THE GLOBAL EXISTENCE OF YANG-MILLS FIELDS ON CURVED SPACE-TIMES

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ABSTRACT. This is an introductory chapter in a series in which we take a systematic study of the Yang-Mills equations on curved space-times. In this first, we provide standard material that consists in writing the proof of the global existence of Yang-Mills fields on arbitrary curved space-times using the Klainerman-Rodnianski parametrix combined with suitable Grönwall type inequalities. While the Chruściel-Shatah argument requires a simultaneous control of the L^∞_{loc} and the H^2_{loc} norms of the Yang-Mills curvature, we can get away by controlling only the H^1_{loc} norm instead, and write a new gauge independent proof on arbitrary, fixed, sufficiently smooth, globally hyperbolic, curved 4-dimensional Lorentzian manifolds. This manuscript is written in an expository way in order to provide notes to Master's level students willing to learn mathematical General Relativity.

1. INTRODUCTION

Gauge field theories, such as the Maxwell equations and the Yang-Mills equations, arise in important physical theories to describe electromagnetism and the weak and strong interactions, and are to some extent mathematically related to the Einstein vacuum equations in General Relativity. Indeed, using Cartan formalism the Einstein vacuum equations can be written as the Yang-Mills equations except to the fact that the background geometry is part of the unknown solution of the evolution problem in General Relativity, while in Yang-Mills theory one can fix the background to be a given space-time.

In a classical paper, [EM1]-[EM2], Eardley and Moncrief proved global existence of solutions of the Yang-Mills equations in the 4-dimensional Minkowski background. This is an introductory chapter in a series in which we aim to extend their global regularity result to curved backgrounds. In this first, we write the proof of the global existence of Yang-Mills fields on arbitrary fixed curved space-time.

The Eardley-Moncrief result made a use of a hyperbolic formulation of the problem. Indeed, while the Yang-Mills equations say that the Yang-Mills curvature is divergence free on the background geometry, one can obtain a hyperbolic formulation by taking the covariant divergence of the Bianchi identity. This leads to a tensorial covariant wave equation on the Yang-Mills curvature with a non-linear term. It is exactly the study of this non-linear term that permits one to answer the question of local well-posdness, and global well-posdness of the equations. In this formulation,

the initial data consists of the Yang-Mills potential, that is a one form valued in the Lie algebra, and the electric field (loosely speaking the time derivative of the potential) on a given spacelike Cauchy hypersurface Σ. The initial data set has to verify itself the Yang-Mills equations, that is the covariant divergence of the electric field vanishes. One looks for a Yang-Mills curvature that satisfies the Yang-Mills equations such that once restricted on this hypersurface Σ the Yang-Mills curvature corresponds to that given by the prescribed potential and electric field.

Eardley and Moncrief proved global existence of solutions of the Yang-Mills equations in the 4-dimensional Minkowski background by proving a local existence result and providing pointwise estimates on the curvature, [EM1]-[EM2]. Their approach depended on the fundamental solution of the wave equation on flat space-time, and the use of the Cronström gauge condition, that has the remarkable advantage of expressing the potential as a function of the curvature directly in terms of an integral, to estimate the non-linear term. Later on, this result was extended by Chruściel and Shatah, [CS], to curved space-times using the same approach, by making use of the Friedlander parametrix for the wave equation in causal domains in curved space-times, [Fried], and the Cronström gauge condition as well. In a recent paper, [KR1], Klainerman and Rodnianski constructed a parametrix for the wave equation which permitted them to give a new gauge independent proof of the Eardley-Moncrief result [EM2] in a Minkowski background.

The Klainerman-Rodnianski's approach relies on their derivation of a covariant representation formula for the wave equation on arbitrary, smooth, globally hyperbolic, curved space-times, in which the integral terms are supported on the past null cone. As the authors pointed out, their parametrix can be immediately adapted to gauge covariant derivatives; this is because the scalar product on the Lie algebra $< , >$ is Ad-invariant. They used it to give a new gauge independent proof of the Eardley-Moncrief result [EM2], of which the only ingredient is the conservation of the energy. As the authors mentioned, one can generalize their proof of the global existence of Yang-Mills fields on the flat Minkowski space-time to arbitrary smooth, globally hyperbolic, curved space-times under the assumption that there exists a timelike verctor field $\frac{\partial}{\partial t}$ of which the deformation tensor is finite, as it has been assumed in previous work by Chruściel and Shatah, [CS].

In this manuscript, we provide standard material, but not so clearly pointed out in literature, that consists in writing the proof of the global existence of Yang-Mills fields on arbitrary curved space-times by using the Klainerman-Rodnianski parametrix combined with suitable Grönwall type inequalities. While the Chruściel-Shatah argument requires a simultaneous control of the L_{loc}^{∞} and the H_{loc}^{2} norms of the Yang-Mills curvature, we can get away by controlling only the H_{loc}^{1} norm instead. However, we were unable to get rid of any control on the gradient of the Yang-Mills curvature, as it is the case in the proof on Minkowski space-time in [KR1]. Hence, this provides a new gauge independent proof and improves the Chruściel-Shatah's result, [CS], for sufficiently smooth, globally hyperbolic, curved 4-dimensional Lorentzian manifolds.

1.1. **The statement.**

More precisely, we will prove the following theorem,

Theorem 1. *Let $(M, \mathbf{g})$ be a curved 4-dimensional Lorentzian manifold. We know by then that at each point $p \in M$, there exists a frame $\{\hat{t}, n, e_a, e_b\}$ where,*

$$\mathbf{g} \quad = \quad -d\hat{t}^2 + dn^2 + de_a^2 + de_b^2$$

We assume that $\mathbf{g}$ is sufficiently smooth, M is globally hyperbolic, and that there exists a timelike vector field $\frac{\partial}{\partial t}$ and $C(t) \in L^1_{loc}$, such that for all $\hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_a, e_b\}$, the components of the deformation tensor $\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t}) = \frac{1}{2}[\nabla^{\hat{\mu}}(\frac{\partial}{\partial t})^{\hat{\nu}} + \nabla^{\hat{\nu}}(\frac{\partial}{\partial t})^{\hat{\mu}}]$ verify,

$$|\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t})|_{L^\infty_{loc}(\Sigma_t)} \leq C(t)$$

where Σ_t are the $t = $ constant hypersurfaces, and coordinate t could be defined only locally. Let, $\Sigma_{t=t_0}$ be a Cauchy hypersurface prescribed by $t = t_0$. Let $F_{\hat{\mu}\hat{\nu}}$ be the components of the Yang-Mills fields in the frame $\{\hat{t}, n, e_a, e_b\}$, defined as the anti-symmetric 2-tensor solution of the Cauchy problem of the Yang-Mills equations $\mathbf{D}^{(A)}_\alpha F^{\alpha\beta} = 0$, where the initial data prescribed on the Cauchy hypersurface $\Sigma_{t=0}$ verifies the Yang-Mills constraint equations,

$$\mathbf{D}^{(A)^\beta} F_{\hat{t}\beta}(t = 0) = 0$$

Then, we have that local solutions to the Yang-Mills equations can be extended globally in t if,

$$E_F^{\frac{\partial}{\partial t}}(t = t_0) < \infty$$

and,

$$E_{\mathbf{D}^{(A)}F}^{\frac{\partial}{\partial t}}(t = t_0) < \infty$$

where,

$$E_F^{\frac{\partial}{\partial t}}(t = 0) \quad = \quad \int_{q \in \Sigma_{t=0}} \frac{1}{2}[|F_{\hat{t}n}|^2 + |F_{\hat{t}a}|^2 + |F_{\hat{t}b}|^2 + |F_{na}|^2 + |F_{nb}|^2 + |F_{ab}|^2](q)$$
$$\cdot \sqrt{-\mathbf{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})} dV_\Sigma(q)$$

and,

$$E_{\mathbf{D}^{(A)}F}^{\frac{\partial}{\partial t}}(t = t_0) \quad = \quad \int_{q \in \Sigma_{t=0}} \frac{1}{2}[|\mathbf{D}^{(A)}F_{\hat{t}n}|^2 + |\mathbf{D}^{(A)}F_{\hat{t}a}|^2 + |\mathbf{D}^{(A)}F_{\hat{t}b}|^2 + |\mathbf{D}^{(A)}F_{na}|^2$$
$$+ |\mathbf{D}^{(A)}F_{nb}|^2 + |\mathbf{D}^{(A)}F_{ab}|^2](q) \cdot \sqrt{-\mathbf{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})} dV_\Sigma(q)$$

where,

$$|\mathbf{D}^{(A)}F_{\hat{\mu}\hat{\nu}}|^2 \quad = \quad |\mathbf{D}^{(A)}_{\hat{t}}F_{\hat{\mu}\hat{\nu}}|^2 + |\mathbf{D}^{(A)}_n F_{\hat{\mu}\hat{\nu}}|^2 + |\mathbf{D}^{(A)}_{e_a}F_{\hat{\mu}\hat{\nu}}|^2 + |\mathbf{D}^{(A)}_{e_b}F_{\hat{\mu}\hat{\nu}}|^2$$

1.2. **Strategy of the proof.**

As we will show, see (19), the Yang-Mills fields satisfy a non-linear hyperbolic
differential equation on the background geometry. Since the scalar product on the
Lie algebra $< , >$ is Ad-invariant, the Klainerman-Rodnianski parametrix can be
immediately generalized (see Appendix) to gauge covariant derivatives to give a
representation formula for solutions of $(\Box_{\mathbf{g}}^{(A)} F)_{\mu\nu} = S_{\mu\nu}$, where $S_{\mu\nu}$ is a source
tensor, and hence it can be used for the Yang-Mills fields, see (198).

We would like to bound all the terms in the representation formula in a way that
we could use Grönwall lemma to deduce that the L^∞ norm of F will stay finite (see
(84)). For this we need a parameter in which the extension of local solutions can
make sense; this would be a timelike vector field $\frac{\partial}{\partial t}$.

The main advantage of the parametrix is that all it's integral terms are supported
on the past null cone. Naively, one can hope that those can be bounded by the
flux of the energy generated from $\frac{\partial}{\partial t}$. Thus, if one can bound the energy flux along
the null cones, the proof might go through. To bound the energy flux, one needs,
as in [CS], to assume that the deformation tensor of a timelike vector field has it's
integral in t finite on bounded domains:

$$|\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t})|_{L^\infty_{loc(\Sigma_t)}} \leq C(t) \in L^1_{loc}$$

Using the divergence theorem on the energy-momentum tensor contracted with $\frac{\partial}{\partial t}$
will lead to an inequality on the energy. The assumption on the deformation tensor
above can show using Grönwall lemma that the local energy will stay finite, see
(40). Using this and the assumption on the deformation tensor again, one can
show that the space-time integral generated from the divergence theorem will stay
finite in t, see (41), from which one can deduce the finiteness of the flux (42).

The integral terms supported on the past null cone in the Klainerman-Rodnianski
parametrix involve a term that is a generalization of the fundamental solution of
the wave equation on flat space-time to curved space-times. This is $\lambda_{\mu\nu}$ that is a
two tensor solution of a transport equation along the null cone given by (104) and
(105). Using the transport equation, one can prove that the L^∞ norm of $s\lambda_{\mu\nu}$,
where s is the geodesic parameter for a null vector field L normal to the null cone
used to define the transport equation for $\lambda_{\mu\nu}$, will be bounded by the initial data for
$\lambda_{\mu\nu}$, see (43). Yet, since we would want to apply Grönwall lemma, the terms which
contain $\lambda_{\mu\nu}$ and $F_{\mu\nu}$ can be bounded as in (4.19), by controlling $s\lambda_{\mu\nu}$, see (43),
and $s^{-1}F_{\mu\nu}$, see (85). The terms which contain $\lambda_{\mu\nu}$ and $[F, F]$ can be bounded as
in (4.21) by using the finiteness of the energy flux, see (88).

However, a major difference with the situation on Minkowski space, is in the way
to deal with the term which contains $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ and F, where $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ is the in-
duced Laplacian on the 2-sphere prescribed by $s = constant$ defined by (144). In
Minkowski space, one can control directly $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ as shown by Rodnianski and
Klainerman in [KR1], because one can close a system of transport equations along
the null cone. On curved space-times, we are unable to close such a system, con-
sequently, we will use the divergence theorem on $\mathbb{S}^2$, see (93), so as to bring the

problem to controlling $\mathbf{D}_a^{(A)}\lambda$ and $\mathbf{D}_a^{(A)}F$, where these are the derivatives tangential to the 2-sphere prescribed by $s = constant$.

To control $\mathbf{D}_a^{(A)}\lambda$ we will follow [KR3], see (4.13). Since the area element on the 2-spheres is at the level of s^2, see (154), one would want to control the L^2 norm of $s\mathbf{D}_a^{(A)}\lambda$ on the null cone, with respect to the measure $dsd\sigma^2$, where $d\sigma^2$ is the usual volume form on $\mathbb{S}^2$. One could try to use the fundamental theorem of calculus directly to control the L^2 norm on $\mathbb{S}^2$ then integrate in s, yet by doing so, we would find ourselves confronted to controlling near the vertex p ($s = 0$) a quantity of the type $(\frac{1}{s} - tr\chi)$, where χ is the null second fundamental form of the null hypersurfaces. This quantity cannot be controlled even in the 4-dimensional Minkowski space where $tr\chi = \frac{2}{s}$. To change the factor in front of $\frac{1}{s}$ from 1 to 2, one would need to apply the fundamental theorem of calculus to control the L^2 norm on the null cone of $s^2\mathbf{D}_a^{(A)}\lambda$ instead of $s\mathbf{D}_a^{(A)}\lambda$, see (56). Since it is the L^2 norm, this means that one would have to consider $s^4|\mathbf{D}_a^{(A)}\lambda|^2$ instead of $s^2|\mathbf{D}_a^{(A)}\lambda|^2$ for applying the fundamental theorem of calculus. However, since what we want to control is the integral on the null cone of $s^2|\mathbf{D}_a^{(A)}\lambda|^2$, which is bigger than that of $s^4|\mathbf{D}_a^{(A)}\lambda|^2$, near $s = 0$, we would need to lower the power on s, for this one can actually control the integral on $\mathbb{S}^2$ of $s^{-1}s^4|\mathbf{D}_a^{(A)}\lambda|^2$ by applying the fundamental theorem of calculus to $s^4|\mathbf{D}_a^{(A)}\lambda|^2$ as described above, see (57) and (58). This would allow then to control the integral on $\mathbb{S}^2$ of $s^3|\mathbf{D}_a^{(A)}\lambda|^2$ in a way that one could then get an estimate on the L^1 norm on $\mathbb{S}^2$ for $s^2|\mathbf{D}_a^{(A)}\lambda|^2$, see (68), which permits one to apply the L^2 maximum principle to control the integral on the null cone of $s^2|\mathbf{D}_a^{(A)}\lambda|^2$ near the vertex p ($s = 0$), see (70). Away from the vertex $s = 0$ the integral is clearly finite and hence, this would give the desired control.

In order to control the L^2 norm of $\mathbf{D}_a^{(A)}F$ on the null cone, we will use the energy momentum tensor of the wave equation T_1 after contracting the free indices of the Yang-Mills fields with respect to a Riemannian metric h, as in [CS], see (73). Since it is a full contraction, we can compute it by choosing a normal frame , i.e. a frame where the Christoffel symbols vanish at that point, and hence we can get the derivatives inside the scalar product as covariant derivatives (and also as gauge covariant derivatives using the fact that the scalar product is Ad-invariant) instead of partial derivatives. Since it is the energy momentum tensor for the wave equation, the boundary term supported on the null cone obtained after contracting T_1 with the normalized timelike vector field, $\frac{\partial}{\partial t}$, and applying the divergence theorem in a region inside the null cone, is at the level of the L^2 norm of $\mathbf{D}_a^{(A)}F$ and $\mathbf{D}_L^{(A)}F$, see (77), and thus it controls the L^2 norm of $\mathbf{D}_a^{(A)}F$. We know by then, from the divergence theorem, that this can be controlled by a quantity that is at the level of a space integral of $T_1^{\hat{t}\hat{t}}$ on the initial spacelike hypersurface and in addition a spacetime integral of $|\mathbf{D}^{(A)}F|(|\mathbf{D}^{(A)}F| + |F| + |F|^2)$, see (80), where $|F|$ and $|F|^2$ arise from the sources of the tensorial gauge hyperbolic wave equation verified by F, and $|\mathbf{D}^{(A)}F|$ in the parenthesis is due to the fact that the deformation tensor of $\frac{\partial}{\partial t}$, as well as the covariant derivative of h, do not vanish. As we wish to get rid of the gradient of F, so as to have a control that involves an integral or a double integral of the square of the L^∞ norm of F, see (83), we recall that the divergence theorem

that we applied previously also permits one to control the space integral of $T_1^{\hat{t}\hat{t}}$ on the spacelike hypersurface, that is at the level of the L^2 norm of the $\mathbf{D}^{(A)}F$, by the same quantity that controls the boundary term on the null cone, (80). This allows one to use Grönwall lemma, after using $a.b \lesssim a^2 + b^2$, and the conservation of the energy that is at the level of the L^2 norm of F, to control the L^2 norm of $\mathbf{D}^{(A)}F$ on the spacelike hypersurfaces by the desired quantity, see (81). Injecting this in the previous control on the L^2 norm of $\mathbf{D}_a^{(A)}F$, (80), and using again $a.b \lesssim a^2 + b^2$ and the conservation of the energy, leads to the desired control (83).

Now, the parametrix (198) permits us to control the value of the Yang-Mills fields contracted with an arbitrary tensor, at a point q in space-time, by the estimates mentioned above. We would want to establish a Grönwall type inequality in t on the L^∞ norm of F on Σ_t^p, the spacelike hypersurfaces prescribed by $t = constant$ in the past of a point p, so as to deduce the finiteness of the fields at the point p. To obtain this, we take the supremum on $q \in \Sigma_t^p$ in the inequality described above, i.e. after using the parametrix and the above estimates, see (84). This can be used to show that $||F||_{L^\infty(\Sigma_t^p)}$ verifies a generalized Grönwall type inequality (96) to which Pachpatte in [Pach], proved a result that ensures that the solutions will stay finite. A local existence result would give that solutions of the Yang-Mills equations will either blow up in finite time, or they will be defined globally in time. Hence, that the non-blow up result that we have established gives that local solutions of the Yang-Mills equations can be extended globally in time t, under the assumptions of theorem (1).

Remark 1.3. The whole manuscript is written in an expository way, where we detail all the calculations, and we show standard material to make these notes self-contained. We also detail well known material in the Appendix.

2. The Field Equations

In this section we present the Yang-Mills curvature, and we derive the Yang-Mills equations from the Yang-Mills Lagrangian. We will also show the Bianchi identities.

2.1. **The Yang-Mills curvature.** Let $(M, \mathbf{g})$ be a four dimensional globally hyperbolic Lorentzian manifold. Let G be a compact Lie group, and $\mathcal{G}$ its Lie algebra such that it has a faithful real matrix representation $\{\theta_a\}$. Let $< , >$ be a positive definite Ad-invariant scalar product on $\mathcal{G}$. The Yang Mills potential can be regarded locally as a $\mathcal{G}$-valued one form A on M, say

$$A = A_\alpha^{(a)} \theta_a dx^\alpha = A_\alpha dx^\alpha$$

in a given system of coordinates. The gauge covariant derivative of a $\mathcal{G}$-valued tensor Ψ is defined as

$$\mathbf{D}_\alpha^{(A)} \Psi = \nabla_\alpha \Psi + [A_\alpha, \Psi] \tag{1}$$

where ∇_α is the space-time covariant derivative of Levi-Cevita on $(M, \mathbf{g})$, and $\nabla_\alpha \Psi$ is the tensorial covariant derivative of Ψ, that is

$$
\begin{aligned}
(\nabla_\alpha \Psi)(X, Y, Z, \dots) &= \partial_\alpha(\Psi(X, Y, Z, \dots)) - \Psi(\nabla_\alpha X, Y, Z, \dots) \\
&\quad - \Psi(X, \nabla_\alpha Y, Z, \dots) - \Psi(X, Y, \nabla_\alpha Z, \dots) \\
&\quad - \dots\dots\dots - \dots\dots\dots
\end{aligned}
\tag{2}
$$

The tensorial second order derivative is defined as

$$
\begin{aligned}
(\nabla_\beta \nabla_\alpha \Psi)(X, Y, Z, \dots) &= \partial_\beta[(\nabla_\alpha \Psi)(X, Y, Z, \dots)] - (\nabla_{\nabla_\beta e_\alpha} \Psi)(X, Y, Z, \dots) \\
&\quad - (\nabla_\alpha \Psi)(\nabla_\beta X, Y, Z, \dots) - (\nabla_\alpha \Psi)(X, \nabla_\beta Y, Z, \dots) \\
&\quad - \dots\dots\dots - \dots\dots\dots
\end{aligned}
\tag{3}
$$

By letting

$$
\begin{aligned}
(\nabla_\beta(\nabla_\alpha \Psi))(X, Y, Z, \dots) &= \partial_\beta[(\nabla_\alpha \Psi)(X, Y, Z, \dots)] - (\nabla_\alpha \Psi)(\nabla_\beta X, Y, Z, \dots) \\
&\quad - (\nabla_\alpha \Psi)(X, \nabla_\beta Y, Z, \dots) - \dots\dots\dots
\end{aligned}
$$

We can then write

$$
(\nabla_\beta \nabla_\alpha \Psi)(X, Y, Z, \dots) = (\nabla_\beta(\nabla_\alpha \Psi))(X, Y, Z, \dots) - (\nabla_{\nabla_\beta e_\alpha} \Psi)(X, Y, Z, \dots)
\tag{4}
$$

The Yang-Mills curvature is a $\mathcal{G}$-valued two form

$$F = F_{\alpha\beta}^{(a)} \theta_a dx^\alpha \wedge dx^\beta = F_{\alpha\beta} dx^\alpha \wedge dx^\beta$$

obtained by commutating in a system of coordinates two gauge covariant derivatives of a $\mathcal{G}$-valued tensor Ψ, where the tensorial second order gauge derivative of Ψ is defined by

$$\mathbf{D}_\alpha^{(A)} \mathbf{D}_\beta^{(A)} \Psi = \mathbf{D}_{\alpha\beta}^2 \Psi = \mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)} \Psi) - \mathbf{D}_{\nabla_\alpha e_\beta}^{(A)} \Psi \tag{5}$$

$$
\begin{aligned}
\mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)} \Psi) &= \nabla_\alpha(\mathbf{D}_\beta^{(A)} \Psi) + [A_\alpha, \mathbf{D}_\beta^{(A)} \Psi] \\
&= \nabla_\alpha(\nabla_\beta \Psi + [A_\beta, \Psi]) + [A_\alpha, \nabla_\beta \Psi + [A_\beta, \Psi]] \\
&= \nabla_\alpha(\nabla_\beta \Psi) + [\nabla_\alpha A_\beta, \Psi] + [A_\beta, \nabla_\alpha \Psi] + [A_\alpha, \nabla_\beta \Psi] + [A_\alpha, [A_\beta, \Psi]]
\end{aligned}
$$

As is a system of coordinates $[e_\alpha, e_\beta] = 0 = \nabla_\alpha e_\beta - \nabla_\beta e_\alpha$ (the metric is assumed to be torsion free), then

$$
\begin{aligned}
&\mathbf{D}_\alpha^{(A)}\mathbf{D}_\beta^{(A)}\Psi - \mathbf{D}_\beta^{(A)}\mathbf{D}_\alpha^{(A)}\Psi \\
=\ & \mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)}\Psi) - \mathbf{D}_\beta^{(A)}(\mathbf{D}_\alpha^{(A)}\Psi) - (\mathbf{D}_{\nabla_\alpha e_\beta}^{(A)}\Psi) + (\mathbf{D}_{\nabla_\alpha e_\beta}^{(A)}\Psi) \\
=\ & \mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)}\Psi) - \mathbf{D}_\beta^{(A)}(\mathbf{D}_\alpha^{(A)}\Psi) + (\mathbf{D}_{(\nabla_\alpha e_\beta - \nabla_\beta e_\alpha)}^{(A)}\Psi) \\
=\ & \mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)}\Psi) - \mathbf{D}_\beta^{(A)}(\mathbf{D}_\alpha^{(A)}\Psi) + 0 \\
=\ & \nabla_\alpha(\nabla_\beta\Psi) + [\nabla_\alpha A_\beta, \Psi] + [A_\beta, \nabla_\alpha\Psi] + [A_\alpha, \nabla_\beta\Psi] + [A_\alpha, [A_\beta, \Psi]] \\
& -\nabla_\beta(\nabla_\alpha\Psi) - [\nabla_\beta A_\alpha, \Psi] - [A_\alpha, \nabla_\beta\Psi] - [A_\beta, \nabla_\alpha\Psi] - [A_\beta, [A_\alpha, \Psi]] \\
=\ & \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta}\Psi_{....\gamma....} + [\nabla_\alpha A_\beta, \Psi] + [A_\alpha, [A_\beta, \Psi]] - [\nabla_\beta A_\alpha, \Psi] - [A_\alpha, \nabla_\beta\Psi] \\
& -[A_\beta, [A_\alpha, \Psi]] \\
=\ & \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta}\Psi_{....\gamma....} + [\nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta], \Psi] \\
=\ & \sum_i R_{a_i}{}^\gamma{}_{\alpha\beta}\Psi_{....\gamma....} + [F_{\alpha\beta}, \Psi]
\end{aligned}
$$

(6)

(7)

where $\Psi = \Psi_{a_1 a_2.....a_i.....}$, and γ is at the ith place. This gives

$$
F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta] \tag{8}
$$

2.2. The Yang-Mills equations.

The Yang-Mills Lagrangian is given by

$$
L = -\frac{1}{4} < F_{\alpha\beta}, F^{\alpha\beta} >
$$

A compact variation $(F(s), U)$, where U is any compact set of M, can be written in terms of a compact variation $(A(s), U)$ of a gauge potential in the following manner:

$$
\dot{F}_{\alpha\beta} = \frac{d}{ds}F_{\alpha\beta}(s)|_{s=0} = \nabla_\alpha\dot{A}_\beta - \nabla_\beta\dot{A}_\alpha + [\dot{A}_\alpha, A_\beta] + [A_\alpha, \dot{A}_\beta]
$$

where

$$
\dot{A} = \frac{d}{ds}A(s)|_{s=0}
$$

The action principle gives

$$
\begin{aligned}
\frac{d}{ds}L(s)|_{s=0} &= -\frac{1}{2} < \dot{F}_{\alpha\beta}, F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} = 0 \\
&= -\frac{1}{2} \int_U < \nabla_\alpha \dot{A}_\beta - \nabla_\beta \dot{A}_\alpha + [\dot{A}_\alpha, A_\beta] + [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} \\
&= -\frac{1}{2} \int_U < \nabla_\alpha \dot{A}_\beta, F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} + \frac{1}{2} \int_U < \nabla_\beta \dot{A}_\alpha, F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} \\
&\quad -\frac{1}{2} \int_U < [\dot{A}_\alpha, A_\beta], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} - \frac{1}{2} \int_U < [A_\alpha, \dot{A}_\beta], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} \\
&= -\int_U < \nabla_\alpha \dot{A}_\beta, F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} - \int_U < [\dot{A}_\alpha, A_\beta], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}}
\end{aligned}
$$

(where we have used the anti-symmetry of F)

$$
= -\int_U < \dot{A}_\beta, \nabla_\alpha F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} - \int_U < [\dot{A}_\alpha, A_\beta], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}}
$$

(where we have integrated by parts, and the boundary terms are zero since F has compact support)

On the other hand

$$
-\int_U < [\dot{A}_\alpha, A_\beta], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}} = \int_U < [\dot{A}_\beta, A_\alpha], F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}}
$$

(By anti-symmetry of F)

$$
= \int_U < \dot{A}_\beta, [A_\alpha, F^{\alpha\beta}] >_{\mathbf{g}} dv_{\mathbf{g}}
$$

because $< , >$ is Ad-invariant. This yields to

$$
0 = < \dot{A}_\beta, \nabla_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] >_{\mathbf{g}} dv_{\mathbf{g}} = \int_U < \dot{A}_\beta, \mathbf{D}_\alpha^{(A)} F^{\alpha\beta} >_{\mathbf{g}} dv_{\mathbf{g}}
$$

So the covariant divergence of the curvature is zero

$$
\mathbf{D}_\alpha^{(A)} F^{\alpha\beta} = 0 \tag{9}
$$

On the other hand, computing

$$
\mathbf{D}^{(A)}_\alpha F_{\mu\nu} + \mathbf{D}^{(A)}_\mu F_{\nu\alpha} + \mathbf{D}^{(A)}_\nu F_{\alpha\mu}
$$

$$
\begin{aligned}
&= \nabla_\alpha F_{\mu\nu} + [A_\alpha, F_{\mu\nu}] + \nabla_\mu F_{\nu\alpha} + [A_\mu, F_{\nu\alpha}] + \nabla_\nu F_{\alpha\mu} + [A_\nu, F_{\alpha\mu}] \\
&= \nabla_\alpha (\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]) + [A_\alpha, \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]] \\
&\quad + \nabla_\mu (\nabla_\nu A_\alpha - \nabla_\alpha A_\nu + [A_\nu, A_\alpha]) + [A_\mu, \nabla_\nu A_\alpha - \nabla_\alpha A_\nu + [A_\nu, A_\alpha]] \\
&\quad + \nabla_\nu (\nabla_\alpha A_\mu - \nabla_\mu A_\alpha + [A_\alpha, A_\mu]) + [A_\nu, \nabla_\alpha A_\mu - \nabla_\mu A_\alpha + [A_\alpha, A_\mu]] \\
&= \nabla_\alpha \nabla_\mu A_\nu - \nabla_\alpha \nabla_\nu A_\mu + [\nabla_\alpha A_\mu, A_\nu] + [A_\mu, \nabla_\alpha A_\nu] \\
&\quad + [A_\alpha, \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]] + \nabla_\mu \nabla_\nu A_\alpha - \nabla_\mu \nabla_\alpha A_\nu \\
&\quad + [\nabla_\mu A_\nu, A_\alpha] + [A_\nu, \nabla_\mu A_\alpha] + [A_\mu, \nabla_\nu A_\alpha - \nabla_\alpha A_\nu + [A_\nu, A_\alpha]] \\
&\quad + \nabla_\nu \nabla_\alpha A_\mu - \nabla_\nu \nabla_\mu A_\alpha + [\nabla_\nu A_\alpha, A_\mu] \\
&\quad + [A_\alpha, \nabla_\nu A_\mu] + [A_\nu, \nabla_\alpha A_\mu - \nabla_\mu A_\alpha + [A_\alpha, A_\mu]]
\end{aligned}
$$

(where $\nabla_\alpha \nabla_\mu A = \nabla_\alpha (\nabla_\mu A) - \nabla_{\nabla_\alpha e_\mu} A$ is the tensorial covariant derivative of A)

$$
\begin{aligned}
&= R_\nu{}^\gamma{}_{\alpha\mu} A_\gamma + R_\alpha{}^\gamma{}_{\mu\nu} A_\gamma + R_\mu{}^\gamma{}_{\nu\alpha} A_\gamma + [A_\alpha, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\alpha]] \\
&\quad + [A_\nu, [A_\alpha, A_\mu]] \\
&= -(R^\gamma{}_{\nu\alpha\mu} + R^\gamma{}_{\alpha\mu\nu} + R^\gamma{}_{\mu\nu\alpha}) A_\gamma + [A_\alpha, [A_\mu, A_\nu]] + [A_\mu, [A_\nu, A_\alpha]] \\
&\quad + [A_\nu, [A_\alpha, A_\mu]] \\
&= 0
\end{aligned}
$$

by Bianchi identity and symmetry of the curvature tensor.

