Structure preserving iterative methods for periodic projected Lyapunov equations and their application in model reduction of periodic descriptor systems

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Abstract In this paper, we develop structure preserving iterative schemes to solve the periodic discrete-time projected Lyapunov equations associated to analysis and design of discrete-time descriptor systems exploiting the reflexive generalized inverses of the periodic matrices associated with these systems. In particular, we extend the Smith method to solve the large scale projected periodic discrete-time algebraic Lyapunov equations in lifted form. A low-rank version of this method is also presented, avoiding the explicit lifted formulation and working directly with the periodic matrix coefficients. Moreover, we consider an application of the Lyapunov solvers in balanced truncation model reduction of periodic discrete-time descriptor systems. Numerical results are given to illustrate the efficiency and accuracy of the proposed methods.

Keywords Periodic descriptor systems · Lifted state space representation · Periodic projected Lyapunov equations · Smith iteration · Model order reduction

1 Introduction

Periodic systems and control theory have received a lot of attention in the last few decades because they have wide applications in many areas of science and engineering, specially in the areas where the periodic control is deserved, such as aerospace realm, control of industrial processes and communication systems, modeling of periodic time-varying filters and networks [13,17,16,29].

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In this paper, we consider linear time-varying (LTV) discrete-time descriptor systems of order $\bar{\mathbf{n}} = (n_0, n_1, \ldots, n_{K-1})$, as

$$E_k x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k, \quad k \in \mathbb{Z}, \tag{1}$$

where $E_k \in \mathrm{R}^{\mu_k \times n_{k+1}}$, $A_k \in \mathrm{R}^{\mu_k \times n_k}$, $B_k \in \mathrm{R}^{\mu_k \times m_k}$, $C_k \in \mathrm{R}^{p_k \times n_k}$ are the system matrices, $x_k \in \mathrm{R}^{n_k}$ is the (generalized) state or descriptor vector, $u_k \in \mathrm{R}^{m_k}$ is the control input, and $y_k \in \mathrm{R}^{p_k}$ is the output. The system matrices are periodic with a period $K \geq 1$, and $\sum_{k=0}^{K-1} \mu_k = \sum_{k=0}^{K-1} n_k = \mathbf{n}$, $\sum_{k=0}^{K-1} m_k = \mathbf{m}$ as well as $\sum_{k=0}^{K-1} p_k = \mathbf{p}$. If all E_k are nonsingular, then (1) can be transformed into a periodic standard system.

Efficient numerical methods for computing poles and zeros, $\mathbb{L}_\infty$-norm, minimal and balanced realizations have been developed for such systems [6, 33–35]. Model reduction of such systems using balanced truncation has been proposed in [1]. All these methods are restricted to problems of small or medium size [1, 30]. Much attention has recently been devoted to the iterative solution of large-scale sparse Lyapunov equations. Iterative solvers for projected generalized Lyapunov equations have been proposed in[24, 25]. On the other hand, an extension of the Smith method and the block-Arnoldi based Krylov subspace method to standard periodic Lyapunov equations has been presented in [11]. But,these methods cannot be directly applied to the projected periodic Lyapunov equations [2].

Analysis and reduced order modeling of periodic discrete-time descriptor systems may require to invert the system matrices in some appropriate sense. A special form of the generalized inverses, known as *(1,2)-inverses*, of periodic systems in the descriptor form has been considered in [31], which works on the corresponding lifted form of the associated system pencil of the periodic system.

In this paper, we discuss the computation of the generalized inverses of periodic discrete-time descriptor systems using the left and right deflating projectors associated with the eigenstructures of the periodic matrix pairs. This technique has been implemented in [25] for continuous-time descriptor system to compute the solution of the corresponding projected Lyapunov equations. We will generalize the idea of [25] for the discrete periodic setting and use those periodic inverses to compute the solutions of the periodic projected Lyapunov equations. Moreover, we reformulate the Smith method to solve the large projected periodic discrete-time algebraic Lyapunov equations in lifted form. The block diagonal structure of the periodic solutions is preserved in every Smith iteration step which is one of the challenging task in many of the iterative computations in the periodic setting. It should be noted that generalized versions of the ADI method and the Smith method have been proposed in [2] for the solution of projected periodic Lyapunov equations. But, the methods fail to preserve the block diagonal structure during the iteration, and it is only achieved upon convergence.

