

Krylov Subspace-Based Model Reduction for a Class of Bilinear Descriptor Systems[☆]

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Abstract

We consider model order reduction for bilinear descriptor systems using an interpolatory projection framework. Such nonlinear descriptor systems can be represented by a series of generalized linear descriptor systems (also called subsystems) by utilizing the Volterra-Wiener approach [Rugh, 1981]. Standard projection techniques for bilinear systems utilize the generalized transfer functions of these subsystems to construct an interpolating approximation. However, the resulting reduced-order system may not match the polynomial parts of the generalized transfer functions. This may result in an unbounded error in terms of \mathcal{H}_2 or \mathcal{H}_∞ norms. In this paper, we derive an explicit expression for the polynomial part of each subsystem by assuming a special structure of the bilinear system which reduces to an index-1 linear descriptor system or differential algebraic equation (DAE) if the bilinear terms are zero. This allows us to propose an interpolatory technique for bilinear DAEs which not only achieves interpolation, but also retains the polynomial parts of the bilinear systems. The approach extends the interpolatory technique for index-1 linear DAEs [Beattie/Gugercin, 2009] to bilinear DAEs. Numerical examples are used to illustrate the theoretical results.

Keywords: Krylov subspace, Bilinear systems, Descriptor systems, Model order reduction, Moment-matching, Transfer functions.

1. Introduction

The importance of model order reduction arises in the analysis of high order mathematical models that describe complex dynamical systems. These high order models are often expensive to analyze and therefore, they are replaced

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5 by reduced-order systems to simulate the approximate behavior of the actual system. Various approaches have been developed for model order reduction, see, e.g., [1, 2, 3, 4]. In case of linear systems, balanced truncation [5], moment-matching methods [6] and the iterative rational Krylov method [7] are well-used and well-established model reduction methods. However, most practical
 10 systems have nonlinearities and model reduction of such systems, particularly models described by differential algebraic equations (DAEs), also called descriptor systems, are less developed and require further research.

In this paper, we investigate Krylov projection methods for bilinear descriptor systems. In general, a bilinear descriptor system has the form

$$\begin{aligned}
 E\dot{x}(t) &= Ax(t) + \sum_{i=1}^m N^{(i)}x(t)u_i(t) + Bu(t), \\
 y(t) &= Cx(t) + Du(t),
 \end{aligned}
 \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. The matrices E , A , $N^{(i)}$, $i = 1, \dots, m$, B , C and D are all real with dimensions determined by those of $x(t)$, $u(t)$ and $y(t)$. Notice that the bilinear terms in the system, involving the product of states and inputs, make it a special class of nonlinear systems. Also, the matrix E might be singular, but it is assumed that the matrix pencil $\alpha E - \beta A$ is regular, that is

$$\det(\alpha E - \beta A) \neq 0, \quad \text{for some } (\alpha, \beta) \in \mathbb{C}^2.$$

The generalized eigenvalues of the matrix pencil $\lambda E - A$ are defined by pairs $(\alpha_i, \beta_i) \in \mathbb{C}^2 \setminus \{0, 0\}$ such that $\det(\alpha_i E - \beta_i A) = 0$. The pairs corresponding
 15 to $\beta_i \neq 0$ are the finite eigenvalues of the matrix pencil, given as $\lambda_i = \alpha_i / \beta_i$, and on the other hand, the pairs corresponding to $\beta_i = 0$, are called infinite eigenvalues of the matrix pencil. In this paper, we also assume that the matrix pencil $\lambda E - A$ is *c-stable*, that is all the finite eigenvalues of the matrix pencil lie in the open left half plane. These assumptions are made in order to ensure
 20 the existence and uniqueness of smooth solutions to the dynamical system for sufficiently smooth inputs. For more details, we refer to [8].

Moreover, if the matrix pencil $\lambda E - A$ is regular, then there exist nonsingular matrices X and Y , transforming the pencil into the Weierstrass canonical form [9, 10]:

$$E = X \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} Y, \quad A = X \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} Y,$$

where the Jordan matrix J is such that its eigenvalues coincides with the finite eigenvalues of the matrix pencil, and N is a nilpotent matrix corresponding to the infinite eigenvalues. If the index of nilpotency of N is $\nu > 0$, then $N^\nu = 0$.
 25 This nilpotency index is often called the *index* of the matrix pencil $\lambda E - A$.

For the nonlinear descriptor system (1), the problem of model order reduction is to derive another system with much smaller state-space dimension $r \ll n$,

similar to (1), i.e.,

$$\begin{aligned} E_r \dot{x}_r(t) &= A_r x_r(t) + \sum_{i=1}^m N_r^{(i)} x_r(t) u_i(t) + B_r u(t), \\ y_r(t) &= C_r x_r(t) + D_r u(t) \end{aligned} \quad (2)$$

such that the output behavior and some important properties of (1) are retained by (2) for an admissible set of input functions $u(t)$. The reduced-order system (2) can be obtained via projections as follows:

- Construct basis matrices $V \in \mathbb{R}^{n \times r}$ and $W \in \mathbb{R}^{n \times r}$ for the subspaces \mathcal{V} and \mathcal{W} respectively.
- Approximate $x(t)$ by $V x_r(t)$.
- Ensure the Petrov-Galerkin condition:

$$W^T \left(EV \dot{x}_r(t) - AV x_r(t) - \sum_{i=1}^m N^{(i)} V x_r(t) u_i(t) - Bu(t) \right) = 0.$$

As a result, the state matrices associated with the reduced-order system (2) are given by

$$\begin{aligned} E_r &= W^T E V, & A_r &= W^T A V, & N_r^{(i)} &= W^T N^{(i)} V, \\ B_r &= W^T B, & C_r &= C V. \end{aligned}$$

Clearly, for a given system, the reduced-order system obtained via projection depends on the choice of V and W , or equivalently, on the subspaces \mathcal{V} and \mathcal{W} . If the matrix E is the identity matrix or nonsingular, these basis matrices and the resulting reduced-order system can be computed by extending the standard balanced truncation and interpolatory projection methods from linear to bilinear systems [11, 12, 13, 14, 15, 16, 17, 18]. The bilinear version of balanced truncation involves the solutions of two generalized Lyapunov equations, which are known to be computationally complex [14]. However, in [19, 20] effective methods for solving these Lyapunov equations are suggested. Its extension to the descriptor case also is an open problem, though. However, in this work, we focus on interpolatory projection methods for descriptor systems.

Recently, for linear descriptor systems it was shown [21] that it is necessary for interpolatory techniques to compute a reduced-order system which not only interpolates the actual transfer function of the system, but also retains its polynomial part in order to ensure a bounded error in terms of the \mathcal{H}_2 -norm. We extend this observation to bilinear descriptor systems. The idea is to compute a reduced-order system for a given bilinear DAE system such that the generalized transfer functions associated with the reduced-order and the actual bilinear systems not only interpolate at some predefined interpolation points, but also match their corresponding polynomial parts. This involves, first, identifying the

generalized transfer functions of the bilinear DAE system which is possible by using the Volterra series representation [22]. Secondly, we construct the basis matrices V and W , where the first k generalized transfer functions are used, similar to the standard interpolatory subspaces [15]. Subsequently, we identify the polynomial part of each generalized transfer function and finally project the bilinear DAE system to obtain the required reduced-order system.

It is not straightforward to identify explicitly the polynomial parts of the generalized transfer functions. In this paper, we assume a special structure of bilinear systems which allows us to compute explicitly a constant polynomial part of each generalized transfer function. The special structure reduces to an index-1 linear DAE system, if the bilinear terms are zero. In Section 2, we first discuss interpolatory techniques for such index-1 linear DAE systems. Its extension with the required modifications to the special class of bilinear DAE systems is shown in Section 3, where an expression for the polynomial part of each generalized transfer function is also derived. In Section 4, we discuss computational issues arising in the interpolatory technique proposed in Section 3. Finally, in Section 5, we present numerical results to illustrate the implementation of our approach.

2. Interpolatory Model Reduction for Linear DAEs

In this section, we briefly review interpolatory projection methods for model reduction of linear descriptor systems. Note that the system (1) reduces to a linear descriptor system for $N^{(i)} = 0$:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{3}$$

For the ease of presentation, we mostly assume single-input single-output systems in this paper, although the results can be extended to the multi-input multi-output case, see Section 4.3. This means that B , C^T represent column vectors and D is a scalar. Also, we denote the transfer function of (3) by $\mathcal{G}(s) := C(sE - A)^{-1}B + D$ which can be decomposed into strictly proper ($\mathcal{G}_{sp}(s)$) and polynomial ($\mathcal{P}(s)$) parts, i.e., $\mathcal{G}(s) = \mathcal{G}_{sp}(s) + \mathcal{P}(s)$.

The problem of reducing the above linear descriptor system by interpolatory projection has been considered recently in [21]. Therein, it was shown that the standard interpolatory techniques for model reduction of linear DAEs will generically produce an interpolating ODE system (reduced-order) and will not necessarily match the polynomial part of the DAE system. This may result in an unbounded \mathcal{H}_2 error. To overcome this issue, an idea was proposed to identify explicitly the polynomial part of the transfer function and to ensure that the reduced-order system retains this polynomial part by using modified interpolatory subspaces for projection [21]. For special descriptor systems of index 1, this can be achieved without modifying the standard interpolatory subspaces [21]. The approach is based on the idea given in [23, 24], where the reduced transfer function $\mathcal{G}_r(s) = C_r(sE_r - A_r)^{-1}B_r + D_r$ interpolates

$\mathcal{G}(s) = C(sE - A)^{-1}B + D$ with $D_r \neq D$, unlike for standard interpolation methods where $D_r = D$. In the following, we review this idea of interpolating with $D_r \neq D$ for linear index-1 descriptor systems [21]. Consider a linear descriptor system of index 1,

$$\begin{aligned} E_{11}\dot{x}_1(t) + E_{12}\dot{x}_2(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \\ 0 &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t), \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t), \end{aligned} \quad (4)$$

where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$. By the index-1 assumption, the matrices A_{22} and $E_{11} - E_{12}A_{22}^{-1}A_{21}$ are invertible. For an index-1 descriptor system, the polynomial part $\mathcal{P}(s)$ is constant and can be determined by the following result.

