u^{b'} = 0 = v^{a''}_{;b''} v^{b''}, \quad g^{a'}_{b;c'} u^{c'} = 0 = g^a_{a'';c''} v^{c''}, \quad (22)$$

which is a restatement of the geodesic equations for $x'(\lambda)$ and $x''(\varepsilon)$ and additionally, one of the defining properties of the parallel propagators [cf. (15) and (18)]. The (0,0) coefficient is given by the limit of (21) as $\lambda \rightarrow 0$ and $\varepsilon \rightarrow 0$ and is the identity map:

$$\Lambda^a_{b(0,0)} = \delta^a_b. \quad (23)$$

Acting on (21) with m λ -derivatives and n ε -derivatives and using (22), we find

$$\begin{aligned} \left(\frac{D}{D\lambda}\right)^m \left(\frac{D}{D\varepsilon}\right)^n \Lambda^a_b &= g^a_{a''} g^{a''}_{b';c'_1 \dots c'_m d'_1 \dots d'_n} u^{c'_1} \dots u^{c'_m} v^{d'_1} \dots v^{d'_n} g^{b'}_b \\ &= \Lambda^a_{b(m,n)} + O(\lambda) + O(\varepsilon). \end{aligned} \quad (24)$$

Taking the $\varepsilon \rightarrow 0$ ($x'' \rightarrow x$) limit of this equation yields

$$\Lambda^a_{b(m,n)} + O(\lambda) = g^a_{b';c'_1 \dots c'_m d_1 \dots d_n} u^{c'_1} \dots u^{c'_m} v^{d_1} \dots v^{d_n} g^{b'}_b, \quad (25)$$

and then taking the $\lambda \rightarrow 0$ ($x' \rightarrow x$) limit yields the expansion coefficients in terms of usual two-point coincidence limits [15],

$$\Lambda^a_{b(m,n)} = \left[g^a_{b';c'_1 \dots c'_m d_1 \dots d_n} \right]_{x' \rightarrow x} u^{c_1} \dots u^{c_m} v^{d_1} \dots v^{d_n}. \quad (26)$$

Recall that in this notation, the coincidence limit turns primed indices associated with the tangent space at x' to unprimed indices associated with those at x . Note that, had we taken the limits in the opposite order, we would have obtained

$$\begin{aligned} \Lambda^a_{b(m,n)} &= \left[g^{a''}_{b;c_1 \dots c_m d'_1 \dots d'_n} \right]_{x'' \rightarrow x} u^{c_1} \dots u^{c_m} v^{d_1} \dots v^{d_n} \\ &= \left[g^{a'}_{b;c_1 \dots c_m d'_1 \dots d'_n} \right]_{x' \rightarrow x} u^{c_1} \dots u^{c_m} v^{d_1} \dots v^{d_n}, \end{aligned} \quad (27)$$

where the second line has inconsequentially renamed x'' to x' . We see that consistency requires the identity

$$\left[g^a_{b';(c'_1 \dots c'_m)(d_1 \dots d_n)} \right]_{x' \rightarrow x} = \left[g^{a'}_{b;(c_1 \dots c_m)(d'_1 \dots d'_n)} \right]_{x' \rightarrow x}. \quad (28)$$

This identity actually holds without the symmetrizations and is a special case of the more general identity

$$\left[T_{a'_1 \dots a'_m b_1 \dots b_n}(x, x') \right]_{x' \rightarrow x} = \left[T_{a_1 \dots a_m b'_1 \dots b'_n}(x', x) \right]_{x' \rightarrow x}, \quad (29)$$

which holds for any bitensor T with a well-defined coincidence limit and which is a consequence of the independence of the path of approach to coincidence.¹ Thus, the two limits taken in (25) and (26) commute [like the two derivatives (24) do].