The rest of the paper is organized as follows. In Section 2, we briefly review discrete-time periodic descriptor systems and their cyclic lifted representations. We also study the causal and noncausal decomposition of the periodic

descriptor systems. In Section 3, we introduce the periodic matrix equations under consideration and review the different techniques for the solutions of those matrix equations associated to the causal and noncausal systems. We also discuss the challenges of iterative techniques used in computing the structure preserving solution of those matrix equations. In Sections 4, we discuss the generalized inverses of periodic discrete-time descriptor systems using the left and right deflating projectors associated with the eigenstructures of the periodic matrix pairs. Iteratively solving of the causal and noncausal lifted Lyapunov equations exploiting the generalized inverses is discussed in Sections 5 and 6, respectively. Low-rank versions of these methods are also presented, which avoid the explicit lifted formulation and work directly with the period matrix coefficients. A balanced truncation model reduction method for periodic descriptor systems is considered in Section 7. Section 8 contains numerical examples that illustrate the properties of the described iterative methods for projected Lyapunov equations and their application to model reduction. Some conclusions are given in Section 9.

2 Periodic descriptor systems

Lifted representations [4,7,33] of periodic systems are the analogous time-invariant representations of the periodic systems. Using the lifting isomorphism one can exploit the theory of time-invariant systems for the analysis and control of periodic systems, provided that the results achieved can be easily re-interpreted in a periodic framework.

2.1 Cyclic lifted representation of periodic systems

We consider here the *cyclic lifted representation* which was introduced first for standard periodic systems in [17]. The essence of the cyclic lifted system is putting inputs, states and outputs of the original LTV descriptor system at cyclic places of those of the lifted LTI system.

The *cyclic lifted representation* of the periodic descriptor system (1) is given by

$$\mathcal{E}\mathcal{X}_{k+1} = \mathcal{A}\mathcal{X}_k + \mathcal{B}\mathcal{U}_k, \quad \mathcal{Y}_k = \mathcal{C}\mathcal{X}_k, \tag{2}$$

where

$$\mathcal{E} = \operatorname{diag}(E_0, E_1, \ldots, E_{K-1}), \quad \mathcal{B} = \operatorname{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & \cdots & 0 & C_0 \\ C_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & C_{K-1} & 0 \end{bmatrix}. \tag{3}$$

The descriptor vector, system input and output of (2) are related to those of (1) via

$$
\mathcal{X}_k = \begin{bmatrix} x_1 \\ \vdots \\ x_{K-1} \\ x_0 \end{bmatrix}, \qquad \mathcal{U}_k = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{K-1} \end{bmatrix}, \qquad \mathcal{Y}_k = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{K-1} \end{bmatrix},
$$

respectively. The transfer function of the lifted system (2) can be rewritten as

$$
\mathcal{H}(z) = \mathcal{C}(z\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}. \tag{4}
$$

The cyclic lifted system (2) describes the eigenstructure and system dynamics of the LTV discrete-time descriptor system (1). Regularity of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ implies the regularity of the cyclic matrix pair $(\mathcal{E}, \mathcal{A})$ [14, 22, 13]. The reverse argument also holds true.

The cyclic matrix pencil $z\mathcal{E} - \mathcal{A}$ is said to be *regular* when $\det(z\mathcal{E} - \mathcal{A}) \neq 0$ for some $z \in \mathbb{C}$. The cyclic lifted system (2) is *asymptotically stable* iff $z\mathcal{E} - \mathcal{A}$ is regular and all its finite eigenvalues lie inside the unit circle. System (1) is *asymptotically stable* if and only if the corresponding cyclic lifted system (2) is *asymptotically stable*.

2.2 Spectral projectors for periodic matrix pairs

In the descriptor setting, the E_k in (1) are singular for $k = 0, 1, \ldots, K - 1$, and analysis and modeling of system (1) can be proceed by separating the causal and the noncausal parts of the periodic descriptor system (1). The periodic Kronecker canonical form [27, 23, 6] of the matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ for $k = 0, 1, \ldots, K - 1$, can be used for this separation.

$$
U_k E_k V_{k+1} = \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & E_k^b \end{bmatrix}, \ U_k A_k V_k = \begin{bmatrix} A_k^f & 0 \\ 0 & I_{n_k^\infty} \end{bmatrix}, \tag{5}
$$

where U_k, V_k are nonsingular. Then the left and right spectral projectors $P_l(k)$ and $P_r(k)$, for $k = 0, 1, \ldots, K - 1$, can be represented as [2, 6, 24],

$$
P_l(k) = U_k^{-1} \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & 0 \end{bmatrix} U_k, \quad P_r(k) = V_k \begin{bmatrix} I_{n_k^f} & 0 \\ 0 & 0 \end{bmatrix} V_k^{-1}, \tag{6}
$$

respectively. The index ρ of the periodic descriptor system (1) and the index ν of the lifted system (2) are related via $\nu \leq K\rho$ [2, 6, 24].

