GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN

Fakultät für Physik

Bachelor Thesis

Non-Linear Response of Water Flux Through a Soil Column

Die nichtlineare Antwort des Abflusses von Wasser durch eine Säule aus Erdreich

prepared by

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1. Motivation

The rain falling today does not "know" anything about the rain of the past months. Why should it do so? The rain is not only determined by the local weather but also by the global climate. Filtering out the seasonal oscillations, the precipitation exhibits so called "white noise" over several decades in the frequency space: there the power spectral density is almost constant. The rivers carrying this water to the seas do not have that property: the discharge, that is the volume flow rate, near the mouth shows a long term correlation in the form of a $1/f$-noise [4, 5]. Consequently the transport process must be non-linear, since linear systems cannot change the noise spectrum.

On the other hand if this process in non-linear, it is not clear whether the seasonal oscillations without any noise already produce the basic $1/f$-characteristics. Nor is it obvious if the transport of the water through the soil or the flow of the river is the dominant non-linearity in this transport process, or if this even changes with the considered time scale.

In this work I will focus on a simple non-linear model that mimics the flow of water through soil. The model can be described by Burgers' equation, that is well known for producing turbulence, but has other applications as well [2]. I will analyze the output that my model produces for an influx representing seasonal changes and present the water content provided by this input as a function of space and time. Especially the comparison between the input and the output reveals remarkable properties of the model.

2. The Model

Here is a model which is quite simple but capable to reproduce salient features of the observations:

Consider a column of length L and a cross-section that shall be negligible against its length[1]. Let the variables t represent time and x the longitudinal coordinate. Let the column consist of soil, that is able to contain water, transport it along the x-direction and to release it in the end at position $x = L$. During that transport process the soil may not be altered in any way: it shall neither be advected nor compressed. Apart from that let gravity g act on the water. The orientation of the column can differ from that of gravity by an angle α as depicted in figure 1. That angle only changes the effectively acting acceleration $g_{\text{eff}} = g \cos\alpha$ and will be considered $\alpha = 0$ without restriction.

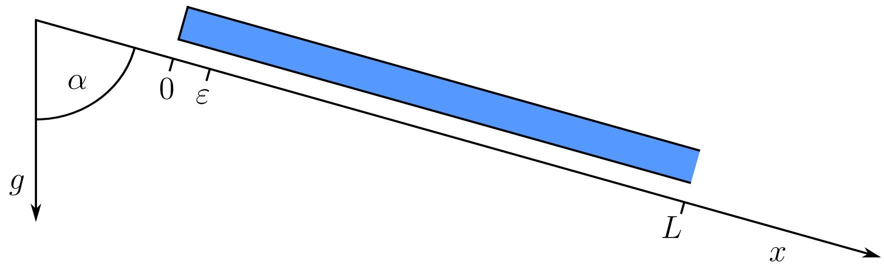

Figure 1: Sketch of the considered geometry.

Now let $\rho(x,t)$ be the water content at location x and time t and $\jmath(x,t)$ the associated current, that satisfy the one dimensional continuity equation

$$\partial_t \rho + \partial_x \jmath = s(x,t) \tag{2.1}$$

with a source $s(x,t)$. Now I assume a special current: In addition to the pure linear diffusion, let the current have a convective term increasing quadratically in ρ:

$$\jmath = -D\,\partial_x \rho + C\,\rho^2\,, \tag{2.2}$$

where C and D are positive constants. The convection is driven by gravity such that $C \sim g$. The quadratic dependency is supposed to model capillary forces: when the water content is very low these forces attach the water to the soil much stronger

[1]Equivalently one could assume the dynamics to be invariant under translation perpendicular to the x-axis.

than gravity is capable of pulling it downwards. For high water content gravity dominates the diffusion and the water is flowing in x-direction. When the water content becomes extremely high, this model will no longer suit to the real physics of the system: if the current is not bounded, the assumption of the soil not being affected by the flow will no longer fit.

That assumed current produces

$$\partial_t \rho = D\,\partial_x^2 \rho - C\,\partial_x\left(\rho^2\right) + s(x,t)\,.$$

Multiplying this equation by $CL^3/D^2\beta^3$, where β is a scaling factor for the system size, and using the rescaling

$$\tilde{x} := \beta\frac{x}{L}\,, \qquad \tilde{t} := \frac{D\beta^2}{L^2}t\,,$$
$$\tilde{\rho}\left(\tilde{x},\tilde{t}\right) := \frac{CL}{D\beta}\rho\left(x,t\right)\,, \qquad \tilde{s}\left(\tilde{x},\tilde{t}\right) := \frac{CL^3}{D^2\beta^3}s\left(x,t\right)$$

yields the nondimensionalized Burgers' equation

$$\partial_{\tilde{t}}\tilde{\rho} = \partial_{\tilde{x}}^2\tilde{\rho} - \partial_{\tilde{x}}\left(\tilde{\rho}^2\right) + \tilde{s}\left(\tilde{x},\tilde{t}\right) \tag{2.3}$$

for $\tilde{x} \in (0,\beta)$ and $\tilde{t} \in (0,\infty)$. The rescaled current is given as

$$\tilde{\jmath} = -\partial_{\tilde{x}}\tilde{\rho} + \tilde{\rho}^2\,. \tag{2.4}$$

We always consider rescaled, dimensionless quantities and suppress the tildes for clarity from now on.

2.1. Cole-Hopf Transformation

In order to linearize Burgers' equation it is useful to apply the Cole-Hopf transformation and write the water content as the logarithmic derivative of a positive potential $\varphi(x,t)$ [1, 3]:

$$\rho = -\,\partial_x \log\varphi = -\,\frac{\partial_x\varphi}{\varphi}\,. \tag{2.5}$$

Here it is important to notice, that this choice features a degree of freedom. A factor $h(t)$ in the potential $\varphi(x,t) = h(t)\psi(x,t)$, that may depend on time, does not alter the water content:

$$\rho = -\frac{\partial_x \varphi}{\varphi} = -\frac{\partial_x h(t)\psi}{h(t)\psi} = -\frac{\partial_x \psi}{\psi}\,.$$

The time derivative of water content becomes

$$\partial_t \rho = \partial_t \left(-\partial_x \log \varphi\right) = -\partial_x \left(\frac{\partial_t \varphi}{\varphi}\right)$$

and the current reads

$$\jmath = -\partial_x \left(-\frac{\partial_x \varphi}{\varphi}\right) + \left(-\frac{\partial_x \varphi}{\varphi}\right)^2 = \frac{\partial_x^2 \varphi}{\varphi}$$

in terms of this new potential.

Via the above transformation the driven Burgers' equation (2.3) turns into

$$\partial_x \left(\frac{\partial_t \varphi}{\varphi}\right) = \partial_x \left(\frac{\partial_x^2 \varphi}{\varphi}\right) - s(x,t)\,.$$

If there is an antiderivative $S(x,t)$ for the source term $s(x,t)$ such that $\partial_x S = s$, then this equation simplifies into the heat equation with an extra term that represents the source:

$$\partial_t \varphi = \partial_x^2 \varphi - S(x,t)\,\varphi\,. \tag{2.6}$$

3. Constant Influx and Steady-State

At first, to simplify matters, I consider a constant influx $s = A^2\,\delta(x-\varepsilon)$, where $\delta(x)$ is the Dirac delta function, located at $x = \varepsilon$ for $0 < \varepsilon < \beta$, $0 < A$. Moreover, I assume the boundary conditions

$$\jmath\,(x=0,t) = 0 \qquad\Longleftrightarrow\qquad \frac{\partial_x^2\varphi}{\varphi}\,(x=0,t) = 0\,, \tag{3.1}$$

i. e. a closed upper boundary where no water can escape, and

$$\rho(x=\beta,t) = 0 \qquad\Longleftrightarrow\qquad -\frac{\partial_x\varphi}{\varphi}\,(x=\beta,t) = 0\,, \tag{3.2}$$

which represents an open lower boundary, where the water leaks. Thus there should be a steady-state solution for equation (2.3), which fulfills $\partial_t\rho \equiv \partial_t\varphi \equiv 0$.

Since the Heaviside step function $\Theta(x)$ fulfills $\partial_x\Theta(x) = \delta(x)$ in terms of distributions, every antiderivative of the source term has the form $S(x) = A^2\,\Theta(x-\varepsilon) + h$, where $h \in \mathbb{R}$. Hence the problem is reduced to the homogeneous linear ODE

$$0 = \varphi'' - \big(A^2\,\Theta(x-\varepsilon) + h\big)\,\varphi \tag{3.3}$$

on the interval $(0,\beta)$ with the transformed boundary conditions (3.1) and (3.2).