The coincidence limits necessary to compute Λ^a_b through third order, via the expression (26), are given by

$$\begin{aligned} [g^a_{b';c}] &= 0 = [g^a_{b';c'}], \\ [g^a_{b';cd}] &= -\frac{1}{2}R^a_{bcd}, & [g^a_{b';cd'}] &= \frac{1}{2}R^a_{bcd}, & [g^a_{b';c'd'}] &= \frac{1}{2}R^a_{bcd} \\ [g^a_{b';cde}] &= -\frac{2}{3}R^a_{bc(d;e)}, & [g^a_{b';cde'}] &= -\frac{1}{3}R^a_{be(c;d)}, \\ [g^a_{b';cd'e'}] &= \frac{1}{3}R^a_{bc(d;e)}, & [g^a_{b';c'd'e'}] &= \frac{2}{3}R^a_{bc(d;e)}. \end{aligned}$$

The limits in the first two lines are well-known results [13, 15], and we discuss the computation of those in the last two lines in Appendix A. Substituting these relations and (26) into (21) and setting $\lambda = \varepsilon = 1$ (or, equivalently, absorbing λ into the definition of u^a and ε into that of v^a), we obtain the holonomy of parallel transport,

$$\Lambda^a_b(\Delta_{u,v}) = \delta^a_b + \frac{1}{2}R^a_{bcd}v^c u^d + \frac{1}{6}R^a_{bcd;e}v^c u^d (v^e + u^e) + O(4). \quad (30)$$

We list the fourth-order corrections to this result in Appendix A.

The term in (30) equal to $\frac{1}{2}R^a_{bcd}v^c u^d$ is half the value that appears in many textbook derivations of the holonomy around an “infinitesimal parallelogram” spanned by two vectors v^a and u^a (see, e.g., [19]). This is not surprising because the geodesic triangle has half its area. We were unable to find an equivalent calculation in the literature against which to check the third-order term in this series, $\frac{1}{6}R^a_{bcd;e}v^c u^d (v^e + u^e)$. It is possible that the result is not new, however. This third-order term provides a quantitative estimate of the error in the leading-order expression for Λ^a_b for finite ε and λ . It also enters at the same order in this expansion as the inhomogeneous part of the generalized holonomy, as we show next.

¹ We thank Jordan Moxon for clarifying this point for us.

3.2 Inhomogeneous part of the holonomy

Next, consider the inhomogeneous part $\Delta\xi^a$ of the generalized holonomy around the triangle. Under the same assumptions as in the calculation of the linear holonomy, we can write the exact solution by composing the solution (12) three times along each leg. Specifically, starting with $\xi^a = 0$ at x , we obtain $\sigma^{a'}(x, x')$ at x' ; then we parallel transport that to x'' and add $\sigma^{a''}(x'', x')$; finally, we parallel transport that to x and add $\sigma^a(x, x'')$. The net result is

$$\begin{aligned}\Delta\xi^a &= g^a{}_{a''} \left(g^{a''}{}_{a'} \sigma^{a'}(x', x) + \sigma^{a''}(x'', x') \right) + \sigma^a(x, x'') \\ &= \Lambda^a{}_b \lambda u^b + g^a{}_{a''} w^{a''} - \varepsilon v^a,\end{aligned}\quad (31)$$

where the second line has used (21) and the definitions of Sec. 2.2. Having already computed $\Lambda^a{}_b$, we now need only to expand the quantity

$$\tilde{w}^a \equiv g^a{}_{a''} w^{a''} = g^a{}_{a''} \sigma^{a''}(x'', x') = \sum_{m,n=0}^{\infty} \frac{\lambda^m \varepsilon^n}{m!n!} \tilde{w}^a{}_{(m,n)}, \quad (32)$$

which is the tangent at x'' to the geodesic between x' and x'' that has been parallel transported back to x . Its coincidence limit is

$$\tilde{w}^a{}_{(0,0)} = 0. \quad (33)$$

The coefficients $\tilde{w}^a{}_{(m,n)}$ can be computed similarly to those of the holonomy in (24). After using the relations (22), derivatives of (32) have the simple form

$$\begin{aligned}\left(\frac{D}{D\lambda}\right)^m \left(\frac{D}{D\varepsilon}\right)^n \tilde{w}^a &= g^a{}_{a''} \sigma^{a''}{}_{c'_1 \dots c'_m d'_1 \dots d'_n} u^{c'_1} \dots u^{c'_m} v^{d'_1} \dots v^{d'_n} \\ &= \tilde{w}^a{}_{(m,n)} + O(\lambda) + O(\varepsilon).\end{aligned}\quad (34)$$