3 Periodic matrix equations and their solutions

Stability analysis and model reduction of periodic systems are strongly related to the matrix equations associated with the systems. For disctere-time periodic descriptor systems, they are well known as *generalized projected periodic discrete-time algebraic Lyapunov equations* (PPDALEs). It has been shown in [2,6] that the periodic Gramians of the asymptotically stable discrete-time descriptor systems (1) satisfy the PPDALEs with special right-hand sides. The causal and noncausal reachability Gramians X_k and $\hat{X}_k$ are the unique symmetric, positive semidefinite periodic solutions of the PPDALEs

$$
\begin{aligned}
A_k X_k A_k^T - E_k X_{k+1} E_k^T &= -P_l(k) B_k B_k^T P_l(k)^T, \\
X_k &= P_r(k) X_k P_r(k)^T,
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
A_k \hat{X}_k A_k^T - E_k \hat{X}_{k+1} E_k^T &= Q_l(k) B_k B_k^T Q_l(k)^T, \\
\hat{X}_k &= Q_r(k) \hat{X}_k Q_r(k)^T,
\end{aligned}
\tag{8}
$$

respectively, where $X_K = X_0$, $\hat{X}_K = \hat{X}_0$. Note that $P_l(k), P_r(k)$, for $k = 0, 1, \ldots, K-1$, are the spectral projectors onto the k-th left and right deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues [1,6], and $Q_l(k) = I - P_l(k)$ and $Q_r(k) = I - P_r(k)$ in (8).

Similarly, the causal and noncausal observability Gramians Y_k and $\hat{Y}_k$ are the unique symmetric, positive semidefinite periodic solutions of the PPDALEs

$$
\begin{aligned}
A_k^T Y_{k+1} A_k - E_{k-1}^T Y_k E_{k-1} &= -P_r^T(k) C_k^T C_k P_r(k), \\
Y_k &= P_l(k-1)^T Y_k P_l(k-1),
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
A_k^T \hat{Y}_{k+1} A_k - E_{k-1}^T \hat{Y}_k E_{k-1} &= Q_r(k)^T C_k^T C_k Q_r(k), \\
\hat{Y}_k &= Q_l(k-1)^T \hat{Y}_k Q_l(k-1),
\end{aligned}
\tag{10}
$$

respectively, where $Y_K = Y_0$, $\hat{Y}_K = \hat{Y}_0$.

The numerical solution of (7) has been considered in [6] for time-varying matrix coefficients. The method proposed there is based on an initial reduction of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ to the generalized periodic Schur form [12,32] and on solving the resulting generalized periodic Sylvester and Lyapunov equations. As a result, the method is computationally expensive and not suitable for large scale problems.

An efficient approach which works with the cyclic lifted representation of (1) and the corresponding lifted form of (7) has been considered in [1]. Following the work of [1], the PPDALEs (7) and (8) are equivalent to the following projected lifted discrete-time algebraic Lyapunov equations (PLDALEs)

$$
\mathcal{A} \mathcal{X} \mathcal{A}^T - \mathcal{E} \mathcal{X} \mathcal{E}^T = -\mathcal{P}_l \mathcal{B} \mathcal{B}^T \mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T,
\tag{11}
$$

$$
\mathcal{A} \hat{\mathcal{X}} \mathcal{A}^T - \mathcal{E} \hat{\mathcal{X}} \mathcal{E}^T = \mathcal{Q}_l \mathcal{B} \mathcal{B}^T \mathcal{Q}_l^T, \quad \hat{\mathcal{X}} = \mathcal{Q}_r \hat{\mathcal{X}} \mathcal{Q}_r^T,
\tag{12}
$$

respectively, where $\mathcal{E}$, $\mathcal{A}$ and $\mathcal{B}$ are as in (3), and

$$\mathcal{X} = \mathrm{diag}(X_1, \ldots, X_{K-1}, X_0), \quad \hat{\mathcal{X}} = \mathrm{diag}(\hat{X}_1, \ldots, \hat{X}_{K-1}, \hat{X}_0).$$

In that case, the projectors in lifted forms are given by

$$\mathcal{P}_l = \mathrm{diag}(P_l(0), P_l(1), \ldots, P_l(K-1)), \qquad \mathcal{Q}_l = I - \mathcal{P}_l,$$
$$\mathcal{P}_r = \mathrm{diag}(P_r(1), \ldots, P_r(K-1), P_r(0)), \qquad \mathcal{Q}_r = I - \mathcal{P}_r.$$

The matrices $\mathcal{X}$ and $\hat{\mathcal{X}}$ are called *the causal and noncausal reachability Gramians* of the lifted system (2). A similar result can also be stated for the causal and noncausal observability Gramians [1].

In practice, one should avoid these direct methods for large-scale problems because the computational complexity for solving a Lyapunov equation of the form (11) or (12) using direct methods is at least of order $(\mathcal{O}(Kn_{\max}^3))$, where $n_{\max} = \max(n_k)$, and they require extensive storage. Therefore, iterative methods have been developed for the solutions of such equations.