It is easy to solve equation (3.3) to the left and right of the source separately:

$$\varphi_l'' = h\,\varphi_l \qquad \text{in } (0,\varepsilon) \tag{3.4}$$

$$\varphi_r'' = \big(A^2 + h\big)\,\varphi_r \qquad \text{in } (\varepsilon,\beta) \tag{3.5}$$

with the boundary conditions

$$\frac{\varphi_l''}{\varphi_l}(0) = 0\,, \qquad \rho_\varepsilon := -\frac{\varphi_l'}{\varphi_l}(\varepsilon) = -\frac{\varphi_r'}{\varphi_r}(\varepsilon)\,, \qquad \frac{\varphi_r'}{\varphi_r}(\beta) = 0\,.$$

Here the connection condition at $x = \varepsilon$ arises from the claim of a continuous water content.

Moreover (3.1) together with (3.4) imply $h = 0$, such that the equation (3.4) has the general solution

$$\varphi_l(x) = m\,(x-\varepsilon) + c\,, \qquad x \in (0,\varepsilon)\,.$$

The general solution to (3.5) can be written as

$$\varphi_r(x) = a\,\mathrm{e}^{A\,(x-\beta)} + b\,\mathrm{e}^{-A\,(x-\beta)}\,, \qquad x \in (\varepsilon, \beta)\,.$$

In order to fulfill the boundary condition at the right boundary the two coefficients have to be equal. Hence the potential is

$$\varphi_r(x) = 2\,a\cosh\big(A\,(\beta - x)\big)$$

and at the location of the source, the water content is

$$-\frac{\varphi_r'}{\varphi_r}(\varepsilon) = A\,\tanh\big(A\,(\beta-\varepsilon)\big) = \rho_\varepsilon.$$

The continuity condition provides

$$-\frac{m}{c} = -\frac{\varphi_l'}{\varphi_l}(\varepsilon) = -\frac{\varphi_r'}{\varphi_r}(\varepsilon) = A\,\tanh\big(A\,(\beta-\varepsilon)\big) = \rho_\varepsilon\,,$$

such that

$$\varphi_l(x) = c\,\big(\rho_\varepsilon(\varepsilon - x) + 1\big)\,.$$

The multiplicative constants a and c cannot be determined any further. This is not a problem, however, since they have no physical relevance, as pointed out in section 2.1 .

In total, the potential is hence given by

$$\varphi(x) = \begin{cases} c\,\big(\rho_\varepsilon(\varepsilon - x) + 1\big)\,, & \text{if } 0 < x < \varepsilon, \\ 2\,a\,\cosh\big(A\,(\beta - x)\big)\,, & \text{if } \varepsilon \le x < \beta. \end{cases} \tag{3.6}$$

That results in the water content

$$\rho(x) = -\frac{\varphi'(x)}{\varphi(x)} = \begin{cases} \left(\varepsilon - x + \frac{1}{\rho_\varepsilon}\right)^{-1}, & \text{if } 0 < x < \varepsilon, \\ A\,\tanh\big(A\,(\beta - x)\big)\,, & \text{if } \varepsilon \le x < \beta\,, \end{cases}$$

which is depicted in figure 2. By construction, the water content is continuous and at the location of the source, its value is $\rho(\varepsilon) = \rho_\varepsilon = A\,\tanh\big(A\,(\beta-\varepsilon)\big)$. Furthermore

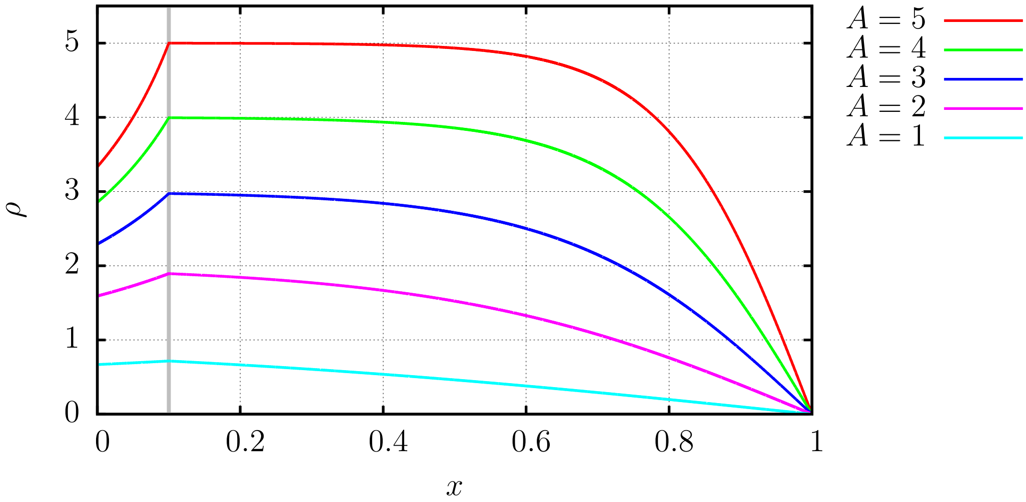

Figure 2: Rescaled steady-state distribution of water, ρ, in the column of soil of length $\beta = 1$ for different constant influxes $s = A^2\,\delta(x - 0.1)$. The gray vertical line marks the location of the point-like source.

the current is given by

$$\jmath(x,t) = A^2\,\Theta(x - \varepsilon)\,,$$

as can immediately be seen from (3.3) utilizing $h = 0$.

Beyond that, one can determine how much water there is in the soil column at all. The total amount of water, M, is dictated by the potential:

$$\begin{aligned} M := \int_0^\beta \rho(x)\,\mathrm{d}x &= -\int_0^\beta \frac{\mathrm{d}}{\mathrm{d}x}\log\varphi\,\mathrm{d}x = -\log\varphi_l(x)\Big|_0^\varepsilon - \log\varphi_r(x)\Big|_\varepsilon^\beta \\ &= \log\left(1 + \varepsilon\rho_\varepsilon\right) + \log\cosh\big(A\,(\beta - \varepsilon)\big) \\ &= \log\Big(1 + A\varepsilon\tanh\big(A\,(\beta-\varepsilon)\big)\Big) + \log\cosh\big(A\,(\beta-\varepsilon)\big) \\ &= \log\Big(\cosh\big(A\,(\beta-\varepsilon)\big) + A\varepsilon\sinh\big(A\,(\beta-\varepsilon)\big)\Big)\,. \end{aligned}$$

Its dependence on the square root of the influx strength, A, is shown in figure 3.

For small input, $A(\beta - \varepsilon) \ll 1$, the amount of water can be approximated by

$$M \approx A^2\,(\beta - \varepsilon)\left(\frac{\beta - \varepsilon}{2} + \varepsilon\right) = A^2\,\frac{\beta^2 - \varepsilon^2}{2}\,.$$

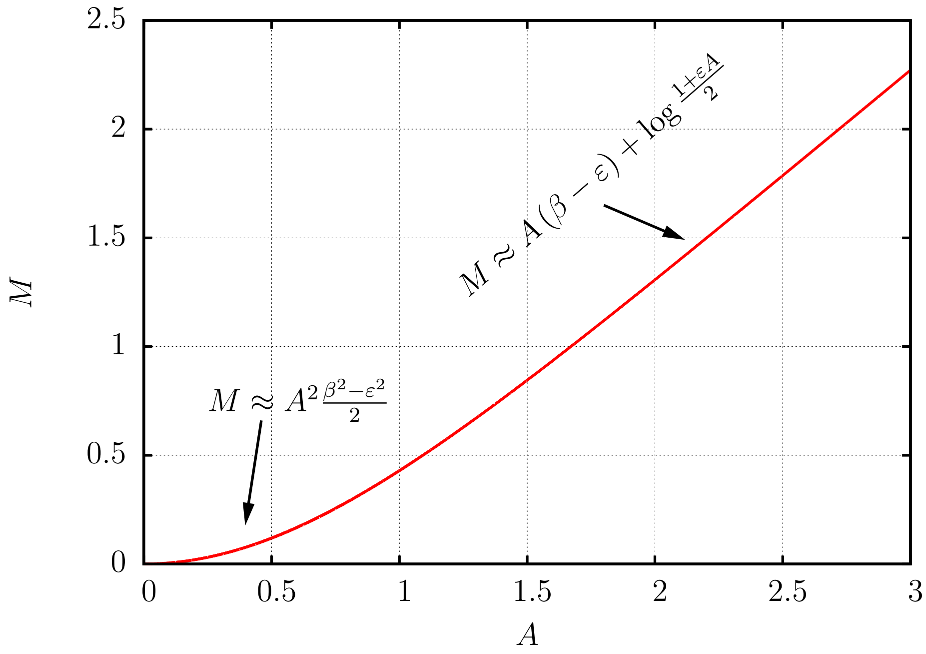

Figure 3: Total amount of water, M, in the column of soil of length $\beta = 1$ plotted as a function of the amplitude A. The influx $s = A^2\,\delta(x-\varepsilon)$ is located at $\varepsilon = 0.1$.