Lemma 2.1. [21]. *Let $\mathcal{G}(s)$ be the transfer function of the linear descriptor system (4) in which A_{22} and $E_{11} - E_{12}A_{22}^{-1}A_{21}$ are both nonsingular. Then the polynomial part of $\mathcal{G}(s)$ can be written as*

$$\mathcal{P} = CMB + D, \quad (5)$$

where

$$M = \lim_{s \rightarrow \infty} (sE - A)^{-1} = \begin{bmatrix} 0 & E_A^{-1}E_{12}A_{22}^{-1} \\ 0 & -A_{22}^{-1}(I + A_{21}E_A^{-1}E_{12}A_{22}^{-1}) \end{bmatrix} \quad (6)$$

80 with $E_A = E_{11} - E_{12}A_{22}^{-1}A_{21}$ and $s := 2\pi jf$ being the Laplace variable in which f is the frequency and j is the imaginary unit.

To ensure a bounded error, the reduced transfer function $\mathcal{G}_r(s) = \hat{\mathcal{G}}_{sp}(s) + \hat{\mathcal{P}}(s)$ should not only interpolate $\mathcal{G}(s)$, but also match the constant part, $\hat{\mathcal{P}}(s) = \mathcal{P}(s) = \mathcal{P}$. This means that the problem reduces to identify $\hat{\mathcal{G}}_{sp}(s)$, which
 85 interpolates $\mathcal{G}_{sp}(s)$. Note that it is easy to identify an interpolating $\hat{\mathcal{G}}_{sp}(s)$, once we have an explicit expression for $\mathcal{G}_{sp}(s)$. However, the goal is to identify an interpolating $\mathcal{G}_r(s)$ without explicitly constructing $\mathcal{G}_{sp}(s)$ utilizing a special structure of the matrices E and A . The following theorem provides a possible solution.

90 **Theorem 2.1.** [21, 23] *Let $\mathcal{G}(s)$ be the transfer function of the linear descriptor system (3). Assume that the interpolation points σ and μ are given such that $sE - A$ and $sE_r - A_r$ are invertible for $s = \sigma, \mu$. Define the projection matrices $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ such that*

- $\text{range}(V) = \mathcal{K}_q((\sigma E - A)^{-1}E, (\sigma E - A)^{-1}B),$
- 95 • $\text{range}(W) = \mathcal{K}_q((\mu E - A)^{-T}E^T, (\mu E - A)^{-T}C^T),$

where $\mathcal{K}_q(\mathcal{A}, \mathcal{B}) = \text{span}\{\mathcal{B}, \mathcal{A}\mathcal{B}, \dots, \mathcal{A}^{q-1}\mathcal{B}\}$. Also, let $F \in \mathbb{C}^{n \times 1}$ and $G \in \mathbb{C}^{n \times 1}$ be solutions to

$$F^T V = (e_1^r)^T \quad \text{and} \quad W^T G = e_1^r \quad (7)$$

in which, e_1^r is the first column of the $r \times r$ identity matrix. Then, projection of the intermediate system $\tilde{\mathcal{G}}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} + \tilde{D}$,

$$\tilde{E} = E, \quad \tilde{A} = A + G\hat{D}F^T, \quad \tilde{B} = B - G\hat{D}, \quad \tilde{C} = C - \hat{D}F^T, \quad \tilde{D} = \mathcal{P},$$

where $\hat{D} = \mathcal{P} - D$, results in a reduced-order system $\mathcal{G}_r(s) = C_r(sE_r - A_r)^{-1}B_r + D_r$, in which:

$$E_r = W^T \tilde{E} V, \quad A_r = W^T \tilde{A} V, \quad B_r = W^T \tilde{B}, \quad C_r = \tilde{C} V_r, \quad D_r = \tilde{D}.$$

Assuming that E_r is invertible, the polynomial parts of $\mathcal{G}_r(s)$ and $\mathcal{G}(s)$ match, that is $D_r = \mathcal{P}$. Also, $\mathcal{G}_r(s)$ satisfies the following interpolation conditions:

$$\mathcal{G}^{(l)}(\sigma) = \mathcal{G}_r^{(l)}(\sigma), \quad \mathcal{G}^{(l)}(\mu) = \mathcal{G}_r^{(l)}(\mu), \quad l = 0, \dots, q-1,$$

Moreover, if $\sigma = \mu$, then

$$\mathcal{G}^{(l)}(\sigma) = \mathcal{G}_r^{(l)}(\sigma), \quad l = 0, \dots, 2q-1.$$

Remark 2.1. The reduced transfer function $\mathcal{G}_r(s)$ is not only interpolating $\mathcal{G}(s)$ with $D_r \neq D$ (unlike for standard interpolation methods), but also matches the polynomial part of $\mathcal{G}(s)$, that is $D_r = \mathcal{P}$. This is possible by first computing the constant polynomial part \mathcal{P} of $\mathcal{G}(s)$, then constructing the intermediate system $\tilde{\mathcal{G}}(s)$ with $\tilde{D} = \mathcal{P}$, and subsequently, applying oblique projection to $\tilde{\mathcal{G}}(s)$ with standard interpolatory subspaces V and W associated with $\mathcal{G}(s)$.

Remark 2.2. Theorem 2.1 does not require the explicit computation of F and G in order to compute the reduced-order system. The expressions for $W^T G$ and $F^T V$ can be substituted directly from (7).

Remark 2.3. In case of Hermite interpolation with m distinct interpolation points (i.e., using σ_i and μ_i , $i = 1, \dots, m$), the conditions on F and G become

$$F^T V = \underbrace{[1, 1, \dots, 1]}_{m \text{ times}} \quad \text{and} \quad W^T G = \underbrace{[1, 1, \dots, 1]}_{m \text{ times}}^T.$$

3. Interpolatory Model Reduction for Bilinear Descriptor Systems

In this section, we extend the interpolatory technique with $D_r \neq D$ as discussed for index-1 linear DAEs to a special class of bilinear descriptor systems. Consider a bilinear descriptor system, where the matrix pencil $\lambda E - A$ has a structure analogous to the index-1 linear DAE, given in (4). That is,

$$\begin{aligned} E_{11}\dot{x}_1(t) + E_{12}\dot{x}_2(t) &= A_{11}x_1(t) + A_{12}x_2(t) + \sum_{i=1}^2 N_{1i}x_i(t)u(t) + B_1u(t), \\ 0 &= A_{21}x_1(t) + A_{22}x_2(t) + \sum_{i=1}^2 N_{2i}x_i(t)u(t) + B_2u(t), \\ y(t) &= C_1x(t) + C_2x_2(t) + Du(t), \end{aligned} \tag{8}$$

where A_{22} and $E_{11} - E_{12}A_{22}^{-1}A_{21}$ are invertible. In frequency domain, the input-output representation of the bilinear system is given by the Volterra series representation of the system. Each term of the Volterra series can be considered as a subsystem of the bilinear descriptor system and involves generalized multivariate transfer functions. The structure of these multivariate transfer functions corresponding to the k th subsystem in the so-called regular form is given as

$$H(s_1, \dots, s_k) = C(s_k E - A)^{-1} N(s_{k-1} E - A)^{-1} N \cdots N(s_1 E - A)^{-1} B + D\delta(k-1), \quad (9)$$

where $\delta(l) = 1$, if $l = 0$, and $\delta(l) = 0$, otherwise. The next lemma shows that each subsystem of (8) has a constant polynomial part.

Lemma 3.1. *Let $H(s_1, \dots, s_k)$ be defined as in (9), that is the regular multivariate Laplace transform of the degree- k kernel associated to Σ , where A_{22} and $E_{11} - E_{12}A_{22}^{-1}A_{21}$ are both nonsingular. Then, the constant polynomial part of $H(s_1, \dots, s_k)$ is given by*

$$D_k = C(MN)^{k-1}MB + D\delta(k-1), \quad (10)$$

where M is as defined in (6).