3.1 Revised iterative methods for solutions of PLDALEs

Iteratively solving (7) and (8) using their corresponding lifted structures, i.e., (11) and (12), has been considered in [2]. A generalized version of the alternating direction implicit (ADI) method and the Smith method is proposed there for the solutions of (11) and (12), respectively. Note that, for the descriptor system (1), the matrix $\mathcal{E} = \mathrm{diag}(E_0, \ldots, E_{K-1})$ is singular. However, both the ADI and Smith iterations fail to converge for the resulting Lyapunov equations since the iteration operator of the ADI and the Smith iterations, in this case, does not have spectral radius less than one. This problem has been circumvented by considering a *generalized Cayley transformation* given by

$$\mathfrak{C}(\mathcal{E}, \mathcal{A}) = \lambda(\mathcal{A} - \mathcal{E}) - (\mathcal{A} + \mathcal{E}) \tag{13}$$

see, e.g., [15]. This transformation transfers the PLDALE (11) to an equivalent projected continuous-time algebraic Lyapunov equation (PCALE)

$$\mathbf{E}\mathcal{X}\mathbf{A}^T + \mathbf{A}\mathcal{X}\mathbf{E}^T = -2\mathcal{P}_l\mathcal{B}\mathcal{B}^T\mathcal{P}_l^T, \quad \mathcal{X} = \mathcal{P}_r\mathcal{X}\mathcal{P}_r^T, \tag{14}$$

where $\lambda\mathbf{E} - \mathbf{A} = \lambda(\mathcal{A} - \mathcal{E}) - (\mathcal{A} + \mathcal{E})$ is the Cayley-transformed pencil. The finite eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ lying inside the unit circle are mapped to the finite eigenvalues of $\lambda\mathbf{E} - \mathbf{A}$ in the open left half-plane, and the eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ at infinity is mapped to $\lambda = 1$.

The solutions of the PLDALE (11) and the PCALE (14) are identical and have the block diagonal structure. Therefore, it is desired that the iterative solution of (14) will have the block diagonal structure analogous to (11) at each ADI iteration step. Unfortunately, the case is not so. It is observed that the generalized ADI method does not preserve the block diagonal structure at every ADI iteration step due to the specific structure of the matrices $\mathbf{E}$

and $\mathbf{A}$, although the approximate Gramian is block diagonal [2]. Therefore, we use the concept of reflexive generalized inverses of periodic matrix pairs associated with the periodic system to preserve the block diagonal structure of the approximate solution at each iteration step.

4 Reflexive generalized inverses of periodic matrices

For periodic systems, the inversion problem has been introduced in [10,19] by exploiting the concept of left and right invertibility of system (1) in the standard case (i.e., with $E_k = I_{n_{k+1}}$). A generalization of that concept, which computes the generalized inverses of periodic descriptor systems via the corresponding lifted representation has been considered in [32]. A special class of generalized inverse, called *reflexive generalized inverse*, of the system pencil has been proposed in [25] to find the solutions of projected continuous-time algebraic Lyapunov equations using Krylov subspace methods, and also in [3] for computing a partial realization for descriptor systems. Similar inverses have been proposed in [31] to compute generalized inverses of rational matrices in descriptor state-space representation, and named there *(1,2)-inverse* as they satisfy the first two Moore-Penrose conditions. Details of these definitions can be found in [5]. Analogous to [25], we can find the *reflexive generalized inverses* for E_k with respect to the projectors $P_l(k)$ and $P_r(k+1)$ as

$$\bar{E}_k = V_{k+1} \begin{bmatrix} I_{n_{k+1}^f} & 0 \\ 0 & 0 \end{bmatrix} U_k, \tag{15}$$

for $k = 0, 1, \ldots, K-1$. Moreover, the reflexive generalized inverses satisfy the relations

$$\bar{E}_k E_k \bar{E}_k = \bar{E}_k, \quad E_k \bar{E}_k = P_l(k), \quad \bar{E}_k E_k = P_r(k+1), \tag{16}$$

for $k = 0, 1, \ldots, K-1$. These reflexive generalized inverses of E_k for $k = 0, 1, \ldots, K-1$ will be exploited in the next two sections to find the block diagonal approximate solutions of (11) and (12) using the Smith iterative method.

5 Generalized Smith method for causal PLDALEs

Consider again the PLDALE (11). Multiplying it from the left and right by $\bar{\mathcal{E}}$, and $(\bar{\mathcal{E}})^T$, we get

$$\mathcal{P}_r \mathcal{X} \mathcal{P}_r^T - \bar{\mathcal{E}} \mathcal{A} \mathcal{X} \mathcal{A}^T (\bar{\mathcal{E}})^T = \bar{\mathcal{E}} \mathcal{P}_l \mathcal{B} \mathcal{B}^T \mathcal{P}_l^T (\bar{\mathcal{E}})^T, \ \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \tag{17}$$

where $\bar{\mathcal{E}} \mathcal{E} = \mathcal{P}_r$ by the definition of the reflexive generalized inverse, and $\bar{\mathcal{E}} = \mathrm{diag}(\bar{E}_0, \bar{E}_1, \cdots, \bar{E}_{K-1})$. Equation (17) can be written in a more usual form as