So we have,

$$
\mathbf{D}^{(A)}_\alpha F_{\mu\nu} + \mathbf{D}^{(A)}_\mu F_{\nu\alpha} + \mathbf{D}^{(A)}_\nu F_{\alpha\mu} = 0 \tag{10}
$$

The equations (9) and (10) form the Yang-Mills equations. The Maxwell equations correspond to the abelian case where $[\,,\,] = 0$, and therefore $\mathbf{D}^{(A)} = \nabla$.

The Cauchy problem for the Yang-Mills equations formulates as the following: given a Cauchy hypersurface Σ in M, and a $\mathcal{G}$-valued one form A_μ on Σ, and a $\mathcal{G}$-valued one form E_i on Σ satisfying $\mathbf{D}^{(A)}_i E^i$, we are looking for a $\mathcal{G}$-valued two form $F_{\mu\nu}$ satisfying the Yang-Mills equations such that once $F_{\mu\nu}$ restricted on M we have $F_{0i} = E_i$, and such that $F_{\mu\nu}$ corresponds to the curvature derived from the Yang-Mills potential A_μ (i.e. $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha + [A_\alpha, A_\beta]$).

3. MOTIVATION

Our motivation for the systematic study of gauge field theories such as the Yang-Mills equations is to have insights into the Einstein vacuum equations in General Relativity. Indeed, using Cartan formalism the Einstein vacuum equations can be viewed as to some extent mathematically related to the Yang-Mills equations.

General Relativity postulates that the space-time is a 4-dimensional Lorentzian manifold $(M, \mathbf{g})$, that satisfies the Einstein vacuum equations $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci curvature, i.e. $R_{\mu\nu} = R^\gamma{}_{\mu\gamma\nu}$, and where $R_{\mu\nu\alpha\beta}$ is the Riemann tensor associated to the metric $\mathbf{g}$, defined by,

$$R(X, Y, U, V) = \mathbf{g}\big(X, \big[\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]} Y\big]\big)$$

where X, Y, U, V are vectorfields in the tangent bundle of M. We will see that using Cartan formalism one can write the Riemann tensor as a Yang-Mills curvature.

3.1. Cartan formalism.

At a point p of the space-time, one can choose a normal frame, which means a frame such that $\mathbf{g}(e_\alpha, e_\beta)(p) = \text{diag}(-1, 1, \ldots, 1)$, and $\frac{\partial}{\partial \sigma} \mathbf{g}(e_\alpha, e_\beta)(p) = 0$, i.e. the first partial derivatives of the metric at p vanish. Cartan formalism consists in defining the connection 1-form,

$$(A)_{\alpha\beta}(X) = \mathbf{g}(\nabla_X e_\beta, e_\alpha) \tag{11}$$

where ∇ is the Levi-Cevita connection. Thus, since $A = A_\mu dx^\mu$, we can write,

$$(A_\mu)_{\alpha\beta} = (A)_{\alpha\beta}(\frac{\partial}{\partial \mu}) = \mathbf{g}(\nabla_\mu e_\beta, e_\alpha)$$

Computing,

$$
\begin{aligned}
R(e_\alpha, e_\beta, \frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}) &= \mathbf{g}\big(e_\alpha, \big(\nabla_{\frac{\partial}{\partial\mu}}\nabla_{\frac{\partial}{\partial\nu}} - \nabla_{\frac{\partial}{\partial\nu}}\nabla_{\frac{\partial}{\partial\mu}} - \nabla_{[\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}]}\big)e_\beta\big) \\
&= \mathbf{g}\big(e_\alpha, \nabla_{\frac{\partial}{\partial\mu}}\nabla_{\frac{\partial}{\partial\nu}}e_\beta - \nabla_{\frac{\partial}{\partial\nu}}\nabla_{\frac{\partial}{\partial\mu}}e_\beta - \nabla_{[\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}]}e_\beta\big) \\
&= \frac{\partial}{\partial\mu}\mathbf{g}(e_\alpha, \nabla_{\frac{\partial}{\partial\nu}}e_\beta) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\mu}}e_\alpha, \nabla_{\frac{\partial}{\partial\nu}}e_\beta) \\
&\quad - [\frac{\partial}{\partial\nu}\mathbf{g}(e_\alpha, \nabla_{\frac{\partial}{\partial\mu}}e_\beta) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\nu}}e_\alpha, \nabla_{\frac{\partial}{\partial\mu}}e_\beta)] \\
&\quad - \mathbf{g}(e_\alpha, \nabla_{[\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}]}e_\beta) \\
&= \frac{\partial}{\partial\mu}\mathbf{g}(e_\alpha, \nabla_{\frac{\partial}{\partial\nu}}e_\beta) - \frac{\partial}{\partial\nu}\mathbf{g}(e_\alpha, \nabla_{\frac{\partial}{\partial\mu}}e_\beta) \\
&\quad + \mathbf{g}(\nabla_{\frac{\partial}{\partial\nu}}e_\alpha, \nabla_{\frac{\partial}{\partial\mu}}e_\beta) - \mathbf{g}(\nabla_{\frac{\partial}{\partial\mu}}e_\alpha, \nabla_{\frac{\partial}{\partial\nu}}e_\beta)
\end{aligned}
$$

$([\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\nu}] = 0$ since they are coordinate vectorfields)

$$
\begin{aligned}
&= \quad \frac{\partial}{\partial \mu}(A_\nu)_{\alpha\beta} - \frac{\partial}{\partial \nu}(A_\mu)_{\alpha\beta} + \mathbf{g}(\nabla_{\frac{\partial}{\partial \nu}} e_\alpha, \nabla_{\frac{\partial}{\partial \mu}} e_\beta) - \mathbf{g}(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha, \nabla_{\frac{\partial}{\partial \nu}} e_\beta) \\
&= \quad \frac{\partial}{\partial \mu}(A_\nu)_{\alpha\beta} - \frac{\partial}{\partial \nu}(A_\mu)_{\alpha\beta} + \mathbf{g}(e^\lambda(\nabla_{\frac{\partial}{\partial \nu}} e_\alpha) e_\lambda, \nabla_{\frac{\partial}{\partial \mu}} e_\beta) - \mathbf{g}(e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha) e_\lambda, \nabla_{\frac{\partial}{\partial \nu}} e_\beta) \\
&= \quad \frac{\partial}{\partial \mu}(A_\nu)_{\alpha\beta} - \frac{\partial}{\partial \nu}(A_\mu)_{\alpha\beta} + e^\lambda(\nabla_{\frac{\partial}{\partial \nu}} e_\alpha) \mathbf{g}(e_\lambda, \nabla_{\frac{\partial}{\partial \mu}} e_\beta) - e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha) \mathbf{g}(e_\lambda, \nabla_{\frac{\partial}{\partial \nu}} e_\beta) \\
&= \quad \frac{\partial}{\partial \mu}(A_\nu)_{\alpha\beta} - \frac{\partial}{\partial \nu}(A_\mu)_{\alpha\beta} + e^\lambda(\nabla_{\frac{\partial}{\partial \nu}} e_\alpha)(A_\mu)_{\lambda\beta} - e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha)(A_\nu)_{\lambda\beta}
\end{aligned}
$$

Computing at the point p,

$$(A_\mu)^\lambda{}_\alpha = \mathbf{g}^{\gamma\lambda}(A_\mu)_{\gamma\alpha} = \mathbf{g}^{\lambda\lambda}(A_\mu)_{\lambda\alpha} = \mathbf{g}(e_\lambda, e_\lambda)^{-1}(A_\mu)_{\lambda\alpha} = \mathbf{g}(e_\lambda, e_\lambda)^{-1}\mathbf{g}(\nabla_\mu e_\alpha, e_\lambda)$$

Since the metric is compatible, we have $\nabla \mathbf{g} = 0$, and thus,

$$
\begin{aligned}
\nabla_\mu \mathbf{g}(e_\alpha, e_\lambda) &= \quad \frac{\partial}{\partial \mu} \mathbf{g}(e_\alpha, e_\lambda) - \mathbf{g}(\nabla_\mu e_\alpha, e_\lambda) - \mathbf{g}(e_\alpha, \nabla_\mu e_\lambda) \\
&= \quad -\mathbf{g}(\nabla_\mu e_\alpha, e_\lambda) - \mathbf{g}(e_\alpha, \nabla_\mu e_\lambda) = 0
\end{aligned}
$$

Therefore,

$$\mathbf{g}(\nabla_\mu e_\alpha, e_\lambda) = -\mathbf{g}(e_\alpha, \nabla_\mu e_\lambda)$$

and thus, the matrix A is anti-symmetric, ie. $(A_\mu)_{\alpha\beta} = (A_\mu)_{\beta\alpha}$.

We have,

$$\nabla_{\frac{\partial}{\partial \mu}} e_\alpha = e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha) e_\lambda$$

Thus,

$$\mathbf{g}(e_\lambda, \nabla_{\frac{\partial}{\partial \mu}} e_\alpha) = \mathbf{g}(e_\lambda, e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha) e_\lambda) = \mathbf{g}(e_\lambda, e_\lambda) e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha)$$

Consequently,

$$e^\lambda(\nabla_{\frac{\partial}{\partial \mu}} e_\alpha) = \mathbf{g}(e_\lambda, e_\lambda)^{-1}\mathbf{g}(e_\lambda, \nabla_{\frac{\partial}{\partial \mu}} e_\alpha) = (A_\mu)^\lambda{}_\alpha$$

and,

$$e^\lambda(\nabla_{\frac{\partial}{\partial \nu}} e_\alpha) = \mathbf{g}(e_\lambda, e_\lambda)^{-1}\mathbf{g}(e_\lambda, \nabla_{\frac{\partial}{\partial \nu}} e_\alpha) = (A_\nu)^\lambda{}_\alpha$$

Therefore,

$$
\begin{aligned}
R(e_\alpha, e_\beta, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \nu}) &= \quad \partial_\mu(A_\nu)_{\alpha\beta} - \partial_\nu(A_\mu)_{\alpha\beta} + (A_\nu)^\lambda{}_\alpha(A_\mu)_{\lambda\beta} - (A_\mu)^\lambda{}_\alpha(A_\nu)_{\lambda\beta} \\
&= \quad \partial_\mu(A_\nu)_{\alpha\beta} - \partial_\nu(A_\mu)_{\alpha\beta} - (A_\nu)_\alpha{}^\lambda(A_\mu)_{\lambda\beta} + (A_\mu)_\alpha{}^\lambda(A_\nu)_{\lambda\beta}
\end{aligned}
$$

(by anti-symmetry of the matrix A).

As commutator of matrices, we have,

$$[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu$$

and thus,

$$([A_\mu, A_\nu])_{\alpha\beta} = (A_\mu)_\alpha{}^\lambda (A_\nu)_{\lambda\beta} - (A_\nu)_\alpha{}^\lambda (A_\mu)_{\lambda\beta}$$

Consequently, we get,

$$R_{\alpha\beta\mu\nu} = \partial_\mu(A_\nu)_{\alpha\beta} - \partial_\nu(A_\mu)_{\alpha\beta} + ([A_\mu, A_\nu])_{\alpha\beta}, \tag{12}$$

We have,

$$\nabla_\mu A_\nu = \partial_\mu(A_\nu) - (A)(\nabla_\mu \frac{\partial}{\partial\nu})$$

and,

$$\nabla_\nu A_\mu = \partial_\nu(A_\mu) - (A)(\nabla_\nu \frac{\partial}{\partial\mu})$$

Thus,

$$
\begin{aligned}
\nabla_\mu A_\nu - \nabla_\nu A_\mu &= \partial_\mu(A_\nu) - \partial_\nu(A_\mu) + (A)(\nabla_\nu \frac{\partial}{\partial\mu}) - (A)(\nabla_\mu \frac{\partial}{\partial\nu}) \\
&= \partial_\mu(A_\nu) - \partial_\nu(A_\mu) + (A)(\nabla_\nu \frac{\partial}{\partial\mu} - \nabla_\mu \frac{\partial}{\partial\nu}) \\
&= \partial_\mu(A_\nu) - \partial_\nu(A_\mu) + (A)([\frac{\partial}{\partial\nu}, \frac{\partial}{\partial\mu}])
\end{aligned}
$$

(because the metric is symmetric)

$$= \partial_\mu(A_\nu) - \partial_\nu(A_\mu)$$

(since $\frac{\partial}{\partial\mu}$, and $\frac{\partial}{\partial\nu}$ are coordinate vectorfields, therefore they commute).
As a result,

$$R_{\alpha\beta\mu\nu} = \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]\right)_{\alpha\beta} \tag{13}$$

Since the curvature tensor of the connection A is,

$$(F_{\mu\nu})_{\alpha\beta} = \left(\nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]\right)_{\alpha\beta} \tag{14}$$

We get,

$$R_{\alpha\beta\mu\nu} = (F_{\mu\nu})_{\alpha\beta} \tag{15}$$

3.2. The Einstein equations in a Yang-Mills form.

The following is a well known proposition, of which we sketch the proof.

Proposition 3.3. *Let $(M, \mathbf{g})$ be a 4-dimensional Lorentzian manifold that is Ricci flat, i.e. $R_{\mu\nu} = 0$. Then, we have $(\mathbf{D}^{(A)\mu} F_{\mu\nu})_{\alpha\beta} = 0$, where A and F are defined as in* (11) *and* (14).

Proof

Computing,

$$
\begin{aligned}
\nabla_\sigma R_{\alpha\beta\mu\nu} &= \frac{\partial}{\partial\sigma}R(e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(\nabla_\sigma e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,\nabla_\sigma e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) \\
&\quad - R(e_\alpha,e_\beta,\nabla_\sigma\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\nabla_\sigma\frac{\partial}{\partial\nu}) \\
&= \frac{\partial}{\partial\sigma}R(e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,e_\beta,\nabla_\sigma\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\nabla_\sigma\frac{\partial}{\partial\nu}) \\
&\quad - R(\nabla_\sigma e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,\nabla_\sigma e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) \\
&= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - R(\nabla_\sigma e_\alpha,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,\nabla_\sigma e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) \\
&= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - R(e^\lambda(\nabla_\sigma e_\alpha)e_\lambda,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - R(e_\alpha,e^\lambda(\nabla_\sigma e_\beta)e_\lambda,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) \\
&= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - e^\lambda(\nabla_\sigma e_\alpha)R(e_\lambda,e_\beta,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) - e^\lambda(\nabla_\sigma e_\beta)R(e_\alpha,e_\lambda,\frac{\partial}{\partial\mu},\frac{\partial}{\partial\nu}) \\
&= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} - (A_\sigma)^\lambda{}_\alpha (F_{\mu\nu})_{\lambda\beta} - (A_\sigma)^\lambda{}_\beta (F_{\mu\nu})_{\alpha\lambda} \\
&= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} + (A_\sigma)_\alpha{}^\lambda (F_{\mu\nu})_{\lambda\beta} - (F_{\mu\nu})_{\alpha\lambda}(A_\sigma)^\lambda{}_\beta
\end{aligned}
$$

(using the anti-symmetry of A). Thus,

$$
\begin{aligned}
\nabla_\sigma R_{\alpha\beta\mu\nu} &= (\nabla_\sigma F_{\mu\nu})_{\alpha\beta} + ([A_\sigma,F_{\mu\nu}])_{\alpha\beta} \\
&= (\mathbf{D}_\sigma^{(A)}F_{\mu\nu})_{\alpha\beta} \tag{16}
\end{aligned}
$$

Computing,

$$
\begin{aligned}
(\mathbf{D}^{(A)^\mu}F_{\mu\nu})_{\alpha\beta} &= \nabla^\mu R_{\alpha\beta\mu\nu} = \mathbf{g}^{\mu\sigma}\nabla_\sigma R_{\alpha\beta\mu\nu} \\
&= \nabla_\sigma \mathbf{g}^{\mu\sigma}R_{\alpha\beta\mu\nu}
\end{aligned}
$$

(because the metric is compatible)

$$
= \nabla_\sigma R_{\alpha\beta}{}^\sigma{}_\nu = \nabla_\sigma R^\sigma{}_{\nu\alpha\beta}
$$

(using the symmtery of the Riemann tensor)

$$
= -\nabla_\alpha R^\sigma{}_{\nu\beta\sigma} - \nabla_\beta R^\sigma{}_{\nu\sigma\alpha}
$$

(where we have used another symmetry of the Riemann tensor)

$$
= \nabla_\alpha R^\sigma{}_{\nu\sigma\beta} - \nabla_\beta R^\sigma{}_{\nu\sigma\alpha} = \nabla_\alpha R_{\nu\beta} - \nabla_\beta R_{\nu\alpha} = 0
$$

(since the Einstein vacuum equations say that $R_{\mu\gamma} = 0 = R^\sigma{}_{\mu\sigma\gamma}$). We get,

$$
(\mathbf{D}^{(A)^\mu}F_{\mu\nu})_{\alpha\beta} = 0 \tag{17}
$$

∎

The second Bianchi identitiy for the Riemann tensor,

$$
\begin{aligned}
0 &= \nabla_\alpha R_{\gamma\sigma\mu\nu} + \nabla_\mu R_{\gamma\sigma\nu\alpha} + \nabla_\nu R_{\gamma\sigma\alpha\mu} \\
&= \mathbf{D}^{(A)}_\alpha F_{\mu\nu} + \mathbf{D}^{(A)}_\mu F_{\nu\alpha} + \mathbf{D}^{(A)}_\nu F_{\alpha\mu} \tag{18}
\end{aligned}
$$

which is the Bianchi identity for the Yang-Mills fields. The equations above (17) and (18) are the Yang-Mills equations except to the fact that the background geometry $(M, \mathbf{g})$ is part of the unknown that we are looking for while trying to solve the Einstein vacuum equations.

This analogy between the Einstein equations in General Relativity and the Yang-Mills equations has been pursued by V. Moncrief in [M], by developing an integral representation formula for the curvature tensor in General Relativity, and then independently by I. Rodnianski and S. Klainerman in [KR1] and [KR3] partly as a desire to adapt the Eardley-Moncrief argument [EM1]-[EM2] to General Relativity.

4. The Proof of the Global Existence of Yang-Mills Fields on Arbitrary, Sufficiently Smooth, Globally Hyperbolic, Curved Lorentzian Manifolds

4.1. **A hyperbolic formulation for the Yang-Mills equations.**

It is known that the Yang-Mills fields can be shown to satisfy a tensorial hyperbolic wave equation with sources, on the background geometry. To see this, we start by taking the covariant divergence of (10), we obtain:

$$
\mathbf{D}^{(A)\,\alpha}\mathbf{D}^{(A)}{}_\alpha F_{\mu\nu} + \mathbf{D}^{(A)\,\alpha}\mathbf{D}^{(A)}{}_\mu F_{\nu\alpha} + \mathbf{D}^{(A)\,\alpha}\mathbf{D}^{(A)}{}_\nu F_{\alpha\mu} = 0
$$

We have:

$$
\begin{aligned}
\mathbf{D}^{(A)\,\alpha}\mathbf{D}^{(A)}{}_\mu F_{\nu\alpha} &= \mathbf{D}^{(A)}{}_\mu \mathbf{D}^{(A)\,\alpha} F_{\nu\alpha} + \nabla^\alpha \nabla_\mu F_{\nu\alpha} - \nabla_\mu \nabla^\alpha F_{\nu\alpha} + [F^\alpha{}_\mu, F_{\nu\alpha}] \\
&= \mathbf{D}^{(A)}{}_\mu \mathbf{D}^{(A)\,\alpha} F_{\nu\alpha} + R_\nu{}^{\gamma\alpha}{}_\mu F_{\gamma\alpha} + R_\alpha{}^{\gamma\alpha}{}_\mu F_{\nu\gamma} + [F^\alpha{}_\mu, F_{\nu\alpha}] \\
&= \mathbf{D}^{(A)}{}_\mu (\mathbf{D}^{(A)\,\alpha} F_{\nu\alpha}) - \mathbf{D}^{(A)\,\nabla_{nu} e_\alpha} F_{\nu\alpha} + R_\nu{}^{\gamma\alpha}{}_\mu F_{\gamma\alpha} + R_\alpha{}^{\gamma\alpha}{}_\mu F_{\nu\gamma} \\
&\quad + [F^\alpha{}_\mu, F_{\nu\alpha}] \\
&= 0 - \mathbf{D}^{(A)\,\nabla_\nu e_\alpha} F_{\nu\alpha} + R_\nu{}^{\gamma\alpha}{}_\mu F_{\gamma\alpha} + R_\alpha{}^{\gamma\alpha}{}_\mu F_{\nu\gamma} + [F^\alpha{}_\mu, F_{\nu\alpha}]
\end{aligned}
$$

(by equation (9)). By choosing a normal frame at each point in space-time, i.e. a frame where $\mathbf{g}(e_\alpha, e_\beta) = \mathrm{diag}(-1, 1, \ldots, 1)$ and $\nabla_\alpha e_\beta = 0$ at that point, to compute the contraction $\mathbf{D}^{(A)\,\nabla_\nu e_\alpha} F_{\nu\alpha}$, we get that it vanishes. So

$$
\begin{aligned}
\mathbf{D}^{(A)\,\alpha}\mathbf{D}^{(A)}{}_\mu F_{\nu\alpha} &= R_{\nu\gamma\alpha\mu} F^{\gamma\alpha} + R_{\alpha\gamma}{}^\alpha{}_\mu F_\nu{}^\gamma + [F^\alpha{}_\mu, F_{\nu\alpha}] \\
&= R_{\nu\gamma\alpha\mu} F^{\gamma\alpha} + R_{\gamma\mu} F_\nu{}^\gamma + [F^\alpha{}_\mu, F_{\nu\alpha}]
\end{aligned}
$$

$$
\mathbf{D}^{(A)\,\alpha}\mathbf{D}^{(A)}{}_\mu F_{\nu\alpha} = R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} + R_{\mu\gamma} F_\nu{}^\gamma + [F^\alpha{}_\mu, F_{\nu\alpha}]
$$

On the other hand, we have:

$$
\begin{aligned}
\mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}{}_\nu F_{\alpha\mu} &= \mathbf{D}^{(A)}{}_\nu \mathbf{D}^{(A)\alpha} F_{\alpha\mu} + \nabla^\alpha \nabla_\nu F_{\alpha\mu} - \nabla_\nu \nabla^\alpha F_{\alpha\mu} + [F^\alpha{}_\nu, F_{\alpha\mu}] \\
&= 0 + R_\alpha{}^{\gamma\alpha}{}_\mu F_{\gamma\mu} + R_\mu{}^{\gamma\alpha}{}_\nu F_{\alpha\gamma} + [F^\alpha{}_\nu, F_{\alpha\mu}] \\
&\quad \text{(by equation (9))} \\
&= R_{\alpha\gamma}{}^\alpha{}_\mu F^\gamma{}_\mu + R_{\mu\gamma\alpha\nu} F^{\alpha\gamma} + [F_{\alpha\nu}, F^\alpha{}_\mu] \\
&= R_{\gamma\nu} F^\gamma{}_\mu + R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} + [F^\alpha{}_\mu, F_{\nu\alpha}]
\end{aligned}
$$

(where we have used the anti-symmetry of F)

$$
= R_{\nu\gamma} F^\gamma{}_\mu + R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} + [F^\alpha{}_\mu, F_{nu\alpha}]
$$

As we have

$$
\mathbf{D}^{(A)}_\alpha F_{\mu\nu} + \mathbf{D}^{(A)}_\mu F_{\nu\alpha} + \mathbf{D}^{(A)}_\nu F_{\alpha\mu} = 0
$$

we get

$$
\mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}{}_\alpha F_{\mu\nu} + 2R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} + R_{\mu\gamma} F_\nu{}^\gamma + R_{\nu\gamma} F^\gamma{}_\mu + 2[F^\alpha{}_\mu, F_{\nu\alpha}] = 0
$$

We obtain:

$$
\Box_{\mathbf{g}}^{(A)} F_{\mu\nu} = \mathbf{D}^{(A)\alpha}\mathbf{D}^{(A)}{}_\alpha F_{\mu\nu} = -2R_{\gamma\mu\nu\alpha} F^{\alpha\gamma} - R_{\mu\gamma} F_\nu{}^\gamma - R_{\nu\gamma} F^\gamma{}_\mu - 2[F^\alpha{}_\mu, F_{\nu\alpha}]
\tag{19}
$$

Due to the equation (19), the held belief is that the Yang-Mills equations are hyperbolic in nature.

4.2. **Energy estimates.**

Consider the energy momentum tensor:

$$
T_{\mu\nu} = <F_{\mu\beta}, F_\nu{}^\beta> - \frac{1}{4}\mathbf{g}_{\mu\nu} <F_{\alpha\beta}, F^{\alpha\beta}>
\tag{20}
$$

We will wright $<F_{\alpha\beta}, F^{\alpha\beta}>$ as $F_{\alpha\beta}.F^{\alpha\beta}$ to lighten the notation.

Taking the covariant divergence of $T_{\mu\nu}$ we obtain:

$$
\begin{aligned}
\nabla^\nu T_{\mu\nu} &= \nabla^\nu (F_{\mu\beta}.F_\nu{}^\beta - \frac{1}{4}\mathbf{g}_{\mu\nu} F_{\alpha\beta}.F^{\alpha\beta}) \\
&= (\mathbf{D}^{(A)^\nu} F_{\mu\beta}).F_\nu{}^\beta + F_{\mu\beta}.\mathbf{D}^{(A)^\nu} F_\nu{}^\beta - \frac{1}{4}\mathbf{g}_{\mu\nu}\mathbf{D}^{(A)^\nu} F_{\alpha\beta}.F^{\alpha\beta} \\
&\quad -\frac{1}{4}\mathbf{g}_{\mu\nu} F_{\alpha\beta}.\mathbf{D}^{(A)^\nu} F^{\alpha\beta}
\end{aligned}
$$

(where we used the fact that the metric is Killing, i.e. $\nabla \mathbf{g} = 0$, and that $<\,,\,>$ is Ad-invariant)

$$
= (\mathbf{D}^{(A)^\nu} F_{\mu\beta}).F_\nu{}^\beta - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{D}^{(A)^\nu} F_{\alpha\beta}.F^{\alpha\beta}
$$

(we used the field equations)

$$
\begin{aligned}
&= (\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta} - \frac{1}{2}(\mathbf{D}^{(A)}_\mu F_{\alpha\beta}).F^{\alpha\beta} \\
&= (\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta} + \frac{1}{2}(\mathbf{D}^{(A)}_\alpha F_{\beta\mu}).F^{\alpha\beta} + \frac{1}{2}(\mathbf{D}^{(A)}_\beta F_{\mu\alpha}).F^{\alpha\beta}
\end{aligned}
$$

(using the Bianchi identities)

$$
\begin{aligned}
&= (\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta} - \frac{1}{2}(\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta} + \frac{1}{2}(\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\beta\alpha} \\
&= (\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta} - \frac{1}{2}(\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta} - \frac{1}{2}(\mathbf{D}^{(A)}_\alpha F_{\mu\beta}).F^{\alpha\beta}
\end{aligned}
$$

(where we used the anti-symmetry of F in the last two equalities)

$$
= 0 \tag{21}
$$

Considering a vector field V^ν we let

$$
J_\mu(V) = V^\nu T_{\mu\nu}
$$

We have

$$
\begin{aligned}
\nabla^\mu J_\mu(V) &= \nabla^\mu(V^\nu T_{\mu\nu}) \\
&= \nabla^\mu(V^\nu)T_{\mu\nu}
\end{aligned}
$$

(since T is divergenceless)

$$
= \frac{1}{2}(\nabla^\mu(V^\nu)T_{\mu\nu} + \nabla^\mu(V^\nu)T_{\mu\nu}) = \frac{1}{2}(\nabla^\mu(V^\nu)T_{\mu\nu} + \nabla^\nu(V^\mu)T_{\mu\nu})
$$

(where we used the symmetry of $T_{\mu\nu}$)

$$
= \pi^{\mu\nu}(V)T_{\mu\nu} \tag{22}
$$

where $\pi^{\mu\nu}(V)$ is the deformation tensor that is,

$$
\pi^{\mu\nu}(V) = \frac{1}{2}(\nabla^\mu V^\nu + \nabla^\nu V^\mu) \tag{23}
$$

Applying the divergence theorem on $J_\mu(V)$ in a region B bounded to the past by a spacelike hypersurface Σ_1 and to the future by a spacelike hypersurface Σ_2, and by a null hypersurface N, we obtain:

$$\int_B \pi^{\mu\nu}(V)T_{\mu\nu}dV_B = \int_{\Sigma_1} J_\mu(V)w^\mu dV_{\Sigma_1} - \int_{\Sigma_2} J_\mu(V)w^\mu dV_{\Sigma_2} - \int_N J_\mu(V)w_N^\mu dV_N$$

(24)

where w^μ are the unit normal to the hypersurfaces Σ, w_N^μ is any null generator of N, dV_Σ are the induced volume forms and dV_N is defined such that the divergence theorem applies.