Taking the $\varepsilon \rightarrow 0$ ($x'' \rightarrow x$) limit simplifies the expression to

$$\tilde{w}^a{}_{(m,n)} + O(\lambda) = \sigma^{a'}{}_{c'_1 \dots c'_m d_1 \dots d_n} u^{c'_1} \dots u^{c'_m} v^{d_1} \dots v^{d_n}, \quad (35)$$

and taking the $\lambda \rightarrow 0$ ($x' \rightarrow x$) limit leaves

$$\tilde{w}^a{}_{(m,n)} = \left[\sigma^{a'}{}_{c'_1 \dots c'_m d_1 \dots d_n} \right]_{x' \rightarrow x} u^{c_1} \dots u^{c_m} v^{d_1} \dots v^{d_n}. \quad (36)$$

As in the previous section, the two limits commute. To compute \tilde{w}^a through fourth order, we first note that the coincidence limits of all third derivatives of the world function vanish. Then, the coincidence limits necessary to evaluate \tilde{w}^a are given by [13, 15]

$$\begin{aligned}\left[\sigma_{;ab} \right] &= g_{ab}, & \left[\sigma_{;ab'} \right] &= -g_{ab}, & \left[\sigma_{;a'b'} \right] &= g_{ab}, \\ \left[\sigma_{;abcd} \right] &= S_{abcd}, & \left[\sigma_{;abcd'} \right] &= -S_{abcd}, & \left[\sigma_{;abc'd'} \right] &= S_{abcd},\end{aligned}$$

$$\left[\sigma_{;ab'c'd'} \right] = -S_{bcda}, \quad \left[\sigma_{;a'b'c'd'} \right] = S_{abcd},$$

where S_{abcd} is the symmetrized Riemann tensor:

$$S_{abcd} = S_{(ab)(cd)} = S_{(cd)(ab)} = -\frac{1}{3} \left(R_{acbd} + R_{adbc} \right).$$

Substituting these results into (36) and the series (32), we obtain

$$\tilde{w}^a = g^a_{a''} w^{a''} = v^a - u^a + \frac{1}{6} R^a_{bcd} (v^b - 2u^b) v^c u^d + O(4). \quad (37)$$

Putting this together with the results in (30) and (31) yields the inhomogeneous part of the generalized holonomy,

$$\Delta \xi^a(\Delta_{u,v}) = \frac{1}{6} R^a_{bcd} (v^b + u^b) v^c u^d + O(4). \quad (38)$$

Appendix A also contains the fourth-order corrections to this result, which give a quantitative error estimate on this leading result for finite ε and λ .

Although we are unaware of another reference which has computed the inhomogeneous part of the generalized holonomy around a geodesic triangle for a torsion-free derivative operator, we can compute a closely related quantity that appears in classical differential geometry as a check of our result. We first note that an expansion similar to that above gives the tangent at x' to the $x'-x''$ leg, parallel transported back to x , as

$$g^a_{a'} w^{a'} = v^a - u^a + \frac{1}{6} R^a_{bcd} (u^b - 2v^b) v^c u^d + O(4), \quad (39)$$

which is simply (37) with $u^a \leftrightarrow v^a$ and an overall minus sign. This result [or the result (37) for $g^a_{a''} w^{a''}$] contracted with itself provides the leading-order correction to the law of cosines in curved space [17, 13]. Expressing the squared geodesic interval between x' and x'' in terms of u^a and v^a , we find this better-known result,

$$w^2 = (v - u)^2 - \frac{1}{3} R_{abcd} v^a u^b v^c u^d + O(5). \quad (40)$$

With the inhomogeneous solution for the triangle, we can now compute the generalized holonomy around an infinitesimal parallelogramoid in the next section.