Proof. Let $F(\mathcal{S}_k) = F(s_1, \dots, s_k)$ be the multivariable function

$$F(\mathcal{S}_k) = (s_k E - A)^{-1} N(s_{k-1} E - A)^{-1} N \cdots N(s_1 E - A)^{-1} B, \quad (11)$$

then the polynomial part of $H(\mathcal{S}_k)$ is given by

$$D_k = C \lim_{\mathcal{S}_k \rightarrow \infty} F(\mathcal{S}_k) + D\delta(k-1). \quad (12)$$

Note that for $k = 1$, (11) reduces to the linear case and using (6) we have

$$\lim_{\mathcal{S}_1 \rightarrow \infty} F(\mathcal{S}_1) = \lim_{s_1 \rightarrow \infty} (s_1 E - A)^{-1} B = MB. \quad (13)$$

It is easy to see from (12) that (10) holds for $k = 1$ (analog to the linear case). Now for $k = j \geq 1$, assume that

$$\lim_{\mathcal{S}_j \rightarrow \infty} F(\mathcal{S}_j) = (MN)^{j-1}MB. \quad (14)$$

We need to show that the above equation holds for $k = j + 1$. Note that

$$F(\mathcal{S}_{j+1}) = (s_{j+1} E - A)^{-1} N F(\mathcal{S}_j).$$

Taking the limit $\mathcal{S}_{j+1} \rightarrow \infty$, we have

$$\begin{aligned} \lim_{\mathcal{S}_{j+1} \rightarrow \infty} F(\mathcal{S}_{j+1}) &= \lim_{s_{j+1} \rightarrow \infty} (s_{j+1} E - A)^{-1} N \lim_{\mathcal{S}_j \rightarrow \infty} F(\mathcal{S}_j) \\ &= \lim_{s_{j+1} \rightarrow \infty} (s_{j+1} E - A)^{-1} N (MN)^{j-1} MB, \end{aligned}$$

where the last equation follows from (14). Now, we define

$$\mathcal{B}_{MN} = N(MN)^{j-1}MB$$

and use (6) to obtain

$$\lim_{\mathcal{S}_{j+1} \rightarrow \infty} F(\mathcal{S}_{j+1}) = \lim_{s_{j+1} \rightarrow \infty} (s_{j+1}E - A)^{-1} \mathcal{B}_{MN} = M\mathcal{B}_{MN} = (MN)^j MB.$$

Thus, (12) implies that (10) holds. \square

Lemma 3.1 suggests that if

$$N = \begin{bmatrix} N_{11} & N_{12} \\ 0 & 0 \end{bmatrix}, \quad \text{where } N_{11} \in \mathbb{R}^{n_1 \times n_1} \text{ and } N_{12} \in \mathbb{R}^{n_1 \times n_2}, \quad (15)$$

110 then $D_k = 0$ for $k > 1$ and $\mathcal{P} = D_1 = CMB + D$. This means that only the first subsystem has polynomial part and all other subsystems have zero polynomial parts. The next subsection addresses the issue of retaining the polynomial part, D_1 , in the first subsystem of the reduced bilinear system.

3.1. Interpolating a bilinear descriptor system and retaining D_1

115 We begin with outlining the standard interpolatory projection, where $D_r = D$.

Theorem 3.1. [15] Consider arbitrary interpolation points $\sigma_i, \mu_i \in \mathbb{C}$ such that $sE - A$ and $sE_r - A_r$ are invertible for $s = \sigma_i, \mu_i, i = 1, \dots, k$. Define the projection matrices V and W as follows:

$$\begin{aligned} \text{range}(V^{(1)}) &= \mathcal{K}_q((\sigma_1 E - A)^{-1} E, (\sigma_1 E - A)^{-1} B), \\ \text{range}(V^{(i)}) &= \mathcal{K}_q((\sigma_i E - A)^{-1} E, (\sigma_i E - A)^{-1} N V^{(i-1)}), \quad i = 2, \dots, k, \\ \text{range}(W^{(1)}) &= \mathcal{K}_q((\mu_1 E - A)^{-T} E^T, (\mu_1 E - A)^{-T} C^T), \\ \text{range}(W^{(i)}) &= \mathcal{K}_q((\mu_i E - A)^{-T} E^T, (\mu_i E - A)^{-T} N^T W^{(i-1)}), \quad i = 2, \dots, k, \\ \text{range}(V) &= \bigcup_{i=1}^k \{ \text{range}(V^{(i)}) \}, \quad \text{range}(W) = \bigcup_{i=1}^k \{ \text{range}(W^{(i)}) \}. \end{aligned}$$

Assume V and W are full column rank matrices. Construct the reduced-order system matrices as:

$$\begin{aligned} E_r &= W^T E V, & A_r &= W^T A V, & N_r &= W^T N V, \\ B_r &= W^T B, & C_r &= C V, & D_r &= D, \end{aligned}$$

then

$$H(\mathcal{S}_k) = H_r(\mathcal{S}_k) + \mathcal{O}((s_1 - \mu_1)^q \cdots (s_k - \mu_k)^q (s_1 - \sigma_1)^q \cdots (s_k - \sigma_k)^q).$$

120 **Remark 3.1.** *It was shown in [15] that the choice of the projection matrices V and W in Theorem 3.1 also yields the matching of additional moments which involve $W^T E V$ and $W^T N V$.*

Our aim is to utilize the basis matrices V and W as given in Theorem 3.1 and extend the approach used in Theorem 2.1 to the bilinear descriptor system (8), with $N_{21} = N_{22} = 0$ as in (15). The following theorem provides a possible solution.

Theorem 3.2. *Let V and W be as defined in Theorem 3.1 and assume that the structure of the matrix N is as in (15). Also, let $F \in \mathbb{C}^{n \times 1}$ and $G \in \mathbb{C}^{n \times 1}$ be solutions to*

$$F^T V = (e_1^r)^T \quad \text{and} \quad W^T G = e_1^r, \quad (16)$$

where r is the order of the reduced-order system. Then, projection of the intermediate system $\tilde{\Sigma}(\tilde{E}, \tilde{A}, \tilde{N}, \tilde{B}, \tilde{C}, \tilde{D})$

$$\begin{aligned} \tilde{E} &= E, & \tilde{A} &= A + G\hat{D}F^T, & \tilde{N} &= N, \\ \tilde{B} &= B - G\hat{D}, & \tilde{C} &= C - \hat{D}F^T, & \tilde{D} &= CMB + D, \end{aligned}$$

130 where $\hat{D} = CMB$, results in a reduced-order system $\Sigma_r(E_r, A_r, N_r, B_r, C_r, D_r)$ in which

$$\begin{aligned} E_r &= W^T \tilde{E} V, & A_r &= W^T \tilde{A} V, & N_r &= W^T \tilde{N} V, \\ B_r &= W^T \tilde{B}, & C_r &= \tilde{C} V_r, & D_r &= \tilde{D}. \end{aligned}$$

Assuming nonsingular E_r , the polynomial parts of the first subsystem associated with the reduced and original bilinear systems are matched. Also

$$H(\mathcal{S}_k) = H_r(\mathcal{S}_k) + \mathcal{O}((s_1 - \mu_1)^q \cdots (s_k - \mu_k)^q (s_1 - \sigma_1)^q \cdots (s_k - \sigma_k)^q).$$

Proof. For the first subsystem, the result reduces to Theorem 2.1. However, for the sake of completeness, we derive its proof in the following:

$$\begin{aligned} H(\sigma_1) - H_r(\sigma_1) &= C \left((\sigma_1 E - A)^{-1} B - V(\sigma_1 E_r - A_r)^{-1} B_r \right) \\ &\quad - \hat{D} + \hat{D} F^T V(\sigma_1 E_r - A_r)^{-1} B_r. \end{aligned} \quad (17)$$

We now consider

$$\begin{aligned} V(\sigma_1 E_r - A_r)^{-1} B_r &= V(\sigma_1 E_r - A_r)^{-1} W^T (B - G\hat{D}) \\ &= V(\sigma_1 E_r - A_r)^{-1} W^T (\sigma_1 E - A - G\hat{D}F^T) (\sigma_1 E - A)^{-1} B. \end{aligned}$$

Note that $P_\sigma = V(\sigma E_r - A_r)^{-1} W^T (\sigma E - A - G\hat{D}F^T)$ is an oblique projector onto $\text{range}(V)$ and let $z \in \text{range}(V)$, then $P_\sigma z = z$. This implies,

$$V(\sigma_1 E_r - A_r)^{-1} B_r = P_{\sigma_1} (\sigma_1 E - A)^{-1} B = (\sigma_1 E - A)^{-1} B. \quad (18)$$

Thus, the equation (17) becomes

$$H(\sigma_1) - H_r(\sigma_1) = -\hat{D} + \hat{D}F^T(\sigma_1E - A)^{-1}B.$$

Now from (16), we have $F^T(\sigma_1E - A)^{-1}B = 1$ and this proves the matching at σ_1 for the first subsystem. Similarly, $H(\mu_1) = H_r(\mu_1)$ holds. Next, we consider the second subsystem

$$\begin{aligned} H(\sigma_1, \sigma_2) - H_r(\sigma_1, \sigma_2) &= C((\sigma_2E - A)^{-1}N(\sigma_1E - A)^{-1}B \\ &\quad - V(\sigma_2E_r - A_r)^{-1}N_r(\sigma_1E_r - A_r)^{-1}B_r) \\ &\quad + \hat{D}F^TV(\sigma_2E_r - A_r)^{-1}N_r(\sigma_1E_r - A_r)^{-1}B_r. \end{aligned} \quad (19)$$

Since

$$\begin{aligned} &\underbrace{V(\sigma_2E_r - A_r)^{-1}N_r(\sigma_1E_r - A_r)^{-1}B_r}_{z_r} \\ &= V(\sigma_2E_r - A_r)^{-1}W^TNV(\sigma_1E_r - A_r)^{-1}B_r, \end{aligned}$$

this leads to

$$Vz_r = V(\sigma_2E_r - A_r)^{-1}W^TN(\sigma_1E - A)^{-1}B,$$

by using (18). Now using (16), the above equation becomes

$$\begin{aligned} Vz_r &= V(\sigma_2E_r - A_r)^{-1}W^T\left((\sigma_2E - A) - G\hat{D}F^T\right)(\sigma_2E - A)^{-1}N(\sigma_1E - A)^{-1}B \\ &= P_{\sigma_2}(\sigma_2E - A)^{-1}N(\sigma_2E - A)^{-1}B = (\sigma_2E - A)^{-1}N(\sigma_1E - A)^{-1}B. \end{aligned}$$

Employing the above relation in (19), we have

$$H(\sigma_1, \sigma_2) - H_r(\sigma_1, \sigma_2) = \hat{D}F^T(\sigma_2E - A)^{-1}N(\sigma_1E - A)^{-1}B.$$

From (16), we know $F^T(\sigma_2E - A)^{-1}N(\sigma_1E - A)^{-1}B = 0$. Hence, $H(\sigma_1, \sigma_2) = H_r(\sigma_1, \sigma_2)$. Similarly, $H(\mu_1, \mu_2) = H_r(\mu_1, \mu_2)$. Analogously, we can deal with
135 subsystems of higher order and higher derivatives. \square

3.2. Interpolating a bilinear descriptor system and retaining the polynomial parts of the first k subsystems

So far, we have discussed how an interpolatory technique can retain the polynomial part of the first subsystem in the reduced bilinear system by assuming that the higher order subsystems have zero polynomial parts. In this section, we consider a general case where the higher order subsystems also have non-zero polynomial parts. The goal is to construct a reduced bilinear system that retains the polynomial parts (non-zero) of the first k subsystems associated with the original bilinear system, in addition to interpolating these subsystems.