Taking $V = \frac{\partial}{\partial t}$, where $\frac{\partial}{\partial t}$ is a timelike vector field.

Taking $B = \Sigma_+ \cap J^-(p)$, we get:

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\frac{\partial}{\partial t})T_{\mu\nu}dV_B = \int_{\Sigma \cap J^-(p)} J_\mu(\frac{\partial}{\partial t})w^\mu dV_{\Sigma_+} - \int_{N^-(p)\cap\Sigma_+} J_\mu(\frac{\partial}{\partial t})w_N^\mu dV_N$$

where $w = -\frac{\partial}{\partial \hat{t}}$, is the normalized timelike vector field, i.e.

$$\mathbf{g}(\frac{\partial}{\partial \hat{t}}, \frac{\partial}{\partial \hat{t}}) = -1$$

(25)

Definition 4.3. Define the energy $E_t^{\frac{\partial}{\partial t}}$ by,

$$E_t^{\frac{\partial}{\partial t}} = \int_{\Sigma_t} J_\mu(\frac{\partial}{\partial t})(\frac{\partial}{\partial \hat{t}})^\mu dV_{\Sigma_t} = \int_{\Sigma_t} J_\mu(\frac{\partial}{\partial t})(\frac{\partial}{\partial \hat{t}})^\mu \sqrt{-\mathbf{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})}dV_{\Sigma_t}$$

(26)

And define the flux $F^{\frac{\partial}{\partial t}}(N^-(p) \cap \Sigma_+)$ by,

$$F^{\frac{\partial}{\partial t}}(N^-(p) \cap \Sigma_+) = -\int_{N^-(p)\cap\Sigma_+} J_\mu(\frac{\partial}{\partial t})w_{N^-(p)}^\mu dV_{N^-(p)}$$

(27)

We get:

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\frac{\partial}{\partial t})T_{\mu\nu}dV_B = -E_{t=0}^{\frac{\partial}{\partial t}}(\Sigma \cap J^-(p)) + F^{\frac{\partial}{\partial t}}(N^-(p) \cap \Sigma_+)$$

(28)

Definition 4.4. We define $\{L, \underline{L}, e_1, e_2\}$ a null frame as in (99), (100), (101), and (102), in the following manner:

We define L as in (5.1), and we define $\underline{L}$ as:

$$\underline{L} = -\mathbf{g}(L, \hat{t})^{-1}(2\hat{t} + \mathbf{g}(L, \hat{t})^{-1}L)$$

(29)

Define e_i, $i \in \{1, 2\}$, such that,

$$\mathbf{g}(e_i, e_j) = \delta_{ij}$$

(30)

$$\mathbf{g}(L, e_i) = \mathbf{g}(\underline{L}, e_i) = 0 \tag{31}$$

We verify that

$$
\begin{aligned}
\mathbf{g}(\underline{L}, \underline{L}) &= \mathbf{g}_{L\hat{t}}^{-2}\mathbf{g}(2\hat{t} + \mathbf{g}_{L\hat{t}}^{-1}L, 2\hat{t} + \mathbf{g}_{L\hat{t}}^{-1}L) = 4\mathbf{g}_{L\hat{t}}^{-2}\mathbf{g}(\hat{t}, \hat{t}) + 4\mathbf{g}_{L\hat{t}}^{-2}\mathbf{g}_{L\hat{t}}^{-1}\mathbf{g}(\hat{t}, L) \\
&= -4\mathbf{g}_{L\hat{t}}^{-2} + 4\mathbf{g}_{L\hat{t}}^{-2} \\
&= 0 \tag{32}
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathbf{g}(L, \underline{L}) &= -\mathbf{g}_{L\hat{t}}^{-1}\mathbf{g}(L, 2\hat{t} + \mathbf{g}_{L\hat{t}}^{-1}L) = -2\mathbf{g}_{L\hat{t}}^{-1}\mathbf{g}(L, \hat{t}) \\
&= -2 \tag{33}
\end{aligned}
$$

We have

$$\hat{t} = -\frac{(\mathbf{g}_{L\hat{t}}^{-1}L + \mathbf{g}_{L\hat{t}}\underline{L})}{2} \tag{34}$$

4.4.1. *Computing explicitly $F^{\frac{\partial}{\partial t}}(N^-(p) \cap \Sigma_+)$.*

$$
\begin{aligned}
& J_\mu(\frac{\partial}{\partial \hat{t}})L^\mu \\
=\ & T_{\mu\hat{t}}L^\mu = T_{L\hat{t}} = F_{L\beta}.F_{\hat{t}}^{\ \beta} - \frac{1}{4}\mathbf{g}_{L\hat{t}}F_{\alpha\beta}.F^{\alpha\beta} \\
=\ & F_{L\underline{L}}.F_{\hat{t}}^{\ \underline{L}} + F_{Le_a}.F_{\hat{t}}^{\ e_a} + F_{Le_b}.F_{\hat{t}}^{\ e_b} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{\underline{L}L}.F^{\underline{L}L} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{La}.F^{La} \\
& -\frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Lb}.F^{Lb} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{\underline{L}a}.F^{\underline{L}a} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{\underline{L}b}.F^{\underline{L}b} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}.F^{ab} \\
=\ & -\frac{1}{2}F_{L\underline{L}}.F_{\hat{t}L} + F_{Le_a}.F_{\hat{t}e_a} + F_{Le_b}.F_{\hat{t}e_b} + \frac{1}{8}\mathbf{g}_{L\hat{t}}F_{L\underline{L}}.F_{L\underline{L}} + \frac{1}{4}\mathbf{g}_{L\hat{t}}F_{La}.F_{\underline{L}a} \\
& +\frac{1}{4}\mathbf{g}_{L\hat{t}}F_{Lb}.F_{\underline{L}b} + \frac{1}{4}\mathbf{g}_{L\hat{t}}F_{\underline{L}a}.F_{La} + \frac{1}{4}\mathbf{g}_{L\hat{t}}F_{\underline{L}b}.F_{Lb} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}.F_{ab} \\
=\ & -\frac{1}{2}F_{L\underline{L}}.F_{\hat{t}L} + F_{Le_a}.F_{\hat{t}e_a} + F_{Le_b}.F_{\hat{t}e_b} + \frac{1}{8}\mathbf{g}_{L\hat{t}}F_{L\underline{L}}.F_{L\underline{L}} + \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{La}.F_{\underline{L}a} \\
& +\frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Lb}.F_{\underline{L}b} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}.F_{ab}
\end{aligned}
$$

We have,

$$
\begin{aligned}
-\frac{1}{2}F_{L\underline{L}}.F_{\hat{t}L} &= (-\frac{1}{2})(-\frac{1}{2})\mathbf{g}(L, \hat{t})F_{L\underline{L}}.F_{LL} = -\frac{1}{4}\mathbf{g}_{L\hat{t}}F_{L\underline{L}}.F_{LL} \\
F_{Le_a}.F_{\hat{t}e_a} + F_{Le_b}.F_{\hat{t}e_b} &= -\frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Le_a}.F_{Le_a} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Le_a}.F_{\underline{L}e_a} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Le_b}.F_{Le_b} \\
& \quad -\frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Le_b}.F_{\underline{L}e_b}
\end{aligned}
$$

Therefore,

$$J_\mu(\frac{\partial}{\partial \hat{t}})L^\mu$$

$$= -\frac{1}{4}\mathbf{g}_{L\hat{t}}F_{L\underline{L}}.F_{L\underline{L}} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Le_a}.F_{Le_a} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Le_a}.F_{\underline{L}e_a} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Le_b}.F_{Le_b}$$

$$-\frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Le_b}.F_{\underline{L}e_b} + \frac{1}{8}\mathbf{g}_{L\hat{t}}F_{L\underline{L}}.F_{L\underline{L}} + \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{La}.F_{\underline{L}a} + \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{Lb}.F_{\underline{L}b}$$

$$-\frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}.F_{ab}$$

$$= -\frac{1}{8}\mathbf{g}_{L\hat{t}}F_{L\underline{L}}.F_{L\underline{L}} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Le_a}.F_{Le_a} - \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}F_{Le_b}.F_{Le_b} - \frac{1}{2}\mathbf{g}_{L\hat{t}}F_{ab}.F_{ab}$$

Thus,

$$F^{\frac{\partial}{\partial \hat{t}}}(N^-(p) \cap \Sigma_+)$$

$$= \int_{N^-(p)\cap\Sigma_+}(\frac{1}{8}\mathbf{g}_{L\hat{t}}|F_{L\underline{L}}|^2 + \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}|F_{Le_a}|^2 + \frac{1}{2}\mathbf{g}_{L\hat{t}}^{-1}|F_{Le_b}|^2 + \frac{1}{2}\mathbf{g}_{L\hat{t}}|F_{ab}|^2)(q)$$

$$\tag{35}$$

where $| \, . \, |$ is the norm deduced from $< \, , \, >$.

4.4.2. *Computing* $E_t^{\frac{\partial}{\partial \hat{t}}}$.

$$E_t^{\frac{\partial}{\partial \hat{t}}} = \int_{\Sigma_t} J_\mu(\frac{\partial}{\partial \hat{t}})(\frac{\partial}{\partial \hat{t}})^\mu \sqrt{-\mathbf{g}(\frac{\partial}{\partial t},\frac{\partial}{\partial t})}dV_{\Sigma_t} = \int_{\Sigma_t} T_{\hat{t}\hat{t}}\sqrt{-\mathbf{g}(\frac{\partial}{\partial t},\frac{\partial}{\partial t})}dV_{\Sigma_t} \tag{36}$$

$$T_{\hat{t}\hat{t}} = F_{\hat{t}\beta}.F_{\hat{t}}^{\ \beta} - \frac{1}{4}\mathbf{g}_{\hat{t}\hat{t}}F_{\alpha\beta}.F^{\alpha\beta}$$

Choosing the frame $\{\hat{t}, n, e_a, e_b\}$

$$T_{\hat{t}\hat{t}} = F_{\hat{t}n}.F_{\hat{t}n} + F_{\hat{t}a}.F_{\hat{t}a} + F_{\hat{t}b}.F_{\hat{t}b} + \frac{1}{2}F_{\hat{t}n}.F^{\hat{t}n} + \frac{1}{2}F_{\hat{t}a}.F^{\hat{t}a}$$

$$+\frac{1}{2}F_{\hat{t}b}.F^{\hat{t}b} + \frac{1}{2}F_{na}.F^{na} + \frac{1}{2}F_{nb}.F^{nb} + \frac{1}{2}F_{ab}.F^{ab}$$

$$= F_{\hat{t}n}.F_{\hat{t}n} + F_{\hat{t}a}.F_{\hat{t}a} + F_{\hat{t}b}.F_{\hat{t}b} - \frac{1}{2}F_{\hat{t}n}.F_{\hat{t}n} - \frac{1}{2}F_{\hat{t}a}.F_{\hat{t}a} - \frac{1}{2}F_{\hat{t}b}.F_{\hat{t}b}$$

$$+\frac{1}{2}F_{na}.F_{na} + \frac{1}{2}F_{nb}.F_{nb} + \frac{1}{2}F_{ab}.F_{ab}$$

$$= \frac{1}{2}F_{tn}.F_{\hat{t}n} + \frac{1}{2}F_{\hat{t}a}.F_{\hat{t}a} + \frac{1}{2}F_{\hat{t}b}.F_{\hat{t}b} + \frac{1}{2}F_{na}.F_{na} + \frac{1}{2}F_{nb}.F_{nb} + \frac{1}{2}F_{ab}.F_{ab}$$

$$= \frac{1}{2}[|F_{\hat{t}n}|^2 + |F_{\hat{t}a}|^2 + |F_{\hat{t}b}|^2 + |F_{na}|^2 + |F_{nb}|^2 + |F_{ab}|^2](q) \geq 0$$

Thus,

$$E^{\frac{\partial}{\partial t}}_{t=t_0}(\Sigma \cap J^-(p)) \leq E^{\frac{\partial}{\partial t}}_{t=t_0} \tag{37}$$

4.4.3. *Finiteness of the flux from finite initial energy.* Let Σ_t^- be the past of Σ_t.

We also have,

$$\int_{\Sigma_+ \cap \Sigma_t^- \cap J^-(p)} \pi^{\mu\nu}(\frac{\partial}{\partial t})T_{\mu\nu}dV_B$$

$$\lesssim \sum_{\hat{\mu},\hat{\nu}\in\{\hat{t},n,e_a,e_b\}} \int_{\Sigma_+ \cap \Sigma_t^- \cap J^-(p)} |\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t})|| < F_{\hat{\mu}\beta}, F_{\hat{\nu}}{}^\beta > -\frac{1}{4}\mathbf{g}_{\hat{\mu}\hat{\nu}} < F_{\alpha\beta}, F^{\alpha\beta} > |dV_B$$

$$\lesssim \sum_{\alpha,\beta,\hat{\mu},\hat{\nu}\in\{\hat{t},n,e_a,e_b\}} \int_{t_0}^t |\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t})|_{L^\infty_{\Sigma_{\overline{t}}\cap J^-(p)}} \int_{\Sigma_{\overline{t}}\cap J^-(p)} (|F_{\hat{\mu}\beta}|^2 + |F_{\hat{\nu}}{}^\beta|^2$$

$$+ |F_{\alpha\beta}|^2 + |F^{\alpha\beta}|^2)\cdot\sqrt{-\mathbf{g}(\frac{\partial}{\partial t},\frac{\partial}{\partial t})}dV_{\Sigma_{\overline{t}}} \tag{38}$$

(by using $a.b \lesssim a^2 + b^2$).

As in [CS] we assume that the deformation tensor of $\frac{\partial}{\partial t}$ is finite. More precisely, we assume that for all $\hat{\mu},\hat{\nu}\in\{\hat{t},n,e_a,e_b\}$, the components of the deformation tensor, $\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t}) = \frac{1}{2}[\nabla^{\hat{\mu}}(\frac{\partial}{\partial t})^{\hat{\nu}} + \nabla^{\hat{\nu}}(\frac{\partial}{\partial t})^{\hat{\mu}}]$, verify,

$$|\pi^{\hat{\mu}\hat{\nu}}(\frac{\partial}{\partial t})|_{L^\infty_{loc(\Sigma_t)}} \leq C(t) \tag{39}$$

where $C(t) \in L^1_{loc}$.

Applying the divergence theorem again in the future of Σ and the past of $\Sigma_t \cap J^-(p)$, we get:

$$E^{\frac{\partial}{\partial t}}_t(\Sigma_t \cap J^-(p)) = \int_{\Sigma_+ \cap \Sigma_t^- \cap J^-(p)} \pi^{\mu\nu}(\frac{\partial}{\partial t})T_{\mu\nu}dV_B + E^{\frac{\partial}{\partial t}}_{t=t_0}(\Sigma_{t_0} \cap J^-(p))$$

$$\lesssim E^{\frac{\partial}{\partial t}}_{t=t_0} + \int_{t=t_0}^t C(\overline{t})E^{\frac{\partial}{\partial t}}_{\overline{t}}(\Sigma_{\overline{t}} \cap J^-(p))d\overline{t} \tag{40}$$

(where we used (38), (39), and the positivity of the energy (37)).

Using Grönwall lemma, we get that $E^{\frac{\partial}{\partial t}}_t(\Sigma_t \cap J^-(p))$ is finite and continuous in t, and therefore

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\frac{\partial}{\partial t})T_{\mu\nu}dV_B \lesssim \int_{\hat{t}=t_0}^t c(\overline{t})E^{\frac{\partial}{\partial t}}_{\overline{t}}(\Sigma_{\overline{t}} \cap J^-(p))d\overline{t}$$

$$\lesssim \int_{t_0}^t c(\overline{t})d\overline{t}$$

$$\lesssim c(t_p) \tag{41}$$

and therefore

$$\int_{\Sigma_+ \cap J^-(p)} \pi^{\mu\nu}(\frac{\partial}{\partial t})T_{\mu\nu}dV_B + E^{\frac{\partial}{\partial t}}_{t=t_0}(\Sigma \cap J^-(p)) \lesssim c(t_p)E^{\frac{\partial}{\partial t}}_{t=t_0}$$

Hence,

$$F^{\frac{\partial}{\partial t}}(N^-(p) \cap \Sigma_+) \lesssim c(t_p)E^{\frac{\partial}{\partial t}}_{t=t_0} \tag{42}$$

This finiteness of the flux will play a key role in the proof.

4.5. Definitions and notations.

Definition 4.6. We define $\lambda_{\alpha\beta}$ as in (104) and (105), by fixing at p a $\mathcal{G}$-valued anti-symmetric 2-tensor $\mathbf{J}_p$, and defining $\lambda_{\alpha\beta}$ as the unique 2-tensor field along $N^-(p)$, the boundary of the causal past of p, that verifies the linear transport equation:

$$\mathbf{D}^{(A)}_L \lambda_{\alpha\beta} + \frac{1}{2}tr\chi\lambda_{\alpha\beta} = 0$$

$$(s\lambda_{\alpha\beta})(p) = \mathbf{J}_{\alpha\beta}(p)$$

where s is the affine parameter on $N^-(p)$ defined as in (5.1), and χ is the null second fundamental form of $N^-(p)$ defined as in (98), and $tr\chi$ defined as in (5.3).

Definition 4.7. We define a timelike foliation Σ_t by considering $t = constant$ hypersurfaces.

Definition 4.8. We define positive definite Riemannian metric as in (147), in the following manner:

$$h(e_\alpha, e_\beta) = \mathbf{g}(e_\alpha, e_\beta) + 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}})$$

where

$$\frac{\partial}{\partial \hat{t}} = (-\mathbf{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}))^{-\frac{1}{2}}\frac{\partial}{\partial t}$$

Definition 4.9. For any $\mathcal{G}$-valued 2-tensor K, we define

$$|K|^2 = h_{\alpha\mu}h_{\beta\nu}|K^{\mu\nu}|.|K^{\alpha\beta}|$$

and

$$|K|_{L^\infty} = ||(h_{\alpha\mu}h_{\beta\nu}|K^{\mu\nu}|.|K^{\alpha\beta}|)^{\frac{1}{2}}||_{L^\infty} = ||(|K|^2)^{\frac{1}{2}}||_{L^\infty}$$

We recall (150), that for any two $\mathcal{G}$-valued tensors K and G, we have

$$| < K_{\alpha\beta}, G^{\alpha\beta} > | \lesssim (|K|^2)^{\frac{1}{2}}.(|G|^2)^{\frac{1}{2}}$$

4.9.1. *Notations.* We denote by $N_\tau^-(p)$ the portion of $N^-(p)$ to the past of Σ_{t_p} and to the future of $\Sigma_{t_p - \tau}$.

t^* and $\bar{t}$ are values of t, where $\frac{\partial}{\partial t}$ is a timelike vector field verifying (39).

$\bar{s}$ and $\hat{s}$ are values of s.

We denote by s_τ, the largest value of s on $N_\tau^-(p)$.

We let $\Sigma_t^p = \Sigma_t \cap J^-(p)$.

4.10. Estimates for $s\lambda_{\alpha\beta}$.

Proposition 4.11. *We have,*

$$\sup_{N_\tau^-(p)} |s\lambda| \le C(p,\tau)|J| \tag{43}$$

Proof

We proved in (5.27) that,

$$\sup_{0 \le \bar{s} \le s} |\bar{s}\lambda|^2 \le C(p,s)|J|^2 \tag{44}$$

Hence,

$$\sup_{N_\tau^-(p)} |s\lambda|^2 \le C(p,s_\tau)|J|^2 \tag{45}$$

(where in this last inequality s_τ is the largest value of s on $N_\tau^-(p)$).

In view of (159),

$$s = t_p - t + O(t_p - t) \tag{46}$$

we get,

$$\sup_{N_\tau^-(p)} |s\lambda| \le C(p,\tau)|J| \tag{47}$$

∎

Proposition 4.12.

$$||\lambda||_{L^2(N_\tau^-(p))} \le (\tau)^{\frac{1}{2}} C(p,\tau)|J| \tag{48}$$

Proof

We have,

$$\int_{\mathbb{S}^2} |s\lambda|^2(u=0,s,\omega)da_s \quad \lesssim \quad \int_{\mathbb{S}^2} C(p,s_\tau)^2|J|^2 da_s$$

$$\text{(by (45))}$$

$$\leq \quad C(p,s_\tau)^2|J|^2 \int_{\mathbb{S}^2} 1 da_s$$

$$\lesssim \quad C(p,s_\tau)^2|J|^2 s^2$$

Thus,

$$\int_{\mathbb{S}^2} |\lambda|^2(u=0,s,\omega)da_s \leq C(p,s_\tau)^2|J|^2$$

$$||\lambda_{\alpha\beta}||^2_{L^2(N_\tau^-(p))} = \int_0^{s_\tau^*} \int_{\mathbb{S}^2} |\lambda|^2(u=0,s,\omega)da_s ds$$

(where s_τ^* is the largest value of s for a fixed ω such that $(u=0,s,\omega) \in N_\tau^-(p)$)

$$\lesssim C(p,s_\tau)^2|J|^2 \int_0^{s_\tau} 1 ds \leq s_\tau C(p,s_\tau)^2|J|^2$$

Thus,

$$||\lambda||_{L^2(N_\tau^-(p))} \leq (s_\tau)^{\frac{1}{2}} C(p,s_\tau)|J|$$

Therefore, since $t_p - t = s + o(s)$,

$$||\lambda||_{L^2(N_\tau^-(p))} \leq (\tau)^{\frac{1}{2}} C(p,\tau)|J| \tag{49}$$

∎

4.13. **Estimates for** $||\mathbf{D}_a^{(A)}\lambda||_{L^2(N_\tau^-(p))}$**.**

Definition 4.14. Let

$$\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2}tr\chi\delta_{ab} \tag{50}$$

and let ζ_a be defined as in (131):

$$\zeta_a \quad = \quad \frac{1}{2}\mathbf{g}(\nabla_a L, \underline{L}) \tag{51}$$

Lemma 4.15.

$$\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} \quad = \quad -tr\chi\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}(\nabla_a tr\chi)\lambda_{\alpha\beta} - \frac{1}{2}\zeta_a tr\chi\lambda_{\alpha\beta}$$

$$+[F_{La}, \lambda_{\alpha\beta}] + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma} \tag{52}$$

Proof

We have,

$$
\begin{aligned}
\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} &= \mathbf{D}_a^{(A)}\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma} \\
&= \mathbf{D}_a^{(A)}(\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_a L}^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] \\
&\quad + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma} \\
&= \mathbf{D}_a^{(A)}(-\frac{tr\chi}{2}\lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_a L}^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] \\
&\quad + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma}
\end{aligned}
$$

on $N_\tau^-(p)$.

We remind that for any vectorfield X, we have
$$
X = -\frac{1}{2}\mathbf{g}(X, \underline{L})L - \frac{1}{2}\mathbf{g}(X, L)\underline{L} + \mathbf{g}(X, e_a)e_a
$$
Taking $X = \nabla_a L$, we get
$$
\nabla_a L = -\frac{1}{2}\mathbf{g}(\nabla_a L, \underline{L})L - \frac{1}{2}\mathbf{g}(\nabla_a L, L)\underline{L} + \mathbf{g}(\nabla_a L, e_b)e_b
$$
we get,
$$
\nabla_a L = \mathbf{g}(\nabla_a L, e_b)e_b - \zeta_a L = \chi_{ab}e_b - \zeta_a L \tag{53}
$$

Thus,

$$
\begin{aligned}
\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} &= -\frac{1}{2}(\nabla_a tr\chi)\lambda_{\alpha\beta} - \frac{1}{2}tr\chi(\mathbf{D}_a^{(A)}\lambda_{\alpha\beta}) + \zeta_a\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} \\
&\quad - \chi_{ab}\mathbf{D}_b^{(A)}\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma} \tag{54}
\end{aligned}
$$

Since $\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2}tr\chi\delta_{ab}$, and since we have $\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} = -\frac{1}{2}tr\chi\lambda_{\alpha\beta}$ on $N_\tau^-(p)$, we get,

$$
\begin{aligned}
\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} &= -\frac{1}{2}(\nabla_a tr\chi)\lambda_{\alpha\beta} - \frac{1}{2}tr\chi(\mathbf{D}_a^{(A)}\lambda_{\alpha\beta}) - \frac{1}{2}\zeta_a tr\chi\lambda_{\alpha\beta} \\
&\quad - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}tr\chi\delta_{ab}(\mathbf{D}_b^{(A)}\lambda_{\alpha\beta}) + [F_{La}, \lambda_{\alpha\beta}] \\
&\quad + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma} \\
&= -tr\chi\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}(\nabla_a tr\chi)\lambda_{\alpha\beta} - \frac{1}{2}\zeta_a tr\chi\lambda_{\alpha\beta} \\
&\quad + [F_{La}, \lambda_{\alpha\beta}] + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma}
\end{aligned}
$$

∎

We want to control $||\mathbf{D}_a^{(A)}\lambda||_{L^2(N_\tau^-(p))}$, where we are summing over $a \in \{1,2\}$, by abuse of notation. Following [KR3], let's compute,

$$\overline{s}\int_{\mathbb{S}^2}\overline{s}^2|\mathbf{D}_a^{(A)}\lambda|^2 d\sigma^2 = \int_{\mathbb{S}^2}\overline{s}^{-1}\overline{s}^4|\mathbf{D}_a^{(A)}\lambda|^2 d\sigma^2 = \int_{\mathbb{S}^2}|\overline{s}^{-1}\int_0^{\overline{s}}\slashed{\nabla}_L|s^2\mathbf{D}_a^{(A)}\lambda|^2 ds|d\sigma^2$$

Lemma 4.16. *Let Ψ be a $\mathcal{G}$-valued tensor, we have,*

$$|\nabla_\sigma|\Psi|^2|(p) \quad \leq \quad C(p)[|\mathbf{D}_\sigma^{(A)}\Psi|.|\Psi| + |\Psi|^2] \tag{55}$$

where $C(p)$ depends on the space-time geometry on the point p.

Proof

$$\begin{aligned}\nabla_\sigma|\Psi|^2 &= \nabla_\sigma(h_{\alpha\mu}h_{\beta\nu}|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|) = \nabla_\sigma(h_{\alpha\mu}h_{\beta\nu}).|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|\\ &\quad + h_{\alpha\mu}h_{\beta\nu}.\nabla_\sigma(|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|)\end{aligned}$$

Therefore,

$$\begin{aligned}|\nabla_\sigma|\Psi|^2| &\leq |(\nabla_\sigma h_{\alpha\mu})h_{\beta\nu}|.|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}| + |h_{\alpha\mu}(\nabla_\sigma h_{\beta\nu})|.|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|\\ &\quad + |h_{\alpha\mu}h_{\beta\nu}|.(|\mathbf{D}_\sigma^{(A)}\Psi^{\mu\nu}| + |\Psi(\nabla_\sigma e^\mu, e^\nu)| + |\Psi(e^\mu, \nabla_\sigma e^\nu)|.|\Psi^{\alpha\beta}|)\\ &\quad + |h_{\alpha\mu}h_{\beta\nu}|.|\Psi^{\mu\nu}|.(|\mathbf{D}_\sigma^{(A)}\Psi^{\alpha\beta}| + |\Psi(\nabla_\sigma e^\alpha, e^\beta)| + |\Psi(e^\alpha, \nabla_\sigma e^\beta)|)\end{aligned}$$

(due to (161)).

Choosing a normal frame (where the Christoffel symbols vanish at that point) to consider the contactions, using (162), and the fact that the metric is smooth, we get,

$$|\nabla_\sigma|\Psi|^2|(p) \quad \leq \quad C(p)[h_{\alpha\mu}h_{\beta\nu}|\mathbf{D}_\sigma^{(A)}\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}| + h_{\alpha\mu}h_{\beta\nu}|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|]$$

Using Cauchy-Schwarz, we obtain the desired estimate.