For very high input, $A(\beta - \varepsilon) \gg 1$, it asymptotically approaches

$$M \approx A\,(\beta - \varepsilon) + \log \frac{1 + \varepsilon A}{2}\,.$$

4. Time-Dependent Influx

Now I assume a time-dependent source term $s(x,t) = B(t)\delta(x-\varepsilon)$ in Burgers' equation (2.3). In that case there will not be a steady-state solution any more.

That is why I want to reconsider the antiderivative $S(x,t)$ and its role in equation (2.6): As already mentioned, any antiderivative has the form

$$S(x,t) = B(t)\Theta(x-\varepsilon) + h(t)\,.$$

Note the time dependence of the additional term $h(t)$, which originates in the derivative only acting in x-direction to reproduce s. The time-dependent problem is given by

$$\partial_t \varphi = \partial_x^2 \varphi - \left[B(t)\Theta(x-\varepsilon) + h(t)\right]\varphi \tag{4.1}$$

with boundary conditions

$$\jmath(x=0,t) = \frac{\partial_x^2\varphi}{\varphi}(x=0,t) = 0\,, \qquad \rho(x=\beta,t) = -\frac{\partial_x\varphi}{\varphi}(x=\beta,t) = 0 \tag{4.2}$$

and some arbitrary but (square-)integrable initial condition

$$\rho(x,t=0) = \rho^0(x) \quad\Longleftrightarrow\quad \varphi(x,t=0) \equiv \varphi^0(x) := \exp\left(-\int_0^x \rho^0(\xi)\,\mathrm{d}\xi\right)\,. \tag{4.3}$$

Here I already set the prefactor of φ^0 to unity, as it does not matter physically. Thanks to this gauge freedom, $h(t)$ can be absorbed into the potential as well: The function

$$\tilde\varphi(x,t) := \varphi(x,t)\exp\left(\int_0^t h(\tau)\,\mathrm{d}\tau\right)$$

satisfies the equation

$$\partial_t\tilde\varphi = \partial_x^2\tilde\varphi - B(t)\Theta(x-\varepsilon)\tilde\varphi\,, \tag{4.4}$$

which is basically equation (4.1) for vanishing h. That renormalization does not alter the boundary conditions. Rather, the logarithmic time derivative at $x = 0$ has to be identified with the current at that point:

$$\frac{\partial_t \tilde{\varphi}}{\tilde{\varphi}}(x=0,t) = \frac{\partial_x^2 \tilde{\varphi}}{\tilde{\varphi}}(x=0,t) = 0\,.$$

The boundary conditions prevent the solution from having a pole or a zero at any boundary and therefore the value of the potential at $x = 0$ stays constantly one. For clarity I will, once again, omit the tilde in the following.

The total deviation

$$f := \varphi - 1 \tag{4.5}$$

from that initial and boundary value is the actually interesting quantity. It satisfies the inhomogeneous PDE

$$\partial_t f = \partial_x^2 f - B(t)\Theta(x-\varepsilon)f - B(t)\Theta(x-\varepsilon) \tag{4.6}$$

with the initial and boundary conditions

$$f(x=0,t) \equiv 0\,, \qquad \partial_x f\,(x=\beta,t) \equiv 0\,, \qquad f(x,t=0) \equiv f^0 := \varphi^0 - 1\,. \tag{4.7}$$

Now I will choose $\beta = \frac{\pi}{2}$ and mirror the problem (4.6) symmetrically around the points $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ and anti-symmetrically around $x = \pi$. Thus the problem is defined on the interval $(0, 2\pi)$ and the solution fulfills

$$\begin{aligned} f\left(\tfrac{\pi}{2} + x\right) &= f\left(\tfrac{\pi}{2} - x\right)\,, \\ f\left(\pi + x\right) &= -f\left(\pi - x\right)\,, \\ f\left(\tfrac{3\pi}{2} + x\right) &= f\left(\tfrac{3\pi}{2} - x\right)\,. \end{aligned} \tag{4.8}$$

The new boundary conditions are homogeneous:

$$f(0,t) \equiv 0\,, \qquad\qquad f(2\pi,t) \equiv 0\,. \tag{4.9}$$

Please note that the two unit step functions have to be mirrored differently in order to get correctly mirrored dynamics: Since f has to be antisymmetric around $x = \pi$, the unit step function in the product with f has to be mirrored symmetrically, the last term has to be mirrored anti-symmetrically such that the extended problem is given by

$$\partial_t f = \partial_x^2 f - B(t)g(x)f - B(t)G(x)\,, \tag{4.10}$$

where (cf. figure 4)

$$g(x) := \chi(x) + \chi(x-\pi) \qquad \text{and} \qquad G(x) := \chi(x) - \chi(x-\pi)$$

are the differently mirrored images of the indicator function of the interval $(\varepsilon, \pi-\varepsilon)$:

$$\chi(x) := \chi_{(\varepsilon,\pi-\varepsilon)}(x) = \left\{\begin{matrix} 1\,, & x \in (\varepsilon, \pi-\varepsilon) \\ 0\,, & x \notin (\varepsilon, \pi-\varepsilon) \end{matrix}\right\} =: [x \in (\varepsilon, \pi-\varepsilon)]\,.$$

In the latter step I introduce the indicator of a proposition P

$$[P] := \begin{cases} 1\,, & P \text{ is true} \\ 0\,, & P \text{ is false} \end{cases} \tag{4.11}$$

to be able to write expressions of that kind in a short way.

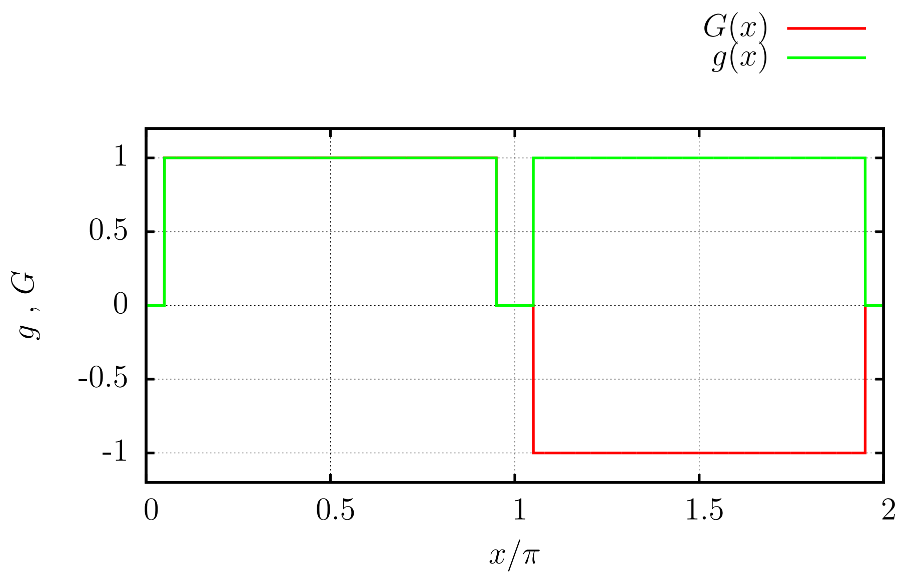

Figure 4: The functions g and G plotted for $\varepsilon = 0.1$. They are equal for many x and only differ on $(\pi+\varepsilon, 2\pi-\varepsilon)$.

4.1. Time-Dependent Problem Expressed in Fourier Series

The extended PDE can be transformed into a system of ODEs by rewriting

$$f(x,t) = \sum_{k\in\mathbb{Z}} f_k(t)\,\mathrm{e}^{\mathrm{i}kx}\,, \qquad g(x) = \sum_{k\in\mathbb{Z}} g_k\,\mathrm{e}^{\mathrm{i}kx}\,, \qquad G(x) = \sum_{k\in\mathbb{Z}} G_k\,\mathrm{e}^{\mathrm{i}kx}\,, \tag{4.12}$$

$$g(x)f(x,t) = \sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} g_l f_m(t)\,\mathrm{e}^{\mathrm{i}(l+m)x} = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} g_{k-m} f_m(t)\,\mathrm{e}^{\mathrm{i}kx} \tag{4.13}$$

in Fourier series. Due to the choice of $\beta = \frac{\pi}{2}$ the problem now is defined on the domain $(0, 2\pi)$, which simplifies the relations for Fourier series, as summarized in the appendix A.