As discussed in the preceding section, the structure of the k th subsystem of the reduced bilinear system can be written as

$$H_r(s_1, \dots, s_k) = C_r(s_k E_r - A_r)^{-1} N_r (s_{k-1} E_r - A_r)^{-1} N_r \cdots \cdots N_r (s_1 E_r - A_r)^{-1} B_r + D_k, \quad (20)$$

in which E_r is nonsingular and D_k is the polynomial part of the k th subsystem of the original bilinear system. This means that the reduced bilinear system ensures matching of the polynomial parts of the first k subsystems corresponding to the original bilinear system. However, we also need to ensure interpolation for these subsystems. The following theorem provides our main result for achieving this.

Theorem 3.3. *Let V and W be as defined in Theorem 3.1 and define intermediate matrices:*

$$\begin{aligned} \tilde{E} &= E, & \tilde{A} &= A + L_A, & \tilde{N} &= N - L_N, \\ \tilde{B} &= B - L_B, & \tilde{C} &= C - L_C, \end{aligned}$$

where L_A, L_N, L_B and L_C are solutions to the following equations:

$$W^T L_B = \left[\hat{D}_1 (e_1^q)^T, D_2 (e_1^{q^2})^T, \dots, D_k (e_1^{q^k})^T \right]^T, \quad (21a)$$

$$L_C V = \left[\hat{D}_1 (e_1^q)^T, D_2 (e_1^{q^2})^T, \dots, D_k (e_1^{q^k})^T \right], \quad (21b)$$

$$L_A V = \left[L_B (e_1^q)^T, L_N \left[V^1 (I_q \otimes (e_1^q)^T), \dots, V^{k-1} (I_{q^{k-1}} \otimes (e_1^q)^T) \right] \right], \quad (21c)$$

$$W^T L_A = \left[L_C^T (e_1^q)^T, L_N^T \left[W^1 (I_q \otimes (e_1^q)^T), \dots, W^{k-1} (I_{q^{k-1}} \otimes (e_1^q)^T) \right] \right]^T \quad (21d)$$

in which $\hat{D}_1 = D_1 - D$ and $e_1^{q^j}$ is the first column of the identity matrix of the size $q^j \times q^j$. Then, projection of the intermediate system results in a reduced-order system:

$$\begin{aligned} E_r &= W^T \tilde{E} V, & A_r &= W^T \tilde{A} V, & N_r &= W^T \tilde{N} V, \\ B_r &= W^T \tilde{B}, & C_r &= \tilde{C} V, \end{aligned}$$

that satisfies

$$H(\mathcal{S}_k) = H_r(\mathcal{S}_k) + \mathcal{O}((s_1 - \mu_1)^q \cdots (s_k - \mu_k)^q (s_1 - \sigma_1)^q \cdots (s_k - \sigma_k)^q).$$

Proof. Consider the first subsystem at $s_1 = \sigma_1$:

$$\begin{aligned} H(\sigma_1) - H_r(\sigma_1) &= C \left((\sigma_1 E - A)^{-1} B - V(\sigma_1 E_r - A_r)^{-1} B_r \right) \\ &\quad + L_C V(\sigma_1 E_r - A_r)^{-1} B_r - \underbrace{(D_1 - D)}_{\hat{D}_1}. \end{aligned} \quad (22)$$

Since $V(\sigma_1 E_r - A_r)^{-1} B_r = V(\sigma_1 E_r - A_r)^{-1} W^T (B - L_B)$ and from (21c), $L_A (\sigma_1 E - A)^{-1} B = L_B$, this yields

$$V(\sigma_1 E_r - A_r)^{-1} B_r = V(\sigma_1 E_r - A_r)^{-1} W^T ((\sigma_1 E - A) - L_A) (\sigma_1 E - A)^{-1} B.$$

Now, introducing the oblique projector

$$P_\sigma = V(\sigma E_r - A_r)^{-1}W^T((\sigma E - A) - L_A)$$

and utilizing $P_\sigma z = z$ for $z \in \text{range}(V)$, we get

$$V(\sigma_1 E_r - A_r)^{-1}B_r = P_{\sigma_1}(\sigma_1 E - A)^{-1}B = (\sigma_1 E - A)^{-1}B. \quad (23)$$

Using this in (22), we get

$$H(\sigma_1) - H_r(\sigma_1) = L_C(\sigma_1 E - A)^{-1}B - \hat{D}_1.$$

From (21b), $L_C(\sigma_1 E - A)^{-1}B = \hat{D}_1$. Hence, $H(\sigma_1) = H_r(\sigma_1)$. Similarly, $H(\mu_1) = H_r(\mu_1)$ holds. Now, consider the second subsystem

$$\begin{aligned} H(\sigma_1, \sigma_2) - H_r(\sigma_1, \sigma_2) &= C((\sigma_2 E - A)^{-1}N(\sigma_1 E - A)^{-1}B \\ &\quad - V(\sigma_2 E_r - A_r)^{-1}N_r(\sigma_1 E_r - A_r)^{-1}B_r) \\ &\quad + L_C V(\sigma_2 E_r - A_r)^{-1}N_r(\sigma_1 E_r - A_r)^{-1}B_r - D_2. \end{aligned} \quad (24)$$

Hence,

$$\begin{aligned} &\underbrace{V(\sigma_2 E_r - A_r)^{-1}N_r(\sigma_1 E_r - A_r)^{-1}B_r}_{\hat{z}_r} \\ &= V(\sigma_2 E_r - A_r)^{-1}W^T(N - L_N)V(\sigma_1 E_r - A_r)^{-1}B_r \\ &= V(\sigma_2 E_r - A_r)^{-1}W^T(N - L_N)(\sigma_1 E - A)^{-1}B, \end{aligned}$$

where the last equation follows from (23). Now, (21c) implies

$$L_A(\sigma_2 E - A)^{-1}N(\sigma_1 E - A)^{-1}B = L_N(\sigma_1 E - A)^{-1}B.$$

Thus,

$$\begin{aligned} V\hat{z}_r &= V(\sigma_2 E_r - A_r)^{-1}W^T((\sigma_2 E - A) - L_A)(\sigma_2 E - A)^{-1}N(\sigma_1 E - A)^{-1}B, \\ &= P_{\sigma_2}(\sigma_2 E - A)N(\sigma_2 E - A)^{-1}B = (\sigma_2 E - A)N(\sigma_1 E - A)^{-1}B. \end{aligned}$$

We make use of the above equality in (24) to obtain

$$H(\sigma_1, \sigma_2) - H_r(\sigma_1, \sigma_2) = L_C(\sigma_2 E - A)^{-1}N(\sigma_1 E - A)^{-1}B - D_2.$$

From (21b), we know that $L_C(\sigma_2 E - A)^{-1}N(\sigma_1 E - A)^{-1}B = D_2$. Therefore, $H(\sigma_1, \sigma_2) = H_r(\sigma_1, \sigma_2)$. Similarly, $H(\mu_1, \mu_2) = H_r(\mu_1, \mu_2)$. Using analogous steps, we can also deal with higher subsystems and higher derivatives. \square

4. Computational Issues and Time-Domain Representation of the Reduced-Order System

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In this section, we discuss the computational issues associated with the intermediate system $\tilde{\Sigma}$, the time-domain representation of the reduced-order system, and an extension of the proposed technique to multi-input multi-output bilinear systems.

160 *4.1. Computational issues*

Before we proceed towards investigating computational issues, we discuss the condition for the existence of a simultaneous solution of two given linear systems in the following lemma which helps us determining the condition for the existence of the solutions of (21a)–(21d).