$$\mathcal{X} - (\bar{\mathcal{E}} \mathcal{A}) \mathcal{X} (\bar{\mathcal{E}} \mathcal{A})^T = \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \ (\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B})^T, \ \mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T, \tag{18}$$

where $\mathcal{X} = \mathcal{P}_r \mathcal{X} \mathcal{P}_r^T$. Note that in the above representation, we use the relation $\mathcal{P}_r \bar{\mathcal{E}} = \bar{\mathcal{E}} \mathcal{P}_l$. Clearly, $\mathcal{P}_l$ and $\mathcal{P}_r$ are the spectral projectors onto the invariant subspace of the matrix $\bar{\mathcal{E}} \mathcal{A}$ corresponding to eigenvalues inside the unit circle. Such an equation can be solved by the Smith method [18,24] given by

$$
\begin{aligned}
\mathcal{X}_1 &= \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \, (\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B})^T, \\
\mathcal{X}_\ell &= \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \, (\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B})^T + (\bar{\mathcal{E}} \mathcal{A}) \mathcal{X}_{\ell-1} (\bar{\mathcal{E}} \mathcal{A})^T.
\end{aligned}
\tag{19}
$$

Then the unique solution $\mathcal{X}$ of (18) can be approximated using the Smith iterations

$$
\mathcal{X}_i = \sum_{\ell=0}^{i-1} (\bar{\mathcal{E}} \mathcal{A})^\ell \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \, \mathcal{B}^T \bar{\mathcal{E}}^T \mathcal{P}_r^T ((\bar{\mathcal{E}} \mathcal{A})^T)^\ell.
\tag{20}
$$

Therefore, the Cholesky factor $\mathcal{R}_i$, where $\mathcal{X}_i = \mathcal{R}_i \mathcal{R}_i^T$, is given by

$$
\mathcal{R}_i = [\, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}, \, (\bar{\mathcal{E}} \mathcal{A}) \, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}, \dots, \, (\bar{\mathcal{E}} \mathcal{A})^{i-1} \, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \,].
\tag{21}
$$

Remark 1 At each iteration step i in (21), we do not have an analogous block diagonal structure as given in (11). Note that $\mathcal{X} = \text{diag}(X_1, \dots, X_{K-1}, X_0)$, and $X_k = R_k R_k^T$ for $k = 0, 1, \dots, K-1$. Hence, we demand to compute the block diagonal Cholesky factor $\mathcal{R}_i = \text{diag}(R_{1,i}^b, \dots, R_{K-1,i}^b, R_{0,i}^b)$ at each iteration step i of (21), where $R_{k,i}^b$ collocates all the iterative counterparts of the computed R_k at the i-th iteration steps for $k = 0, 1, \dots, K-1$, and $X_k = R_k R_k^T \approx R_{k,i}^b (R_{k,i}^b)^T$. Unfortunately, the iterations do not result in this form. This is because in each iteration step i, except for the first iteration, in the right-hand side of (21), we have a different block cyclic matrix.

5.1 Structure preserving solutions of PLDALEs

The problem of preserving the block diagonal structure at the iterative computation of the Cholesky factor $\mathcal{R}_i$ can be circumvented by introducing a cyclic permutation matrix in each iteration step i of (21). First, consider the trivial permutation matrix

$$
\Pi = \begin{bmatrix}
J_{m_0} & 0 & \cdots & 0 & 0 \\
\vdots & J_{m_1} & 0 & & 0 \\
& & \ddots & & \vdots \\
0 & & & J_{m_{K-2}} & 0 \\
0 & 0 & \cdots & & J_{m_{K-1}}
\end{bmatrix},
\tag{22}
$$

where J_{m_i} denotes the square identity matrix of size m_i, and m_i equals the number of columns of B_i.

We introduce a new permutation matrix $\sigma^i \Pi$ at each iteration step i in the computation of (20), where $\sigma^i \Pi$ changes in a cyclic manner by a forward block-column shift at each iteration step.

For example, suppose that $K = 3$, and $k = 0, 1, 2$. Then for $i = 1, \sigma^0 \Pi = \Pi$ is given by

$$\Pi = \begin{bmatrix} J_{m_0} & 0 & 0 \\ 0 & J_{m_1} & 0 \\ 0 & 0 & J_{m_2} \end{bmatrix}. \tag{23}$$

For $i = 2$, we get

$$\sigma^1 \Pi = \sigma \Pi = \sigma(\Pi) = \begin{bmatrix} 0 & J_{m_0} & 0 \\ 0 & 0 & J_{m_1} \\ J_{m_2} & 0 & 0 \end{bmatrix}, \tag{24}$$

which is just a forward shift of the last block-column of Π in (23). Clearly for $i = 3$, the permutation matrix takes the form

$$\sigma^2 \Pi = \sigma(\sigma \Pi) = \sigma(\sigma(\Pi)) = \begin{bmatrix} 0 & 0 & J_{m_0} \\ J_{m_1} & 0 & 0 \\ 0 & J_{m_2} & 0 \end{bmatrix}, \tag{25}$$

which is nothing but a forward shift of the last block-column of $\sigma \Pi$. Note that similar types of permutations are also considered in [11] for solutions of periodic Lyapunov equations. One nice property of this permutation matrix is that it satisfies the periodicity property, i.e., $\sigma^K \Pi = \Pi$, where K is the period of system (1).