The solution of equation (4.10) has vanishing Fourier coefficients for even indices, the remaining ones are purely imaginary: From the symmetries in (4.8) and the equation (A.6) one obtains

$$f(x,t) = \sum_{k\in 2\mathbb{N}+1} 2\mathrm{i}\, f_k(t)\sin kx\,.$$

Due to the symmetry and antisymmetry of g and G around the points $p = \frac{\pi}{2}$ and $p = \pi$ respectively, their Fourier coefficients have the characteristics according to (A.5):

$$g_k = g_{-k}\,[k\in 2\mathbb{Z}]\,, \qquad\qquad G_k = -G_{-k}\,[k\in 2\mathbb{Z}+1]\,. \tag{4.14}$$

More precisely, the Fourier coefficients of g and G can be determined exactly: Using the Fourier coefficients

$$\begin{aligned}
\chi_k &= \frac{1}{2\pi}\langle\chi, \mathrm{e}^{\mathrm{i}k\,\cdot}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\chi(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{2\pi}\int_\varepsilon^{\pi-\varepsilon}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \frac{1}{2\pi\mathrm{i}k}\left(\mathrm{e}^{-\mathrm{i}k\varepsilon} - \mathrm{e}^{-\mathrm{i}k(\pi-\varepsilon)}\right) = \frac{1}{\pi k}\,\mathrm{e}^{-\mathrm{i}k\frac{\pi}{2}}\sin\left(k\left(\frac{\pi}{2}-\varepsilon\right)\right) \\
&= \frac{(-\mathrm{i})^k}{2}\left(1-\frac{2\varepsilon}{\pi}\right)\operatorname{sinc}\left(k\left(\frac{\pi}{2}-\varepsilon\right)\right)\,,
\end{aligned}$$

which includes the case $k = 0$, since $\operatorname{sinc} x := \frac{\sin x}{x}$ with removed singularity in the origin, and with

$$\big\langle \chi(\,\cdot\,-\pi),\, \mathrm{e}^{\mathrm{i}k\,\cdot}\big\rangle = \frac{1}{2\pi}\int_0^{2\pi} \chi(x-\pi)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi(y)\,\mathrm{e}^{-\mathrm{i}k(y+\pi)}\mathrm{d}y = \mathrm{e}^{-\mathrm{i}k\pi}\chi_k$$
$$= (-1)^k\,\chi_k$$

it follows that

$$\begin{aligned} g_k &= \chi_k\,\big(1+(-1)^k\big) \\ &= (-1)^{\frac{k}{2}}\left(1-\frac{2\varepsilon}{\pi}\right)\operatorname{sinc}\left(\frac{k}{2}\,(\pi-2\varepsilon)\right)\ [k\in 2\mathbb{Z}] \end{aligned} \tag{4.15}$$

and that

$$\begin{aligned} G_k &= \chi_k\,\big(1-(-1)^k\big) \\ &= -\mathrm{i}\,(-1)^{\frac{k-1}{2}}\left(1-\frac{2\varepsilon}{\pi}\right)\operatorname{sinc}\left(\frac{k}{2}\,(\pi-2\varepsilon)\right)\ [k\in 2\mathbb{Z}+1]\,. \end{aligned} \tag{4.16}$$

For later use I give the limit $\varepsilon \to 0$ for g_k and G_k, respectively:

$$\lim_{\varepsilon\to 0} g_k = [k=0] = \delta_{k,0}\,, \qquad \lim_{\varepsilon\to 0} G_k = -\frac{2\mathrm{i}}{k\pi}[k\in 2\mathbb{Z}+1]\,,$$

since for $k \neq 0$:

$$\begin{aligned} \lim_{\varepsilon\to 0}\operatorname{sinc}\left(\frac{k}{2}\,(\pi-2\varepsilon)\right) &= \lim_{\varepsilon\to 0}\frac{2}{k(\pi-2\varepsilon)}\sin\left(\frac{k}{2}\,(\pi-2\varepsilon)\right) \\ &= \frac{2}{k\pi}(-1)^{\frac{k-1}{2}}\,[k\in 2\mathbb{Z}+1]\ , \end{aligned}$$

but

$$[k\in 2\mathbb{Z}+1][k\in 2\mathbb{Z}] \equiv 0\,,$$

while for $k = 0$:

$$\lim_{\varepsilon\to 0}\operatorname{sinc}\left(\frac{0}{2}\,(\pi-2\varepsilon)\right) = \lim_{\varepsilon\to 0} 1 = 1\,.$$

Using all the above, the problem (4.10) can be expressed in terms of Fourier coefficients:

$$\dot{f}_k(t) = -k^2 f_k(t) - B(t) \sum_{m\in 2\mathbb{Z}+1} g_{k-m} f_m(t) - B(t) G_k\,, \qquad k \in 2\mathbb{Z}+1 \tag{4.17}$$

with the initial condition

$$f_k(0) = f_k^0\,.$$

This is not trivial to solve because all modes are coupled.

4.2. Source Located in Boundary

In the limit where the source is located in the boundary, i. e. $\varepsilon \to 0$, the problem (4.17) can be solved analytically. Then the modes decouple because $g_k \to [k = 0]$, as presented in the previous section.

So the decoupled problem reads

$$\begin{cases} \dot{f}_k(t) = -\left(k^2 + B(t)\right) f_k(t) + \dfrac{2\mathrm{i}}{k\pi} B(t) \\ f_k(0) = f_k^0 \end{cases} \qquad \text{for } k \in 2\mathbb{Z}+1\,. \tag{4.18}$$

Using the abbreviation $\mathcal{B}(t) = \int_0^t B(\tau)\mathrm{d}\tau$ the fundamental solution is given by

$$\mathrm{e}^{-k^2 t - \mathcal{B}(t)}\,, \qquad k \in 2\mathbb{Z}+1\,.$$

For a given $k \in 2\mathbb{Z}+1$ variation of constants provides the ansatz

$$f_k(t) = r_k(t)\,\mathrm{e}^{-k^2 t - \mathcal{B}(t)} \tag{4.19}$$

to solve the inhomogeneous problem (4.18). Inserting the ansatz results in

$$\begin{cases} \dot{r}_k(t) = \dfrac{2\mathrm{i}}{k\pi} B(t)\,\mathrm{e}^{k^2 t + \mathcal{B}(t)} \\ r_k(0) = f_k(0) = f_k^0 \end{cases}\,,$$

which can be integrated immediately. The solution

$$r(t) = f_k^0 + \frac{2\mathrm{i}}{k\pi} \int_0^t B(\tau)\, \mathrm{e}^{k^2\tau + \mathcal{B}(\tau)} \mathrm{d}\tau\,,$$

together with (4.19), yields

$$f_k(t) = f_k^0\, \mathrm{e}^{-k^2 t - \mathcal{B}(t)} + \frac{2\mathrm{i}}{k\pi}\, \mathrm{e}^{-k^2 t - \mathcal{B}(t)} \int_0^t B(\tau)\, \mathrm{e}^{k^2\tau + \mathcal{B}(\tau)} \mathrm{d}\tau \tag{4.20}$$

$$= f_k^0\, \mathrm{e}^{-k^2 t - \mathcal{B}(t)} + \frac{2\mathrm{i}}{k\pi} \int_0^t B(\tau)\, \mathrm{e}^{k^2(\tau - t) + \mathcal{B}(\tau) - \mathcal{B}(t)} \mathrm{d}\tau\,. \tag{4.21}$$

4.3. Solution for Special Driving

In the following I assume a constant influx that is perturbed periodically:

$$B(t) = A^2 + P \sin \Omega t \tag{4.22}$$

with positive constants A, P and Ω, and $A^2 \geq P$ for the sake of a non-negative input signal. The frequency Ω can be understood to model a period of about one year. Thus this input is a first approximation to the real rain which usually displays seasonal oscillations.