Lemma 4.1. *Consider the matrices $\mathcal{A}_i, \mathcal{B}_i \in \mathbb{R}^{n \times r}$, $i = \{1, 2\}$ and $X \in \mathbb{R}^{n \times n}$, where $n \geq r$, that satisfy the following two linear equations:*

$$\mathcal{A}_1^T X = \mathcal{B}_1^T, \quad (25a)$$

$$X \mathcal{A}_2 = \mathcal{B}_2. \quad (25b)$$

165 *If $\mathcal{A}_1^T \mathcal{B}_2 = \mathcal{B}_1^T \mathcal{A}_2$, then there exists an X that satisfies (25a) and (25b), else it is not possible to determine an X , satisfying both (25a) and (25b).*

Proof. We first recall an important property of the Kronecker product and vectorization

$$\text{vec}(\tilde{X} \tilde{Y} \tilde{Z}) = (\tilde{Z}^T \otimes \tilde{X}) \text{vec}(\tilde{Y}),$$

where $\text{vec}(\cdot)$ denotes the reshaping of a matrix into a column vector by putting each column of the matrix at the bottom of the previous column. See, e.g, [25, 26] for more details on the Kronecker product. Using the $\text{vec}(\cdot)$ operation on both sides of (25a) and (25b) leads to

$$\begin{bmatrix} I_n \otimes \mathcal{A}_1^T \\ \mathcal{A}_2^T \otimes I_n \end{bmatrix} X_v = \begin{bmatrix} \text{vec}(\mathcal{B}_1^T) \\ \text{vec}(\mathcal{B}_2) \end{bmatrix}, \quad (26)$$

where $X_v = \text{vec}(X)$ and I_q is the identity matrix of size $q \times q$. Next, we define the matrix

$$M = \begin{bmatrix} \mathcal{A}_2^T \otimes I_r & 0 \\ P \otimes I_r & 0 \\ 0 & I_r \otimes \mathcal{A}_1^T \\ 0 & I_r \otimes Q \end{bmatrix},$$

where $P, Q \in \mathbb{R}^{(n-r) \times n}$ such that the matrix M is invertible. Multiplying M on both sides of (26) yields

$$\underbrace{\begin{bmatrix} \mathcal{A}_2^T \otimes \mathcal{A}_1^T \\ P \otimes \mathcal{A}_1^T \\ \mathcal{A}_2^T \otimes \mathcal{A}_1^T \\ \mathcal{A}_2^T \otimes Q \end{bmatrix}}_{\mathcal{A}} X_v = \begin{bmatrix} (\mathcal{A}_2^T \otimes I_r) \text{vec}(\mathcal{B}_1^T) \\ (P \otimes I_r) \text{vec}(\mathcal{B}_1^T) \\ (I_r \otimes \mathcal{A}_1^T) \text{vec}(\mathcal{B}_2) \\ (I_r \otimes Q) \text{vec}(\mathcal{B}_2) \end{bmatrix} = \underbrace{\begin{bmatrix} \text{vec}(\mathcal{B}_1^T \mathcal{A}_2) \\ (P \otimes I_r) \text{vec}(\mathcal{B}_1^T) \\ \text{vec}(\mathcal{A}_1^T \mathcal{B}_2) \\ (I_r \otimes Q) \text{vec}(\mathcal{B}_2) \end{bmatrix}}_{\mathcal{B}} \quad (27)$$

Now by using the Kronecker-Capelli theorem [27], the linear system in (27) has a solution if and only if:

$$\text{rank}(\mathcal{A}) = \text{rank}([\mathcal{A}, \mathcal{B}]). \quad (28)$$

Clearly, the first and third row blocks of the matrix \mathcal{A} are the same. Therefore, P and Q can be chosen such that the rank of the matrix \mathcal{A} is equal to $r(2n - r)$

which is equal to the number of rows, having removed the third row block of the matrix \mathcal{A} . In order to have the same rank for the matrix $[\mathcal{A}, \mathcal{B}]$, the first and the third row blocks of the matrix \mathcal{B} should also be the same. This leads to the following condition:

$$\mathcal{B}_1^T \mathcal{A}_2 = \mathcal{A}_1^T \mathcal{B}_2. \quad (29)$$

One can verify that if the above condition (29) is fulfilled, then $\text{rank}([\mathcal{A}, \mathcal{B}])$ is also equal to $r(2n - r)$. This means that the system (27) has a solution, if $r \leq n$.

170 On the other hand, if the condition (29) is not satisfied, then $\text{rank}([\mathcal{A}, \mathcal{B}])$ is also equal to $r(2n - r) + 1$, implying that the system (27) does not have any solution. Hence, it is not possible to determine an X that satisfies both (25a) and (25b). \square

Coming back to computational issues related to the solutions of (21a)–(21d), L_B and L_C are independent of other unknowns, therefore they can be easily computed. However, the main issue lies in the computation of L_A and L_N . These matrices require the simultaneous solution of (21c) and (21d) for given L_B and L_C . Next, using Lemma 4.1, we derive a necessary and sufficient condition, called the *compatibility condition* to ensure the existence of the simultaneous solution. This follows by equating the right hand sides of (21c) and (21d) after pre-multiplying by W^T and post-multiplying by V , respectively:

$$\begin{aligned} W^T [L_B(e_1^q)^T, L_N [V^1(I_q \otimes (e_1^q)^T), \dots, V^{k-1}(I_{q^{k-1}} \otimes (e_1^q)^T)]] \\ = [L_C^T(e_1^q)^T, L_N^T [W^1(I_q \otimes (e_1^q)^T), \dots, W^{k-1}(I_{q^{k-1}} \otimes (e_1^q)^T)]]^T V. \end{aligned} \quad (30)$$

175 The following theorem guarantees that the above compatibility condition is satisfied.

Theorem 4.1. *Assume L_B and L_C satisfy (21a) and (21b), respectively, and let $L_N \in \mathbb{R}^{n \times n}$ satisfy*

$$W^T L_N V = \mathcal{T} \begin{bmatrix} D_2 & \cdots & D_{k+1} \\ \vdots & \ddots & \vdots \\ D_{k+1} & \cdots & D_{2k} \end{bmatrix} \mathcal{T}^T, \quad (31)$$

where $\mathcal{T} = \sum_{i=0}^{k-1} (e_{1+q^i}^r)^T \otimes e_{i+1}^k$, $r = q + \dots + q^k$ is the order of the reduced-order system and D_j is the polynomial part of the j th subsystem. Then, the compatibility condition (30) is satisfied.

Proof. Consider the first row of the block matrix given in (30):

$$(W^1)^T [L_B(e_1^q)^T, L_N [V^1(I_q \otimes (e_1^q)^T), \dots, V^{k-1}(I_{q^{k-1}} \otimes (e_1^q)^T)]] = e_1^q L_C V.$$

To show that the above equation holds, we use (21a) and (31)

$$\begin{aligned}
& (W^1)^T [L_B(e_1^q)^T, L_N [V^1(I_q \otimes (e_1^q)^T), \dots, V^{k-1}(I_{q^{k-1}} \otimes (e_1^q)^T)]] \\
&= [e_1^q \hat{D}_1(e_1^q)^T, e_1^q D_2(e_1^q)^T(I_q \otimes (e_1^q)^T), \dots, e_1^q D_k(e_1^{q^{k-1}})^T(I_{q^{k-1}} \otimes (e_1^q)^T)] \\
&= e_1^q [\hat{D}_1(e_1^q)^T, D_2(e_1^q)^T \otimes (e_1^q)^T, \dots, D_k(e_1^{q^{k-1}})^T \otimes (e_1^q)^T] \\
&= e_1^q [\hat{D}_1(e_1^q)^T, D_2(e_1^{q^2})^T, \dots, D_k(e_1^{q^k})^T] \\
&= e_1^q L_C V,
\end{aligned}$$

where the last equality follows from (21b). Now consider the i th row of the block matrix in (30):

$$\begin{aligned}
& (W^i)^T [L_B(e_1^q)^T, L_N [V^1(I_q \otimes (e_1^q)^T), \dots, V^{k-1}(I_{q^{k-1}} \otimes (e_1^q)^T)]] \\
&= [L_N^T W^{i-1}(I_{q^{i-1}} \otimes (e_1^q)^T)]^T V \\
&= [e_1^{q^i} D_i(e_1^q)^T, e_1^{q^i} D_{i+1}(e_1^q)^T(I_q \otimes (e_1^q)^T), \dots, \\
&\quad e_1^{q^i} D_{i+k-1}(e_1^{q^{k-1}})^T(I_{q^{k-1}} \otimes (e_1^q)^T)] \\
&= (I_{q^{i-1}} \otimes (e_1^q)^T)^T e_1^{q^{i-1}} [D_i(e_1^q)^T, D_{i+1}(e_1^{q^2})^T, \dots, D_{i+k-1}(e_1^{q^k})^T] \\
&=: \mathcal{R}.
\end{aligned}$$

Using the condition on L_N given in (31), we obtain

$$\begin{aligned}
\mathcal{R} &= (I_{q^{i-1}} \otimes (e_1^q)^T)^T [(W^{i-1})^T L_N V^1, (W^{i-1})^T L_N V^2, \dots, (W^{i-1})^T L_N V^k] \\
&= (I_{q^{i-1}} \otimes (e_1^q)^T)^T [(W^{i-1})^T L_N V^1, (W^{i-1})^T L_N V^2, \dots, (W^{i-1})^T L_N V^k] \\
&= [L_N^T W^{i-1}(I_{q^{i-1}} \otimes (e_1^q)^T)]^T V.
\end{aligned}$$

180 This means that each row of the block matrix corresponding to the left and right side of the compatibility condition given in (30) is equal. Therefore, if L_N is chosen to satisfy the assumption (31), then it is ensured that (30) holds. \square

Remark 4.1. *It is interesting to see that in order to compute the reduced-order system, we do not need to compute explicitly the matrices L_A , L_N , L_B and L_C . We only require the expressions for $W^T L_B$, $L_C V$, $W^T L_A V$ and $W^T L_N V$. One can substitute $W^T L_B$ and $L_C V$ directly from (21a) and (21b). The expression of $W^T L_N V$ can be easily identified by using (31). Similarly, one can obtain the expression of $W^T L_A V$ without explicitly computing L_A by pre-multiplying (21c) by W^T , and using (31) and (21a).*

190 Now, we summarize the complete methodology of computing the reduced-order system for the system (8) in Algorithm 1.

Remark 4.2. *As shown in [15], two-sided projections might lead to much better approximation, since more multi-moments are matched for higher order subsystems. The same holds for the proposed modified Krylov subspace technique. To*

Algorithm 1 Interpolatory Model Reduction for Bilinear Systems.