Hence (20) takes the new form

$$\mathcal{X}_i = \sum_{\ell=0}^{i-1} (\bar{\mathcal{E}}\mathcal{A})^\ell \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}(\sigma^\ell \Pi) \, (\sigma^\ell \Pi)^{\mathrm{T}} \mathcal{B}^{\mathrm{T}} \bar{\mathcal{E}}^{\mathrm{T}} \mathcal{P}_r^{\mathrm{T}} ((\bar{\mathcal{E}}\mathcal{A})^{\mathrm{T}})^\ell.$$

Therefore, in this new representation the Cholesky factor $\mathcal{R}_i$ has the form

$$\mathcal{R}_i = [\, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \Pi, \ (\bar{\mathcal{E}}\mathcal{A}) \, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}(\sigma \Pi), \dots, \ (\bar{\mathcal{E}}\mathcal{A})^{i-1} \, \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}(\sigma^{i-1} \Pi) \,]. \tag{26}$$

In (26) we have $\mathcal{P}_r \bar{\mathcal{E}} \mathcal{B} \Pi = \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}$. It is to be noted that the matrix $\mathcal{R}_i$ in (26) is not block diagonal but each block computed at the i-th iteration step of $\mathcal{R}_i$ in (26) preserves the block diagonal structure. Therefore, we construct the block diagonal Cholesky factor $\mathrm{diag}(R_{1,i}^b, \dots, R_{K-1,i}^b, R_{0,i}^b)$ of $\mathcal{X}_i$ which is obtained from $\mathcal{R}_i$ by additional block permutation at each iteration step i, $i = 2, 3, \dots$, and $X_k = R_k R_k^T \approx R_{k,i}^b (R_{k,i}^b)^T$ for $k = 0, 1, \dots, K - 1$. The whole iterative process is summarized in Algorithm 1.

The approximate solution satisfies $\mathcal{R}_i = \mathcal{P}_r \mathcal{R}_i$ in every iteration step i. Algorithm 1 is to be stopped as soon as the *normalized residual norm* given by

$$\eta(\mathcal{R}_i) = \frac{\|\mathcal{A}\mathcal{R}_i \mathcal{R}_i^T \mathcal{A}^T - \mathcal{E}\mathcal{R}_i \mathcal{R}_i^T \mathcal{E}^T + \mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T\|_F}{\|\mathcal{P}_l \mathcal{B}\mathcal{B}^T \mathcal{P}_l^T\|_F} \tag{27}$$

satisfies the condition $\eta(\mathcal{R}_i) < tol$ with a user-defined tolerance *tol* or a stagnation of residual norms is observed.

Algorithm 1 Generalized Smith method for causal PLDALE

Input: $\mathcal{A}, \bar{\mathcal{E}}, \mathcal{B}, \mathcal{P}_r$, cyclic permutation matrix Π.
Output: A low-rank Cholesky factor $\mathcal{R}_i$ such that $\mathcal{X}_i \approx \mathcal{R}_i \mathcal{R}_i^T$.

1: $W_1 = \mathcal{P}_r \bar{\mathcal{E}} \mathcal{B}$
2: $Z_1 = W_1$;
3: $\mathcal{R}_1 = Z_1$
4: **for** $i = 2, 3, \ldots$ **do**
5:    $W_i = (\bar{\mathcal{E}} \mathcal{A}) W_{i-1}$
6:    $Z_i = W_i(\sigma^{i-1} \Pi)$;
7:    $\mathcal{R}_i = [\mathcal{R}_{i-1}, \ Z_i]$
8: **end for**

Remark 2 When the numbers of columns of B_k, i.e., m_k, $k = 0, 1, \ldots, K-1$, are big, $\mathcal{R}_i$ may face rank deficiency, because, in each iteration step, Algorithm 1 will add as many columns as there are in $\mathcal{B}$ to the previous $\mathcal{R}_i$. Hence, we propose the rank-revealing QR decomposition (RRQR) [8] of $\mathcal{R}_i$ with tolerance τ to compute a low-rank factor of $\mathcal{R}_i$. But in general, the RRQR will destroy the constructed block diagonal structure of $\mathcal{R}_i$. Hence, we use the RRQR decomposition on the factored form of $\mathcal{R}_i$, i.e., on $R_{k,i}^b$ at each iteration step i, for $k = 0, 1, \ldots, K-1$. Note that