4 Generalized holonomy for parallelogramoids

There are two possible parallelogramoid loops which can be defined from two vectors χ^a and τ^a at a point x (see the right half of Fig. 1 or Fig. 4 below). The points y' , y'' , z_1 , and z_2 are defined in terms of the vectors χ^a and τ^a at x by

$$\chi^a = -\sigma^{;a}(x, y'), \quad \tau^{a'} = g^{a'}_a(x, y') \tau^a = -\sigma^{;a'}(y', z_1),$$

$$\tau^a = -\sigma^{i'a}(x, y''), \quad \chi^{a''} = g^{a''}_a(x, y'') \chi^a = -\sigma^{i'a''}(y'', z_2).$$

We can define vectors ψ_1^a and ψ_2^a at x to be the tangents to the “diagonal” geodesics connecting x to z_1 and z_2 , respectively, with unit affine parameter intervals along the segments. They have the form

$$\psi_1^a = -\sigma^{i'a}(x, z_1), \quad \psi_2^a = -\sigma^{i'a}(x, z_2),$$

and we can show that they are related to χ^a and τ^a by

$$\psi_1^a + O(3) = \psi_2^a + O(3) = \chi^a + \tau^a \equiv \psi^a, \quad (41)$$

with the following argument: First, let us identify the x - z_2 - y'' triangle with the x - x' - x'' triangle of Sec. 2.2. The tangent vectors ψ_2^a , τ^a , and $\chi^{a''}$ should then be identified with u^a , v^a , and $-w^{a''}$, respectively. The result (37) then tells us that $-\chi^a = -g^{a''}_a \chi^{a''} = \tau^a - \psi_2^a + O(3)$. An analogous result holds for the x - z_1 - y'' triangle, from which we obtain (41).

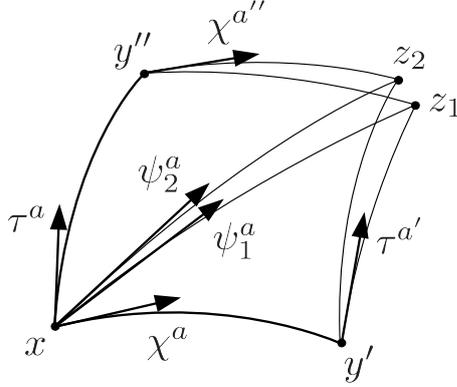


Fig. 4 The two parallelogramoid loops, $x \rightarrow y' \rightarrow z_1 \rightarrow y'' \rightarrow x$ and $x \rightarrow y' \rightarrow z_2 \rightarrow y'' \rightarrow x$, as in Fig. 1. Each parallelogramoid can be split into two triangles by defining a geodesic that runs along the diagonal from x to z_1 or z_2 .

The result of parallel or affine transport around the $x \rightarrow y' \rightarrow z_i \rightarrow y'' \rightarrow x$ parallelogramoid loop (with $z_i = z_1$ or z_2) will be the same as the result of transport around the $x \rightarrow y' \rightarrow z_i \rightarrow x$ triangle loop followed by transport around the $x \rightarrow z_i \rightarrow y'' \rightarrow x$ triangle loop, because transport along the last leg of the first triangle is the inverse of transport along the first leg of the second. We find that, at this order, the same holonomies are obtained from either choice of the parallelogramoid loop. Using the relations (30) and (41), the linear holonomy A^a_b is given by

$$\begin{aligned} A^a_b(\diamond_{\chi, \tau}) &= A^a_c(\triangle_{\chi, \psi}) A^c_b(\triangle_{\psi, \tau}) + O(4) \\ &= \delta^a_b + R^a_{bcd} \tau^c \chi^d + \frac{1}{2} R^a_{bcd;e} \tau^c \chi^d (\tau^e + \chi^e) + O(4). \end{aligned} \quad (42)$$

Similarly, from (38) and (41), the inhomogeneous part of the generalized holonomy is given by

$$\begin{aligned}\Delta\xi^a(\diamond_{\chi,\tau}) &= \Lambda^a_b(\Delta_{\psi,\tau}) \Delta\xi^b(\Delta_{\chi,\psi}) + \Delta\xi^a(\Delta_{\psi,\tau}) + O(4) \\ &= \Delta\xi^a(\Delta_{\chi,\psi}) + \Delta\xi^a(\Delta_{\psi,\tau}) + O(4) \\ &= \frac{1}{2}R^a_{bcd}(\tau^b + \chi^b)\tau^c\chi^d + O(4).\end{aligned}\quad (43)$$

We see that, through this order, the generalized holonomy for the parallelogramoid(s) can be found by simply adding the results ($\Lambda^a_b - \delta^a_b$ or $\Delta\xi^a$) from the generalized holonomies of two appropriate triangles (which does not hold at higher orders). We note that, through this order, the same result would also be found by traversing the $x \rightarrow y' \rightarrow z_1 \rightarrow z_2 \rightarrow y'' \rightarrow x$ pentagon.