- 1: **Input:** $E, A, N, B, C, D, [\sigma_1, \dots, \sigma_k], [\mu_1, \dots, \mu_k], q$.
- 2: **Output:** E_r, A_r, N_r, B_r, C_r .
- 3: Construct V and W according to Theorem 3.1.
- 4: Compute the polynomial part of the k th subsystem

$$D_k = C(MN)^{k-1}MB + D\delta(k-1).$$

- 5: Identify the expression of $W^T L_B, L_C V, W^T L_A V$ and $W^T L_N V$ as:

$$W^T L_B = \left[\hat{D}_1(e_1^q)^T, D_2(e_1^{q^2})^T, \dots, D_k(e_1^{q^k})^T \right]^T =: R_B, \quad \hat{D}_1 = D_1 - D,$$

$$W^T L_C = \left[\hat{D}_1(e_1^q)^T, D_2(e_1^{q^2})^T, \dots, D_k(e_1^{q^k})^T \right]^T =: R_C,$$

$$W^T L_N V = \mathcal{T} \begin{bmatrix} D_2 & \cdots & D_{k+1} \\ \vdots & \ddots & \vdots \\ D_{k+1} & \cdots & D_{2k} \end{bmatrix} \mathcal{T}^T =: R_N,$$

where $\mathcal{T} = \sum_{i=0}^{k-1} (e_{1+q^i}^r)^T \otimes e_{i+1}^k, r = \sum_{i=1}^k q^i + \dots + q^k$, and

$$W^T L_A V = \left[R_B(e_1^q)^T, R_N(:, 1 : q)(I_q \otimes (e_1^q)^T), \dots, \right. \\ \left. R_N(:, q + \dots + q^{k-1} + (1 : q^k))(I_{q^{k-1}} \otimes (e_1^q)^T) \right] =: R_A.$$

- 6: Compute the reduced model as:

$$E_r = W^T E V, \quad A_r = W^T A V + R_A, \quad N_r = W^T N V - R_N, \\ B_r = W^T B - R_B, \quad C_r = C V - R_C.$$

see this, we consider an example similar to the one used in [15]. Let us assume the projection subspaces V and W are as follows:

$$\text{span}(V) = \text{span}\{A^{-1}B, \dots, (A^{-1}E)^5 A^{-1}B, A^{-1}N A^{-1}B, A^{-1}N(A^{-1}E)A^{-1}B\}, \\ \text{span}(W^T) = \text{span}\{C A^{-1}, \dots, C(A^{-1}E)^5 A^{-1}, C A^{-1}N A^{-1}, C(A^{-1}E)A^{-1}N A^{-1}\}.$$

According to Theorem 3.3, the reduced-order system preserves 12 multi-moments of the first subsystem

$$C(A^{-1}E)^{l_1} A^{-1}B + D\delta(l_1) = C_r^T (A_r^{-1}E_r)^{l_1} A_r^{-1}B_r + D_1\delta(l_1),$$

where $l_1 = 0, \dots, 11$. For the second subsystem, 29 multi-moments are matched

$$C(A^{-1}E)^{l_2} A^{-1}N(A^{-1}E)^{l_1} A^{-1}B = C_r^T (A_r^{-1}E_r)^{l_2} A_r^{-1}N_r (A_r^{-1}E_r)^{l_1} A_r^{-1}B_r \\ + D_2\delta(l_1)\delta(l_2),$$

where $l_1, l_2 = 0, 1, \dots, 5$ or $l_1 = 6, l_2 = 0, 1$ and $l_1 = 0, 1, l_2 = 6$. For the third subsystem, 37 multi-moments are matched

$$C(A^{-1}E)^{l_3} A^{-1}N \dots N(A^{-1}E)^{l_1} A^{-1}B \\ = C_r^T (A_r^{-1}E_r)^{l_3} A_r^{-1}N_r \dots N_r (A_r^{-1}E_r)^{l_1} A_r^{-1}B_r + D_3\delta(l_1)\delta(l_2)\delta(l_3),$$

where $l_1 = 0, 1, \dots, 5, l_2 = 0, l_3 = 0, 1$ or $l_1 = 0, 1, l_2 = 0, l_3 = 2, 3, 4, 5$ or $l_1 = 0, 1, l_2 = 1, l_3 = 0, 1$. For the fourth subsystem, 4 multi-moments are matched

$$\begin{aligned} & C((A^{-1}E)^{l_4}A^{-1}N \cdots N(A^{-1}E)^{l_1}A^{-1}B \\ & = C_r((A_r^{-1}E_r)^{l_4}A_r^{-1}N_r \cdots N_r(A_r^{-1}E_r)^{l_1}A_r^{-1}B_r + D_4\delta(l_1)\delta(l_2)\delta(l_3)\delta(l_4), \end{aligned}$$

where $l_1 = 0, 1, l_2 = 0, l_3 = 0, l_4 = 0, 1$.

4.2. Time-domain representation

Till now, we have shown how to achieve interpolation for the leading k subsystems along with retaining their polynomial parts. In this subsection, we derive the time-domain representation of the reduced bilinear system whose k th order subsystem is of the form given in (20). The following theorem summarizes our results.

Theorem 4.2. *Given a bilinear system, whose k th order transfer function has the form given in (20). Then, the time-domain representation of this bilinear system can be written as*

$$\begin{aligned} E_r \dot{x}_r(t) &= A_r x_r(t) + N_r x_r(t) u(t) + B_r u(t), \\ y_r(t) &= C_r x_r(t) + \sum_{k=1}^{\infty} D_k u^k(t). \end{aligned} \quad (32)$$

Proof. We begin with the k th order transfer function

$$\begin{aligned} H_r(s_1, \dots, s_k) &= C_r(s_k E_r - A_r)^{-1} N_r (s_{k-1} E_r - A_r)^{-1} N_r \cdots \\ &\quad \cdots N_r (s_1 E_r - A_r)^{-1} B_r + D_k, \\ &= \hat{C}(s_k I_r - \hat{A})^{-1} \hat{N} (s_{k-1} I_r - \hat{A})^{-1} \hat{N} \cdots \\ &\quad \cdots \hat{N} (s_1 I_r - \hat{A})^{-1} \hat{B} + D_k, \end{aligned} \quad (33)$$

where

$$\hat{A} = E_r^{-1} A_r, \quad \hat{N} = E_r^{-1} N_r, \quad \hat{B} = E_r^{-1} B_r \quad \text{and} \quad \hat{C} = C_r. \quad (34)$$

By utilizing the multivariate inverse Laplace transform on (33), we obtain the regular Volterra kernel as:

$$h_k(t_1, t_2, \dots, t_k) = \hat{C} e^{\hat{A}t_k} \hat{N} e^{\hat{A}t_{k-1}} \hat{N} \cdots \hat{N} e^{\hat{A}t_1} \hat{B} + D_k \delta(t_k) \delta(t_{k-1}) \cdots \delta(t_1). \quad (35)$$

As discussed in [22], the output $y_r(t)$ of a nonlinear system can be described in terms of the Volterra kernel $h_k(t_1, t_2, \dots, t_k)$ and input $u(t)$ as follows:

$$y_r(t) = \sum_{k=1}^{\infty} \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} h_k(t_1, t_2, \dots, t_k) u(t - \sum_{i=1}^k t_i) \cdots u(t - t_k) dt_k \cdots dt_1.$$

Substituting (35) in the above equation, we can write

$$y_r(t) = y_r^{(1)}(t) + y_r^{(2)}(t),$$

where

$$y_r^{(1)}(t) = \sum_{k=1}^{\infty} \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} \hat{C} e^{\hat{A}t_k} \hat{N} \cdots \hat{N} e^{\hat{A}t_2} \hat{N} e^{\hat{A}t_1} \hat{B} u(t - \sum_{i=1}^k t_i) \cdots u(t - t_k) dt_k \cdots dt_1,$$

$$y_r^{(2)}(t) = \sum_{k=1}^{\infty} \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_k} D_k \delta(t_k) \delta(t_{k-1}) \cdots \delta(t_1) u(t - \sum_{i=1}^k t_i) \cdots u(t - t_k) dt_k \cdots dt_1.$$

The response $y_r^{(1)}(t)$ is simply the Volterra series representation of a bilinear ODE system with zero initial condition [22]. This means that corresponding to $y_r^{(1)}(t)$, we have

$$\begin{aligned} \dot{x}_r(t) &= \hat{A}x_r(t) + \hat{N}x_r(t)u(t) + \hat{B}u(t), \\ y_r^{(1)}(t) &= \hat{C}x_r(t), \quad x_r(0) = 0. \end{aligned} \tag{36}$$

For $y_r^{(2)}(t)$, we use the properties of the Dirac delta function [28] which leads to

$$y_r^{(2)}(t) = \sum_{k=1}^{\infty} D_k u(t) \cdots u(t) = \sum_{k=1}^{\infty} D_k u^k(t).$$

By combining the responses $y_r^{(1)}(t)$ and $y_r^{(2)}(t)$ and substituting the expression for $\hat{A}, \hat{N}, \hat{B}$ and \hat{C} from (34), we obtain a bilinear system as in (32) and this proves the theorem. \square

Since the output equation in (32) contains the sum of an input dependent infinite series, we need to compute the summation at each time step. This increases the computational cost and may destroy the effect of the model reduction procedure. In the following, we discuss some cases where this infinite summation can be computed cheaply.

Case 1: For the particular structure of N as in (15), $D_k = 0$ for all $k > 1$.

Thus $\sum_{k=1}^{\infty} D_k u^k(t)$ reduces to $D_1 u(t)$, which is computationally cheap.