$$R_{k,i}^b = [R_{k,1}, R_{k,2}, \ldots, R_{k,i}] \tag{28}$$

at the i-th iteration steps for $k = 0, 1, \ldots, K-1$. We then use the RRQR decomposition

$$[\hat{V}_i, \hat{Q}_i, r_{cc}] = \mathrm{RRQR}((R_{k,i}^b)^T, \tau_{cc}),$$

where $\tau_{cc} = \sqrt{\epsilon}$ [2], update

$$R_{k,i}^b = \hat{Q}_i \hat{V}_i^T [\, I_{r_{cc}}, \, 0\,]^T,$$

for $k = 0, 1, \ldots, K-1$, construct $\mathcal{R}_i = \mathrm{diag}(R_{1,i}^b, \ldots, R_{K-1,i}^b, R_{0,i}^b)$ at each iteration step i in Step 7 of Algorithm 1, and use it in the $i+1$-st iteration step.

5.2 Cyclic computation of causal PLDALEs

In fact, the iteration (26) implemented in Algorithm 1 not only proves that each block matrix in $\mathcal{R}_i$ stay block diagonal at each iteration step i, it also enables us to rewrite (21) in such a way that one can directly compute the periodic Cholesky factors for different k, $k = 0, 1, \ldots, K-1$. From simple algebraic manipulation of (26), we observe that the periodic matrices E_k, A_k, and B_k appear in a cyclic manner in the computation of the periodic Cholesky factors $R_{k,i}^b$ in every iteration step i for different values of k, $k = 0, 1, \ldots, K-1$. Observing these cyclic relations and handing them technically, we can compute the periodic Cholesky factors $R_{k,i}^b$, $k = 0, 1, \ldots, K-1$, $i = 1, 2, \ldots$, directly. We represent some of those computations in the following.

For $i = 1$, we get

$$R_{0,1} = R_{K,1} = P_r(K)\bar{E}_{K-1}B_{K-1} = P_r(0)\bar{E}_{K-1}B_{K-1},$$
$$R_{1,1} = P_r(1)\bar{E}_0 B_0, \qquad \% \; \bar{E}_K = \bar{E}_0, \; B_K = B_0$$
$$\vdots$$
$$R_{K-1,1} = P_r(K-1)\bar{E}_{K-2}B_{K-2}.$$

For $i = 2$, we get

$$R_{0,2} = R_{K,2} = \bar{E}_{K-1}A_{K-1}P_r(K-1)\bar{E}_{K-2}B_{K-2} = \bar{E}_{K-1}A_{K-1}R_{K-1,1},$$
$$R_{1,2} = \bar{E}_0 A_0 P_r(0)\bar{E}_{K-1}B_{K-1} = \bar{E}_0 A_0 R_{0,1},$$
$$\vdots$$
$$R_{K-1,2} = \bar{E}_{K-2}A_{K-2}P_r(K-2)\bar{E}_{K-3}B_{K-3} = \bar{E}_{K-2}A_{K-2}R_{K-2,1}.$$

For $i = 3$, we get

$$R_{0,3} = R_{K,3} = \bar{E}_{K-1}A_{K-1}\bar{E}_{K-2}A_{K-2}P_r(K-2)\bar{E}_{K-3}B_{K-3}$$
$$= \bar{E}_{K-1}A_{K-1}R_{K-1,2},$$
$$R_{1,3} = \bar{E}_0 A_0 \bar{E}_{K-1}A_{K-1}P_r(K-1)\bar{E}_{K-2}B_{K-2} = \bar{E}_0 A_0 R_{0,2},$$
$$\vdots$$
$$R_{K-1,3} = \bar{E}_{K-2}A_{K-2}\bar{E}_{K-3}A_{K-3}P_r(K-3)\bar{E}_{K-4}B_{K-4}$$
$$= \bar{E}_{K-2}A_{K-2}R_{K-2,2},$$

and so on. The whole computation is summarized in Algorithm 2. Note that in the above computations and also in Algorithm 2, we use the periodicity of the coefficient matrices and that of the projectors. Here, $P_r(K) = P_r(0)$, $P_r(K-1) = P_r(-1)$, $\bar{E}_K = \bar{E}_0$, $\bar{E}_{-1} = \bar{E}_{K-1}$, $\bar{E}_{K+k} = \bar{E}_k$, and the similar for others. Obviously, $X_K = X_0 = R_K R_K^T = R_0 R_0^T$. It should be also noted that in Algorithm 2, $R_{k,i}^b$ means the computed R_k at the i-th iteration steps. Also note that $R_{k,i}^b$ collects all these iterative counterparts for an individual k, where $k = 0, 1, \ldots, K-1$. That means for $k = 0$, we compute $R_{0,i}^b = [R_{0,1}, R_{0,2}, \ldots, R_{0,i}]$, and similarly the others.