5 Implications for Ref. [5]

The results derived for the generalized holonomy around parallelogramoids above—and summarized in Eqs. (8) and (9)—may appear problematic for the generalized holonomy of gravitational-wave spacetimes near future null infinity (the case considered in [5]). Consider a linearized gravitational wave spacetime in transverse traceless coordinates, for which the metric is given by

$$ds^2 = -dt^2 + (\delta_{ij} + r^{-1}h_{ij}^{\text{TT}})dx^i dx^j. \quad (44)$$

The nonzero components of the Riemann tensor at leading order in $1/r$ are given by

$$R_{titj} = \frac{\ddot{h}_{ij}^{\text{TT}}}{r} + O(r^{-2}), \quad (45)$$

where a dot was used to denote ∂_t .

Next, consider two observers at fixed large r who are separated by $\delta x^i = r\delta\theta^i$, where $\sqrt{\delta\theta^i\delta\theta_i} = d\theta$ is small angle separating the observers. Impose that their 4-velocities are given by $\mathbf{u} = \partial_t$, and allow an increment of time δt to elapse along their worldlines. The two worldlines and the separation δx^i between the two observers can be used to define a closed curve around which we can compute the generalized holonomy. The results (8) and (9) imply that to leading order in δt and $\delta\theta^i$, the homogeneous and inhomogeneous parts are given by

$$\Lambda_{ti} = \ddot{h}_{ij}^{\text{TT}}\delta\theta^j\delta t, \quad \Delta\xi_t = -\frac{r}{2}\ddot{h}_{ij}^{\text{TT}}\delta\theta^i\delta\theta^j\delta t, \quad \Delta\xi_i = -\frac{1}{2}\ddot{h}_{ij}^{\text{TT}}\delta\theta^j(\delta t)^2. \quad (46)$$

Note that the homogeneous solution goes to a constant at large r and that part of the inhomogeneous solution scales as r . A completely analogous result for the homogeneous solution was derived in [20] using the Bondi framework near null infinity and the Newman-Penrose formalism. This result implies that the holonomy of the affine transport may not be well suited for investigating

angular momentum ambiguities around these types of curves near null infinity, when the spacetime is dynamical. The linear holonomy associated with parallel transport might be better suited for these situations.

A different scaling for the homogeneous and inhomogeneous solutions was found in [5] for spacetimes that undergo stationary-to-stationary transitions and for a class of observers following geodesic worldlines over long times separated by large (or small) angles. In this context, the holonomy scales as $1/r$, and the inhomogeneous solution approaches a constant near future null infinity for the curves considered. For these larger curves, the higher-order terms in the expansion become relevant; thus, computing the holonomy by directly integrating the equations of affine transport (1) becomes a more efficient and accurate method for computing the holonomy. It is a noteworthy property of stationary-to-stationary spacetimes that holonomies around these large curves have can have a different scaling with r than infinitesimal holonomies within the same loop do. The requirement of stationarity at early and late times causes cancellations between the holonomies in small regions such that the net result falls off more rapidly with r than the holonomies around the small area elements do.

6 Conclusions and Discussion

In this paper, we used covariant bitensor methods to derive a series solution for the generalized holonomy (both the homogeneous and inhomogeneous parts) around a geodesic triangle to any order, in terms of coincidence limits of derivatives of the parallel propagator and Synge’s world function. We presented explicit results through fourth order in the vectors that define the geodesic triangle. Through third order in these vectors, the generalized holonomy (minus the identity map) around a parallelogramoid is just the sum of the generalized holonomies (again minus the identity maps) around the two triangles above and below its diagonal. The lowest-order part of the linear holonomy around the parallelogramoid reproduces the standard textbook treatments. The inhomogeneous part of the generalized holonomy is a higher-order quantity for a connection without torsion. We also computed higher-order corrections to both of these quantities, which could be useful for estimating the errors in the leading-order expressions for larger curves.