Case 2: There are some applications where the input $u(t)$ can be considered constant or unity ($u(t) = \alpha$ or $u(t) = 1$). These scenarios may appear, for example in the parameter varying systems [29]. In such a case

$$\sum_{k=1}^{\infty} D_k u^k(t) = D_1 \alpha + D_2 \alpha^2 + D_3 \alpha^3 \cdots .$$

Substituting the expression of D_k from Lemma 3.1 in the above equation, we get

$$\begin{aligned}\sum_{k=1}^{\infty} D_k u^k(t) &= (CMB + D)\alpha + C(MN)MB\alpha^2 + C(MN)^2MB\alpha^3 + \dots \\ &= \alpha C(I + \alpha MN + \alpha^2(MN)^2 + \dots)MB + \alpha D.\end{aligned}$$

Now, if we assume $\|\alpha MN\|_2 < 1$, we have

$$\sum_{k=1}^{\infty} D_k u^k(t) = (C(I - \alpha MN)^{-1}MB + D)\alpha.$$

Thus, we can identify an expression of the convergent series for constant inputs.

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Case 3: In this case, we assume convergence for $\|D_k\|$, i.e. $\sum_{k=j+1}^{\infty} \|D_k\| < \tau \ll 1$.

Then for bounded inputs, we can truncate the infinite summation after the j th term. That is

$$\sum_{k=1}^{\infty} D_k u^k(t) \approx \sum_{k=1}^j D_k u^k(t).$$

Thus, we can save the computations associated with $\sum_{k=j+1}^{\infty} D_k u^k(t)$.

4.3. Interpolation of multi-input multi-output bilinear DAEs

Thus far, we have concentrated on interpolation of single-input single-output bilinear systems for simplicity of notation. Nonetheless, it can be extended to multi-input multi-output bilinear systems, where the notation becomes much more difficult to handle. Therefore, we consider interpolation of the first 2 subsystems only in order to give a glimpse how the proposed methodology can be applied to MIMO bilinear DAEs, but nonetheless one can consider interpolation of the leading first k subsystem as well. We consider a MIMO bilinear system (1) whose the leading four multivariate subsystems can given as follows:

$$\begin{aligned}H_1(\mathcal{S}_1) &= C(s_1 E - A)^1 B, \\ H_2(\mathcal{S}_2) &= [H_2^{(1)}(\mathcal{S}_2), \dots, H_2^{(m)}(\mathcal{S}_2)], \\ H_3(\mathcal{S}_3) &= [H_3^{(1,1)}(\mathcal{S}_3), \dots, H_3^{(1,m)}(\mathcal{S}_3), H_3^{(2,1)}(\mathcal{S}_3), \dots, H_3^{(m,m)}(\mathcal{S}_3)], \\ H_4(\mathcal{S}_4) &= [H_4^{(1,1,1)}(\mathcal{S}_4), \dots, H_4^{(1,1,m)}(\mathcal{S}_4), H_4^{(2,1,1)}(\mathcal{S}_4), \dots, H_4^{(m,m,m)}(\mathcal{S}_4)],\end{aligned}$$

where

$$H_i^{(l_1, \dots, l_{i-1})}(\mathcal{S}_i) = C\phi(s_i)N_{l_{i-1}}\phi(s_{i-1})\dots N_{l_1}\phi(s_1)B \quad (37)$$

with $\phi(s_i) := (s_i E - A)^{-1}$. Moreover, we denote the polynomial parts of $H_i^{(l_1, \dots, l_{i-1})}(\mathcal{S}_i)$ by $D_i^{(l_1, \dots, l_{i-1})}$, which can be given as follows by using Lemma 3.1:

$$\begin{aligned} D_1 &= CMB + D, \\ D_i^{(l_1, \dots, l_{i-1})} &= CMN_{l_{i-1}}M \cdots N_{l_1}MB, \end{aligned} \quad (38)$$

where M is the same as defined in (6). Furthermore, we need to use a more general nested structure to determine the projection matrices. For this, we assume arbitrary interpolation points $\sigma_i, \mu_i \in \mathbb{C}$ such that $sE - A$ and $sE_r - A_r$ are invertible for $s = \sigma_i, \mu_i$, and define the projection matrices V and W as follows:

$$\begin{aligned} \text{range}(V^{(1)}) &= \mathcal{K}_\alpha((\sigma_1 E - A)^{-1}E, (\sigma_1 E - A)^{-1}B), \\ \text{range}(V_i^{(2)}) &= \mathcal{K}_\alpha((\sigma_i E - A)^{-1}E, (\sigma_i E - A)^{-1}N_i V^1), \quad i = 1, \dots, m, \\ \text{range}(W^{(1)}) &= \mathcal{K}_\beta((\mu_1 E - A)^{-T}E^T, (\mu_1 E - A)^{-T}C^T), \\ \text{range}(W_i^{(2)}) &= \mathcal{K}_\alpha((\mu_i E - A)^{-T}E^T, (\mu_i E - A)^{-T}N_i^T W^1), \quad i = 1, \dots, m, \\ \text{range}(V) &= \text{range}(V^{(1)}) + \bigcup_{i=1}^m \{ \text{range}(V_m^{(2)}) \}, \\ \text{range}(W) &= \text{range}(W^{(1)}) + \bigcup_{i=1}^m \{ \text{range}(W_m^{(2)}) \}. \end{aligned}$$

In order to ensure the same number of columns in V and W , we choose α and β such that $m\alpha = p\beta$, where p and m are the numbers of outputs and inputs, respectively. Next, we consider L_A, L_N, L_B and L_C which are solutions to the following set of equations:

$$\begin{aligned} W^T L_B &= \left[(e_1^\alpha)^T \otimes \hat{D}_1, (e_1^{\alpha^2})^T \otimes D_2^{(1)}, \dots, (e_1^{\alpha^2})^T \otimes D_2^{(m)} \right]^T, \\ L_C V &= \left[\hat{D}_1 \otimes (e_1^\beta)^T, D_2^{(1)} \otimes (e_1^{\beta^2})^T, \dots, D_2^{(m)} \otimes (e_1^{\beta^2})^T \right], \\ L_A V &= \left[L_B \otimes (e_1^\alpha)^T, L_{N^{(1)}} V^1 (I_{m\alpha} \otimes (e_1^\alpha)^T), \dots, L_{N^{(m)}} V^1 (I_{m\alpha} \otimes (e_1^\alpha)^T) \right], \\ W^T L_A &= \left[L_C^T \otimes (e_1^\beta)^T, L_{N^{(1)}}^T W^1 (I_{p\beta} \otimes (e_1^\alpha)^T), \dots, L_{N^{(m)}}^T W^1 (I_{p\beta} \otimes (e_1^\alpha)^T) \right]^T, \end{aligned}$$

in which $\hat{D}_1 = D_1 - D$. Then, the reduced-order system can be determined as follows:

$$\begin{aligned} \hat{E} &= W^T E V, & \hat{A} &= W^T (A + L_A) V, & \hat{N}_i &= W^T (N_i - L_{N^{(i)}}) V, \\ \hat{B} &= W^T (B - L_B), & \hat{C} &= (C - L_C) V. \end{aligned}$$

Similar to the SISO bilinear systems, the explicit computation of the matrices $L_A, L_{N^{(i)}}, L_B$ and L_C can also be avoided in order to determine reduced-order systems. However, for brevity of the paper, we skip the derivation for the MIMO systems. 215 However, for brevity of the paper, we skip the derivation for the MIMO case, but it can be done analogously as shown for the SISO case.

5. Numerical Experiments

In this section, we present numerical results for model reduction of structured bilinear DAE systems using different approaches. The reduced-order system can be computed either by direct implementation of Theorem 3.1, without matching the polynomial part in the reduced-order system (classical interpolatory technique) or by our proposed methodology which achieves the matching of the polynomial part in addition to interpolation. All the numerical results were simulated in MATLAB[®] version 7.11.0.584(R2010b) 64-bit (glnza64) on Intel(R) Core(TM)2 Quad CPU Q9550 @ 2.83GHz, 6 MB cache, 4GB RAM, openSUSE 12.1 (x86-64).

5.1. Artificial Example

The bilinear DAE system, that is to be reduced, is generated randomly of order $n = 100$ and with partitioning $n_1 = 90$, $n_2 = 10$. It is ensured that the structure of the matrix pencil $\lambda E - A$ is similar to the index-1 pencil of a linear DAE. The polynomial parts of the first 4 subsystems of the bilinear system are $D_1 = 0.1472$, $D_2 = 5 \cdot 10^{-3}$, $D_3 = 1.92 \cdot 10^{-4}$, $D_4 = 7.35 \cdot 10^{-6}$, where D_i is the polynomial part of the i th subsystem. The interpolation points are selected as $\sigma = \mu = [0, 0.5]$ with multiplicity $q = 1$ resulting in a reduced-order system of order $r = 4$. We truncate the infinite summation in Lemma 4.1 after 4 terms since $\|D_i\|$ decreases exponentially.

We compute the reduced-order systems by using the classical interpolation technique and the proposed methodology, having the same interpolation points and multiplicities. The time-domain responses of the actual and the reduced bilinear systems, obtained by using the implicit Euler method, are shown in Figure 1a for an exponential input. The relative errors associated with the two approaches are shown in Figure 1b.