∎

We can compute,

$$\begin{aligned}\slashed{\mathbf{D}}_L^{(A)}(s^2\mathbf{D}_a^{(A)}\lambda_{\alpha\beta}) &= 2s\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + s^2\mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta}\\ &= s^2(2s^{-1}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta})\end{aligned}$$

Thus, using (55), we get

$$|\nabla_L|s^2\mathbf{D}_a^{(A)}\lambda|^2| \quad \lesssim \quad |s^2(2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda)|.|s^2\mathbf{D}_a^{(A)}\lambda| + |s^2\mathbf{D}_a^{(A)}\lambda|^2 \tag{56}$$

Therefore,

$$\overline{s}\int_{\mathbb{S}^2}\overline{s}^2|\mathbf{D}_a^{(A)}\lambda|^2 d\sigma^2$$

$$\lesssim \quad \int_{\mathbb{S}^2}\overline{s}^{-1}|\overline{s}^2\mathbf{D}_a^{(A)}\lambda|^2 d\sigma^2$$

$$\lesssim \quad \int_{\mathbb{S}^2}\overline{s}^{-1}\int_0^{\overline{s}}|\nabla_L|\overline{s}^2\mathbf{D}_a^{(A)}\lambda|^2|dsd\sigma^2 \tag{57}$$

$$\lesssim \quad \int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}}|s^2(2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda)|.|s^2\mathbf{D}_a^{(A)}\lambda| + |s^2\mathbf{D}_a^{(A)}\lambda|^2 ds]d\sigma^2$$

$$\lesssim \quad \int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}}\sup_{s\in[0,\overline{s}]}s^4|\mathbf{D}_a^{(A)}\lambda|^2 ds$$

$$+\overline{s}^{-1}\int_0^{\overline{s}}\epsilon^{-\frac{1}{2}}s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|.\epsilon^{\frac{1}{2}}s^{\frac{3}{2}}|\mathbf{D}_a^{(A)}\lambda|ds]d\sigma^2$$

$$\lesssim \quad \overline{s}^{-1}\sup_{s\in[0,\overline{s}]}(s)\int_{\mathbb{S}^2}\int_0^{\overline{s}}\sup_{s\in[0,\overline{s}]}(s^3|\mathbf{D}_a^{(A)}\lambda|^2)dsd\sigma^2 + \overline{s}^{-1}\epsilon\int_{\mathbb{S}^2}\int_0^{\overline{s}}\sup_{s\in[0,\overline{s}]}(s^3|\mathbf{D}_a^{(A)}\lambda|^2)dsd\sigma^2$$

$$+\frac{1}{\epsilon}\int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds]^2 d\sigma^2$$

$$\lesssim \quad \overline{s}\int_{\mathbb{S}^2}\sup_{s\in[0,\overline{s}]}(s^3|\mathbf{D}_a^{(A)}\lambda|^2)d\sigma^2 + \epsilon\int_{\mathbb{S}^2}\sup_{s\in[0,\overline{s}]}(s^3|\mathbf{D}_a^{(A)}\lambda|^2)d\sigma^2$$

$$+\frac{1}{\epsilon}\int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds]^2 d\sigma^2 \tag{58}$$

Taking the supremum in this last inequality on $\overline{s}\in[0,\hat{s}]$, where $\overline{s}$ and $\hat{s}$ are values of s, we get,

$$\sup_{\overline{s}\in[0,\hat{s}]}\int_{\mathbb{S}^2}\overline{s}^3|\mathbf{D}_a^{(A)}\lambda|^2(0,\overline{s},\omega)d\sigma^2$$

$$\lesssim \quad (\hat{s}+\epsilon)\sup_{\overline{s}\in[0,\hat{s}]}\int_{\mathbb{S}^2}\sup_{s\in[0,\overline{s}]}(s^3|\mathbf{D}_a^{(A)}\lambda|^2)d\sigma^2$$

$$+\sup_{\overline{s}\in[0,\hat{s}]}\int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds]^2 d\sigma^2$$

Choosing $\hat{s}$ and ϵ small enough depending on the space-time geometry on p, we obtain:

$$\sup_{\overline{s}\in[0,\hat{s}]}\int_{\mathbb{S}^2}\overline{s}^3|\mathbf{D}_a^{(A)}\lambda|^2(0,\overline{s},\omega)d\sigma^2$$

$$\lesssim \quad \sup_{\overline{s}\in[0,\hat{s}]}\int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds]^2 d\sigma^2 \tag{59}$$

We want to control

$$\sup_{\overline{s}\in[0,\hat{s}]}\int_{\mathbb{S}^2}[\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds]^2 d\sigma^2$$

$$= \quad ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds||_{L^2_\omega}^2$$

where the L^p_ω denotes the L^p norm on $s =$ constant, with the canonical induced volume form $d\sigma^2$ induced on $\mathbb{S}^2$.

$$||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds||_{L^2_\omega}^2$$

$$= \quad ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda - tr\chi\mathbf{D}_a^{(A)}\lambda + tr\chi\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds||_{L^2_\omega}^2$$

$$= \quad ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - tr\chi\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}(\nabla_a tr\chi)\lambda_{\alpha\beta}$$

$$-\frac{1}{2}\zeta_a tr\chi\lambda_{\alpha\beta} + [F_{La}, \lambda_{\alpha\beta}] + R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma}|_h ds||_{L^2_\omega}^2$$

where we used (52), and where $|\ |_h$ means that we consider a full contraction with respect to the metric h, in the indices α, β. Hence,

$$||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda + \mathbf{D}_L^{(A)}\mathbf{D}_a^{(A)}\lambda|ds||_{L^2_\omega}^2$$

$$\lesssim \quad ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|2s^{-1}\mathbf{D}_a^{(A)}\lambda - tr\chi\mathbf{D}_a^{(A)}\lambda - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda|ds||_{L^2_\omega}^2$$

$$+||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|-\frac{1}{2}(\nabla_a tr\chi)\lambda|ds||_{L^2_\omega}^2$$

$$+||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|-\frac{1}{2}\zeta_a tr\chi\lambda|ds||_{L^2_\omega}^2$$

$$+||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|[F_{La}, \lambda_{\alpha\beta}]|_h ds||_{L^2_\omega}^2$$

$$+||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}} s^{\frac{5}{2}}|R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma}|_h ds||_{L^2_\omega}^2$$

$$= \quad I_1 + I_2 + I_3 + I_4 + I_5 \tag{60}$$

where I_i , $i \in \{1, ..., 5\}$, are defined in order.

4.16.1. *Estimating I_1.*

$$I_1 = ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \int_0^{\overline{s}} s^{\frac{5}{2}} |2s^{-1}\mathbf{D}_a^{(A)}\lambda - tr\chi \mathbf{D}_a^{(A)}\lambda - \hat{\chi}_{ab}\mathbf{D}_b^{(A)}\lambda| ds||_{L_\omega^2}^2$$

$$\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \sup_{s\in[0,\overline{s}]} |s^{\frac{5}{2}}\mathbf{D}_a^{(A)}\lambda| \int_0^{\overline{s}} |2s^{-1} - tr\chi - \hat{\chi}_{ab}| ds||_{L_\omega^2}^2$$

(we remind that we were summing over a with abuse of notation)

$$\lesssim ||\sup_{s\in[0,\hat{s}]} (s^{\frac{3}{2}}\mathbf{D}_a^{(A)}\lambda)||_{L_\omega^2}^2 ||\sup_{\overline{s}\in[0,\hat{s}]} \int_0^{\overline{s}} |2s^{-1} - tr\chi - \hat{\chi}_{ab}| ds||_{L_\omega^\infty}^2$$

$$\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \int_0^{\overline{s}} 1.|2s^{-1} - tr\chi - \hat{\chi}_{ab}| ds||_{L_\omega^\infty}^2 ||\sup_{s\in[0,\hat{s}]} (s^{\frac{3}{2}}\mathbf{D}_a^{(A)}\lambda)||_{L_\omega^2}^2$$

$$\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s} \int_0^{\overline{s}} |2s^{-1} - tr\chi - \hat{\chi}_{ab}|^2 ds||_{L_\omega^\infty} ||\sup_{s\in[0,\hat{s}]} (s^{\frac{3}{2}}\mathbf{D}_a^{(A)}\lambda)||_{L_\omega^2}^2$$

$$\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \int_0^{\overline{s}} |2s^{-1} - tr\chi - \hat{\chi}_{ab}|^2 ds||_{L_\omega^\infty} ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}||_{L_\omega^\infty} ||\sup_{s\in[0,\hat{s}]} (s^{\frac{3}{2}}\mathbf{D}_a^{(A)}\lambda)||_{L_\omega^2}^2$$

$$\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \int_0^{\overline{s}} [|2s^{-1} - tr\chi|^2 + |\hat{\chi}_{ab}|^2] ds||_{L_\omega^\infty} ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}||_{L_\omega^\infty} ||\sup_{s\in[0,\hat{s}]} (s^{\frac{3}{2}}\mathbf{D}_a^{(A)}\lambda)||_{L_\omega^2}^2$$

We know that

$$|2s^{-1} - tr\chi| = O(s^2) \tag{61}$$

and

$$\int_0^{\overline{s}} |\hat{\chi}_{ab}|^2] ds \lesssim 1 \tag{62}$$

(see proposition 3.1 in [Wang]).

We get,

$$I_1 \lesssim \hat{s} ||\sup_{s\in[0,\hat{s}]} (s^{\frac{3}{2}}\mathbf{D}_a^{(A)}\lambda)||_{L_\omega^2}^2 \tag{63}$$

4.16.2. *Estimating I_2.*

$$
\begin{aligned}
I_2 &\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{5}{2}}|-\frac{1}{2}(\nabla_a tr\chi)\lambda|ds||_{L_\omega^2}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}\lambda||_{L^\infty}^2||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{3}{2}}|-\frac{1}{2}(\nabla_a tr\chi)|ds||_{L_\omega^2}^2 \\
&\lesssim |J|^2.||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}|\overline{s}\nabla_a tr\chi|\int_0^{\overline{s}}s^{\frac{1}{2}}ds||_{L_\omega^2}^2 \\
&\lesssim |J|^2.|\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\overline{s}^{\frac{3}{2}}|^2||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}\nabla_a tr\chi||_{L_\omega^2}^2 \\
&\lesssim \hat{s}||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}\nabla_a tr\chi||_{L_\omega^2}^2
\end{aligned}
$$

We have

$$||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}\nabla_a tr\chi||_{L_\omega^2}^2 \lesssim 1$$

(see proposition 3.2 in [Wang]).

We get,

$$I_2 \lesssim \hat{s} \tag{64}$$

4.16.3. *Estimating I_3.*

$$
\begin{aligned}
I_3 &= ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{5}{2}}|-\frac{1}{2}\zeta_a tr\chi\lambda|ds||_{L_\omega^2}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}\lambda||_{L_\omega^2}^2||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{3}{2}}|-\frac{1}{2}\zeta_a tr\chi|ds||_{L_\omega^2}^2 \\
&\lesssim |J|^2.||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}tr\chi||_{L_\omega^\infty}^2||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\int_0^{\overline{s}}s^{\frac{1}{2}}|\zeta_a|ds||_{L_\omega^2}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}(\int_0^{\overline{s}}sds)^{\frac{1}{2}}(\int_0^{\overline{s}}|\zeta_a|^2ds)^{\frac{1}{2}}||_{L_\omega^2}^2 \\
&\quad \text{(from (61))} \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]}\overline{s}^{-1}\overline{s}||_{L_\omega^\infty}^2\int_{\mathbb{S}^2}\int_0^{\hat{s}}|\zeta_a|^2dsd\sigma^2
\end{aligned}
$$

$$I_3 \lesssim \int_{\mathbb{S}^2}\int_0^{\hat{s}}|\zeta_a|^2dsd\sigma^2 \tag{65}$$

4.16.4. *Estimating* I_4.

$$\begin{aligned}
I_4 &= ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \int_0^{\overline{s}} s^{\frac{5}{2}} |[F_{La}, \lambda_{\alpha\beta}]|_h ds||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \int_0^{\overline{s}} s^{\frac{5}{2}} s^{-1} |[F_{La}, s\lambda_{\alpha\beta}]|_h ds||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} |\overline{s}\lambda|_h||_{L^\infty_\omega}^2 ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1}\overline{s}^{\frac{1}{2}} \int_0^{\overline{s}} 1.s|F_{La}|ds||_{L^2_\omega}^2 \\
&\lesssim |J|^2 ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-\frac{1}{2}}\overline{s}^{\frac{1}{2}} (\int_0^{\overline{s}} s^2|F_{La}|^2 ds)^{\frac{1}{2}}||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} (\int_0^{\overline{s}} s^2|F_{La}|^2 ds)^{\frac{1}{2}}||_{L^2_\omega}^2
\end{aligned}$$

$$I_4 \lesssim \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 \tag{66}$$

4.16.5. *Estimating* I_5.

$$\begin{aligned}
I_5 &= ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \int_0^{\overline{s}} s^{\frac{5}{2}} |R_\alpha{}^\gamma{}_{La}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{La}\lambda_{\alpha\gamma}|_h ds||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \int_0^{\overline{s}} s^{\frac{5}{2}} |h^{\sigma\gamma}|.|R_{\alpha\sigma La}\lambda_{\gamma\beta} + R_{\beta\sigma La}\lambda_{\alpha\gamma}|_h ds||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} |\overline{s}\lambda|_h||_{L^\infty_\omega}^2 ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1} \int_0^{\overline{s}} s^{\frac{3}{2}} |R_{\alpha\beta La}|_h ds||_{L^2_\omega}^2 \\
&\lesssim |J|^2.||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-1}\overline{s}^{\frac{1}{2}} \int_0^{\overline{s}} s|R_{\alpha\beta La}|_h ds||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-\frac{1}{2}} \int_0^{\overline{s}} 1.s|R_{\alpha\beta La}|_h ds||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} \overline{s}^{-\frac{1}{2}}\overline{s}^{\frac{1}{2}} [\int_0^{\overline{s}} s^2|R_{\alpha\beta La}|_h^2 ds]^{\frac{1}{2}}||_{L^2_\omega}^2 \\
&\lesssim ||\sup_{\overline{s}\in[0,\hat{s}]} [\int_0^{\overline{s}} s^2|R_{\alpha\beta La}|_h ds]^{\frac{1}{2}}||_{L^2_\omega}^2
\end{aligned}$$

$$I_5 \lesssim \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \tag{67}$$

4.16.6. *Estimating* $||\mathbf{D}_a^{(A)}\lambda||_{L^2(N_\tau^-(p))}$.

Injecting (63), (64), (65), (66), (67) in (60), and then in (59), we obtain:

$$\sup_{\overline{s} \in [0,\hat{s}]} \int_{\mathbb{S}^2} \overline{s}^3 |\mathbf{D}_a^{(A)} \lambda|^2 (0, \overline{s}, \omega) d\sigma^2$$

$$\lesssim \quad \hat{s} \| \sup_{s \in [0,\hat{s}]} |s^{\frac{3}{2}} \mathbf{D}_a^{(A)} \lambda|_h \|_{L^2_\omega}^2 + \hat{s} + \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2$$

$$+ \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2$$

There exists $C(p)$ (constant depending on p) such that for $\hat{s} \lesssim C(p)$, we have:

$$\sup_{\overline{s} \in [0,\hat{s}]} \int_{\mathbb{S}^2} \overline{s}^3 |\mathbf{D}_a^{(A)} \lambda|^2 (0, \overline{s}, \omega) d\sigma^2$$

$$\lesssim \quad \hat{s} + \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2$$

$$+ \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2$$

For $\hat{s} \lesssim C(p)$, we have,

$$\hat{s} \int_{\mathbb{S}^2} \hat{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2 (0, \hat{s}, \omega) d\sigma^2$$

$$\lesssim \quad \sup_{\overline{s} \in [0,\hat{s}]} \int_{\mathbb{S}^2} \overline{s}^3 |\mathbf{D}_a^{(A)} \lambda|^2 (0, \overline{s}, \omega) d\sigma^2$$

$$\lesssim \quad \hat{s} + \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 + \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2$$

Hence,

$$\int_{\mathbb{S}^2} \hat{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2 (0, \hat{s}, \omega) d\sigma^2$$

$$\lesssim \quad 1 + \frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2$$

$$+ \frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2 + \frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2 \qquad (68)$$

Integrating, we obtain

$$\int_0^{C(p)} \int_{\mathbb{S}^2} \hat{s}^2 |\mathbf{D}_a^{(A)} \lambda|^2 (0, \hat{s}, \omega) d\sigma^2 d\hat{s}$$

$$\lesssim \int_0^{C(p)} 1 d\hat{s} + \int_0^{C(p)} [\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2] d\hat{s}$$

$$+ \int_0^{C(p)} [\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2] d\hat{s} + \int_0^{C(p)} [\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |R_{\alpha\beta La}|_h^2 ds d\sigma^2] d\hat{s}$$

$$\lesssim 1 + (\int_0^{C(p)} [\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}} + (\int_0^{C(p)} [\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}}$$

$$+ (\int_0^{C(p)} [\frac{1}{\hat{s}} \int_{\mathbb{S}^2} \int_0^{\hat{s}} (s^2 |R_{\alpha\beta La}|_h^2)^{\frac{1}{2}} ds d\sigma^2]^2 d\hat{s} \tag{69}$$

Using the L^2 maximum principle, and letting $t(C(p))$ be the value of t for which $s(t) = C(p)$, we get in view of (154):

$$||\mathbf{D}_a^{(A)} \lambda||_{L^2(N_{t(C(p))}^-(p))}$$

$$\lesssim 1 + (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{\hat{s}} |\zeta_a|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}} + (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{\hat{s}} s^2 |F_{La}|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}}$$

$$+ (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{\hat{s}} (s^2 |R_{\alpha\beta La}|_h^2)^{\frac{1}{2}} ds d\sigma^2]^2 d\hat{s}$$

$$\lesssim 1 + (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{s_\tau} |\zeta_a|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}} + (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{s_\tau} s^2 |F_{La}|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}}$$

$$+ (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{s_\tau} (s^2 |R_{\alpha\beta La}|_h^2)^{\frac{1}{2}} ds d\sigma^2]^2 d\hat{s}$$

$$\lesssim 1 + (\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{s_\tau} |\zeta_a|^2 ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}} + [C(p)(F^{\frac{\partial}{\partial t}}(N_\tau^-(p)))^2]^{\frac{1}{2}} \tag{70}$$

(since the metric is smooth).

We have,

$$(\int_0^{C(p)} [\int_{\mathbb{S}^2} \int_0^{s_\tau} (|\zeta_a|^2) ds d\sigma^2]^2 d\hat{s})^{\frac{1}{2}} \lesssim 1 \tag{71}$$

(see proposition 3.1 in the Appendix of [Wang]) .

Thus,

$$||\mathbf{D}_a^{(A)} \lambda||_{L^2(N_{t(C(p))}^-(p))} \lesssim 1$$

Since the metric is smooth, $\lambda_{\alpha\beta}$ is smooth away from $s = 0$; we finally obtain:

$$||\mathbf{D}_a^{(A)}\lambda||_{L^2(N_\tau^-(p))} \lesssim 1 \qquad (72)$$

4.17. **Estimates for** $||\mathbf{D}_a^{(A)}F||_{L^2(N_\tau^-(p))}$.

We want to control $||\mathbf{D}^{(A)}{}_aF||_{L^2(N_\tau^-(p))}$. For this, as in [CS], we take the energy momentum tensor for the wave equation, after considering a full contraction with respect to the Riemannian metric h, and define the 2-tensor:

$$T_1^{\alpha\beta} = h^{\mu\nu}h^{\rho\sigma}[< \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > -\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >] \quad (73)$$

Let $\hat{t}_\beta = (\frac{\partial}{\partial t})_\beta$

We have

$$T_1^{\alpha\beta}\hat{t}_\beta = h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[< \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > -\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >]$$

We would like to compute $\nabla_\alpha(T_1^{\alpha\beta}\hat{t}_\beta)$. Since it is a full contraction, we can compute it by choosing a normal frame , i.e. a frame where the Christoffel sympbols vanish at that point, and hence we can get the derivatives inside the scalar product as covariant derivatives and also as gauge covariant derivatives using the fact that the scalar product is Ad-invariant, instead of partial derivatives. We obatin

$$\nabla_\alpha(T_1^{\alpha\beta}\hat{t}_\beta)$$
$$= \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[< \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > -\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >]$$
$$+ h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[< \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > + < \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\beta}F_{\nu\sigma} >$$
$$-\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >] - \frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\alpha^{(A)}\mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >]$$

We have

$$-\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}_\alpha^{(A)}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >= -\frac{1}{2} < \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >$$

Computing

$$h^{\mu\nu}h^{\rho\sigma} < \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}_\lambda^{(A)}F_{\nu\sigma} >= h^{\rho\sigma}h^{\mu\nu} < \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\nu\sigma}, \mathbf{D}_\lambda^{(A)}F_{\mu\rho} >$$

We get

$$\nabla_\alpha(T_1^{\alpha\beta}\hat{t}_\beta)$$

$$= \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[< \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > -\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}^{(A)}_\lambda F_{\nu\sigma} >]$$

$$+ h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[< \mathbf{D}^{(A)}_\alpha \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > + < \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)}_\alpha \mathbf{D}^{(A)\beta}F_{\nu\sigma} >$$

$$- < \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)\lambda}F_{\nu\sigma}, \mathbf{D}^{(A)}_\lambda F_{\mu\rho} >]$$

$$= \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[< \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > -\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}^{(A)}_\lambda F_{\nu\sigma} >]$$

$$+ h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[< \Box^{(A)}_{\mathbf{g}} F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} >$$

$$+ < \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)}_\alpha \mathbf{D}^{(A)\beta}F_{\nu\sigma} - \mathbf{D}^{(A)\beta}\mathbf{D}^{(A)}_\alpha F_{\nu\sigma} >]$$

$$= \nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)[< \mathbf{D}^{(A)\alpha}F_{\mu\rho}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > -\frac{1}{2}\mathbf{g}^{\alpha\beta} < \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}^{(A)}_\lambda F_{\nu\sigma} >]$$

$$+ h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta[-2 < R_{\gamma\mu\rho\alpha}F^{\alpha\gamma}, \mathbf{D}^{(A)\beta}F_{\nu\sigma} >$$

$$- < R_{\mu\gamma}F_\rho{}^\gamma, \mathbf{D}^{(A)\beta}F_{\nu\sigma} > - < R_{\rho\gamma}F^\gamma{}_\mu, \mathbf{D}^{(A)\beta}F_{\nu\sigma} >$$

$$-2 < [F^\alpha{}_\mu, F_{\rho\alpha}], \mathbf{D}^{(A)\beta}F_{\nu\sigma} >$$

$$+ < \mathbf{D}^{(A)\alpha}F_{\mu\rho}, [F_\alpha{}^\beta, F_{\nu\sigma}] + R_\nu{}^\gamma{}_\alpha{}^\beta F_{\gamma\sigma} + R_\sigma{}^\gamma{}_\alpha{}^\beta F_{\nu\gamma} >]$$

(where we used (19)).

From (29), we have

$$\hat{t} = -\frac{1}{2}(\mathbf{g}(L,\hat{t})\underline{L} + \mathbf{g}(L,\hat{t})^{-1}L)$$

we get,

$$T_1^{\hat{t}L} = -\frac{1}{2}\mathbf{g}(L,\hat{t})T^{\underline{L}L} - \frac{1}{2}\mathbf{g}(L,\hat{t})^{-1}T^{LL} \tag{74}$$

Computing

$$< \mathbf{D}^{(A)\lambda}F_{\mu\rho}, \mathbf{D}^{(A)}_\lambda F_{\nu\sigma} >$$

$$= < \mathbf{D}^{(A)L}F_{\mu\rho}, \mathbf{D}^{(A)}_L F_{\nu\sigma} > + < \mathbf{D}^{(A)\underline{L}}F_{\mu\rho}, \mathbf{D}^{(A)}_{\underline{L}} F_{\nu\sigma} >$$

$$+ < \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}^{(A)}_a F_{\nu\sigma} >$$

$$= \mathbf{g}_{L\underline{L}} < \mathbf{D}^{(A)L}F_{\mu\rho}, \mathbf{D}^{(A)\underline{L}}F_{\nu\sigma} > + \mathbf{g}_{L\underline{L}} < \mathbf{D}^{(A)\underline{L}}F_{\mu\rho}, \mathbf{D}^{(A)L}F_{\nu\sigma} >$$

$$+ < \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}^{(A)}_a F_{\nu\sigma} >$$

$$= -4 < \mathbf{D}^{(A)L}F_{\mu\rho}, \mathbf{D}^{(A)\underline{L}}F_{\nu\sigma} > + < \mathbf{D}^{(A)a}F_{\mu\rho}, \mathbf{D}^{(A)}_a F_{\nu\sigma} >$$

(using (33)).

We get,

$$
\begin{aligned}
T_1^{\underline{L}L} & \\
&= h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)\underline{L}} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} > -\frac{1}{2} \mathbf{g}^{\underline{L}L} < \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_\lambda^{(A)} F_{\nu\sigma} >] \\
&= h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)\underline{L}} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} > +\frac{1}{4}(-4 < \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)\underline{L}} F_{\nu\sigma} > \\
& \quad + < \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} >)] \\
&= \frac{1}{4} h^{\mu\nu} h^{\rho\sigma} < \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} >
\end{aligned}
\tag{75}
$$

and we have,

$$
\begin{aligned}
T_1^{LL} &= h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} > -\frac{1}{2} \mathbf{g}^{LL} < \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_\lambda^{(A)} F_{\nu\sigma} >] \\
&= h^{\mu\nu} h^{\rho\sigma} < \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} >
\end{aligned}
\tag{76}
$$

Injecting (75) and (76) in (74), we obtain

$$
\begin{aligned}
T_1^{\hat{t}L} &= -\frac{1}{2} \mathbf{g}(L,\hat{t}) T^{\underline{L}L} - \frac{1}{2} \mathbf{g}(L,\hat{t})^{-1} T^{LL} \\
&= -\frac{1}{2} [\frac{1}{4} \mathbf{g}(L,\hat{t}) h^{\mu\nu} h^{\rho\sigma} < \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} > \\
& \quad + \mathbf{g}(L,\hat{t})^{-1} h^{\mu\nu} h^{\rho\sigma} < \mathbf{D}^{(A)L} F_{\mu\rho}, \mathbf{D}^{(A)L} F_{\nu\sigma} >] \\
&= -\frac{1}{2} [\frac{1}{4} \mathbf{g}(L,\hat{t}) |\mathbf{D}^{(A)a} F|^2 + \mathbf{g}(L,\hat{t})^{-1} |\mathbf{D}^{(A)L} F|^2]
\end{aligned}
\tag{77}
$$

On the other hand, we have

$$
\begin{aligned}
T_1^{\hat{t}\hat{t}} &= h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)\hat{t}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}} F_{\nu\sigma} > -\frac{1}{2} \mathbf{g}^{\hat{t}\hat{t}} < \mathbf{D}^{(A)\lambda} F_{\mu\rho}, \mathbf{D}_\lambda^{(A)} F_{\nu\sigma} >] \\
&= h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)\hat{t}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}} F_{\nu\sigma} > +\frac{1}{2}(< \mathbf{D}^{(A)\hat{t}} F_{\mu\rho}, \mathbf{D}_{\hat{t}}^{(A)} F_{\nu\sigma} > \\
& \quad + < \mathbf{D}^{(A)\hat{n}} F_{\mu\rho}, \mathbf{D}_{\hat{n}}^{(A)} F_{\nu\sigma} > + < \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}_a^{(A)} F_{\nu\sigma} >)] \\
&= h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)\hat{t}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}} F_{\nu\sigma} > -\frac{1}{2} < \mathbf{D}^{(A)\hat{t}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}} F_{\nu\sigma} > \\
& \quad +\frac{1}{2} < \mathbf{D}^{(A)\hat{n}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{n}} F_{\nu\sigma} > +\frac{1}{2} < \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}^{(A)a} F_{\nu\sigma} >)] \\
&= \frac{1}{2} h^{\mu\nu} h^{\rho\sigma} [< \mathbf{D}^{(A)\hat{t}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{t}} F_{\nu\sigma} > + < \mathbf{D}^{(A)\hat{n}} F_{\mu\rho}, \mathbf{D}^{(A)\hat{n}} F_{\nu\sigma} > \\
& \quad + < \mathbf{D}^{(A)a} F_{\mu\rho}, \mathbf{D}^{(A)a} F_{\nu\sigma} >] \\
&= \frac{1}{2} [|\mathbf{D}^{(A)\hat{t}} F|^2 + |\mathbf{D}^{(A)\hat{n}} F|^2 + |\mathbf{D}^{(A)a} F|^2]
\end{aligned}
\tag{78}
$$

Denoting by $N_{t_p-\tau,t}^-(p)$ the portion of $N^-(p)$ that is to the future of $\Sigma_{t_p-\tau}$ and to the past of Σ_t. Denoting the gradient of F by $\mathbf{D}^{(A)}F$, and defining,

$$|\mathbf{D}^{(A)}F|^2 \;=\; h^{\alpha\beta}h^{\mu\gamma}h^{\nu\sigma} <\mathbf{D}_\alpha^{(A)}F_{\gamma\sigma}, \mathbf{D}_\beta^{(A)}F_{\mu\nu}> \tag{79}$$

Applying the divergence theorem to $T_1^{\alpha\beta}\hat{t}_\beta$ in $J^-(p)\cap\Sigma_{t_p-\tau}^+\cap\Sigma_t^-$, using (74) and the fact that the metric is sufficiently smooth so that $\nabla_\alpha(h^{\mu\nu}h^{\rho\sigma}\hat{t}_\beta)$ is finite, using (77), (78), and applying Cauchy-Schwarz, we obtain:

$$
\begin{aligned}
&||\mathbf{D}^{(A)}F||_{L^2(\Sigma_t\cap J^-(p))}^2 + ||\mathbf{D}_a^{(A)}F||_{L^2(N_{t_p-\tau,t}^-(p))}^2 \\
\lesssim\;\; &||\mathbf{D}^{(A)}F||_{L^2(\Sigma_{t_p-\tau}\cap J^-(p))}^2 \\
&+ \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} |\mathbf{D}^{(A)}F|(|\mathbf{D}^{(A)}F|+|F|+|F|^2)dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t
\end{aligned}
\tag{80}
$$

We get,

$$
\begin{aligned}
&\int_{\Sigma_t\cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_t\cap J^-(p)} + ||\mathbf{D}_a^{(A)}F||_{L^2(N_{t_p-\tau,t}^-(p))}^2 \\
\lesssim\;\; &||\mathbf{D}^{(A)}F||_{L^2(\Sigma_{t_p-\tau}\cap J^-(p))}^2 + \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t \\
&+ \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} |\mathbf{D}^{(A)}F|(|F|^2+|F|)dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t \\
\lesssim\;\; &||\mathbf{D}^{(A)}F||_{L^2(\Sigma_{t_p-\tau}\cap J^-(p))}^2 + \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t \\
&+ \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} (|\mathbf{D}^{(A)}F|^2+|F|^4+|F|^2)dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t \\
\lesssim\;\; &||\mathbf{D}^{(A)}F||_{L^2(\Sigma_{t_p-\tau}\cap J^-(p))}^2 + \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} (|F|^4+|F|^2)dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t \\
&+ \int_{t_p-\tau}^t \int_{\Sigma_{\overline t}\cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline t}\cap J^-(p)}d\overline t
\end{aligned}
$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}} \cap J^-(p)} |F|^4 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t}$$

$$+ \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}} \cap J^-(p)} |F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t} + \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}} \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t}$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^{t} (|F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} \int_{\Sigma_{\overline{t}}} |F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)}) d\overline{t}$$

$$+ \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}}} |F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t} + \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}} \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t}$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} E_{t=0}^{\frac{\partial}{\partial t}}(\Sigma \cap J^-(p)) d\overline{t} + \int_{t_p-\tau}^{t} E_{t=0}^{\frac{\partial}{\partial t}}(\Sigma \cap J^-(p)) d\overline{t}$$

$$+ \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}} \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t} \tag{81}$$

From (81), we get

$$\int_{\Sigma_t \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_t \cap J^-(p)}$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} + \int_{t_p-\tau}^{t} (\int_{\Sigma_{\overline{t}} \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)}) d\overline{t}$$

Using Grönwall lemma, we obtain

$$\int_{\Sigma_t \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_t \cap J^-(p)} \lesssim C(t) + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} \tag{82}$$

where $C(t)$ is a finite constant that depends on t.