Let the source be located at $x = \varepsilon \to 0$ as discussed in the previous section. With the antiderivative of the input signal

$$\mathcal{B}(t) = \int_0^t B(\tau)\, \mathrm{d}\tau = A^2 t + \frac{P}{\Omega}\, (1 - \cos \Omega t)$$

equation (4.21) takes the form

$$\begin{aligned} f_k(t) &= f_k^0\, \mathrm{e}^{-\left(k^2 + A^2\right)t}\, \mathrm{e}^{\frac{P}{\Omega}(1 - \cos\Omega t)} \\ &\quad + \frac{2\mathrm{i}}{k\pi} \int_0^t \left(A^2 + P \sin \Omega\tau\right)\, \mathrm{e}^{-\left(k^2 + A^2\right)(t - \tau)}\, \mathrm{e}^{\frac{P}{\Omega}(\cos\Omega t - \cos\Omega\tau)} \mathrm{d}\tau \\ &= f_k^0\, \mathrm{e}^{-\left(k^2 + A^2\right)t}\, \mathrm{e}^{\frac{P}{\Omega}(1 - \cos\Omega t)} \\ &\quad + \frac{2\mathrm{i}}{k\pi}\, \mathrm{e}^{-\left(k^2 + A^2\right)t}\, \mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} \underbrace{\int_0^t \left(A^2 + P \sin \Omega\tau\right)\, \mathrm{e}^{\left(k^2 + A^2\right)\tau}\, \mathrm{e}^{-\frac{P}{\Omega}\cos\Omega\tau} \mathrm{d}\tau}_{=:I}\,. \end{aligned} \tag{4.23}$$

The long time behavior of the solution does not depend on the initial condition f^0 since the corresponding term decays exponentially. The second term depends on the input and the physical properties of the model. Therefore, the latter contains the information about the long time behavior of the flow and I will not discuss the former in the following.

The integral I is still not easy to solve. Let $N \gg \max\left\{\frac{P}{\Omega}, 1\right\}$ be given, then for long times

$$t > \frac{N}{A^2} > t_k^* := \frac{N}{k^2 + A^2} \tag{4.24}$$

it can be approximated. For this purpose the integral is divided in three parts, each of which will be taken care of separately:

$$\begin{aligned} I = & \underbrace{\int_0^{t_k^*} \left(A^2 + P \sin \Omega\tau\right) \mathrm{e}^{\left(k^2+A^2\right)\tau} \, \mathrm{e}^{-\frac{P}{\Omega}\cos\Omega\tau} \mathrm{d}\tau}_{=:I_1} \\ & + \underbrace{\int_{t_k^*}^{t} A^2 \, \mathrm{e}^{\left(k^2+A^2\right)\tau} \, \mathrm{e}^{-\frac{P}{\Omega}\cos\Omega\tau} \mathrm{d}\tau}_{=:I_2} + \underbrace{\int_{t_k^*}^{t} P \sin\Omega\tau \, \mathrm{e}^{\left(k^2+A^2\right)\tau} \, \mathrm{e}^{-\frac{P}{\Omega}\cos\Omega\tau} \mathrm{d}\tau}_{=:I_3} \\ = & \; I_1 + I_2 + I_3 \, . \end{aligned}$$

The first part of the integral can be estimated very roughly via

$$I_1 < \left(A^2 + P\right) \frac{N}{k^2 + A^2} \, \mathrm{e}^{N} \, \mathrm{e}^{\frac{P}{\Omega}} \, , \tag{4.25}$$

which is independent of time. To simplify matters in the second and third parts, I assume that the term involving the cosine in the exponential function can be neglected against the growing exponential factor[2]. Applying this approximation to the second part results in

$$I_2 \approx \int_{t_k^*}^{t} A^2 \, \mathrm{e}^{\left(k^2+A^2\right)\tau} \mathrm{d}\tau = \frac{A^2}{k^2 + A^2} \left(\mathrm{e}^{\left(k^2+A^2\right)t} - \mathrm{e}^{N}\right) \, . \tag{4.26}$$

[2]At this point this is an uncontrolled approximation. Its quality will be checked a posteriori in section 4.4, where we discuss to what quality the obtained solution approximates the solution of equation (4.21).

For the third and last part of I one finds

$$
\begin{aligned}
I_3 &\approx P \int_{t_k^*}^{t} \sin\Omega\tau \, \mathrm{e}^{\left(k^2+A^2\right)\tau} \mathrm{d}\tau = P \,\mathrm{Im} \int_{t_k^*}^{t} \mathrm{e}^{\left(k^2+A^2+\mathrm{i}\Omega\right)\tau} \mathrm{d}\tau \\
&= P \,\mathrm{Im} \frac{1}{k^2+A^2+\mathrm{i}\Omega} \left(\mathrm{e}^{\left(k^2+A^2+\mathrm{i}\Omega\right)t} - \mathrm{e}^{\left(k^2+A^2+\mathrm{i}\Omega\right)t_k^*} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= P \,\mathrm{Im} \frac{k^2+A^2-\mathrm{i}\Omega}{\left(k^2+A^2\right)^2+\Omega^2} \left(\mathrm{e}^{\left(k^2+A^2\right)t} \left(\cos\Omega t + \mathrm{i}\sin\Omega t\right) - \mathrm{e}^{N} \left(\cos\Omega t_k^* + \mathrm{i}\sin\Omega t_k^*\right) \right) \\
&= \frac{P \,\mathrm{e}^{\left(k^2+A^2\right)t}}{\left(k^2+A^2\right)^2+\Omega^2} \Big(\left(k^2+A^2\right) \sin\Omega t - \Omega\cos\Omega t \Big) \\
&\quad - \frac{P \,\mathrm{e}^{N}}{\left(k^2+A^2\right)^2+\Omega^2} \Big(\left(k^2+A^2\right) \sin\Omega t_k^* - \cos\Omega t_k^* \Big) .
\end{aligned}
$$

Since the entire integral I in the solution (4.23) is multiplied with $\mathrm{e}^{-(k^2+A^2)t}$, every term in I that is bounded can be neglected for $t > \frac{N}{A^2}$.

In total the long term solution reads

$$
f_k(t) = \frac{2\mathrm{i}}{k\pi} \mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} \left(\frac{A^2}{k^2+A^2} + P \frac{(k^2+A^2)\sin\Omega t - \Omega\cos\Omega t}{\left(k^2+A^2\right)^2+\Omega^2} \right) , \tag{4.27}
$$

$$
\begin{aligned}
f(x,t) &= \sum_{k\in 2\mathbb{N}+1} 2\mathrm{i} f_k(t) \sin kx \\
&= -\frac{4}{\pi} \mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} \sum_{k\in 2\mathbb{N}+1} \frac{\sin kx}{k} \left(\frac{A^2}{k^2+A^2} + P \frac{(k^2+A^2)\sin\Omega t - \Omega\cos\Omega t}{\left(k^2+A^2\right)^2+\Omega^2} \right) \\
&= -\frac{4}{\pi} \mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} \sum_{k\in 2\mathbb{N}+1} \frac{\sin kx}{k} \left(\frac{A^2}{k^2+A^2} + P \,\mathrm{Im} \frac{\mathrm{e}^{\mathrm{i}\Omega t}}{k^2+A^2+\mathrm{i}\Omega} \right) .
\end{aligned} \tag{4.28}
$$

4.4. Limitation of the Solution

As a consequence of the approximations made in the previous section, the solution (4.27) will only be suitable for a reduced set of parameters. In order to determine this range of parameters, I test the solution using the original problem in the form (4.18):

$$
\dot{f}_k(t) + \left(k^2 + B(t)\right) f_k(t) = \frac{2\mathrm{i}}{k\pi} B(t) , \qquad k \in 2\mathbb{Z}+1 . \tag{4.29}
$$

Differentiation of the solution with respect to time yields

$$\begin{aligned}\dot{f}_k(t) &= -P\sin(\Omega t) f_k(t) + \frac{2\mathrm{i}}{k\pi}\,\mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} P\,\mathrm{Im}\,\frac{\mathrm{i}\Omega\,\mathrm{e}^{\mathrm{i}\Omega t}}{k^2+A^2+\mathrm{i}\Omega} \\ &= \big(A^2 - B(t)\big)\, f_k(t) + \frac{2\mathrm{i}}{k\pi}\,\mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} P\,\mathrm{Im}\,\frac{\mathrm{i}\Omega\,\mathrm{e}^{\mathrm{i}\Omega t}}{k^2+A^2+\mathrm{i}\Omega}\,,\end{aligned}$$

which implies

$$\begin{aligned}\dot{f}_k(t) + \big(k^2 + B(t)\big)\, f_k(t) &= \big(k^2 + A^2\big)\, f_k(t) + \frac{2\mathrm{i}}{k\pi}\,\mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} P\,\mathrm{Im}\,\frac{\mathrm{i}\Omega\,\mathrm{e}^{\mathrm{i}\Omega t}}{k^2+A^2+\mathrm{i}\Omega} \\ &= \frac{2\mathrm{i}}{k\pi}\,\mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} \underbrace{\left(A^2\frac{k^2+A^2}{k^2+A^2} + P\,\mathrm{Im}\,\mathrm{e}^{\mathrm{i}\Omega t}\frac{k^2+A^2+\mathrm{i}\Omega}{k^2+A^2+\mathrm{i}\Omega}\right)}_{=A^2+P\sin\Omega t = B(t)} \\ &= \frac{2\mathrm{i}}{k\pi} B(t)\,\mathrm{e}^{\frac{P}{\Omega}\cos\Omega t} \\ &\not\equiv \frac{2\mathrm{i}}{k\pi} B(t)\,.\end{aligned}$$

This means, that only in the limit $\frac{P}{\Omega} \to 0$ the solution (4.28) is a valid solution to (4.29). Therefore, I want to assume $\frac{P}{\Omega} \ll 1$ in the following.