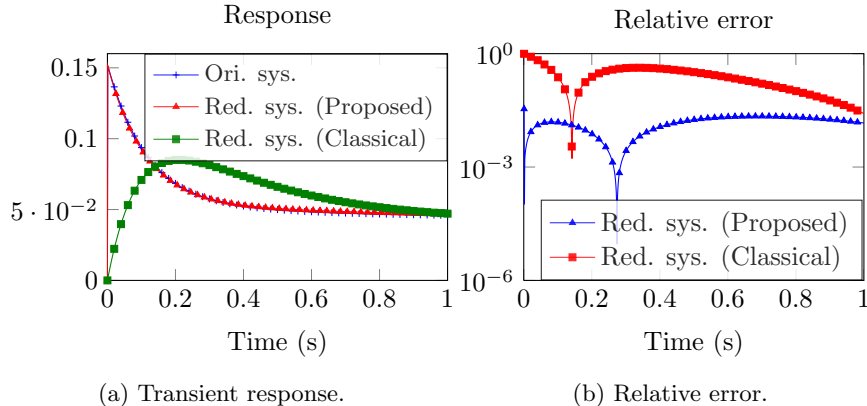


Figure 1: Comparison of the time-domain simulations of the original and reduced-order systems for an input $u(t) = e^{-10t}$.

Certainly, the reduced-order system obtained from the direct implementation shows completely different dynamics whereas the proposed methodology captures the dynamics of the original system well.

5.2. Nonlinear RC Circuit

As a second example, we consider a nonlinear RC circuit that represents a modified form of the transmission line circuit proposed in [30]. The circuit includes resistors, capacitors and diodes as shown in Figure 2.

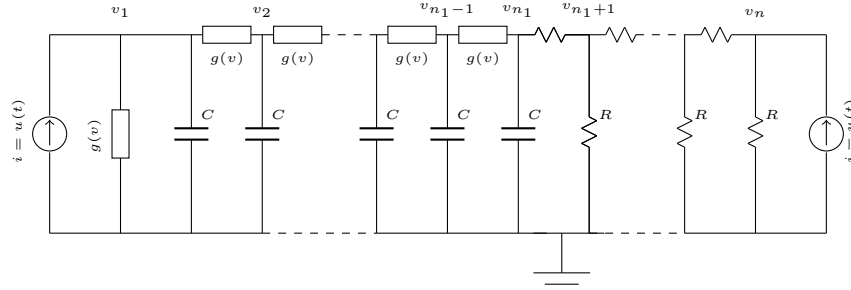


Figure 2: Nonlinear transmission line circuit.

All the resistances and capacities are set to 1 and all the diodes ensure $i_D = e^{40v_D} + v_D - 1$, where i_D represents the current and v_D is the voltage across the diodes. The input $u(t)$ is the current source i and the output $y(t)$ represents the average voltage over all nodes ranging from 1 to n . Using Kirchhoff's current law at each node, we have

$$\begin{aligned} \dot{v}_1 &= -2v_1 + v_2 + 2 - e^{40v_1} - e^{40(v_1-v_2)} + u(t), \\ \dot{v}_k &= -2v_k + v_{k-1} + v_{k+1} + e^{40(v_{k-1}-v_k)} - e^{40(v_k-v_{k+1})}, \quad (2 \leq k \leq n_1 - 1) \\ \dot{v}_{n_1} &= -2v_{n_1} + v_{n_1-1} + v_{n_1+1} - 1 + e^{40(v_{n_1-1}-v_{n_1})}, \\ 0 &= 3v_k - v_{k-1} - v_{k+1}, \quad (n_1 + 1 \leq k \leq n - 1) \\ 0 &= -2v_n + v_{n-1} + u(t). \end{aligned}$$

In order to represent the above nonlinear system as a quadratic-bilinear system, we set v_1 to $v_{k,k+1}$ ($v_{k,k+1} = v_k - v_{k+1}$), $k = 1, \dots, n_1 - 1$, and v_{n_1+1} to v_n as the state variables, and perform some changes of variables by defining $y_1 = e^{40v_1} - 1$ and $y_k = e^{40(v_{k-1,k})} - 1$, $2 \leq k \leq n_1$. Together with the differential equations

of all y_k 's, the above changes lead to the following set of equations:

$$\begin{aligned}
\dot{v}_1 &= -v_1 - v_{1,2} - y_1 - y_2 + u(t), \\
\dot{v}_{1,2} &= -v_1 - 2v_{1,2} + v_{2,3} - y_1 - 2y_2 + y_3 + u(t), \\
\dot{v}_{k,k+1} &= -2v_{k,k+1} + v_{k-1,k} + v_{k+1,k+2} + y_k - 2y_{k+1} + y_{k+2}, \quad (2 \leq k \leq n_1 - 2) \\
\dot{v}_{n_1-1,n_1} &= -2v_{n_1-1,n_1} + v_{n_1-2,n_1-1} + v_{n_1} - v_{n_1+1} + y_{n_1-1} - 2y_{n_1}, \\
0 &= 3v_k - v_{k-1} - v_{k+1}, \quad (n_1 + 1 \leq k \leq n - 1) \\
0 &= -2v_n + v_{n-1} + u(t), \\
\dot{y}_1 &= 40(y_1 + 1)(-v_1 - v_{1,2} - y_1 - y_2 + u(t)), \\
\dot{y}_2 &= 40(y_2 + 1)(-v_1 - 2v_{1,2} + v_{2,3} - y_1 - 2y_2 + y_3 + u(t)), \\
\dot{y}_k &= 40(y_k + 1)(-2v_{k-1,k} + v_{k-2,k-1} + v_{k,k+1} + y_{k-1} - 2y_k + y_{k+1}), \\
\dot{y}_{n_1} &= 40(y_{n_1} + 1)(-2v_{n_1-1,n_1} + v_{n_1-2,n_1-1} + v_{n_1} - v_{n_1+1} + y_{n_1-1} - 2y_{n_1}).
\end{aligned}$$

250 In the above set of equations, we fixed v_{n_1} to $v_1 - \sum_{i=2}^{n_1} v_{k-1,k}$. This means that the circuit can be modelled by a quadratic-bilinear descriptor system of order $\tilde{n} = n_1 + n$, having an index-1 matrix pencil associated with the quadratic system of index-1. Next, we utilize the Carleman bilinearization, ensuring that the resulting bilinearized system also has an index-1 matrix pencil [31]. The order of the bilinearized DAE system is $\mathcal{N} = (n_1 + n)(2n_1 + 1)$. The polynomial part of the first subsystem of the bilinearized system is $D_1 = 0.0333$ and higher order subsystems have zero polynomial parts.

260 For our experiment, we choose $n_1 = 10$ and $n = 30$. The bilinearized system is therefore of order $\mathcal{N} = 840$. Using Theorem 3.1, we compute the projection matrices such that the reduced-order system guarantees interpolation of the first two subsystems at $\sigma = \mu = [10, 50, 300]$. The multiplicities of all the interpolation points are set to 1. The reduced-order systems of the bilinearized system are computed using the classical and the proposed methodology using the same interpolation points and multiplicities, since we do not have specific criteria yet to choose these interpolation points and their multiplicities which can ensure a stable reduced-order system for both the modified and the classical method. For our result, it is possible to get stable reduced-order systems using this methodology for the same interpolation points and same multiplicities in case of one-sided projection, i.e. $W = V$.

270 The time responses of the resulting reduced-order bilinear systems are shown in Figure 3a by utilizing the implicit Euler method, and also the absolute errors ($|y - \hat{y}|$) are shown in Figure 3b. Clearly, the proposed interpolatory technique shows a substantial improvement in the transient response of the system.

6. Conclusions

275 We proposed interpolatory techniques for a special class of bilinear descriptor systems with a particular attention to their polynomial parts. An expression that explicitly identifies the polynomial part of each subsystem associated with

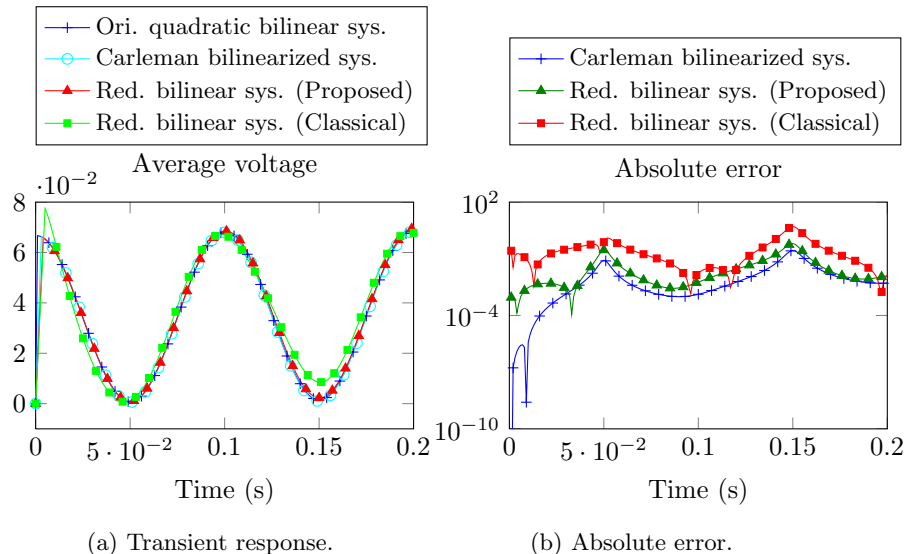


Figure 3: The comparison of time-domain responses of the original, Carleman bilinearized and the reduced-order systems for an input $u(t) = \cos(20\pi t) + 1$.

the bilinear system has been derived. This extends the expression for the polynomial part of linear index-1 DAE systems discussed in [21] to bilinear systems. Also, we have derived conditions on interpolatory subspaces that not only guarantees interpolation of the first k subsystems, but also retains the polynomial part of the bilinear system. By means of numerical examples, we have shown the efficiency of the proposed model reduction technique.

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