Injecting (82) in (81), we have

$$||\mathbf{D}_a^{(A)}F||^2_{L^2(N^-_{t_p-\tau,t}(p))}$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} + \int_{t_p-\tau}^{t} \int_{\Sigma_{\overline{t}} \cap J^-(p)} |\mathbf{D}^{(A)}F|^2 dVol_{\Sigma_{\overline{t}} \cap J^-(p)} d\overline{t}$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} + \int_{t_p-\tau}^{t} (C(t) + \int_{t_p-\tau}^{t^*} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t}) dt^*$$

$$\lesssim \quad c(t) + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} + \int_{t_p-\tau}^{t} \int_{t_p-\tau}^{t^*} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} dt^*$$

Finally,

$$||\mathbf{D}_a^{(A)}F||^2_{L^2(N^-_{t_p-\tau,t}(p))} \lesssim c(t) + \int_{t_p-\tau}^{t} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} + \int_{t_p-\tau}^{t} \int_{t_p-\tau}^{t^*} |F|^2_{L^\infty_{\Sigma_{\overline{t}}^p}} d\overline{t} dt^* \tag{83}$$

4.18. **The proof.**

Let $p \in \Sigma_{t_p}$.

Let $q \in \Sigma_t$ where $t_p - \tau \le t \le t_p$.

Let $\Omega_q = J^-(q) \cap J^+(\Sigma_{t_p - \tau})$.

Let $\Sigma_t^p = \Sigma_t \cap J^-(p)$.

Using an adaptation of the Klainerman-Rodnianski parametrix in [KR1] to the Yang-Mills setting, see Appendix (198), we have

$$4\pi < \mathbf{J}_{\alpha\beta}, F^{\alpha\beta} > (q)$$

$$= -\int_{\Omega_q} < \lambda_{\alpha\beta}\delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > + \int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta} + 2\zeta_a \mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta}$$

$$+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} > + C_{t_p - \tau}$$

where $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ is the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$, of $\lambda_{\alpha\beta}$, as in (144), and where $C_{t_p - \tau}$ depends on the value of F on $\Sigma_{t_p - \tau}$. Let $\hat{\mu}, \hat{\nu} \in \{\hat{t}, n, e_a, e_b\}$. Hence,

$$|F_{\hat{\mu}\hat{\nu}}(q)|$$

$$\lesssim \int_{\Omega_q} | < \lambda_{\alpha\beta}\delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > | + \int_{\Omega_q} |\delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > |$$

$$+ \int_{\Omega_q} |\delta(u) < \zeta_a \mathbf{D}_a^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > | + \int_{\Omega_q} |\delta(u) < \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta}, F^{\alpha\beta} > |$$

$$+ \int_{\Omega_q} |\delta(u) < [F_{L\underline{L}}, \lambda_{\alpha\beta}], F^{\alpha\beta} > | + \int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta}, F^{\alpha\beta} > |$$

$$+ \int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} > | + C_{t_p - \tau}$$

We have,

$$\int_{\Omega_q} | < \lambda_{\alpha\beta}\delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > | = \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), \square_{\mathbf{g}}^{(A)} F_{\mu\nu} > |$$

$$= \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), -2R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} - R_{\mu\gamma}F_\nu{}^\gamma - R_{\nu\gamma}F^\gamma{}_\mu - 2[F^\alpha{}_\mu, F_{\nu\alpha}] >$$

$$(\text{using } (19))$$

$$\lesssim \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} > | + \int_{\Omega_q} | < \lambda\delta(u), R_{\mu\gamma}F_\nu{}^\gamma > |$$

$$+ \int_{\Omega_q} | < \lambda\delta(u), R_{\nu\gamma}F^\gamma{}_\mu > | + \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] > |$$

Finally,

$$\sup_{q \in \Sigma_t \cap J^-(p)} |F_{\hat{\mu}\hat{\nu}}(q)|$$

$$\lesssim \sup_{q \in \Sigma_t \cap J^-(p)} [\int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} > | + \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\mu\gamma}F_\nu{}^\gamma > |$$

$$+ \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\nu\gamma}F^\gamma{}_\mu > | + \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] >$$

$$+ | \int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > | + \int_{\Omega_q} |\delta(u) < \zeta_a \mathbf{D}_a^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > |$$

$$+ | \int_{\Omega_q} \delta(u) < \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta}, F^{\alpha\beta} > | + \int_{\Omega_q} |\delta(u) < [F_{L\underline{L}}, \lambda_{\alpha\beta}], F^{\alpha\beta} > |$$

$$+ | \int_{\Omega_q} \delta(u) < \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta}, F^{\alpha\beta} > | + \int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} > |]$$

$$+ C_{t_p - \tau} \tag{84}$$

In what follows, we note $\Sigma_t \cap J^-(p)$ as Σ_t^p.

Lemma 4.19. *We have,*

$$\sup_{q \in \Sigma_t^p} [\int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} > | + \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\mu\gamma}F_\nu{}^\gamma > |$$

$$+ \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\nu\gamma}F^\gamma{}_\mu > | + \int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta}, F^{\alpha\beta} > |$$

$$+ \int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} > |]$$

$$\lesssim \tau^{\frac{3}{2}} + [\int_{t_p-\tau}^t ||F||_{L^\infty(\Sigma_t^p)}^2 dt]\tau^{\frac{3}{2}}$$

Proof

We have,

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} > |$$

$$\lesssim \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \int_{S_t} | < s\lambda^{\mu\nu}, R_{\gamma\mu\nu\alpha}s^{-1}F^{\alpha\gamma} > |\phi da_t dt$$

$$\lesssim \sup_{q \in \Sigma_t^p} ||\phi||_{L^\infty(N_{t_p-\tau,t}^-(q))} \sup_{q \in \Sigma_t^p} ||s\lambda||_{L^\infty(N_{t_p-\tau,t}^-(q))} \sup_{q \in \Sigma_t^p} ||R||_{L^\infty(N_{t_p-\tau,t}^-(q))}$$

$$\cdot \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t (||F||_{L^\infty(\Sigma_t)} \int_{S_t} s^{-1} da_t) dt.$$

We have,

$$\sup_{q \in \Sigma_t^p} ||R||_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim 1$$

(since the metric is smooth)

$$||s\lambda||_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim |J|$$

thus

$$\sup_{q \in \Sigma_t} ||s\lambda||_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim 1$$

$$\sup_{q \in \Sigma_t^p} ||\phi||_{L^\infty(N_{t_p-\tau,t}^-(q))} \lesssim 1$$

(since ϕ is smooth and bounded)

$$\sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t (||F||_{L^\infty(\Sigma_t^p)} \int_{S_t} s^{-1} da_t) dt$$

$$\lesssim \sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (||F||_{L^\infty(\Sigma_t^p)}^2) dt]^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (\int_{S_t} s^{-1} da_t)^2 dt]^{\frac{1}{2}} \qquad (85)$$

We get,

$$\int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} > | \lesssim [\int_{t_p-\tau}^t (||F||_{L^\infty(\Sigma_t)}^2) dt]^{\frac{1}{2}} \sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (\int_{S_t} s^{-1} da_t)^2 dt]^{\frac{1}{2}}$$

$$[\int_{t_p-\tau}^t (||F||_{L^\infty(\Sigma_t)}^2) dt]^{\frac{1}{2}} \lesssim 1 + \int_{t_p-\tau}^t ||F||_{L^\infty(\Sigma_t)}^2 dt$$

Recall that

$$A_t(p) = O((t_p - t)^2)$$

(see (154)), and

$$t_p - t = s + o(s)$$

(see (159)).

Thus,

$$A_t(p) = O(s^2)$$

We get,

$$\sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (\int_{S_t} s^{-1}(t) da_t)^2 dt]^{\frac{1}{2}} \lesssim [\int_{t_p-\tau}^{t_p} s_p(t)^2 dt]^{\frac{1}{2}} \lesssim \tau^{\frac{3}{2}}$$

Thus,

$$\int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\gamma\mu\nu\alpha}F^{\alpha\gamma} > | \lesssim \tau^{\frac{3}{2}} + [\int_{t_p-\tau}^{t} ||F||^2_{L^\infty(\Sigma_t^p)}dt]\tau^{\frac{3}{2}} \qquad (86)$$

In the same manner this controls the terms

$$\int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\mu\gamma}F_\nu{}^\gamma > |, \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), R_{\nu\gamma}F^\gamma{}_\mu > |,$$

$$\int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta}, F^{\alpha\beta} > |, \text{and} \int_{\Omega_q} |\delta(u) < \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} > |$$

■

Lemma 4.20. *We have,*

$$\sup_{q\in\Sigma_t^p} \int_{\Omega_q} |\delta(u) < \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta}, F^{\alpha\beta} > | \lesssim \tau^{\frac{1}{2}} + [\int_{t_p-\tau}^{t} ||F||^2_{L^\infty(\Sigma_t^p)}dt]\tau^{\frac{1}{2}}$$

Proof

The term

$$\sup_{q\in\Sigma_t^p} \int_{\Omega_q} |\delta(u) < \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta}, F^{\alpha\beta} > | \lesssim \sup_{q\in\Sigma_t^p} \int_{t_p-\tau}^{t} \int_{S_t} | < \frac{1}{2}\hat{\mu}s\lambda_{\alpha\beta}, s^{-1}F^{\alpha\beta} > |\phi da_t dt$$

$$\lesssim \sup_{q\in\Sigma_t^p} ||s\lambda||_{L^\infty(N_{t_p-\tau,t}^-(q))} \sup_{q\in\Sigma_t^p} \int_{t_p-\tau}^{t} (||F||_{L^\infty(\Sigma_t^p)} \int_{S_t} |\hat{\mu}|s^{-1}\phi da_t)dt$$

We get,

$$\sup_{q\in\Sigma_t^p} \int_{\Omega_q} |\delta(u) < \frac{1}{2}\hat{\mu}\lambda_{\alpha\beta}, F^{\alpha\beta} > | \lesssim [\int_{t_p-\tau}^{t} (||F||^2_{L^\infty(\Sigma_t^p)})dt]^{\frac{1}{2}} \sup_{q\in\Sigma_t^p} [\int_{t_p-\tau}^{t} (\int_{S_t} |\hat{\mu}|s^{-1}\phi da_t)^2 dt]^{\frac{1}{2}}$$

And we have

$$\sup_{q\in\Sigma_t^p} \int_{t_p-\tau}^{t} (\int_{S_t} |\hat{\mu}|s^{-1}\phi da_t)^2 dt \lesssim \int_{t_p-\tau}^{t} \sup_{q\in\Sigma_t^p} (\int_{S_t} |\hat{\mu}|^2\phi da_t)(\int_{S_t} s^{-2}\phi da_t)dt$$

We have $\hat{\mu} = o(s^{-1})$ (see proposition 3.1 in [Wang])

Thus,

$$(\sup_{q\in\Sigma_t^p} \int_{S_t} |\hat{\mu}|^2\phi da_t) \lesssim \sup_{q\in\Sigma_t^p} ||\phi||_{L^\infty(N_{t_p-\tau,t}^-(q))} (\sup_{q\in\Sigma_t^p} \int_{S_t} s^{-2}da_t) \lesssim 1$$

$$(\sup_{q\in\Sigma_t^p} \int_{S_t} s^{-2}\phi da_t) \lesssim 1$$

Thus,

$$\sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (\int_{S_t} |\hat{\mu}| s^{-1} \phi da_t)^2 dt]^{\frac{1}{2}} \lesssim \tau^{\frac{1}{2}}$$

We obtain,

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} |\delta(u) < \frac{1}{2} \hat{\mu} \lambda_{\alpha\beta}, F^{\alpha\beta} >| \lesssim \tau^{\frac{1}{2}} + [\int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_t^p)} dt] \tau^{\frac{1}{2}} \qquad (87)$$

∎

Next, we want to control the term

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | < \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] >| = \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \int_{S_t} | < \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] > \phi da_t) dt$$

Lemma 4.21. *We have,*

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | < \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] > \lesssim (\tau)^{\frac{1}{2}} + [\int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_t^p)} dt](\tau)^{\frac{1}{2}}$$

Proof

Following the remark of Eardley and Moncrief in [EM2], we have

$$|[F^\alpha{}_\mu, F_{\nu\alpha}]| \lesssim ||F||_{L^\infty(S_t)} (|F_{L\underline{L}}| + |F_{La}| + |F_{Lb}| + |F_{ab}|)$$

on $N^-_{t_p-\tau,t}(q) \cap \Sigma_t = S_t$, and therefore,

$$\sup_{q \in \Sigma_t^p} \int_{\Omega_q} | < \lambda^{\mu\nu} \delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] >$$

$$\lesssim \sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t \int_{S_t} | < s\lambda^{\mu\nu} \delta(u), [s^{-1} F^\alpha{}_\mu, F_{\nu\alpha}] > \phi da_t) dt$$

$$\lesssim \sup_{q \in \Sigma_t^p} ||\phi||_{L^\infty(N^-_{t_p-\tau,t}(q))} \sup_{q \in \Sigma_t^p} ||s\lambda||_{L^\infty(N^-_{t_p-\tau,t}(q))}$$

$$\sup_{q \in \Sigma_t^p} \int_{t_p-\tau}^t ||F||_{L^\infty(\Sigma_t)} \int_{S_t} (s^{-1}|F_{L\underline{L}}| + s^{-1}|F_{La}| + s^{-1}|F_{Lb}| + s^{-1}|F_{ab}|) dt da_t$$

$$\lesssim [\int_{t_p-\tau}^t (||F||^2_{L^\infty(\Sigma_t^p)}) dt]^{\frac{1}{2}}$$

$$. \sup_{q \in \Sigma_t^p} [\int_{t_p-\tau}^t (\int_{S_t} (s^{-1}|F_{L\underline{L}}| + s^{-1}|F_{La}| + s^{-1}|F_{Lb}| + s^{-1}|F_{ab}|) da_t)^2 dt]^{\frac{1}{2}}$$

(since ϕ is smooth and bounded near p).

And we have,

$$[\int_{S_t} (s^{-1}|F_{L\underline{L}}| + s^{-1}|F_{La}| + s^{-1}|F_{Lb}| + s^{-1}|F_{ab}|)da_t]^2$$

$$\lesssim \quad (\int_{S_t} (s^{-2})da_t)(\int_{S_t} (|F_{L\underline{L}}|^2 + |F_{La}|^2 + |F_{Lb}|^2 + |F_{ab}|^2)da_t)$$

(by Cauchy-Schwarz inequality)

and

$$F^{\frac{\partial}{\partial t}}(N_\tau^-(q)) = \int_{N_\tau^-(q)} \frac{1}{8}|F_{L\underline{L}}|^2 + \frac{1}{2}|F_{Le_a}|^2 + \frac{1}{2}|F_{Le_b}|^2 + \frac{1}{2}|F_{ab}|^2$$

Thus,

$$\int_{t_p-\tau}^{t} \int_{S_t} (s^{-1}|F_{L\underline{L}}| + s^{-1}|F_{La}| + s^{-1}|F_{Lb}| + s^{-1}|F_{ab}|)dtda_t$$

$$\lesssim \quad (\int_{t_p-\tau}^{t} 1dt)^{\frac{1}{2}}(F^{\frac{\partial}{\partial t}}(N_{t_p-\tau,t}^-(q)))^{\frac{1}{2}}$$

$$\lesssim \quad (\tau)^{\frac{1}{2}}(E_{t=0}^{\frac{\partial}{\partial t}})^{\frac{1}{2}} \tag{88}$$

Thus,

$$\sup_{q\in\Sigma_t^p} \int_{\Omega_q} | < \lambda^{\mu\nu}\delta(u), [F^\alpha{}_\mu, F_{\nu\alpha}] > \lesssim (\tau)^{\frac{1}{2}} + [\int_{t_p-\tau}^{t} ||F||_{L^\infty(\Sigma_t^p)}^2 dt](\tau)^{\frac{1}{2}} \tag{89}$$

∎

Lemma 4.22. *We have,*

$$\sup_{q\in\Sigma_t\cap J^-(p)} \int_{\Omega_q} |\delta(u) < [F_{L\underline{L}}, \lambda_{\alpha\beta}], F^{\alpha\beta} > | \lesssim (\tau)^{\frac{1}{2}} + [\int_{t_p-\tau}^{t} ||F||_{L^\infty(\Sigma_t^p)}^2 dt](\tau)^{\frac{1}{2}}$$

$$\tag{90}$$

Proof

By same as previously, the term

$$\sup_{q\in\Sigma_t\cap J^-(p)} \int_{\Omega_q} |\delta(u) < [F_{L\underline{L}}, \lambda_{\alpha\beta}], F^{\alpha\beta} > |$$

$$\lesssim \quad [\int_{t_p-\tau}^{t} (||F||_{L^\infty(\Sigma_t^p)}^2)dt]^{\frac{1}{2}} \sup_{q\in\Sigma_t^p} [\int_{t_p-\tau}^{t} (\int_{S_t} (s^{-1}|F_{L\underline{L}}|)da_t)^2 dt]^{\frac{1}{2}}$$

$$\lesssim \quad (\tau)^{\frac{1}{2}} \sup_{q\in\Sigma_t^p} (F^{\frac{\partial}{\partial t}}(N_{t_p-\tau,t}^-(q)))^{\frac{1}{2}}$$

$$\lesssim \quad (\tau)^{\frac{1}{2}}(E_{t=0}^{\frac{\partial}{\partial t}})^{\frac{1}{2}}$$

Lemma 4.23. *We have,*

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) < \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} >| \lesssim (\tau)^4 + [\int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_t^p)} dt](\tau)^4$$

Proof

The term

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) < \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} >|$$

$$= \sup_{q \in \Sigma_t \cap J^-(p)} \int_{t_p-\tau}^t (||F||_{L^\infty(S_t)} \int_{S_t} |\zeta_a \mathbf{D}_a^{(A)} \lambda| \phi da_t) dt$$

$$\lesssim [\int_{t_p-\tau}^t (||F||^2_{L^\infty(\Sigma_t^p)}) dt]^{\frac{1}{2}} \sup_{q \in \Sigma_t \cap J^-(p)} [\int_{t_p-\tau}^t ([\int_{S_t} |\zeta_a|^2 \phi da_t][\int_{S_t} |\mathbf{D}_a^{(A)} \lambda|^2 \phi da_t]) dt]^{\frac{1}{2}}$$

We have

$$\zeta_a = O(s)$$

(see proposition 3.1 in [Wang]).

Thus,

$$\int_{S_t} |\zeta_a|^2 \phi da_t \lesssim \int_{S_t} s^2 da_t \lesssim s^4 \lesssim \tau^4$$

Therefore,

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) < \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} >|$$

$$\lesssim [\int_{t_p-\tau}^t (||F||^2_{L^\infty(\Sigma_t^p)}) dt]^{\frac{1}{2}} \tau^4 \sup_{q \in \Sigma_t^p \cap J^-(p)} [\int_{t_p-\tau}^t \int_{S_t} |\mathbf{D}_a^{(A)} \lambda|^2 \phi da_t dt]^{\frac{1}{2}}$$

We showed previously that $||\mathbf{D}_a^{(A)} \lambda||_{L^2(N_\tau^-(q))} \lesssim 1$, thus

$$\sup_{q \in \Sigma_t^p \cap J^-(p)} ||\mathbf{D}_a^{(A)} \lambda||_{L^2(N_\tau^-(q))} \lesssim 1$$

Finally,

$$\sup_{q \in \Sigma_t \cap J^-(p)} \int_{\Omega_q} |\delta(u) < \zeta_a \mathbf{D}_a^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} >| \quad \lesssim \quad (\tau)^4 + [\int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_t^p)} dt](\tau)^4$$

$$(91)$$

We are left with the term $\sup_{q \in \Sigma_t^p} | \int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > |$. We recall that $\hat{\Delta}^{(A)} \lambda_{\alpha\beta}$ is the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$,

Lemma 4.24. *We have,*

$$\sup_{q \in \Sigma_t^p} | \int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > |$$

$$\lesssim \quad 1 + \int_{t_p - \tau}^{t} ||F||_{L^\infty(\Sigma_t^p)}^2 dt + \int_{t_p - \tau}^{t} \int_{t_p - \tau}^{t} ||F||_{L^\infty(\Sigma_t^p)}^2 d\bar{t} dt$$

Proof

$$\sup_{q \in \Sigma_t^p} | \int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > | \quad = \quad \sup_{q \in \Sigma_t^p} \int_{t_p - \tau}^{t_p} \int_{S_t} < \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > \phi da_t dt|$$

$$\lesssim \quad \sup_{q \in \Sigma_t^p} \int_{t_p - \tau}^{t_p} | \int_{S_t} < \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > \phi da_t | dt$$

Definition 4.25. We define a restriction of the covariant derivative of $\nabla_b e_a$ to the span of $\{e_a\}$, $a \in \{1, 2\}$ at $q \in N^-(p) \backslash \{p\}$ as being $\overline{\nabla}_b e_a$.

Definition 4.26. We define,

$$\rlap{/}{\mathbf{D}}_b^{(A)} \rlap{/}{\mathbf{D}}_a^{(A)} \lambda_{\alpha\beta} = \mathbf{D}_b^{(A)} (\mathbf{D}_a^{(A)} \lambda)_{\alpha\beta} - \mathbf{D}_{\overline{\nabla}_b e_a}^{(A)} \lambda_{\alpha\beta} \tag{92}$$

whereas,

$$\mathbf{D}_b^{(A)} \mathbf{D}_a^{(A)} \lambda_{\alpha\beta} = \mathbf{D}_b^{(A)} (\mathbf{D}_a^{(A)} \lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_b e_a}^{(A)} \lambda_{\alpha\beta}$$

We have

$$\hat{\Delta}^{(A)} \lambda_{\alpha\beta}$$
$$= \quad (\rlap{/}{\mathbf{D}}^{(A)a} \rlap{/}{\mathbf{D}}_a^{(A)} \lambda)(e_\alpha, e_\beta)$$
$$= \quad \partial^a [(\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)] + [A^a, (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)]$$
$$\quad -(\mathbf{D}_a^{(A)} \Psi)(\nabla^a e_\alpha, e_\beta) - (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, \nabla^a e_\beta) - (\mathbf{D}_{\overline{\nabla}^a e_a}^{(A)} \Psi)(e_\alpha, e_\beta)$$

Hence,

$$< \hat{\Delta}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} >$$
$$= \quad < \partial^a [(\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)] + [A^a, (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, e_\beta)], F^{\alpha\beta} >$$
$$\quad - < (\mathbf{D}_a^{(A)} \Psi)(\nabla^a e_\alpha, e_\beta), F^{\alpha\beta} > - < (\mathbf{D}_a^{(A)} \Psi)(e_\alpha, \nabla^a e_\beta), F^{\alpha\beta} >$$
$$\quad - < (\mathbf{D}_{\overline{\nabla}^a e_a}^{(A)} \Psi)(e_\alpha, e_\beta), F^{\alpha\beta} >$$

To compute $< (\mathbf{D}_{\overline{\nabla}^a e_a}^{(A)} \Psi)(e_\alpha, e_\beta), F^{\alpha\beta} >$, since it is a full contraction on the 2-spheres S_t, we can choose a normal frame with respect to the induced metric on S_t, i.e. a frame where the restricted covariant derivative of elements of the frame $\mathbf{D}_{\overline{\nabla}^a e_a}^{(A)}$ vanish at that point. Hence, this term vanishes.

Whereas to the terms

$$< -(\mathbf{D}_a^{(A)}\Psi)(\nabla^a e_\alpha, e_\beta), F^{\alpha\beta} >$$

and

$$< (\mathbf{D}_a^{(A)}\Psi)(e_\alpha, \nabla^a e_\beta), F^{\alpha\beta} >$$

since they are full contractions with respect to the space-time metric $\mathbf{g}$, we can compute those with respect to a normal frame where $\nabla_\alpha e_\beta = 0$ at that point. We can then express ∇^a as a combination of covariant derivatives at that frame ∇^α, and hence we get that $\nabla^a e_\alpha$ vanish.

Consequently,

$$< \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > \quad = \quad < \partial^a[(\mathbf{D}_a^{(A)}\Psi)(e_\alpha, e_\beta)] + [A^a, (\mathbf{D}_a^{(A)}\Psi)(e_\alpha, e_\beta)], F^{\alpha\beta} >$$

Similarly,

$$< \mathbf{D}_a^{(A)}\lambda_{\alpha\beta}, \mathbf{D}^{(A)^a}F^{\alpha\beta} >=< \mathbf{D}_a^{(A)}\lambda_{\alpha\beta}, \partial^a F^{\alpha\beta} + [A^a, F^{\alpha\beta}] >$$

Using the fact that the scalar product $< , >$ is Ad-invariant, we get

$$< \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > \quad = \quad \not\nabla^a < \not{\mathbf{D}}_a^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > - < \mathbf{D}_a^{(A)}\lambda_{\alpha\beta}, \mathbf{D}^{(A)^a}F^{\alpha\beta} >$$

Integrating on S_t, then applying the divergence theorem, and using the fact that we have no boundary terms since it is an integral on S_t, we get

$$\left| \int_{S_t} \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > \phi da_t \right|$$
$$= \quad \left| \int_{S_t} - < \mathbf{D}_a^{(A)}\lambda_{\alpha\beta}, \mathbf{D}^{(A)^a}F^{\alpha\beta} > \phi da_t \right| \qquad (93)$$
$$\lesssim \quad (\int_{S_t} |\mathbf{D}_a^{(A)}\lambda|^2 \phi da_t)^{\frac{1}{2}} (\int_{S_t} |\mathbf{D}^{(A)}F|^2 \phi da_t)^{\frac{1}{2}}$$

Thus,

$$\left| \int_{t_p-\tau}^{t_p} \int_{S_t} < \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F_{\hat\mu\hat\nu} > \right| \phi da_t dt$$
$$\lesssim \quad ||\mathbf{D}_a^{(A)}\lambda||_{L^2(N_\tau^-(q))} \left(\int_{t_p-\tau}^{t_p} \int_{S_t} |\mathbf{D}^{(A)}F|^2 \phi da_t dt \right)^{\frac{1}{2}}$$

We proved that $||\mathbf{D}_a^{(A)}\lambda||_{L^2(N_\tau^-(p))} \lesssim 1$. We also have

$$(\int_{t_p-\tau}^{t_p} \int_{S_t} |\mathbf{D}^{(A)}F|^2 \phi da_t dt)^{\frac{1}{2}} \quad \lesssim \quad 1 + \int_{t_p-\tau}^{t_p} \int_{S_t} |\mathbf{D}^{(A)}F|^2 \phi da_t dt$$
$$\lesssim \quad 1 + ||\mathbf{D}_a^{(A)}F||^2_{L^2(N_{t_p-\tau,t}^-(q))}$$

We get,

$$\sup_{q\in\Sigma_t^p} |\int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > | \lesssim 1 + \sup_{q\in\Sigma_t^p} ||\mathbf{D}_a^{(A)}F||^2_{L^2(N^-_{t_p-\tau,t}(q))} \qquad (94)$$

We proved that,

$$||\mathbf{D}_a^{(A)}F||^2_{L^2(N^-_{t_p-\tau,t}(q))} \lesssim c(t) + \int_{t_p-\tau}^t |F|^2_{L^\infty_{\Sigma_{\bar{t}}^q}} d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} |F|^2_{L^\infty_{\Sigma_{\bar{t}}^q}} d\bar{t}dt^*$$

where $c(t)$ is a finite constant for all t .

$$\sup_{q\in\Sigma_t^p} ||\mathbf{D}_a^{(A)}F||^2_{L^2(N^-_{t_p-\tau,t}(q))}$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_{\bar{t}}^p)} d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} ||F||^2_{L^\infty(\Sigma_{\bar{t}}^p)} d\bar{t}dt^*$$

Thus,

$$\sup_{q\in\Sigma_t^p} |\int_{\Omega_q} \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta} > |$$

$$\lesssim \quad 1 + \int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_{\bar{t}}^p)} d\bar{t} + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} ||F||^2_{L^\infty(\Sigma_{\bar{t}}^p)} d\bar{t}dt^* \qquad (95)$$

∎

Finally, summing over all the indices we obtain,

$$||F||_{L^\infty(\Sigma_t^p)} \lesssim 1 + \int_{t_p-\tau}^t ||F||^2_{L^\infty(\Sigma_t^p)} dt + \int_{t_p-\tau}^t \int_{t_p-\tau}^{t^*} ||F||^2_{L^\infty(\Sigma_{\bar{t}}^p)} d\bar{t}dt^* \qquad (96)$$

Using the result of Pachpatte in [Pach], we get,

$$||F||_{L^\infty(\Sigma_t^p)} \lesssim 1, \text{ for all } t \in [t_p - \tau, t_p]$$

From a local existence result, at each point p of the space-time, we have either the Yang-Mills fields blow up or they can be extended as solutions to the Yang-Mills equations up to that point. This combined with pointwise estimate above prove that the solutions can be extended up to the point p, and this can be done to any point p in the space-time, under the assumption of global hyperbolicity.