4.5. Analysis of the Solution

4.5.1. Comparison With Steady State

It is easy to see, that the approximated solution becomes time-independent for a constant influx $B(t) = A^2$, i. e. setting $P = 0$ in (4.28) results in

$$f(x) = -\sum_{k\in 2\mathbb{N}+1} \frac{4}{\pi k}\,\frac{A^2}{k^2+A^2}\,\sin kx\,. \tag{4.30}$$

This series is equivalent to the steady state solution (3.6) if the further assumptions $\beta = \frac{\pi}{2}$, $\varepsilon = 0$, the rescaling and shifting

$$f^{\mathrm{s}}(x) := \frac{\varphi^{\mathrm{s}}(x)}{\varphi^{\mathrm{s}}(0)} - 1 = \frac{\cosh\!\big(A\left(\frac{\pi}{2} - x\right)\big)}{\cosh\frac{\pi A}{2}} - 1\,,$$

as well as the mirroring from (4.8) are applied to the steady state potential

$$\varphi^{\mathrm{s}}(x) = \cosh\!\big(A\left(\tfrac{\pi}{2} - x\right)\big)\,.$$

Because of the symmetries it is sufficient to calculate the Fourier coefficients for $k \in 2\mathbb{N}+1$ by

$$2\mathrm{i} f_k^{\mathrm{s}} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f^{\mathrm{s}}(x)\sin(kx)\,\mathrm{d}x = \frac{4}{\pi\cosh\frac{\pi A}{2}} \underbrace{\int_0^{\frac{\pi}{2}} \cosh\bigl(A\left(\tfrac{\pi}{2}-x\right)\bigr)\sin(kx)\,\mathrm{d}x}_{=:I_4} - \frac{4}{\pi k}\,.$$

Partially integrating I_4 twice reproduces the integral:

$$\begin{aligned} I_4 &= -\frac{1}{k}\cosh\bigl(A\left(\tfrac{\pi}{2}-x\right)\bigr)\cos(kx)\Big|_0^{\frac{\pi}{2}} - \frac{A}{k}\int_0^{\frac{\pi}{2}} \sinh\bigl(A\left(\tfrac{\pi}{2}-x\right)\bigr)\cos(kx)\,\mathrm{d}x \\ &= \frac{1}{k}\cosh\frac{\pi A}{2} - \underbrace{\frac{A}{k}\sinh\bigl(A(\tfrac{\pi}{2}-x\bigr)\sin(kx)\Big|_0^{\frac{\pi}{2}}}_{=0} - \frac{A^2}{k^2} I_4 \\ &= \frac{1}{k}\cosh\frac{\pi A}{2} - \frac{A^2}{k^2} I_4\,. \end{aligned}$$

Thus the integral itself is given by

$$I_4 = \frac{k}{k^2+A^2}\cosh\frac{\pi A}{2}\,.$$

This implies

$$2\mathrm{i} f_k^{\mathrm{s}} = \frac{4}{\pi}\left(\frac{k}{k^2+A^2} - \frac{1}{k}\right) = -\frac{4}{\pi k}\,\frac{A^2}{k^2+A^2}\,,$$

which coincides with (4.30). In summary, for $x \in \left[0,\frac{\pi}{2}\right]$ the following holds:

$$\sum_{k\in 2\mathbb{N}+1} \frac{4}{\pi k}\,\frac{A^2}{k^2+A^2}\sin kx = 1 - \frac{\cosh\bigl(A\left(\frac{\pi}{2}-x\right)\bigr)}{\cosh\frac{\pi A}{2}}\,. \tag{4.31}$$

Since this equation remains valid when replacing $A^2 \to A^2 + \mathrm{i}\Omega$, it can be exploited to calculate the series in the time-dependent long term solution:

$$
\begin{aligned}
f(x,t) &= -\frac{4}{\pi} \sum_{k \in 2\mathbb{N}+1} \frac{\sin kx}{k} \left(\frac{A^2}{k^2 + A^2} + P \operatorname{Im} \frac{\mathrm{e}^{\mathrm{i}\Omega t}}{k^2 + A^2 + \mathrm{i}\Omega} \right) \\
&= -\sum_{k \in 2\mathbb{N}+1} \frac{4}{\pi k} \frac{A^2}{k^2 + A^2} \sin kx \\
&\quad - P \operatorname{Im} \left(\frac{\mathrm{e}^{\mathrm{i}\Omega t}}{A^2 + \mathrm{i}\Omega} \sum_{k \in 2\mathbb{N}+1} \frac{4}{\pi k} \frac{A^2 + \mathrm{i}\Omega}{k^2 + A^2 + \mathrm{i}\Omega} \sin kx \right) \\
&= -\left(1 - \frac{\cosh\big(A\left(\frac{\pi}{2} - x\right)\big)}{\cosh \frac{\pi A}{2}} \right) \\
&\quad - P \operatorname{Im} \left(\frac{\mathrm{e}^{\mathrm{i}\Omega t}}{A^2 + \mathrm{i}\Omega} \left(1 - \frac{\cosh\big(\sqrt{A^2 + \mathrm{i}\Omega}\left(\frac{\pi}{2} - x\right)\big)}{\cosh \frac{\pi}{2}\sqrt{A^2 + \mathrm{i}\Omega}} \right) \right) .
\end{aligned}
\tag{4.32}
$$

4.5.2. Outflux

We now analyze the current

$$
\jmath = \frac{\partial_x^2 f}{f + 1}
$$

provided by (4.32), especially the current leaving the system at $x = \frac{\pi}{2}$. With

$$
\partial_x^2 f(x,t) = A^2 \frac{\cosh\big(A\left(\frac{\pi}{2} - x\right)\big)}{\cosh \frac{\pi A}{2}} + P \operatorname{Im} \mathrm{e}^{\mathrm{i}\Omega t} \frac{\cosh\big(\sqrt{A^2 + \mathrm{i}\Omega}\left(\frac{\pi}{2} - x\right)\big)}{\cosh\left(\frac{\pi}{2}\sqrt{A^2 + \mathrm{i}\Omega}\right)}
$$

and $\operatorname{sech} x := \frac{1}{\cosh x}$ the outflux $J(t) := \jmath\left(\frac{\pi}{2}, t\right)$ can be written compactly:

$$
J(t) = \frac{A^2 + P \cosh\left(\frac{\pi A}{2}\right) \operatorname{Im} \mathrm{e}^{\mathrm{i}\Omega t} \operatorname{sech}\left(\frac{\pi}{2}\sqrt{A^2 + \mathrm{i}\Omega}\right)}{1 - P \cosh\left(\frac{\pi A}{2}\right) \operatorname{Im} \frac{\mathrm{e}^{\mathrm{i}\Omega t}}{A^2 + \mathrm{i}\Omega} \left(1 - \operatorname{sech}\left(\frac{\pi}{2}\sqrt{A^2 + \mathrm{i}\Omega}\right)\right)} .
\tag{4.33}
$$

This result already shows, that in general there is a phase shift between the outflux $J(t)$ and the influx $B(t)$: the imaginary part of $\operatorname{sech}\left(\frac{\pi}{2}\sqrt{A^2 + \mathrm{i}\Omega}\right)$ in the numerator does not vanish for all A and Ω. Its complex phase can be absorbed into the exponential function. The corresponding term in the denominator provides another phase shift.