Because the inhomogeneous part of the generalized holonomy scales with area to the three-halves power times the Riemann tensor for a torsion-free connection, whereas the linear holonomy associated with parallel transport scales with the area times Riemann, the inhomogeneous solution will generally be less relevant for investigating the effects of spacetime curvature in infinitesimal regions than the linear holonomy is. This scaling of the inhomogeneous solution with area seems relevant for proposals to find spacetimes with “torsion without torsion” (see, e.g., [12]). In the language of this paper, these torsion-without-torsion solutions are spacetimes in which there is a non-vanishing inhomogeneous solution per unit area for a torsion-free connection.

Based on the results computed in this paper, as the unit area tends to zero, the inhomogeneous solution per unit area should vanish. Thus, it seems that torsion without torsion would only be relevant for larger nonlocal spacetime regions.

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A The generalized holonomy for small geodesic triangles to fourth order in distance

The (symmetrized) coincidence limits needed to compute the holonomy of parallel transport around the triangle are given by

$$\begin{aligned} \left[g^a{}_{b';(cdef)} \right] &= 0, \\ \left[g^a{}_{b';(cde)f'} \right] &= -\frac{1}{4} R^a{}_{bf(c;de)} + \frac{1}{4} R^a{}_{bg(c)R^g{}_{de}f}, \\ \left[g^a{}_{b';(cd)(e'f')} \right] &= \frac{1}{12} \left(R^a{}_{b(c(e;f)d)} - R^a{}_{b(e(c;d)f)} \right) + \frac{1}{2} R^a{}_{g(c(e)R_f d)b}{}^g, \\ &\quad + \frac{1}{4} \left(R^a{}_{bg(c)R_d(e)f}{}^g - R^a{}_{bg(e)R_f(c)d}{}^g \right), \\ \left[g^a{}_{b';c(d'e'f')} \right] &= \frac{1}{4} R^a{}_{bc(d;e'f')} - \frac{1}{4} R^a{}_{bg(d)R^g{}_{e'f'}c}, \\ \left[g^a{}_{b';(c'd'e'f')} \right] &= 0. \end{aligned}$$

These [and some of the coincidence limits above (30)] have been obtained by differentiating the coincidence expansions presented in Refs. [21, 22] and applying Synge's rule [13, 15], while also employing the Bianchi identities and commuting derivatives of the Riemann tensor (to simplify the resulting expressions). Following the calculation in Sec. 3.1, the holonomy through fourth order is

$$\begin{aligned} \Lambda^a{}_b &= \delta^a{}_b + \frac{1}{2} R^a{}_{bvu} + \frac{1}{6} R^a{}_{bvu;(v+u)} + \frac{1}{8} R^a{}_{cvu} R^c{}_{bvu} \\ &\quad + \frac{1}{48} \left\{ \left(R^a{}_{bv u;(2v+u)v} + R^a{}_{bcv} R^c{}_{(2v-3u)vu} \right) - (v \leftrightarrow u) \right\} + O(5), \end{aligned} \quad (47)$$

where u and v appearing in index slots denote contractions of the Riemann tensor or its derivatives with u^a and v^a .

To compute the inhomogeneous part of the generalized holonomy at fourth order, we employ the following (symmetrized) coincidence limits [13]:

$$\begin{aligned} \left[\sigma^{;a}{}_{(bcde)} \right] &= 0 = \left[\sigma^{;a}{}_{(b'c'd'e')} \right], & \left[\sigma^{;a}{}_{(bc)(d'e')} \right] &= \frac{1}{6} R^a{}_{(de)(b;c)} + \frac{1}{2} R^a{}_{(de)(b;c)} \\ \left[\sigma^{;a}{}_{(bcd)e'} \right] &= -\frac{1}{2} R^a{}_{(b|e|c;d)}, & \left[\sigma^{;a}{}_{b(c'd'e')} \right] &= -\frac{1}{2} R^a{}_{(c|b|d;e)}. \end{aligned}$$

Using the results of Sec. 3.1, the inhomogenous part has the form

$$\Delta \xi^a(\Delta_{u,v}) = \left\{ \left(\frac{1}{6} R^a{}_{vvu} + \frac{1}{12} R^a{}_{vuv;v} + \frac{1}{24} R^a{}_{uvu;v} \right) - (v \leftrightarrow u) \right\} + O(5). \quad (48)$$

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