5. Appendix: Kirchoff-Sobolev Parametrix for $\Box_{\mathbf{g}}^{(A)} F_{\mu\nu}$

We assume $(M, \mathbf{g})$ to be globally hyperbolic, i.e. it admits a Cauchy surface Σ, which means a space-like hypersurface $\Sigma \subset M$, that is intersected precisely once by every inextendible causal curve. We also assume that the null cones are regular past the space-like hypersurface Σ.

We sketch an adaptation to the Yang-Mills setting of the original construction by S. Klainerman and I. Rodnianski in [KR1] of the Kirchoff-Soboloev parametrix.

Recall that the original construction presented in [KR1] was done in the context of a one tensor with values in the tangent bundle verifying the tensorial wave equation, and cannot be applied directly as it is to the the Yang-Mills equations. However, as noted in [KR1] this construction could be systematically generalized to $\mathcal{G}$-valued tensors of arbitrary order verifying the gauge covariant tensorial wave equations with a compatible Ad-invariant scalar product $< , >$ on $\mathcal{G}$, leading to a representation formula suitable to present a gauge invariant proof of the global existence of Yang-Mills fields on the 4-dimensional Minkowski background. We are sketching the adaptation in this appendix so as to use it to give a proof of the global existence of Yang-Mills fields on curved backgrounds.

Definition 5.1. Let p be a point to the future of Σ. The affine parameter s on $N^-(p)$ is defined by fixing a future unit time-like vector $\mathbf{T}_p$ at p and considering for every ω in $\mathbb{S}^2$, the null vector l_ω in $T_p(M)$, such that

$$\mathbf{g}(l_\omega, \mathbf{T}_p) = 1 \tag{97}$$

and associate to it the null geodesic $\gamma_\omega(s)$ such that $\gamma_\omega(0) = p$, $\frac{\partial}{\partial s}\gamma_\omega(0) = l_\omega$, and $L = \frac{\partial}{\partial s}\gamma_\omega(s)$ where s is chosen so that $\nabla_L L = 0$. Thus, $L(s) = 1, s(p) = 0$.

Definition 5.2. Let $N^-(p)$ be the boundary of the causal past of p. Let χ denote the null second fundamental form of $N^-(p)$, that is, for all $q \in N^-(p)\backslash\{p\}$,

$$\chi(X,Y)(q) = \mathbf{g}(\nabla_X L, Y)(q) \tag{98}$$

for all X, Y in $T_q N^-(p)$.

Lemma 5.3. χ is symmetric and thus χ is diagonalisable, moreover $\chi(L,X) = \chi(L,L) = 0$, for all $X \in T_q N^-(p)$, consequently, we can define $tr\chi = \chi(L,L) + \chi(e_1,e_1) + \chi(e_2,e_2) = \chi_{11} + \chi_{22}$.

Proof

Given a point $q \in N^-(p)\backslash\{p\}$, we can define a null frame $\{L, \underline{L}, e_1, e_2\}$ - where e_1 and e_2 are tangent to $N^-(p) \cap \{s = constant\}$ 2-surfaces - that forms a basis of $T_q M$, such that at $q \in N^-(p)\backslash\{p\}$,

$$\begin{aligned}
\mathbf{g}(L,L) &= \mathbf{g}(\underline{L},\underline{L}) = 0 & (99)\\
\mathbf{g}(L,\underline{L}) &= -2 & (100)\\
\mathbf{g}(e_a,e_b) &= \delta_{ab}, \quad a,b \in \{1,2\} & (101)\\
\mathbf{g}(L,e_a) &= \mathbf{g}(\underline{L},e_a) = 0, \quad a,b \in \{1,2\} & (102)
\end{aligned}$$

This null frame can be extended locally in a neighbourhood of $q \in N^-(p)\backslash\{p\}$ such that $L, \underline{L}, e_1, e_2$ are vector fields in the neighbourhood and $\mathbf{g}(L,L) = \mathbf{g}(\underline{L},\underline{L}) = 0$ in the neighbourhood.

Let, $X, Y \in T_q N^-(p)$. Since the metric is Killing, we have,

$$\begin{aligned}
0 &= \nabla_X \mathbf{g}(L,Y) = X\mathbf{g}(L,Y) - \mathbf{g}(\nabla_X L, Y) - \mathbf{g}(L, \nabla_X Y)\\
&= -\mathbf{g}(\nabla_X L, Y) - \mathbf{g}(L, \nabla_X Y) \qquad\qquad\qquad (103)\\
&\quad (\text{since } \mathbf{g}(L,Y) = 0 \text{ for } Y \in T_q N^-(p))
\end{aligned}$$

Thus, $\mathbf{g}(\nabla_X L, Y) = -\mathbf{g}(L, \nabla_X Y)$, and since we have $[X, Y] \in T_q N^-(p)$, we get,

$$0 = \mathbf{g}(L, [X, Y]) = \mathbf{g}(L, \nabla_X Y - \nabla_Y X)$$

which gives, $\mathbf{g}(L, \nabla_X Y) = \mathbf{g}(L, -\nabla_Y X)$, we finally get,

$$\mathbf{g}(\nabla_X L, Y) = \mathbf{g}(L, \nabla_Y X)$$

Again,

$$\mathbf{g}(\nabla_Y L, X) = -\mathbf{g}(L, \nabla_Y X)$$

(by inverting the roles of X and Y in before)

This gives,

$$\mathbf{g}(\nabla_X L, Y) = \mathbf{g}(\nabla_Y L, X)$$

Consequently χ is symmetric, and hence $\chi(L, X) = \chi(X, L) = \mathbf{g}(\nabla_L L, X) = 0$ because $\nabla_L L = 0$ by construction. ∎

Definition 5.4. Let $\mathbf{J}_p$ be a fixed $\mathcal{G}$-valued anti-symmetric 2-tensor at p, and let $\lambda_{\alpha\beta}$ be the unique 2-tensor field along $N^-(p)$, that verifies the linear transport equation:

$$\mathbf{D}_L^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} tr\chi \lambda_{\alpha\beta} = 0 \qquad (104)$$

$$(s\lambda_{\alpha\beta})(p) = \mathbf{J}_{\alpha\beta}(p) \qquad (105)$$

$\lambda_{\alpha\beta}$ can be extended smoothly to be defined in a similar way in a neighborhood away from $N^-(p) \backslash \{p\}$.

Definition 5.5. For small $\epsilon > 0$, let $T_\epsilon : (1 - \epsilon, 1 + \epsilon) \longmapsto M$ be the timelike geodesic from p such that $T_\epsilon(1) = p$ and $T_\epsilon^{'}(1) = \mathbf{T}_p$. We define u, optical function, as $u_{|N^-(q)} = t - 1$ for each $q = T_\epsilon(t)$, where $N^-(q)$ is the boundary of the past set of q, assumed to be regular.

Definition 5.6. The following integral in Σ^+, future of Σ, for any $\mathcal{G}$-valued 2-tensors $\lambda_{\alpha\beta}$ and $\Lambda_{\alpha\beta}$ supported in Σ^+, is defined as,

$$\int_{\Sigma^+} < \lambda_{\alpha\beta} \delta(u), \Lambda^{\alpha\beta} > = < \delta(u), < \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} >>$$

in the sense of the distribution, where u is defined in a neighborhood D_ϵ of $N^-(p) \cap \Sigma^+$ as in above, and

$$< \delta(u), < \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} >> = \int_0^\infty \int_{\mathbb{S}^2} < \lambda_{\alpha\beta}, \Lambda^{\alpha\beta} > (t = 1, s, \omega) ds dA_{S^2} \qquad (106)$$

where dA_{S^2} is the induced volume form on the 2-surfaces defined by $s = constant$, and $t = 1$. This integral depends only on $< \lambda_{\alpha\beta}, \Lambda_{\mu\nu} >$ on $N^-(p)$ and the normalisation condition

$$\mathbf{g}(l_\omega, \mathbf{T}_p) = 1 \quad \left(\mathbf{g}(l_\omega, l_\omega) = 0 \right).$$

Therefore, for any continuous function f supported in Σ^+, we can define $\int_{N^-(p)} f$ as $< \delta(u), f >$ in the sense of the distribution.

5.7. Computing $\int_{J^-(p) \cap \Sigma^+} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} >$.

Now, our goal is to compute $\int_{J^-(p) \cap \Sigma^+} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} >$ for F supported in Σ^+.

Definition 5.8. We define a timelike foliation near p, by extending locally the parameter t near p by starting with a fixed spacelike hypersurface Σ_1 passing through p and orthogonal to the future unit timelike vectorfield $\mathbf{T}_p$ and considering the timelike geodesics orthogonal to Σ_1.

Definition 5.9. We define $\Omega_\epsilon = (J^-(p) \cap \Sigma^+) \setminus \cup_{t \in [1-\epsilon, 1]} \Sigma_t$.

We have

$$\int_{J^-(p) \cap \Sigma^+} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > \qquad (107)$$

On the other hand,

$$\int_{\Omega_\epsilon} < \lambda_{\alpha\beta} \delta(u), \square_{\mathbf{g}}^{(A)} F^{\alpha\beta} > = \int_{\Omega_\epsilon} < \lambda_{\alpha\beta} \delta(u), \mathbf{D}^{(A)\gamma} \mathbf{D}_\gamma^{(A)} F^{\alpha\beta} > .$$

Lemma 5.10. *Given any two $\mathcal{G}$-valued tensors K and G, since $< \, , \, >$ is Ad-invariant, we have,*

$$\nabla_\gamma < K_{\alpha\beta}, G^{\alpha\beta} > = < \mathbf{D}_\gamma^{(A)} K_{\alpha\beta}, G^{\alpha\beta} > + < K_{\alpha\beta}, \mathbf{D}_\gamma^{(A)} G^{\alpha\beta} > \qquad (108)$$

Proof

Now, given any two $\mathcal{G}$-valued tensors $K_{\alpha\beta}$ and $G_{\alpha\beta}$, we have

$$\partial_\gamma < K_{\alpha\beta}, G^{\alpha\beta} > - < \partial_\gamma K_{\alpha\beta}, G^{\alpha\beta} > - < K_{\alpha\beta}, \partial_\gamma G^{\alpha\beta} > = 0 \qquad (109)$$

Since $< K_{\alpha\beta}, G^{\alpha\beta} >$ does not depend on the choice of the basis, one can choose a normal frame as in (11) to compute (109). In such a frame $\partial_\gamma K_{\alpha\beta} = \nabla_\gamma K_{\alpha\beta}$ and $\partial_\gamma G_{\alpha\beta} = \nabla_\gamma K_{\alpha\beta}$, and by abuse of notation, we will wright $\partial_\gamma < K_{\alpha\beta}, G^{\alpha\beta} >$ as $\nabla_\gamma < K_{\alpha\beta}, G^{\alpha\beta} >$. Hence, we have

$$\nabla_\gamma < K_{\alpha\beta}, G^{\alpha\beta} > - < \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} > - < K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} > = 0$$

So we have

$$
\begin{aligned}
\nabla_\gamma < K_{\alpha\beta}, G^{\alpha\beta} > \quad &= \quad < \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} > + < K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} > \\
&= \quad < \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} > - < K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] > \\
&\quad + < K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] > + < K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} > \\
&= \quad < \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} > - < [K_{\alpha\beta}, A_\gamma], G^{\alpha\beta} > \\
&\quad + < K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] > + < K_{\alpha\beta}, \nabla_\gamma G^{\alpha\beta} >
\end{aligned}
$$

(since $< \, , \, >$ is Ad-invariant)

$$
\begin{aligned}
&= < \nabla_\gamma K_{\alpha\beta}, G^{\alpha\beta} > + < [A_\gamma, K_{\alpha\beta}], G^{\alpha\beta} > + < K_{\alpha\beta}, [A_\gamma, G^{\alpha\beta}] + \nabla_\gamma G^{\alpha\beta} > \\
&= < \mathbf{D}_\gamma^{(A)} K_{\alpha\beta}, G^{\alpha\beta} > + < K_{\alpha\beta}, \mathbf{D}_\gamma^{(A)} G^{\alpha\beta} >
\end{aligned}
$$

$\blacksquare$

Let $D_\epsilon = (\{-\epsilon' \le u(t) \le \epsilon'\}\backslash \Sigma_{1-\epsilon}^+) \cap \Sigma_0^+$, for ϵ' chosen small enough so that $u(t)$ would be defined on $[-\epsilon', \epsilon']$. Also, recall that λ is smooth in a neighborhood away from p, and thus, by choosing ϵ' small enough, λ is smooth in D_ϵ.

Given this, we have,

$$\int_{D_\epsilon} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}^{\gamma(A)}\mathbf{D}_\gamma^{(A)}F^{\alpha\beta} >$$

$$= \int_{D_\epsilon} \nabla^\gamma < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - \int_{D_\epsilon} \nabla^\gamma < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$$

$$+ \int_{D_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$$

so,

$$\int_{D_\epsilon} < \lambda_{\alpha\beta}\delta(u), \Box_{\mathbf{g}}^{(A)}F^{\alpha\beta} >$$

$$= \int_{D_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$$

$$+ \int_{D_\epsilon} \nabla^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >] \quad (110)$$

By divergence theorem,

$$\int_{D_\epsilon} \nabla^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]$$

$$= - \int_{\Sigma_t} T^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]_{t=0}^{t=1-\epsilon}$$

$$+ \int_{(N^-(T(-\epsilon'))\backslash\Sigma_{1-\epsilon}^+)\cap\Sigma_0^+} L^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]$$

$$+ \int_{(N^-(T(\epsilon'))\backslash\Sigma_{1-\epsilon}^+)\cap\Sigma_0^+} L^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]$$

Since the distributions δ and δ' are supported on $N^-(T(0)) = N^-(p)$, we get,

$$\int_{(N^-(T(-\epsilon'))\backslash\Sigma_{1-\epsilon}^+)\cap\Sigma_0^+} L^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >] = 0$$

$$\int_{(N^-(T(\epsilon'))\backslash\Sigma_{1-\epsilon}^+)\cap\Sigma_0^+} L^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >] = 0$$

$$\int_{D_\epsilon} \nabla^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]$$

$$= \int_{\Omega_\epsilon} \nabla^\gamma[< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]$$

This yields to,

$$\int_{\Omega_\epsilon} \nabla^\gamma [< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]$$

$$= -\int_{J^-(p)\cap\Sigma_t} T^\gamma [< \lambda_{\alpha\beta}\delta(u), \mathbf{D}_\gamma^{(A)}F^{\alpha\beta} > - < \mathbf{D}_\gamma^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]_{t=0}^{t=1-\epsilon}$$

where $\Sigma_0 = \Sigma$, and where T is defined on Σ as being the unit normal timelike vectorfield on Σ.

We get,

$$\int_{\Omega_\epsilon} < \lambda_{\alpha\beta}\delta(u), \Box_{\mathbf{g}}^{(A)}F^{\alpha\beta} > = \int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$$

$$-[\int_{J^-(p)\cap\Sigma_t} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_T^{(A)}F^{\alpha\beta} >]_{t=0}^{t=1-\epsilon}$$

$$+[\int_{J^-(p)\cap\Sigma_t} < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]_{t=0}^{t=1-\epsilon}$$

$$(111)$$

5.11. Computing $\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$.

Now, we would like to compute $\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >= \int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F >$ in (111).

We start by computing $\Box_{\mathbf{g}}^{(A)}(\lambda\delta(u))$. We have:

$$\begin{aligned}
\Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)) &= \mathbf{g}^{\mu\nu}\mathbf{D}_\mu^{(A)}\mathbf{D}_\nu^{(A)}(\lambda\delta(u)) \\
&= \mathbf{g}^{\mu\nu}\mathbf{D}_\mu^{(A)}(\nabla_\nu(\lambda\delta(u)) + [A_\nu, (\lambda\delta(u))]) \\
&= \mathbf{g}^{\mu\nu}\mathbf{D}_\mu^{(A)}(\nabla_\nu(\lambda)\delta(u) + [A_\nu, (\lambda\delta(u))] + \lambda\nabla_\nu(\delta(u))) \\
&= \mathbf{g}^{\mu\nu}\mathbf{D}_\mu^{(A)}[(\nabla_\nu(\lambda) + [A_\nu, \lambda])\delta(u) + \lambda\nabla_\nu(\delta(u))] \\
&= \mathbf{g}^{\mu\nu}\mathbf{D}_\mu^{(A)}[\mathbf{D}_\nu^{(A)}(\lambda)\delta(u) + \lambda\delta'(u)\nabla_\nu(u)] \\
&= \mathbf{g}^{\mu\nu}\mathbf{D}_\mu^{(A)}\mathbf{D}_\nu^{(A)}(\lambda)\delta(u) + \delta'(u)\nabla_\mu(u)\mathbf{g}^{\mu\nu}\mathbf{D}_\nu^{(A)}(\lambda) \\
&\quad + \delta'(u)\mathbf{g}^{\mu\nu}\nabla_\mu(\lambda\nabla_\nu(u)) + \mathbf{g}^{\mu\nu}\delta''(u)\nabla_\mu(u)\nabla_\nu(u)\lambda
\end{aligned}$$

(by the symmetry of the metric tensor). Thus,

$$\begin{aligned}
\Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)) &= \Box_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + \delta'(u)(\Box_{\mathbf{g}}(u)\lambda + 2g^{\mu\nu}\nabla_\nu u\mathbf{D}_\mu^{(A)}\lambda) \\
&\quad + \delta''(u)(\mathbf{g}^{\mu\nu}\nabla_\mu u\nabla_\nu u)\lambda
\end{aligned}$$

$$(112)$$

Now, we want to compute $\Box_{\mathbf{g}}(u) = \nabla^\alpha\nabla_\alpha u$, at $q \in N^-(p)\backslash\{p\}$.

Lemma 5.12. *We have, $\Box_{\mathbf{g}}u = tr\chi$, at $q \in N^-(p)\backslash\{p\}$.*

Proof

Now, let $\nabla u = \nabla^\nu u \partial_\nu$, defined in a neighbourhood D_ϵ of $N^- \cap \Sigma^+$. Since

$$L = \frac{d}{ds}\gamma_\omega(s)$$

where $\gamma_\omega(s)$ is the null geodesic initiating at p, we have $L \in T_q(N^-(p))$ for $q \in N^-(p)\backslash\{p\}$ and since u is constant on $N^-(m)$, for $m \in T_\epsilon(t)$, $t \in [1-\epsilon, 1+\epsilon]$, we have

$$L(u) = 0 = du(L) = \mathbf{g}(\nabla u, L) \tag{113}$$

And since $e_a \in T_q(N^-(p))$, $a \in \{1,2\}$. We also have,

$$e_a(u) = du(e_a) = \mathbf{g}(\nabla u, e_a) = 0 \tag{114}$$

(113) and (114) give that,

$$\nabla u(p) = f(p)L$$

Hence,

$$\nabla^\nu u \nabla_\nu u = (\nabla u)u = fL(u) = 0 \tag{115}$$

At a point p of the space-time, one can choose a normal frame, which means a frame such that $\mathbf{g}(e_\alpha, e_\beta)(p) = \text{diag}(-1, 1, \ldots, 1)$, and $\frac{\partial}{\partial\sigma}\mathbf{g}(e_\alpha, e_\beta)(p) = 0$. Hence, in such a frame $\Gamma^i_{kl} = \frac{1}{2}\mathbf{g}^{im}(\frac{\partial \mathbf{g}_{mk}}{\partial x^l} + \frac{\partial \mathbf{g}_{ml}}{\partial x^k} - \frac{\partial \mathbf{g}_{kl}}{\partial x^m}) = 0$. Computing in such a frame,

$$
\begin{aligned}
(\nabla_{\nabla u}\nabla u)^\gamma(p) &= (\nabla_{\nabla^\nu u \partial_\nu}(\nabla^\mu u \partial_\mu))^\gamma = (\nabla^\nu u \nabla_\nu(\nabla^\mu u \partial_\mu))^\gamma \\
&= (\nabla^\nu u(\nabla_\nu \nabla^\mu u)\partial_\mu)^\gamma + (\nabla^\nu u \nabla^\mu u \nabla_\nu \partial_\mu)^\gamma \\
&= \nabla^\nu u(\nabla_\nu \nabla^\mu u)\delta_\mu{}^\gamma + \nabla^\nu u \nabla^\mu u \Gamma^\gamma_{\nu\mu} \\
&= \nabla^\nu u(\nabla^\mu \nabla_\nu u)\delta_\mu{}^\gamma + \nabla^\nu u \nabla^\mu u \Gamma^\gamma_{\nu\mu} \\
&\quad \text{(using that the metric is compatible. i.e. } \nabla \mathbf{g} = 0) \\
&= \frac{1}{2}\nabla^\mu(\nabla_\nu u \nabla^\nu u)\delta_\mu{}^\gamma \\
&= 0
\end{aligned}
$$

(in view of (115)).

Therefore, ∇u is parallel, this gives,

$$\nabla u = cL$$

where c is a constant. We have,

$$T(u)(p) = 1 = \mathbf{g}(\nabla u, \mathbf{T}_p) = \mathbf{g}(cL, \mathbf{T}_p)$$

In view of (97), we have,

$$\mathbf{g}(cL, \mathbf{T}_p) = c$$

Thus,

$$c = 1$$

which gives,

$$\nabla u = L = \nabla^\nu u \partial_\nu \tag{116}$$

Computing,

$$\Box_{\mathbf{g}} u = \nabla^\alpha \nabla_\alpha u = \nabla^L \nabla_L u + \nabla^{\underline{L}} \nabla_{\underline{L}} u + \nabla^a \nabla_a u$$

$a = \{1, 2\}$, at $q \in N^-(p) \backslash \{p\}$. Therefore, at $q \in N^-(p) \backslash \{p\}$,

$$
\begin{aligned}
\Box_{\mathbf{g}} u &= \mathbf{g}^{L\alpha} \nabla_\alpha \nabla_L u + \mathbf{g}^{\underline{L}\alpha} \nabla_\alpha \nabla_{\underline{L}} u + \mathbf{g}^{a\alpha} \nabla_\alpha \nabla_a u \\
&= -\frac{1}{2} \nabla_{\underline{L}} \nabla_L u - \frac{1}{2} \nabla_L \nabla_{\underline{L}} u + \nabla_a \nabla_a u \\
&= -\frac{1}{2} \nabla_{\underline{L}} (\nabla_L u) + \frac{1}{2} \nabla_{\nabla_{\underline{L}} L} u - \frac{1}{2} \nabla_L (\nabla_{\underline{L}} u) + \frac{1}{2} \nabla_{\nabla_L \underline{L}} u + \nabla_a (\nabla_a u) - \nabla_{\nabla_a e_a} u
\end{aligned}
$$

and at $q \in N^-(p) \backslash \{p\}$,

$$
\begin{aligned}
\nabla_L u &= L(u) = du(L) = \mathbf{g}(L, L), \\
\nabla_{\underline{L}} u &= \underline{L}(u) = du(\underline{L}) = \mathbf{g}(L, \underline{L}), \\
\nabla_a u &= e_a(u) = du(e_a) = \mathbf{g}(L, e_a).
\end{aligned}
$$

We get

$$\nabla_{\underline{L}} (\nabla_L u) = \underline{L} \mathbf{g}(L, L) = \mathbf{g}(\nabla_{\underline{L}} L, L) + \mathbf{g}(L, \nabla_{\underline{L}} L)$$

thus

$$
\begin{aligned}
\nabla_{\underline{L}} (\nabla_L u) &= 2g(\nabla_{\underline{L}} L, L) \\
\nabla_L (\nabla_{\underline{L}} u) &= L g(L, \underline{L}) = 0 \\
\nabla_a (\nabla_a u) &= e_a \mathbf{g}(L, e_a) = 0
\end{aligned}
$$

Therefore,

$$\Box_{\mathbf{g}} u = -\mathbf{g}(\nabla_{\underline{L}} L, L) + \frac{1}{2} \nabla_{\nabla_{\underline{L}} L} u + \frac{1}{2} \nabla_{\nabla_L \underline{L}} u - \nabla_{\nabla_a e_a} u$$

We recall that in the frame $\{L, \underline{L}, e_a, e_b\}$ a vector field X can be written as:

$$X = -\frac{1}{2} \mathbf{g}(X, \underline{L}) L - \frac{1}{2} \mathbf{g}(X, L) \underline{L} + \mathbf{g}(X, e_a) e_a \tag{117}$$

Thus, taking $X = \nabla_L \underline{L}$, we get

$$\nabla_L \underline{L} = -\frac{1}{2} \mathbf{g}(\nabla_L \underline{L}, \underline{L}) L - \frac{1}{2} \mathbf{g}(\nabla_L \underline{L}, L) \underline{L} + \mathbf{g}(\nabla_L \underline{L}, e_a) e_a$$

Therefore,

$$\frac{1}{2} \nabla_{\nabla_L \underline{L}} u = \frac{1}{2} \mathbf{g}(L, \nabla_L \underline{L}) = -\frac{1}{4} \mathbf{g}(\mathbf{g}(\nabla_L \underline{L}, L) \underline{L}, L) = \frac{1}{2} \mathbf{g}(\nabla_L \underline{L}, L) = 0$$

$$\frac{1}{2} \nabla_{\nabla_{\underline{L}} L} u = \frac{1}{2} \mathbf{g}(L, \nabla_{\underline{L}} L) = -\frac{1}{4} \mathbf{g}(\mathbf{g}(\nabla_{\underline{L}} L, L) \underline{L}, L) = \frac{1}{2} \mathbf{g}(\nabla_{\underline{L}} L, L) = 0$$

We are left with $-\nabla_{\nabla_a e_a} u$

Taking $X = \nabla_a e_a$, we obtain,

$$
\begin{aligned}
\nabla_a e_a &= -\frac{1}{2}\mathbf{g}(\nabla_a e_a, \underline{L})L - \frac{1}{2}\mathbf{g}(\nabla_a e_a, L)\underline{L} + \mathbf{g}(\nabla_a e_a, e_b)e_b \\
&= \frac{1}{2}\mathbf{g}(e_a, \nabla_a \underline{L})L + \frac{1}{2}\mathbf{g}(e_a, \nabla_a L)\underline{L} + \mathbf{g}(\nabla_a e_a, e_b)e_b \\
&= \frac{1}{2}tr\underline{\chi}L + \frac{1}{2}tr\chi\underline{L} + \mathbf{g}(\nabla_a e_a, e_b)e_b
\end{aligned}
\tag{118}
$$

Finally,

$$
-\nabla_{\nabla_a e_a} u = -\mathbf{g}(L, \nabla_a e_a) = -\mathbf{g}(L, \frac{1}{2}tr\chi\underline{L}) = tr\chi
$$

This yields to

$$
\Box_{\mathbf{g}} u = tr\chi
\tag{119}
$$

∎

Going back to (112), we have now shown that

$$
\Box_{\mathbf{g}}^{(A)}(\lambda\delta(u))
\tag{120}
$$
$$
= \Box_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + 2\delta'(u)(\frac{TrX}{2}\lambda + \mathbf{g}^{\mu\nu}\nabla_\nu u \mathbf{D}_\mu^{(A)}\lambda) + \delta''(u)(\mathbf{g}^{\mu\nu}\nabla_\mu u \nabla_\nu u)\lambda
$$

at $q \in N^-(p)\backslash\{p\}$.

We recall that u is constant on $N^-(p)$ and $L \in T_q N^-(p)$ for $q \in N^-(p)\backslash\{p\}$.

(113) and (116) yield to

$$
L(u) = 0 = \mathbf{g}^{\mu\nu}\nabla_\mu u \nabla_\nu u
$$

at q. Also,

$$
\mathbf{g}^{\mu\nu}\nabla_\nu u \mathbf{D}_\mu^{(A)}\lambda = L^\mu \mathbf{D}_\nu^{(A)}\lambda
$$

(since $L = \mathbf{g}^{\mu\nu}\nabla_\mu u \partial_\nu$ gives $L^\nu = \mathbf{g}^{\mu\nu}\nabla_\mu u$).