A very interesting limit is that of very slow changing in the influx:

$$\Omega \ll A^2 \qquad\qquad \Longrightarrow \qquad\qquad A^2 + \mathrm{i}\Omega \approx A^2\,.$$

In that case the outflux is very simple:

$$J(t) \approx \frac{A^2 + P\sin\Omega t}{1 - P\,\sigma(A)\sin\Omega t}\,, \tag{4.34}$$

where

$$\sigma(y) = \frac{\cosh\frac{\pi y}{2} - 1}{y^2} = \frac{\pi^2}{4}\sum_{n=0}^{\infty}\frac{1}{(2n+2)!}\left(\frac{\pi y}{2}\right)^{2n} = \frac{\pi^2}{8} + \frac{\pi^4}{384}\,y^2 + \mathcal{O}(y^4)$$

is a well defined positive function. The similarity of $J(t)$ and $B(t)$ is already obvious. Since $P \ll \Omega \ll A^2$, the outflux (4.34) has no poles for sufficiently small A. Consequently the denominator can be expanded into a geometric series:

$$\begin{aligned}\frac{1}{1 - P\,\sigma(A)\sin\Omega t} &= \sum_{n=0}^{\infty}\big(P\,\sigma(A)\sin\Omega t\big)^n \\ &= 1 + P\,\sigma(A)\sin\Omega t + \mathcal{O}(P^2)\,,\end{aligned}$$

The outflux up to the second order in P reads

$$\begin{aligned}J(t) &= A^2 + \big(1 + A^2\sigma(A)\big)P\sin\Omega t + \mathcal{O}(P^2) \\ &= B(t) + PA^2\sigma(A)\sin\Omega t + \mathcal{O}(P^2)\,.\end{aligned} \tag{4.35}$$

The terms of higher order in P should be considered small against A^2. Up to second order in P the determined solution faithfully describes the mass balance of the system: the mean output is $A2$, i.e. it agrees with the mean input. This provides further confidence in the faithfulness of the predictions which presumably can be trusted to linear order in P. This leading order results suggest an amplification of the perturbation: Since σ is positive, the factor of amplification is $1 + A^2\sigma(A) > 1$.

5. Discussion and Summary

In the preceding section the current

$$\jmath = \frac{\partial_x^2 f}{f+1}$$

was determined. Accordingly the water content as a function of space and time is given as

$$\rho = -\frac{\partial_x f}{f+1}\,.$$

These two functions are plotted in the figures 5 to 7. All these figures have in common, that for increasing A, but unchanged $P = 10^{-2}$ and $\Omega = \frac{1}{4}$, the amplification of the perturbation increases, even though the relative perturbation P/A^2 becomes smaller and smaller.

For even higher A the density and the flux display poles as can already be seen in equation (4.34). That effect could be inherited from the simple model discussed in section 2. We do not expect that it describes the real physics:

- The soil will eventually be advected or compressed due to the high current.
- The water density in a random point is bounded by the physical pressure of the environment. The pressure is not at all taken into account.

On the other hand, the model shows another interesting phenomenon: the water can be stored in the soil column for some time, before finally being entirely released (cf. figure 6). The possibility to store and later release water might be the physical background of the observed singularities. For a system with this property, a noisy input signal, e. g. rain, could cause severe floods in response to an in comparison relatively small stimulus in the input flux.

In order to further substantiate this expectation it would be of considerable interest to expand the results of the present thesis to cover the response to a noisy input with a broad spectrum of input frequencies.

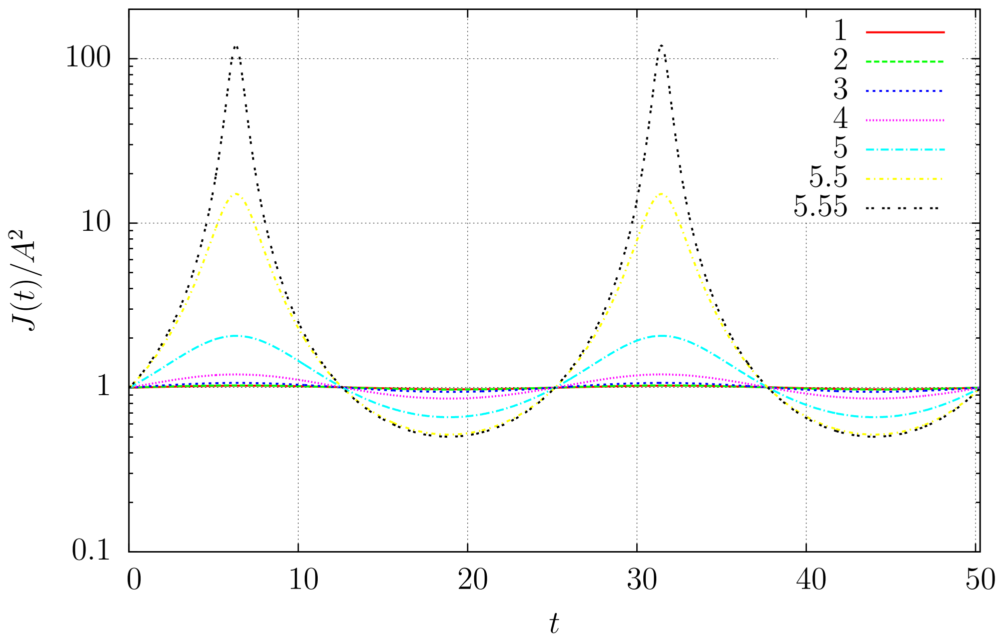

Figure 5: The outflux $J(t)$ scaled by the average outflux A^2 for different A (as indicated in the legend) and fixed $P = 10^{-2}$.

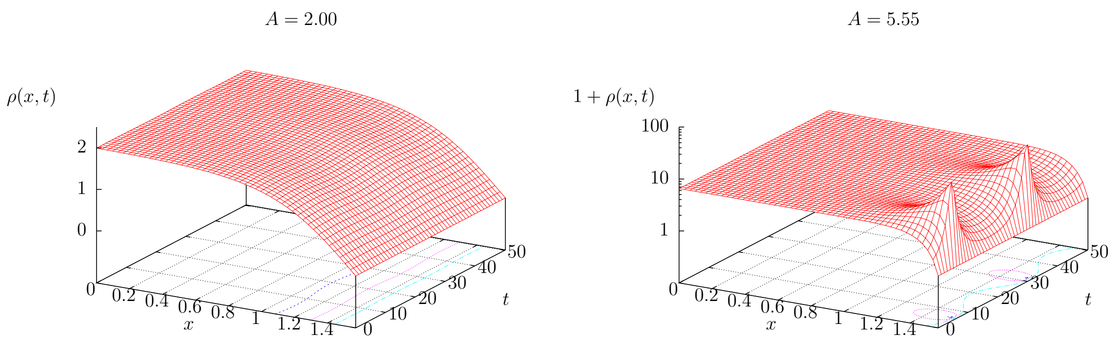

Figure 6: The density ρ as a function of x and t for $A = 2.00$ (left) and $A = 5.55$ (right), respectively. The amplitude of the perturbation is set to $P = 10^{-2}$. Beware the vastly different scales of the output signal.