Thus (120) becomes,

$$
\Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)) = \Box_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + 2\delta'(u)(\mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda)
\tag{121}
$$

Hence,

$$
\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F > = \int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda)\delta(u) + 2\delta'(u)(\mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda), F >
\tag{122}
$$

5.13. **Evaluating $\int_{\Omega_\epsilon} \delta'(u) < \mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{tr\chi}{2}\lambda_{\alpha\beta}, F^{\alpha\beta} >$.**

We have $\nabla_{\underline{L}}\delta(u) = \delta'(u)\nabla_{\underline{L}}(u)$.

$$
\nabla_{\underline{L}}(u) = du(\underline{L}) = \mathbf{g}(\nabla u, \underline{L}) = \mathbf{g}(L, \underline{L}) = -2
$$

Thus,

$$
\nabla_{\underline{L}}\delta(u) = -2\delta'(u)
$$

or

$$\delta^{'}(u) = -\frac{1}{2}\nabla_{\underline{L}}\delta(u)$$

This yields to,

$$2\int_{\Omega_\epsilon}\delta^{'}(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F >$$

$$= 2(\frac{-1}{2})\int_{\Omega_\epsilon}\nabla_{\underline{L}}\delta(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F >$$

$$= -\int_{J^-(p)\cap\Sigma_t}\mathbf{g}(\underline{L}, T)\delta(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F >$$

$$+ \int_{\Omega_\epsilon}\delta(u) < \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda), F >$$

$$+ \int_{\Omega_\epsilon}\delta(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, \mathbf{D}_{\underline{L}}^{(A)}F >$$

$$+ \int_{\Omega_\epsilon}\delta(u)\nabla_\alpha\underline{L}^\alpha < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F > \tag{123}$$

(by integration by parts)

The integrals

$$-\int_{J^-(p)\cap\Sigma_t}\mathbf{g}(\underline{L}, T)\delta(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F >$$

$$\int_{\Omega_\epsilon}\delta(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, \mathbf{D}_{\underline{L}}^{(A)}F >$$

and

$$\int_{\Omega_\epsilon}\delta(u)\nabla_\alpha\underline{L}^\alpha < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F >$$

depend only on the values of the integrated function on $N^-(p)$ and on the normalisation condition $\mathbf{g}(L, T)(p) = 1$. As $\mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda = 0$ on $N^-(p)$, and due to the presence of $\delta(u)$, these three terms vanish. Thus, (123) can be written as,

$$2\int_{\Omega_\epsilon}\delta^{'}(u) < \mathbf{D}_L^{(A)}\lambda + \frac{tr\chi}{2}\lambda, F >= \int_{\Omega_\epsilon}\delta(u) < \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}^{(A)}{}_L\lambda) + \mathbf{D}_{\underline{L}}^{(A)}(\frac{tr\chi}{2}\lambda), F >$$

$$\tag{124}$$

and thus,

$$\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F >$$

$$= \int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda)\delta(u), F > \tag{125}$$

$$+ \int_{\Omega_\epsilon}\delta(u) < \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}^{(A)}{}_L\lambda) + \mathbf{D}_{\underline{L}}^{(A)}(\frac{tr\chi}{2}\lambda), F >$$

5.14. **Evaluating** $\int_{\Omega_\epsilon} < \square_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta})\delta(u), F^{\alpha\beta} >$.

Now, in its turn, we would like to compute the tensorial

$$\square_{\mathbf{g}}^{(A)}(\lambda) = \mathbf{g}^{\alpha\beta}\mathbf{D}^{(A)}{}^2_{\alpha\beta}\lambda$$

where

$$\mathbf{g}^{\alpha\beta}\mathbf{D}^{(A)}{}^2_{\alpha\beta}\lambda = \mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha^{(A)}\mathbf{D}_\beta^{(A)}\lambda$$

which we will distinguish from $\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)}\lambda)$ as $\mathbf{D}^{(A)}{}^2_{\alpha\beta}$ is the tensorial second order derivative defined by

$$\mathbf{D}^{(A)}{}^2_{\alpha\beta}\lambda_{\mu\nu} = (\mathbf{D}_\alpha^{(A)}(\mathbf{D}_\beta^{(A)}\lambda))_{\mu\nu} - (\mathbf{D}_{\nabla_\alpha e_\beta}^{(A)}\lambda)_{\mu\nu} \tag{126}$$

where the tensorial derivative $\mathbf{D}_\alpha^{(A)}\lambda$ is defined by,

$$(\mathbf{D}_\alpha^{(A)}\lambda)(X,Y) = \mathbf{D}_\alpha^{(A)}(\lambda(X,Y)) - \lambda(\nabla_\alpha X, Y) - \lambda(X, \nabla_\alpha Y) \tag{127}$$

for any $X, Y \in TM$. We have,

$$\square_{\mathbf{g}}^{(A)}\lambda_{\alpha\beta} = -\frac{1}{2}\mathbf{D}^{(A)}{}^2_{L\underline{L}}\lambda_{\alpha\beta} - \frac{1}{2}\mathbf{D}^{(A)}{}^2_{\underline{L}L}\lambda_{\alpha\beta} + \delta^{ab}\mathbf{D}^{(A)}{}^2_{ab}\lambda_{\alpha\beta} \tag{128}$$

$$
\begin{aligned}
\mathbf{D}^{(A)}{}^2_{L\underline{L}}\lambda_{\alpha\beta} - \mathbf{D}^{(A)}{}^2_{\underline{L}L}\lambda_{\alpha\beta} &= \nabla_L\nabla_{\underline{L}}\lambda_{\alpha\beta} - \nabla_{\underline{L}}\nabla_L\lambda_{\alpha\beta} + [F_{L\underline{L}}, \lambda_{\alpha\beta}] \\
&= R_\alpha{}^\gamma{}_{L\underline{L}}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{L\underline{L}}\lambda_{\alpha\gamma} + [F_{L\underline{L}}, \lambda_{\alpha\beta}]
\end{aligned}
$$

$$\mathbf{D}^{(A)}{}^2_{\underline{L}L}\lambda_{\mu\nu} = \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda_{\alpha\beta}) - \mathbf{D}_{\nabla_{\underline{L}}L}^{(A)}\lambda_{\alpha\beta} \tag{129}$$

Lemma 5.15. *We have,*

$$\nabla_{\underline{L}}L = 2\zeta_a e_a - 2\omega L \tag{130}$$

where,

$$\zeta_a = \frac{1}{2}\mathbf{g}(\nabla_a L, \underline{L}) \tag{131}$$

$$\omega = -\frac{1}{4}\mathbf{g}(\nabla_{\underline{L}}L, L) \tag{132}$$

Proof Using (117),

$$
\begin{aligned}
\nabla_{\underline{L}}L &= -\frac{1}{2}\mathbf{g}(\nabla_{\underline{L}}L, \underline{L})L - \frac{1}{2}\mathbf{g}(\nabla_{\underline{L}}L, L)\underline{L} + \mathbf{g}(\nabla_{\underline{L}}L, e_a)e_a, \ a \in 1, 2 \\
&= \frac{1}{2}\mathbf{g}(\nabla_{\underline{L}}L, L)L + 0 - \mathbf{g}(L, \nabla_{\underline{L}}e_a)e_a
\end{aligned}
$$

Let q_m be the 1-parameter group generated by $\underline{L}$, θ_r the 1-parameter group generated by e_a. Let $\Omega(r,m) = \theta_{-r} \circ q_{-m} \circ \theta_r \circ q_m$. We have,

$$[\underline{L}, e_a](p) = \frac{\partial^2}{\partial r \partial m}\Omega(0,0)(p)$$

θ_r maps $J^-(p) \cap \Sigma_t$ into itself for all t, and q_m maps $J^-(p) \cap \Sigma_t$ into say $Q_m(t)$, where the vector field e_a can still be constructed to be tangent to $Q_m(t)$ for all

t in a neighborhood of p, and therefore θ_r maps $Q_m(t)$ into itself. We get that $\theta_{-r} \circ q_{-m} \circ \theta_r \circ q_m$ maps $J^-(p) \cap \Sigma_t$ into itself for each t and consequently, $\Omega(r,m)$ maps $J^-(p) \cap \Sigma_t$ into itself. Therefore $[\underline{L}, e_a]$ is tangential to $J^-(p) \cap \Sigma_t$. Hence,

$$\mathbf{g}(L, [\underline{L}, e_a]) = 0$$

Thus,

$$\begin{aligned}
\nabla_{\underline{L}} L &= \frac{1}{2}\mathbf{g}(\nabla_{\underline{L}}\underline{L}, L)L - \mathbf{g}(L, \nabla_{e_a}\underline{L})e_a \\
&= \frac{1}{2}\mathbf{g}(\nabla_{\underline{L}}\underline{L}, L)L + \mathbf{g}(\nabla_{e_a}L, \underline{L})e_a
\end{aligned}$$

With the notation (131) and (132), we get (130).

∎

Therefore (129) can be written as,

$$\mathbf{D}^{(A)}{}^2_{L\underline{L}}\lambda_{\alpha\beta} \tag{133}$$
$$= \mathbf{D}^{(A)}_{\underline{L}}(\mathbf{D}^{(A)}_L \lambda_{\alpha\beta}) - 2\zeta_a \mathbf{D}^{(A)}_a \lambda_{\alpha\beta} + 2\omega \mathbf{D}^{(A)}_L \lambda_{\alpha\beta}$$
$$+ R_\alpha{}^\gamma{}_{L\underline{L}}\lambda_{\gamma\beta} + R_\beta{}^\gamma{}_{L\underline{L}}\lambda_{\alpha\gamma} + [F_{L\underline{L}}, \lambda_{\alpha\beta}]$$

We define,

$$\Delta\lambda_{\alpha\beta} = \delta^{ab}\mathbf{D}^{(A)}{}^2_{ab}\lambda_{\alpha\beta}$$

Injecting (133) in (128), we obtain,

$$\begin{aligned}
\Box_{\mathbf{g}}^{(A)}\lambda_{\alpha\beta} &= -\frac{1}{2}\mathbf{D}^{(A)}_{\underline{L}}(\mathbf{D}^{(A)}_L \lambda_{\alpha\beta}) - \frac{1}{2}\mathbf{D}^{(A)}{}^2_{L\underline{L}}\lambda_{\alpha\beta} + \zeta_a \mathbf{D}^{(A)}_a \lambda_{\alpha\beta} - \omega \mathbf{D}^{(A)}_L \lambda_{\alpha\beta} \\
&\quad -\frac{1}{2}R_\alpha{}^\gamma{}_{L\underline{L}}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{L\underline{L}}\lambda_{\alpha\gamma} + [F_{L\underline{L}}, \lambda_{\alpha\beta}] + \Delta\lambda_{\alpha\beta} \tag{134}
\end{aligned}$$

Recall (130), then (129) can be written as,

$$\mathbf{D}^{(A)}{}^2_{\underline{L}L}\lambda_{\alpha\beta} = \mathbf{D}^{(A)}_{\underline{L}}(\mathbf{D}^{(A)}_L \lambda_{\alpha\beta}) - 2\zeta_a \mathbf{D}^{(A)}_{e_a}\lambda_{\alpha\beta} + 2\omega \mathbf{D}^{(A)}_L \lambda_{\alpha\beta} \tag{135}$$

Injecting (135) in (134), we get

$$\Box_{\mathbf{g}}^{(A)}\lambda_{\alpha\beta} \tag{136}$$
$$= -\mathbf{D}^{(A)}_{\underline{L}}(\mathbf{D}^{(A)}_L \lambda_{\alpha\beta}) + \zeta_a \mathbf{D}^{(A)}_a \lambda_{\alpha\beta} - \omega \mathbf{D}^{(A)}_L \lambda_{\alpha\beta} - \frac{1}{2}R_\alpha{}^\gamma{}_{L\underline{L}}\lambda_{\gamma\beta}$$
$$- \frac{1}{2}R_\beta{}^\gamma{}_{L\underline{L}}\lambda_{\alpha\gamma} + [F_{L\underline{L}}, \lambda_{\alpha\beta}] + \Delta\lambda_{\alpha\beta} + \frac{1}{2}\mathbf{D}^{(A)}_{\nabla_{\underline{L}}L}\lambda_{\alpha\beta}$$

5.16. **Revisiting** $\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$.

We showed (126) that is,

$$
\begin{aligned}
\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F > \;=\;& \int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda)\delta(u), F > \\
&+ \int_{\Omega_\epsilon} \delta(u) < \mathbf{D}_{\underline{L}}^{(A)}(\mathbf{D}_L^{(A)}\lambda) + \mathbf{D}_{\underline{L}}^{(A)}(\frac{trX}{2}\lambda), F > \\
\mathbf{D}_{\underline{L}}^{(A)}(\frac{tr\chi}{2}\lambda_{\alpha\beta}) \;=\;& \nabla_{\underline{L}}(\frac{tr\chi}{2})\lambda_{\alpha\beta} + \frac{tr\chi}{2}\mathbf{D}_{\underline{L}}^{(A)}\lambda_{\alpha\beta}
\end{aligned}
$$

Recall (129), we also have $\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{tr\chi}{2}\lambda_{\alpha\beta} = 0$ at $q \in N^-(p)\backslash\{p\}$, using (137) we obtain,

$$
\begin{aligned}
&\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} > \\
=\;& \int_{\Omega_\epsilon} \delta(u) < \zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - \omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma} \\
&+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] + \Delta\lambda_{\alpha\beta} + \frac{1}{2}tr\chi\mathbf{D}_{\underline{L}}^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mathbf{D}_{\nabla_{\underline{L}}L}^{(A)}\lambda_{\alpha\beta} + \nabla_{\underline{L}}(\frac{tr\chi}{2})\lambda_{\alpha\beta}, F^{\alpha\beta} > \\
=\;& \int_{\Omega_\epsilon} \delta(u) < \zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} - 2\omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}tr\underline{\chi}\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} \\
&+ \frac{1}{2}tr\underline{\chi}\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma} + [F_{L\underline{L}}, \lambda_{\alpha\beta}] + \Delta\lambda_{\alpha\beta} \\
&+ \frac{1}{2}tr\chi\mathbf{D}_{\underline{L}}^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mathbf{D}_{\nabla_{\underline{L}}L}^{(A)}\lambda_{\alpha\beta} + \nabla_{\underline{L}}(\frac{tr\chi}{2})\lambda_{\alpha\beta}, F^{\alpha\beta} >
\end{aligned}
$$

Let μ be the mass aspect function defined by

$$
\mu = \nabla_{\underline{L}}tr\chi + \frac{1}{2}tr\chi tr\underline{\chi} + 2\omega tr\chi \tag{137}
$$

We have,

$$
\begin{aligned}
&\int_{\Omega_\epsilon} < \Box_{\mathbf{g}}^{(A)}(\lambda\delta(u)), F > \\
=\;& \int_{\Omega_\epsilon} \delta(u) < \zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + \omega\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}tr\underline{\chi}\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma} \\
&+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] + \Delta\lambda_{\alpha\beta} + \frac{1}{2}tr\chi\mathbf{D}_{\underline{L}}^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mathbf{D}_{\nabla_{\underline{L}}L}^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mu\lambda_{\alpha\beta}, F^{\alpha\beta} > \\
=\;& \int_{\Omega_\epsilon} \delta(u) < 2\zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}tr\underline{\chi}\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma} \\
&+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] + \Delta\lambda_{\alpha\beta} + \frac{1}{2}tr\chi\mathbf{D}_{\underline{L}}^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mu\lambda_{\alpha\beta}, F^{\alpha\beta} > \tag{138}
\end{aligned}
$$

(where we used (130))

5.17. **Estimating** $\lim_{\epsilon\to 0} | - \int_{J^-(p)\cap\Sigma_{1-\epsilon}} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_T^{(A)}F^{\alpha\beta} > |$.

Computing $\delta^{ab}\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta}$ we have,

$$\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta} = \mathbf{D}^{(A)}_a(\mathbf{D}^{(A)}_b\lambda_{\alpha\beta}) - \mathbf{D}^{(A)}_{\nabla_a e_b}\lambda_{\alpha\beta} \tag{139}$$

Definition 5.18. We define a restriction of the covariant derivative to the span of $\{e_a\}$, $a \in \{1, 2\}$ at $q \in N^-(p)\backslash\{p\}$ as being,

$$\slashed{\nabla}_a e_b = \nabla_a e_b - \frac{1}{2}\chi_{ab}\underline{L} - \frac{1}{2}\underline{\chi}_{ab}L \tag{140}$$

where

$$\underline{\chi}_{ab} = \mathbf{g}(\nabla_a\underline{L}, e_b) \tag{141}$$

We have

$$\mathbf{g}(\slashed{\nabla}_a e_b, L) = \mathbf{g}(\nabla_a e_b, L) - \frac{1}{2}\chi_{ab}\mathbf{g}(\underline{L}, L) - \frac{1}{2}\underline{\chi}_{ab}\mathbf{g}(L, L) = \mathbf{g}(\nabla_a e_b, L) + \chi_{ab}$$

We have $\mathbf{g}(e_b, L) = 0$ along $N^-(p)$ and since e_a is tangent to $N^-(p)$ at q, we get

$$e_a\mathbf{g}(e_b, L) = 0 = \mathbf{g}(\nabla_a e_b, L) + \mathbf{g}(e_b, \nabla_a L)$$

so

$$\mathbf{g}(\nabla_a e_b, L) = -\mathbf{g}(e_b, \nabla_a L) = -\chi_{ab}$$

and hence

$$\mathbf{g}(\slashed{\nabla}_a e_b, L) = 0$$

Therefore, $\slashed{\nabla}_a e_b$ is tangent to $N^-(p)$.

Computing,

$$\mathbf{g}(\slashed{\nabla}_a e_b, \underline{L}) = \mathbf{g}(\nabla_a e_b, \underline{L}) - \frac{1}{2}\chi_{ab}\mathbf{g}(\underline{L}, \underline{L}) - \frac{1}{2}\underline{\chi}_{ab}\mathbf{g}(L, \underline{L}) = \mathbf{g}(\nabla_a e_b, \underline{L}) + \underline{\chi}_{ab}$$

Similarly, we get

$$\mathbf{g}(\nabla_a e_b, \underline{L}) = -\underline{\chi}_{ab}$$

and therefore

$$\mathbf{g}(\slashed{\nabla}_a e_b, \underline{L}) = 0$$

Finally, we get that $\slashed{\nabla}_a e_b$ is in the span of $\{e_a\}$, $a \in \{1, 2\}$.

Going back to $\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta}$, we have

$$\mathbf{D}^{(A)}_{\nabla_a e_b}\lambda_{\alpha\beta} = \mathbf{D}^{(A)}_{\slashed{\nabla}_a e_b + \frac{1}{2}\chi_{ab}\underline{L} + \frac{1}{2}\underline{\chi}_{ab}L}\lambda_{\alpha\beta} = \mathbf{D}^{(A)}_{\slashed{\nabla}_a e_b}\lambda_{\alpha\beta} + \frac{1}{2}\chi_{ab}\mathbf{D}^{(A)}_{\underline{L}}\lambda_{\alpha\beta} + \frac{1}{2}\underline{\chi}_{ab}\mathbf{D}^{(A)}_L\lambda_{\alpha\beta}$$

Hence, we have at $q \in N^-(p)\backslash\{p\}$,

$$\delta^{\alpha\beta}\mathbf{D}^{(A)2}_{ab}\lambda_{\alpha\beta} = \delta^{\alpha\beta}\mathbf{D}^{(A)}_a(\mathbf{D}^{(A)}_b\lambda_{\alpha\beta}) - \delta^{ab}\mathbf{D}^{(A)}_{\slashed{\nabla}_a e_b}\lambda_{\alpha\beta} - \frac{1}{2}tr\chi\mathbf{D}^{(A)}_{\underline{L}}\lambda_{\alpha\beta} - \frac{1}{2}tr\underline{\chi}\mathbf{D}^{(A)}_L\lambda_{\alpha\beta}$$

$\overline{\nabla}_a e_b$ is in the span of $e_a \in T_q N^-(p)$ for $a \in \{1, 2\}$, we therefore define

$$\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} = \mathbf{D}^{(A)}_a(\mathbf{D}^{(A)}_b \lambda_{\alpha\beta}) - \mathbf{D}^{(A)}_{\overline{\nabla}_a e_b} \lambda_{\alpha\beta} \tag{142}$$

where $\hat{\mathbf{D}}$ is the restriction of the gauge covariant derivative $\mathbf{D}^{(A)}$ along to the span of $e_a \in T_q N^-(p)$, $a \in \{1, 2\}$. We get

$$\begin{aligned}
\delta^{ab}\mathbf{D}^{(A)}{}^2_{ab}\lambda_{\alpha\beta} &= \delta^{ab}\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} - \frac{1}{2}\delta^{ab}\chi_{ab}\mathbf{D}^{(A)}_{\underline{L}}\lambda_{\alpha\beta} - \frac{1}{2}\delta^{ab}\underline{\chi}_{ab}\mathbf{D}^{(A)}_L\lambda_{\alpha\beta} \\
&= \delta^{ab}\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} - \frac{1}{2}tr\chi\mathbf{D}^{(A)}_{\underline{L}}\lambda_{\alpha\beta} - \frac{1}{2}tr\underline{\chi}\mathbf{D}^{(A)}_L\lambda_{\alpha\beta}
\end{aligned} \tag{143}$$

Definition 5.19. Let $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ be the induced Laplacian on the span of $\{e_a\}$, $a \in \{1, 2\}$ defined by,

$$\hat{\Delta}^{(A)}\lambda_{\alpha\beta} = \delta^{ab}\hat{\mathbf{D}}^2_{ab}\lambda_{\alpha\beta} = \Delta^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}tr\underline{\chi}\mathbf{D}^{(A)}_L\lambda_{\alpha\beta} + \frac{1}{2}tr\chi\mathbf{D}^{(A)}_{\underline{L}}\lambda_{\alpha\beta}$$
$$\tag{144}$$

We obtain after injecting (144) and (138) in (111),

$$\begin{aligned}
&\int_{\Omega_\epsilon} < \lambda\delta(u), \Box^{(A)}_{\mathbf{g}} F >_{\mathbf{g}} \\
&= \int_{\Omega_\epsilon} < \hat{\Delta}^{(A)}\lambda_{\alpha\beta} + 2\zeta_a\mathbf{D}^{(A)}_a\lambda_{\alpha\beta} + \frac{1}{2}\mu\lambda_{\alpha\beta} + [F_{L\underline{L}}, \lambda_{\alpha\beta}] \\
&\quad - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} > \\
&\quad - [\int_{J^-(p)\cap\Sigma_t} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}^{(A)}_T F^{\alpha\beta} >]^{t=1-\epsilon}_{t=0} \\
&\quad + [\int_{J^-(p)\cap\Sigma_t} < \mathbf{D}^{(A)}_T(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >]^{t=1-\epsilon}_{t=0}
\end{aligned}$$

We have by definition,

$$\int_{N^-(p)} < \lambda_{\alpha\beta}, \mathbf{D}^{(A)}_T F^{\alpha\beta} >= \int_{\mathbb{S}^2} \int_0^\infty < \lambda_{\alpha\beta}, \mathbf{D}^{(A)}_T F^{\alpha\beta} > (u = 0, s, \omega) ds dA_s$$

We have

$$\frac{dt}{ds} = L(t) = dt(L) = \mathbf{g}(-T, L)$$

We define,

$$\phi = \mathbf{g}(T, L)^{-1} \tag{145}$$

the null lapse function. We have,

$$ds = -\phi dt \tag{146}$$

We denote by $dA_{S_t(p)}$ the area element of the 2-surface $S_t(p) = N^-(p) \cap \Sigma_t$

Thus,

$$\int_0^\infty \int_{\mathbb{S}^2} < \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} > (u=0,s,\omega) dA_s ds$$

$$= \int_{t=1}^{t=-\infty} \int_{S_t} < \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} > (-\phi) dA_t dt$$

$$= \int_{t=-\infty}^{t=1} \int_{S_t} < \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} > \phi dA_t dt$$

We get,

$$-\int_{\mathbf{J}^-(p)\cap\Sigma_{1-\epsilon}} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_T^{(A)} F^{\alpha\beta} > = -\int_{N^-(p)\cap\Sigma_{1-\epsilon}} < \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} > \phi dA_{S_{t=1-\epsilon}(p)}$$

Definition 5.20. We define positive definite Riemannian metric in the following manner:

$$h(e_\alpha, e_\beta) = \mathbf{g}(e_\alpha, e_\beta) + 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) \tag{147}$$

where

$$\frac{\partial}{\partial \hat{t}} = (-\mathbf{g}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}))^{-\frac{1}{2}} \frac{\partial}{\partial t} \tag{148}$$

Definition 5.21. For any $\mathcal{G}$-valued 2-tensor K, we let

$$|K|^2 = h_{\alpha\mu} h_{\beta\nu} |K^{\mu\nu}|.|K^{\alpha\beta}| \tag{149}$$

Lemma 5.22. *For any two $\mathcal{G}$-valued tensors K and G, we have*

$$| < K_{\alpha\beta}, G^{\alpha\beta} > | \lesssim (|K|^2)^{\frac{1}{2}}.(|G|^2)^{\frac{1}{2}} \tag{150}$$

Proof

$$\begin{aligned}
| < K_{\alpha\beta}, G^{\alpha\beta} > | &= |\mathbf{g}_{\alpha\mu}\mathbf{g}_{\beta\nu} < K^{\mu\nu}, G^{\alpha\beta} > | \\
&\leq |\mathbf{g}_{\alpha\mu}|.|\mathbf{g}_{\beta\nu}|.| < K^{\mu\nu}, G^{\alpha\beta} > | \\
&\leq h_{\alpha\mu} h_{\beta\nu} | < K^{\mu\nu}, G^{\alpha\beta} > | \\
&\leq h_{\alpha\mu} h_{\beta\nu} |K^{\mu\nu}|.|G^{\alpha\beta}| \\
&\leq (h_{\alpha\mu} h_{\beta\nu} |K^{\mu\nu}|.|K^{\alpha\beta}|)^{\frac{1}{2}}.(h_{\alpha\mu} h_{\beta\nu} |G^{\mu\nu}|.|G^{\alpha\beta}|)^{\frac{1}{2}}
\end{aligned}$$

(by applying Cauchy-Schwarz)

∎

Hence,

$$|\int_{N^-(p)\cap\Sigma_{1-\epsilon}} < \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} > \phi dA_{S_{1-\epsilon}(p)}|$$

$$\lesssim ||\phi||_{L^\infty} (\int_{N^-(p)\cap\Sigma_{1-\epsilon}} |\lambda|^2 dA_{S_{1-\epsilon}(p)})^{\frac{1}{2}} (\int_{N^-(p)\cap\Sigma_{1-\epsilon}} |\mathbf{D}_T^{(A)} F|^2 dA_{S_{1-\epsilon}(p)})^{\frac{1}{2}}$$

(by Cauchy-Schwarz)

$$\lesssim ||\phi||_{L^\infty}||\mathbf{D}_T^{(A)}F||_{L^\infty}||\lambda||_{L^2(N^-(p)\cap\Sigma_{1-\epsilon})}|A_{S_{1-\epsilon}(p)}|^{\frac{1}{2}} \tag{151}$$

where $A_{S_{1-\epsilon}(p)}$ denote the area of $S_{1-\epsilon}(p) = N^-(p) \cap \Sigma_{t=1-\epsilon}$ and where,

$$||\lambda||_{L^2(\Sigma_{1-\epsilon}(p))} = (\int_{S_{1-\epsilon}=N^-(p)\cap\Sigma_{1-\epsilon}} h_{\alpha\mu}h_{\beta\nu}|\lambda^{\mu\nu}|.|\lambda^{\alpha\beta}|dA_{S_{1-\epsilon}(p)})^{\frac{1}{2}} \tag{152}$$

and

$$||\mathbf{D}_T^{(A)}F||_{L^\infty} = ||(h_{\alpha\mu}h_{\beta\nu}|\mathbf{D}_T^{(A)}F^{\mu\nu}|.|\mathbf{D}_T^{(A)}F^{\alpha\beta}|)^{\frac{1}{2}}||_{L^\infty} \tag{153}$$

We want to study now the behavior of $|A_{S_{1-\epsilon}(p)}|^{\frac{1}{2}}$ near $\epsilon = 0$.

Lemma 5.23. *We have,*

$$\begin{aligned} |A_{S_t}| &= 4\pi s^2 + o(s^2) \\ &= 4\pi(1-t)^2 + o((1-t)^2) \end{aligned} \tag{154}$$

Proof

We have,

$$\frac{d}{dt}|A_{S_t(p)}| = \int_{S_t(p)} \phi tr\chi dA \tag{155}$$

We also have,

$$\frac{d}{ds}(tr\chi) + \frac{1}{2}(tr\chi)^2 = -|\hat{\chi}|^2 - Ric(L,L) \tag{156}$$

where $\hat{\chi}$ is the traceless part of χ and $(str\chi)(p) = 2, \hat{\chi}(p) = 0$. This yields to

$$\lim_{q\longmapsto p} |tr\chi(q) - \frac{2}{s}| = 0 \tag{157}$$

(see [Wang]).

$$\phi = \mathbf{g}(T,L)^{-1} = (\nabla_T u)^{-1}.$$

Since u is smooth and

$$\phi(p) = 1 = (\nabla_T u)^{-1}(p)$$

we also have that ϕ smooth and bounded near p. Thus,

$$\lim_{q\longmapsto p} |\phi(q) - 1| = 0 \tag{158}$$

Since, $ds = -\phi dt$, we get,

$$\int_0^s 1.ds = \int_1^t -\phi dt$$

Hence, using (158), we get

$$s = 1 - t + o(1-t) \tag{159}$$

(155), (157), (158) and (159) yield to (154).