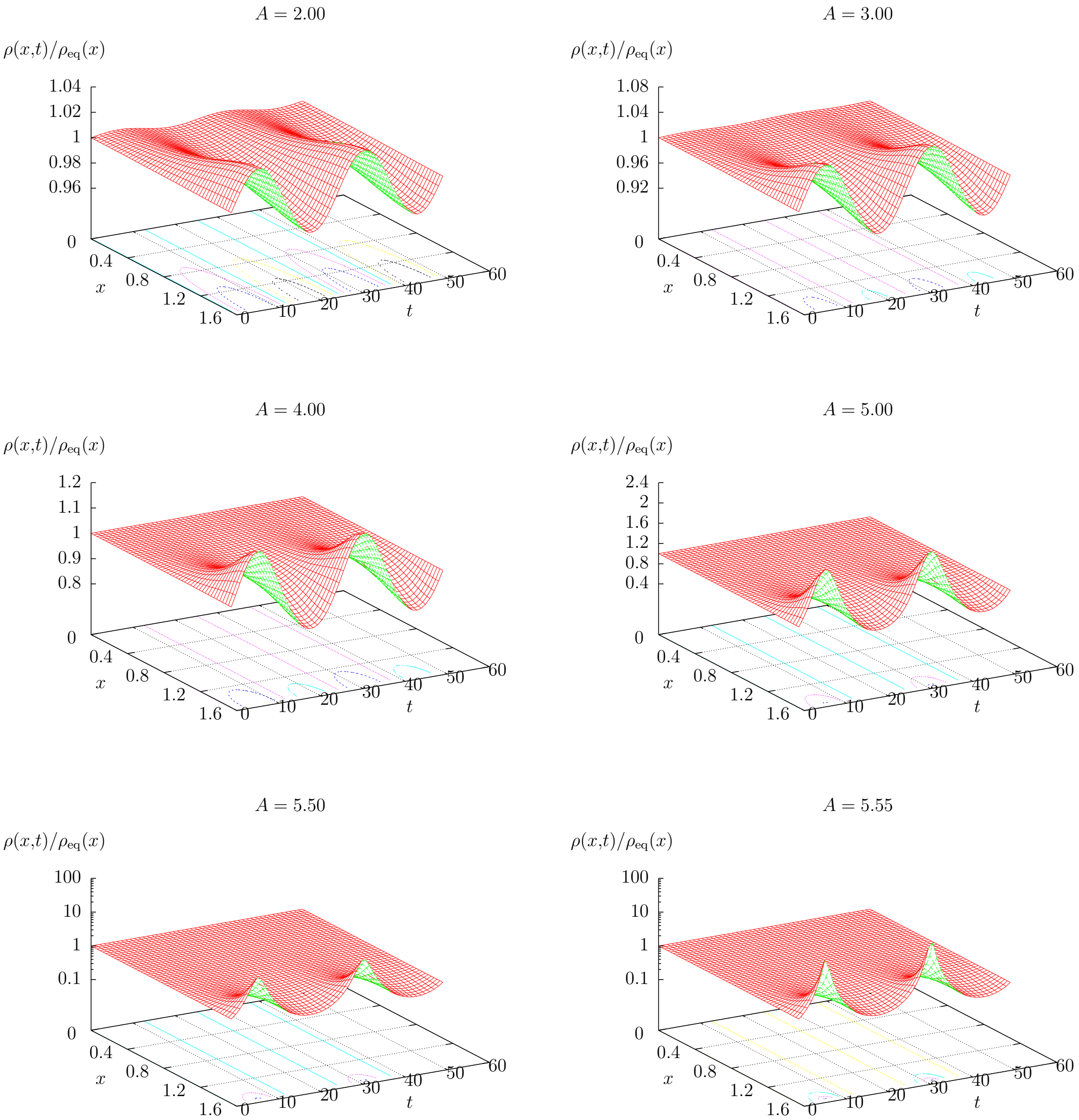

Figure 7: The ratio of the density $-\partial_x f(x,t)/(1+f(x,t))$ and its steady state counterpart $A\ \tanh\big(A\,(\frac{\pi}{2}-x)\big)$ for different average flux A^2 ($A = 2.00$, 3.00, 4.00, 5.00, 5.50, and 5.55, respectively; as indicated above the respective panels), and fixed amplitude $P = 10^{-2}$ of the periodic perturbation. Note the rapid increase of the deviations with increasing average flux A.

Appendix

A. Properties of Fourier Series

In order to solve the time dependent problem I use a Fourier representation of the fields. Some of the general properties that are not obvious will be presented in the following.

For a square integrable function $u \in L^2\left([0, X]\right)$ the Fourier expansion

$$u(x) = \sum_{k \in \mathbb{Z}} u_k \, \mathrm{e}^{\frac{2\pi \mathrm{i} k x}{X}}$$

identifies the function with a square summable sequence $\{u_k\}_{k \in \mathbb{Z}}$. Since the canonical inner product in $L^2\left([0, X]\right)$ is given by

$$\langle u, v \rangle = \int_0^X u(x)\overline{v(x)} \, \mathrm{d}x \,, \tag{A.1}$$

the functions $\left\{ \mathrm{e}^{\frac{2\pi}{X} \mathrm{i} k \, \cdot} \right\}_{k \in \mathbb{Z}}$ are orthogonal:

$$\left\langle \mathrm{e}^{\frac{2\pi}{X} \mathrm{i} k \, \cdot}, \, \mathrm{e}^{\frac{2\pi}{X} \mathrm{i} l \, \cdot} \right\rangle = \int_0^X \mathrm{e}^{\frac{2\pi}{X} \mathrm{i}(k-l)x} \, \mathrm{d}x = X \, \delta_{kl} \,. \tag{A.2}$$

The Fourier expansion expresses the fact, that this indexed family of functions is an orthogonal basis of $L^2([0, X])$ and this implies that the members of the representing sequence, also called Fourier coefficients, can be calculated by

$$u_k = \frac{1}{X} \int_0^X u(x) \, \mathrm{e}^{-\frac{2\pi}{X} \mathrm{i} k x} \, \mathrm{d}x \,. \tag{A.3}$$

By periodic extension, any function in $L^2([0, X])$ can be identified with a periodic function. The scalar product can then be calculated over any full period: For $p = r + kX$ where $r \in [0, X)$ and $k \in \mathbb{Z}$ holds

$$\begin{aligned}
\int_p^{p+X} u(x)\overline{v(x)} \, \mathrm{d}x = \int_r^{r+X} u(x)\overline{v(x)} \, \mathrm{d}x &= \left(\int_r^X + \int_X^{r+X} \right) u(x)\overline{v(x)} \, \mathrm{d}x \\
&= \left(\int_r^X + \int_0^r \right) u(x)\overline{v(x)} \, \mathrm{d}x = \int_0^X u(x)\overline{v(x)} \, \mathrm{d}x \\
&= \langle u, v \rangle \,.
\end{aligned}$$

In the following I will assume a function in $L^2([0,X])$ to be periodically extended whenever it is convenient. On the other hand, any bounded function with period X has a representation in $L^2([0,X])$.

Suppose a function $w \in L^2([0,X])$ to be symmetric or antisymmetric around a point p:

$$w(p+x) = \pm w(p-x)\,. \tag{A.4}$$

Then the Fourier coefficients have a certain symmetry property as well:

$$\begin{aligned} w_k &= \frac{1}{X}\int_0^X w(x)\,\mathrm{e}^{-\frac{2\pi}{X}\mathrm{i}kx}\mathrm{d}x = \frac{1}{X}\int_{-p}^{-p+X} w(p+y)\,\mathrm{e}^{-\frac{2\pi}{X}\mathrm{i}k(p+y)}\mathrm{d}y \\ &= \pm\frac{1}{X}\int_{-p}^{-p+X} w(p-y)\,\mathrm{e}^{-\frac{2\pi}{X}\mathrm{i}k(p+y)}\mathrm{d}y = \pm\,\mathrm{e}^{-\frac{2\pi}{X}\mathrm{i}kp}\frac{1}{X}\int_{-p}^{-p+X} w(p-y)\,\mathrm{e}^{-\frac{2\pi}{X}\mathrm{i}ky}\mathrm{d}y \\ &= \pm\,\mathrm{e}^{-\frac{4\pi}{X}\mathrm{i}kp}\frac{1}{X}\int_{2p-X}^{2p} w(z)\,\mathrm{e}^{\frac{2\pi}{X}\mathrm{i}kz}\mathrm{d}z = \pm\,\mathrm{e}^{-\frac{4\pi}{X}\mathrm{i}kp}\,w_{-k}\,. \end{aligned} \tag{A.5}$$

The case $X = 2\pi$ together with a real valued function u and the symmetries

$$u(\pi+x) = -u(\pi-x) \qquad \text{and} \qquad u(\tfrac{\pi}{2}+x) = u(\tfrac{\pi}{2}-x)$$

plays a major role in my work. Here, the equation (A.5) takes the form

$$u_k = -u_{-k} \qquad\qquad \text{and} \qquad\qquad u_k = (-1)^k\,u_{-k}\ ,$$

respectively. This can only be true, if the u_k vanish for $k \in 2\mathbb{Z}$. The entire Fourier representation of u immediately simplifies to

$$u(x) = \sum_{k\in\mathbb{Z}} u_k\ \mathrm{e}^{\mathrm{i}kx} = \sum_{k\in 2\mathbb{N}+1} 2\mathrm{i}\,u_k \sin kx\,. \tag{A.6}$$

Real-valued functions have a further symmetry in the Fourier coefficients:

$$\overline{u_k} = u_{-k}\ ,$$

which can easily be seen from (A.3). Thus for an odd real-valued function the Fourier coefficients are purely imaginary: $\overline{u_k} = u_{-k} = -u_k\,.$

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Erklärung nach §13(8) der Prüfungsordnung für den Bachelor-Studiengang Physik und den Master-Studiengang Physik an der Universität Göttingen:

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Göttingen, den 24. September 2010

(Artur Wachtel)