■

Injecting (154) in (151), we get,

$$| - \int_{N^-(p) \cap \Sigma_{1-\epsilon}} < \lambda_{\alpha\beta}, \mathbf{D}_T^{(A)} F^{\alpha\beta} > \phi dA_{S_{1-\epsilon}(p)}| \quad \lesssim \quad \epsilon ||\phi||_{L^\infty} ||\mathbf{D}_T^{(A)} F||_{L^\infty} ||\lambda||_{L^2(S_{1-\epsilon})}$$

(160)

and

$$||\lambda||_{L^2(S_{1-\epsilon})} = (\int_{S_{1-\epsilon}} |\lambda|^2 dA)^{\frac{1}{2}} \quad \lesssim \quad ||\lambda||_{L^\infty(S_{1-\epsilon})} |S_{1-\epsilon}|^{\frac{1}{2}} \lesssim \epsilon ||\lambda||_{L^\infty(S_{1-\epsilon})}$$

Now, we would want to study the behavior of $||\lambda||_{L^\infty(S_{1-\epsilon})}$ when $\epsilon \longmapsto 0$.

Lemma 5.24. *Let $\Psi_{\alpha\beta}$ be a $\mathcal{G}$-valued tensor, and $|\Psi_{\alpha\beta}| = < \Psi_{\alpha\beta}, \Psi_{\alpha\beta} >^{\frac{1}{2}}$. Then,*

$$|\nabla_\sigma |\Psi_{\alpha\beta}|| \quad \leq \quad |\mathbf{D}^{(A)} \Psi_{\alpha\beta}| + |\Psi(\nabla_\sigma e_\alpha, e_\beta)| + |\Psi(e_\alpha, \nabla_\sigma e_\beta)| \qquad (161)$$

Proof

We can compute

$$
\begin{aligned}
\nabla_\sigma |\Psi_{\alpha\beta}| &= \frac{2 < \nabla_\sigma(\Psi_{\alpha\beta}), \Psi_{\alpha\beta} >}{2 < \Psi_{\alpha\beta}, \Psi_{\alpha\beta} >^{\frac{1}{2}}} \\
&= \frac{2 < \nabla_\sigma(\Psi_{\alpha\beta}), \Psi_{\alpha\beta} > -2 < \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] > +2 < \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] >}{2 < \Psi_{\alpha\beta}, \Psi_{\alpha\beta} >^{\frac{1}{2}}} \\
&= \frac{2 < \nabla_\sigma(\Psi_{\alpha\beta}), \Psi_{\alpha\beta} > -2 < [\Psi_{\alpha\beta}, A_\sigma], \Psi_{\alpha\beta} > +2 < \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] >}{2 < \Psi_{\alpha\beta}, \Psi_{\alpha\beta} >^{\frac{1}{2}}}
\end{aligned}
$$

(since $< , >$ is Ad-invariant)

$$
\begin{aligned}
&= \frac{2 < \nabla_\sigma \Psi_{\alpha\beta}, \Psi_{\alpha\beta} > +2 < [A_\sigma, \Psi_{\alpha\beta}], \Psi_{\alpha\beta} > +2 < \Psi_{\alpha\beta}, [A_\sigma, \Psi_{\alpha\beta}] >}{2 < \Psi_{\alpha\beta}, \Psi_{\alpha\beta} >^{\frac{1}{2}}} \\
&\quad + \frac{2 < \Psi(\nabla_\sigma e_\alpha, e_\beta), \Psi_{\alpha\beta} > +2 < \Psi(e_\alpha, \nabla_\sigma e_\beta), \Psi_{\alpha\beta} >}{2 < \Psi_{\alpha\beta}, \Psi_{\alpha\beta} >^{\frac{1}{2}}}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\nabla_\sigma |\Psi_{\alpha\beta}|| &\leq \frac{|\mathbf{D}_\sigma^{(A)} \Psi_{\alpha\beta}||\Psi_{\alpha\beta}|}{|\Psi_{\alpha\beta}|} + \frac{|\Psi(\nabla_\sigma e_\alpha, e_\beta)||\Psi_{\alpha\beta}|}{|\Psi_{\alpha\beta}|} + \frac{|\Psi(e_\alpha, \nabla_\sigma e_\beta)||\Psi_{\alpha\beta}|}{|\Psi_{\alpha\beta}|} \\
&\leq |\mathbf{D}^{(A)} \Psi_{\alpha\beta}| + |\Psi(\nabla_\sigma e_\alpha, e_\beta)| + |\Psi(e_\alpha, \nabla_\sigma e_\beta)|
\end{aligned}
$$

■

Lemma 5.25. *We have,*

$$\nabla_\sigma h(e_\alpha, e_\beta) \quad = \quad 2\mathbf{g}(e_\alpha, \nabla_\sigma \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) + 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \nabla_\sigma \frac{\partial}{\partial \hat{t}}) \quad (162)$$

Proof

$$
\begin{aligned}
\nabla_\sigma h(e_\alpha, e_\beta) &= \partial_\sigma h(e_\alpha, e_\beta) - h(\nabla_\sigma e_\alpha, e_\beta) - h(e_\alpha, \nabla_\sigma e_\beta) \\
&= \nabla_\sigma \mathbf{g}(e_\alpha, e_\beta) + 2\nabla_\sigma[\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}})] \\
&= 2\partial_\sigma[\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}})] - 2\mathbf{g}(\nabla_\sigma e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) \\
&\quad -2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(\nabla_\sigma e_\beta, \frac{\partial}{\partial \hat{t}})
\end{aligned}
$$

(since the metric $\mathbf{g}$ is Killing)

$$
\begin{aligned}
&= 2\partial_\sigma \mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) + 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\partial_\sigma \mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) \\
&\quad -2\mathbf{g}(\nabla_\sigma e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) - 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(\nabla_\sigma e_\beta, \frac{\partial}{\partial \hat{t}}) \\
&= 2[\partial_\sigma \mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}) - \mathbf{g}(\nabla_\sigma e_\alpha, \frac{\partial}{\partial \hat{t}})].\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) \\
&\quad +2[\partial_\sigma \mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) - \mathbf{g}(\nabla_\sigma e_\beta, \frac{\partial}{\partial \hat{t}})].\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}})
\end{aligned}
$$

Using the fact that $\nabla g = 0$, we get,

$$
\nabla_\sigma h(e_\alpha, e_\beta) = 2\mathbf{g}(e_\alpha, \nabla_\sigma \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \frac{\partial}{\partial \hat{t}}) + 2\mathbf{g}(e_\alpha, \frac{\partial}{\partial \hat{t}}).\mathbf{g}(e_\beta, \nabla_\sigma \frac{\partial}{\partial \hat{t}})
$$

$\blacksquare$

Let,

$$
\hat{t}_\alpha = (\frac{\partial}{\partial \hat{t}})_\alpha = \mathbf{g}_{\mu\alpha}(\frac{\partial}{\partial \hat{t}})^\mu
$$

Hence, we can write (147) as,

$$
h_{\alpha\beta} = \mathbf{g}_{\alpha\beta} + 2(\frac{\partial}{\partial \hat{t}})_\alpha (\frac{\partial}{\partial \hat{t}})_\beta \tag{163}
$$

and (162) as,

$$
\nabla_\sigma h_{\alpha\beta} = 2[\nabla_\sigma \hat{t}_\alpha.\hat{t}_\beta + \hat{t}_\alpha.\nabla_\sigma \hat{t}_\beta] \tag{164}
$$

Lemma 5.26. *Let Ψ be a $\mathcal{G}$-valued tensor, we have,*

$$
|\nabla_\sigma |\Psi|^2|(p) \leq C(p)[|\mathbf{D}_\sigma^{(A)}\Psi|^2 + |\Psi|^2] \tag{165}
$$

where $C(p)$ depends on the space-time geometry on the point p.

Proof

$$\nabla_\sigma |\Psi|^2 \;=\; \nabla_\sigma(h_{\alpha\mu}h_{\beta\nu}|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|) = \nabla_\sigma(h_{\alpha\mu}h_{\beta\nu}).|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|$$
$$+ h_{\alpha\mu}h_{\beta\nu}.\nabla_\sigma(|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|)$$

Therefore,

$$|\nabla_\sigma |\Psi|^2| \;\leq\; |(\nabla_\sigma h_{\alpha\mu})h_{\beta\nu}|.|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}| + |h_{\alpha\mu}(\nabla_\sigma h_{\beta\nu})|.|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|$$
$$+ |h_{\alpha\mu}h_{\beta\nu}|.(|\mathbf{D}_\sigma^{(A)}\Psi^{\mu\nu}| + |\Psi(\nabla_\sigma e^\mu, e^\nu)| + |\Psi(e^\mu, \nabla_\sigma e^\nu)|.|\Psi^{\alpha\beta}|)$$
$$+ |h_{\alpha\mu}h_{\beta\nu}|.|\Psi^{\mu\nu}|.(|\mathbf{D}_\sigma^{(A)}\Psi^{\alpha\beta}| + |\Psi(\nabla_\sigma e^\alpha, e^\beta)| + |\Psi(e^\alpha, \nabla_\sigma e^\beta)|)$$

(due to (161)).

Using (162), applying Cauchy-Schwarz, using the fact that the metric is smooth, and the inequality $a.b \lesssim a^2 + b^2$, we get,

$$|\nabla_\sigma |\Psi|^2|(p) \;\leq\; C(p)[h_{\alpha\mu}h_{\beta\nu}|\mathbf{D}_\sigma^{(A)}\Psi^{\mu\nu}|.|\mathbf{D}_\sigma^{(A)}\Psi^{\alpha\beta}| + h_{\alpha\mu}h_{\beta\nu}|\Psi^{\mu\nu}|.|\Psi^{\alpha\beta}|]$$
$$\lesssim\; |\mathbf{D}_\sigma^{(A)}\Psi|^2 + |\Psi|^2$$

∎

Finally, we get

Lemma 5.27. *We have,*

$$\sup_{0\leq\overline{s}\leq s} |\overline{s}\lambda|^2 \leq C(p,s)|J|^2 \tag{166}$$

Proof

We also have at $q \in N^-(p)\backslash\{p\}$,

$$\mathbf{D}_L^{(A)}(s\lambda_{\alpha\beta}) = L(s)\lambda_{\alpha\beta} + s\mathbf{D}_L^{(A)}\lambda_{\alpha\beta} = \lambda_{\alpha\beta} + s(-\frac{1}{2}tr\chi\lambda_{\alpha\beta}) = -\frac{s}{2}\lambda_{\alpha\beta}(tr\chi - \frac{2}{s})$$

As $|tr\chi - \frac{2}{s}| \longmapsto 0$, we get

$$\mathbf{D}_L^{(A)}(s\lambda_{\alpha\beta}) = O(1)s\lambda_{\alpha\beta} \tag{167}$$

Hence,

$$|\nabla_L|s\lambda|^2| \;\lesssim\; |\mathbf{D}_L^{(A)}(s\lambda)|^2 + |s\lambda|^2$$
$$\lesssim\; |s\lambda|^2$$

(due to (167)).

For all $(u = 0, \overline{s}, \omega) \in N_\tau^-(p)$,

$$\int_0^s \nabla_L|s\lambda|^2 d\overline{s} = |s\lambda|^2(s) - |s\lambda|^2(p) \leq O(s)C(p)\sup_{0\leq\overline{s}\leq s} |s\lambda|^2$$

As
$$|s\lambda|^2(p) = |J|^2$$
choosing s small depending on p we have (5.27).

∎

Therefore $s\lambda_{\alpha\beta}$ remains bounded near p, and it is also smooth away from p, so $\epsilon||\lambda||_{L^\infty(S_{1-\epsilon})}$ remains bounded and therefore $||\lambda||_{L^2(S_{1-\epsilon})}$ remains bounded. Therefore (160) gives,

$$\lim_{\epsilon\to 0} |-\int_{J^-(p)\cap\Sigma_{1-\epsilon}} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_T^{(A)}F^{\alpha\beta} > | = 0 \qquad (168)$$

5.28. **Estimating** $\lim_{\epsilon\to 0}\int_{J^-(p)\cap\Sigma_{1-\epsilon}} < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$.

Examining now,

$$\int_{J^-(p)\cap\Sigma_{1-\epsilon}} < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >$$

$$= \int_{J^-(p)\cap\Sigma_{1-\epsilon}} < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta})\delta(u) + \nabla_T u\delta'(u)\lambda_{\alpha\beta}, F^{\alpha\beta} >$$

$$= \int_{J^-(p)\cap\Sigma_{1-\epsilon}} \delta(u) < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}), F^{\alpha\beta} > + \int_{J^-(p)\cap\Sigma_{1-\epsilon}} \delta'(u)\phi^{-1} < \lambda_{\alpha\beta}, F^{\alpha\beta} >$$

$$= I_\epsilon^1 + I_\epsilon^2 \qquad (169)$$

At $q \in N^-(p)\backslash\{p\}$ define,

$$N = \phi L + T \qquad (170)$$

We have,

$$\begin{aligned} \mathbf{g}(N, N) &= \mathbf{g}(\phi L + T, \phi L + T) = \phi\mathbf{g}(T, L) + \phi\mathbf{g}(L, T) + \mathbf{g}(T, T) \\ &= 2\phi\phi^{-1} - 1 = 2 - 1 = 1 \end{aligned}$$

means that N is unit. For all $X \in T_q S_{1-\epsilon}(p)$ tangent to $S_{1-\epsilon}(p)$, i.e. $X \in T_q N^-(p)\cap T_q\Sigma_{1-\epsilon}$, we have

$$\mathbf{g}(N, X) = \mathbf{g}(\phi L + T, X) = \phi\mathbf{g}(L, X) + \mathbf{g}(T, X) \qquad (171)$$

$$\mathbf{g}(L, X) = 0 \qquad (172)$$

(since $X \in T_q N^-(p)$), and

$$\mathbf{g}(T, X) = 0 \qquad (173)$$

(since $X \in T_q\Sigma_{1-\epsilon}$).

(172) and (173) show that N is the unit normal to $S_{1-\epsilon} = N^-(p)\cap\Sigma_{1-\epsilon}$, it can be extended locally to define a vectorfield.

Thus, we have

$$
\begin{aligned}
\nabla_N \delta(u) &= \nabla_N(u).\delta^{'}(u) = \nabla_{\phi L + T}(u).\delta^{'}(u) \\
&= (\phi \nabla_L(u) + \nabla_T(u)).\delta^{'}(u) = \nabla_T(u).\delta^{'}(u) \\
&= \phi^{-1}\delta^{'}(u)
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_\epsilon^2 &= \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \nabla_N \delta(u) <\lambda_{\alpha\beta}, F^{\alpha\beta}> \\
&= -\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) \nabla_N <\lambda_{\alpha\beta}, F^{\alpha\beta}> \\
&\quad -\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \mathrm{div}(N).\delta(u) <\lambda_{\alpha\beta}, F^{\alpha\beta}>
\end{aligned}
$$

(by integration by parts)

We choose e_1, e_2 tangent to $S_{1-\epsilon}$. Since N, e_1, e_2 are unit we have

$$
\mathrm{div} N = \mathbf{g}(\nabla_N N, N) + \mathbf{g}(\nabla_a N, e_a), \quad a \in \{1, 2\}
$$

$\mathbf{g}(N, N) = 1$ gives $\mathbf{g}(\nabla_N N, N) = 0$. We get $\mathrm{div} N = \mathbf{g}(\nabla_a N, e_a)$ and N is unit normal to $S_{1-\epsilon}$, and $N \in T_q \Sigma_{1-\epsilon}$ since,

$$
\mathbf{g}(N, T) = \mathbf{g}(\phi L + T, T) = \phi \mathbf{g}(L, T) + \mathbf{g}(T, T) = \phi \phi^{-1} - 1 = 1 - 1 = 0
$$

so we get $\mathrm{div} N = tr\theta$, where θ is the second fundamental form of the surface $S_{1-\epsilon}$ embedded in $\Sigma_{1-\epsilon}$, defined as,

$$
\theta(X, Y) = \mathbf{g}(\nabla_X N, Y) \tag{174}
$$

for all $X, Y \in T_q S_{1-\epsilon}$. Thus (174) becomes,

$$
\begin{aligned}
I_\epsilon^2 &= -\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u)(\nabla_N <\lambda_{\alpha\beta}, F^{\alpha\beta}> + tr\theta <\lambda_{\alpha\beta}, F^{\alpha\beta}>) \\
&= -\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u)(<\mathbf{D}_N^{(A)}\lambda_{\alpha\beta}, F^{\alpha\beta}> + tr\theta <\lambda_{\alpha\beta}, F^{\alpha\beta}>) \\
&\quad -\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) <\lambda_{\alpha\beta}, \mathbf{D}_N^{(A)} F^{\alpha\beta}> \tag{175}
\end{aligned}
$$

We showed (168), in the same manner, we have,

$$
\lim_{\epsilon \to 0} \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) <\lambda_{\alpha\beta}, \mathbf{D}_N^{(A)} F^{\alpha\beta}> = 0 \tag{176}
$$

Thus, injecting (176) and (175) in (169), we get,

$$
\begin{aligned}
\lim_{\epsilon \to 0} I_\epsilon^1 + I_\epsilon^2 &= \lim_{\epsilon \to 0} \Big(\int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) [< \mathbf{D}_T^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > \\
&\quad - < \mathbf{D}_N^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > + tr\theta < \lambda_{\alpha\beta}, F^{\alpha\beta} >] \Big) \\
&= \lim_{\epsilon \to 0} - \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) (< \mathbf{D}_{N-T}^{(A)} \lambda_{\alpha\beta}, F^{\alpha\beta} > + tr\theta < \lambda_{\alpha\beta}, F^{\alpha\beta} >)
\end{aligned}
$$
$$(177)$$

We recall that $N = \phi L + T$, thus $\phi L = N - T$, therefore

$$
\mathbf{D}_{N-T}^{(A)} \lambda_{\alpha\beta} = \phi \mathbf{D}_L^{(A)} \lambda_{\alpha\beta}
$$

and we recall that by construction of λ, we have $\mathbf{D}_L^{(A)} \lambda_{\alpha\beta} + \frac{1}{2} tr\chi \lambda_{\alpha\beta} = 0$ at $q \in N^-(p) \backslash \{p\}$.

We obtain,

$$
\mathbf{D}_{N-T}^{(A)} \lambda_{\alpha\beta} = -\frac{1}{2} tr\chi \phi \lambda_{\alpha\beta}
\tag{178}
$$

at $q \in N^-(p) \backslash \{p\}$.

Hence, from (178) we can write (177) as,

$$
\lim_{\epsilon \to 0} I_\epsilon^1 + I_\epsilon^2 = -\lim_{\epsilon \to 0} \int_{J^-(p) \cap \Sigma_{1-\epsilon}} \delta(u) (-\frac{1}{2} \phi tr\chi + tr\theta) < \lambda_{\alpha\beta}, F^{\alpha\beta} >
\tag{179}
$$

$$
\begin{aligned}
\theta_{aa} &= \mathbf{g}(\nabla_a N, e_a) = \mathbf{g}(\nabla_a(\phi L + T), e_a) \\
&= \mathbf{g}(e_a(\phi) L + \phi \nabla_a L + \nabla_a T, e_a) \\
&= e_a(\phi) \mathbf{g}(L, e_a) + \phi \mathbf{g}(\nabla_a L, e_a) + \mathbf{g}(\nabla_a T, e_a).
\end{aligned}
$$

We have $\mathbf{g}(L, e_a) = 0$, therefore,

$$
\theta_{aa} = \phi \chi_{aa} + k_{aa}
\tag{180}
$$

where,

$$
k_{aa} = \mathbf{g}(\nabla_a T, e_a)
\tag{181}
$$

Injecting (180) and (181) in (179) we get,

$$
\lim_{\epsilon \to 0} I_\epsilon^1 + I_\epsilon^2 = -\lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u) (\frac{1}{2} \phi tr\chi + k_{aa}). < \lambda_{\alpha\beta}, F^{\alpha\beta} >
\tag{182}
$$

where the repeated index k_{aa} means summation $\sum_{a=1,2} k_{aa} = \delta^{ab} k_{ab}$.

We get,

$$\lim_{\epsilon \to 0} I_\epsilon^1 + I_\epsilon^2 \tag{183}$$

$$= -\frac{1}{2} \lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\phi tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} > - \lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\delta^{ab}k_{ab} < \lambda_{\alpha\beta}, F^{\alpha\beta} >$$

Lemma 5.29. *We have,*

$$\lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\delta^{ab}k_{ab} < \lambda_{\alpha\beta}, F^{\alpha\beta} >= 0$$

Proof :

$$|\int_{\Sigma_{1-\epsilon}} \delta^{ab}k_{ab}\delta(u) < \lambda_{\alpha\beta}, F^{\alpha\beta} > | = \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \delta^{ab}k_{ab} < \lambda_{\alpha\beta}, F^{\alpha\beta} > \phi dA_{S_{t=1-\epsilon}}|$$

$$\lesssim ||k||_{L^\infty}||F||_{L^\infty}||\phi||_{L^\infty}||\lambda||_{L^2(S_{1-\epsilon}(p))}|A_{S_{1-\epsilon}(p)}|^{\frac{1}{2}}$$

$$\lesssim \epsilon ||k||_{L^\infty}||F||_{L^\infty}||\phi||_{L^\infty}||\lambda||_{L^2(S_{1-\epsilon}(p))}$$

And as we showed previously $||\lambda||_{L^2(S_{1-\epsilon}(p))}$ remains bounded as $\epsilon \longmapsto 0$. Thus,

$$\lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\delta^{ab}k_{ab} < \lambda_{\alpha\beta}, F^{\alpha\beta} >= 0 \tag{184}$$

∎

We are left to estimate $-\frac{1}{2}\lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\phi tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} >$ in (183).

Lemma 5.30. *We have,*

$$-\frac{1}{2} \lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\phi tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} >= -\frac{1}{2} \lim_{\epsilon \to 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > dA$$

Proof :

$$-\frac{1}{2} \lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\phi tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} >= -\frac{1}{2} \lim_{\epsilon \to 0} \int_{N^-(p) \cap \Sigma_{t=1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} >_{\mathbf{g}} dA$$

We have

$$|\int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) > dA|$$

$$= |\int_{S_{1-\epsilon}(p)} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) > dA|$$

$$\lesssim ||\phi||_{L^\infty}^2 ||A||_{L^2(S_{1-\epsilon}(p))}||F - F(p)||_{L^\infty(S_{1-\epsilon}(p))}|S_{1-\epsilon}(p)|^{\frac{1}{2}}||tr\chi||_{L^\infty(S_{1-\epsilon}(p))}$$

As,

$$|S_{1-\epsilon}(p)| \sim 4\pi\epsilon^2 \quad \text{as} \quad \epsilon \longmapsto 0$$

and

$$|tr\chi| \sim \frac{2}{\epsilon} \quad \text{as} \quad \epsilon \longmapsto 0$$

we get,

$$|S_{1-\epsilon}(p)|^{\frac{1}{2}}||tr\chi||_{L^\infty(S_{1-\epsilon}(p))} \sim 8\pi \quad \text{as} \quad \epsilon \longmapsto 0$$

This yields to

$$|\int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) > dA|$$
$$\lesssim ||\phi||_{L^\infty}^2 ||A||_{L^2(S_{1-\epsilon}(p))}||F - F(p)||_{L^\infty(S_{1-\epsilon}(p))}$$

Since,

$$\lim_{\epsilon\to 0} ||F - F(p)||_{L^\infty(S_{1-\epsilon}(p))} = 0$$

and as we showed $||\lambda_{\alpha\beta}||_{L^2(S_{1-\epsilon}(p))}$ remains bounded as $\epsilon \longmapsto 0$, we get

$$\lim_{\epsilon\to 0} |\int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} - F^{\alpha\beta}(p) >_{\mathbf{g}} dA| = 0$$

Therefore,

$$\lim_{\epsilon\to 0} \int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 trX < \lambda_{\alpha\beta}, F^{\alpha\beta} > dA$$
$$= \lim_{\epsilon\to 0} \int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 trX < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > dA \qquad (185)$$

∎

Lemma 5.31. *We have,*

$$\lim_{\epsilon\to 0} \int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 trX < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) > dA = 8\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) >$$

Proof

We have,

$$\lim_{\epsilon\to 0} \int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) >$$
$$= \lim_{\epsilon\to 0} \int_{N^-(p)\cap\Sigma_{1-\epsilon}} \phi^2 (s^{-1}tr\chi) < (s\lambda_{\alpha\beta}), F^{\alpha\beta}(p) > \qquad (186)$$

As,

$$|tr\chi - \frac{2}{s}| = O(s^2) \qquad (187)$$

where O depends on the geometry of the space-time (see for example proposition 3.2 in the thesis of Q. Wang [Wang]), we get,

$$\lim_{s\to 0} \sup_{S_{1-\epsilon}(p)} |s^{-1}tr\chi - \frac{2}{s^2}| = 0 \qquad (188)$$

and we know that,

$$\lim_{s \to 0} \sup_{S_{1-\epsilon}(p)} |\phi - 1| = 0 \tag{189}$$

and,

$$\lim_{s \to 0} (s\lambda_{\alpha\beta}) = \mathbf{J}_p \tag{190}$$

This yields to

$$|s^{-1} tr\chi| \sim_{\epsilon \to 0} \frac{2}{\epsilon^2} \tag{191}$$

$$|\phi| \sim_{\epsilon \to 0} 1 \tag{192}$$

$$(s\lambda_{\alpha\beta}) \sim_{\epsilon \to 0} \mathbf{J}_p \tag{193}$$

and therefore,

$$\int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 (s^{-1} tr\chi) < (s\lambda_{\alpha\beta}), F^{\alpha\beta}(p) > dA$$

$$\sim_{\epsilon \to 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \frac{2}{\epsilon^2} < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > dA \tag{194}$$

and since,

$$|N^-(p) \cap \Sigma_{1-\epsilon}| = |S_{1-\epsilon}(p)| \sim_{\epsilon \to 0} 4\pi\epsilon^2 \tag{195}$$

we get,

$$\int_{N^-(p) \cap \Sigma_{1-\epsilon}} \frac{2}{\epsilon^2} < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > dA = \frac{2}{\epsilon^2} (4\pi)\epsilon^2 < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > +O(\epsilon) \tag{196}$$

where $O(\epsilon) \longmapsto 0$ as $\epsilon \longmapsto 0$

Given (186), this yields to

$$\lim_{\epsilon \to 0} \int_{N^-(p) \cap \Sigma_{1-\epsilon}} \phi^2 tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta}(p) >= 8\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > \tag{197}$$

$$\blacksquare$$

From (5.30) we get,

$$-\frac{1}{2} \lim_{\epsilon \to 0} \int_{\Sigma_{1-\epsilon}} \delta(u)\phi tr\chi < \lambda_{\alpha\beta}, F^{\alpha\beta} >= -4\pi < (\mathbf{J}_p)_{\alpha\beta}, F(p)^{\alpha\beta} >$$

5.32. **The parametrix.**

Finally, combining (107), (111), (138), (168), (169), (183), (5.29), (5.30) and (5.31), we get,

$$\int_\Omega < \lambda_{\alpha\beta}\delta(u), \Box_{\mathbf{g}}^{(A)}F^{\alpha\beta} >$$

$$= \lim_{\epsilon \to 0}[\int_{\Omega_\epsilon} \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta} + 2\zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mu\lambda_{\alpha\beta}$$

$$+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} >] + 0 - 4\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) >$$

$$+ \int_{J^-(p)\cap\Sigma_t} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_T^{(A)}F^{\alpha\beta} >|_{t=0} - \int_{J^-(p)\cap\Sigma_t} < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} >|_{t=0}$$

Therefore,

$$4\pi < (\mathbf{J}_p)_{\alpha\beta}, F^{\alpha\beta}(p) > = -\int_\Omega < \lambda\delta(u), \Box_{\mathbf{g}}^{(A)}F^{\alpha\beta} >$$

$$+ \int_\Omega \delta(u) < \hat{\Delta}^{(A)}\lambda_{\alpha\beta} + 2\zeta_a\mathbf{D}_a^{(A)}\lambda_{\alpha\beta} + \frac{1}{2}\mu\lambda_{\alpha\beta}$$

$$+ [F_{L\underline{L}}, \lambda_{\alpha\beta}] - \frac{1}{2}R_\alpha{}^\gamma{}_{\underline{L}L}\lambda_{\gamma\beta} - \frac{1}{2}R_\beta{}^\gamma{}_{\underline{L}L}\lambda_{\alpha\gamma}, F^{\alpha\beta} >$$

$$+ \int_{J^-(p)\cap\Sigma} < \lambda_{\alpha\beta}\delta(u), \mathbf{D}_T^{(A)}F^{\alpha\beta} >$$

$$- \int_{J^-(p)\cap\Sigma} < \mathbf{D}_T^{(A)}(\lambda_{\alpha\beta}\delta(u)), F^{\alpha\beta} > \qquad (198)$$

where $\hat{\Delta}^{(A)}\lambda_{\alpha\beta}$ is the induced Laplacian on the span of $\{e_a\}$, $a \in \{1,2\}$, of $\lambda_{\alpha\beta}$, defined by (144), and where the last two terms are the contribution of the initial data, the first term is the contribution of the nonlinear term in the tensorial wave equation, and the middle term is related to the geometry of the problem.

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