Algorithm 2 Smith method for cyclic computations of causal PLDALEs.

Input: $(\bar{E}_k, A_k, B_k)$, spectral projectors $P_r(k)$ for $k = 0, \ldots, K-1$.
Output: Low-rank periodic Cholesky factor R_k such that $X_k \approx R_k^b (R_k^b)^T$.

1: **for** $k = 0 : K - 1$ **do**
2:     $R_{k,1} = P_r(k) \bar{E}_{-1} B_{-1}$ % note that $\bar{E}_{-1} = \bar{E}_{K-1}, B_{-1} = B_{K-1}$
3:     $R_{k,1}^b = R_{k,1}$
4: **end for**

5: **for** $i = 2, 3, \ldots$ **do**
6:     **for** $k = 0 : K - 1$ **do**
7:        $R_{k,i} = \bar{E}_{k-1} A_{k-1} R_{k-1,i-1}$
8:        $R_{k,i}^b = [R_{k,i-1}^b \quad R_{k,i}]$
9:        $R_k^b = \text{RRQR}(R_{k,i}^b, \ \tau_k)$
10:     **end for**
11: **end for**

One can define a stoping criteria for Algorithm 2 by setting a *normalized residual norm* given by

$$
\eta(R_k^b) = \frac{\|A_k R_k^b (R_k^b)^T A_k^T - E_k R_{k+1}^b (R_{k+1}^b)^T E_k^T + P_l(k) B_k B_k^T P_l(k)^T\|_F}{\|P_l(k) B_k B_k^T P_l(k)^T\|_F},
\tag{29}
$$

which satisfy the condition $\eta(R_k^b) < tol_c$ for $k = 0, 1, \ldots, K-1$, where tol_c is a user predefined tolerance. A similar computation can also be stated for the causal periodic observability Gramians Y_k of (9).

6 Smith method for noncausal PLDALEs

For nonsingular $\mathcal{A}$, the PLDALE (12) is equivalent to the PLDALE

$$
\begin{aligned}
\hat{\mathcal{X}} - (\mathcal{A}^{-1}\mathcal{E})\hat{\mathcal{X}}(\mathcal{A}^{-1}\mathcal{E})^T &= \mathcal{Q}_r \mathcal{A}^{-1} \mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T, \\
\hat{\mathcal{X}} &= \mathcal{Q}_r \hat{\mathcal{X}} \mathcal{Q}_r^T.
\end{aligned}
\tag{30}
$$

Such an equation can be solved by the Smith method [21] given by

$$
\begin{aligned}
\hat{\mathcal{X}}_1 &= \mathcal{Q}_r \mathcal{A}^{-1} \mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T, \\
\hat{\mathcal{X}}_i &= \mathcal{Q}_r \mathcal{A}^{-1} \mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T + (\mathcal{A}^{-1}\mathcal{E})\hat{\mathcal{X}}_{i-1}(\mathcal{A}^{-1}\mathcal{E})^T.
\end{aligned}
\tag{31}
$$

Note that in the noncausal case we do not need to compute the reflexive generalized inverses of the periodic matrices E_k, and A_k, for $k = 0, 1, \ldots, K-1$, since (31) requires only the inversion of the cyclic lifted nonsingular matrix $\mathcal{A}$.

In this case, $\mathcal{Q}_r \mathcal{A}^{-1}\mathcal{E} = \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r$ is nilpotent with the nilpotency index ν [2], and after ν iterations we obtain

$$
\hat{\mathcal{X}}_\nu = \sum_{i=0}^{\nu-1} (\mathcal{A}^{-1}\mathcal{E})^i \mathcal{Q}_r \mathcal{A}^{-1} \mathcal{B}\mathcal{B}^T \mathcal{A}^{-T} \mathcal{Q}_r^T ((\mathcal{A}^{-1}\mathcal{E})^T)^i = \hat{\mathcal{X}}.
\tag{32}
$$

Therefore, the Cholesky factor $\hat{\mathcal{R}}$ of the solution $\hat{\mathcal{X}} = \hat{\mathcal{R}}\hat{\mathcal{R}}^T$ of (30) and also of the PLDALE (12) takes the form

$$\hat{\mathcal{R}}_\nu = [\,\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \ldots, (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}]. \tag{33}$$

Note that the above iteration does not preserve the block diagonal structure at every iteration step in the computation of the Cholesky factor $\hat{\mathcal{R}}_\nu$ [2]. By introducing a cyclic permutation matrix in each iteration step like the causal case, one can easily preserve the block diagonal structure at every iteration step in(33).

6.1 Structure preserving solutions for noncausal PLDALEs

Let us again consider a cyclic permutation matrix $\hat{\mathcal{P}}$ of the form

$$\hat{\mathcal{P}} = \begin{bmatrix} 0 & \cdots & 0 & J_{m_0} \\ J_{m_1} & & & 0 \\ & \ddots & & \vdots \\ 0 & & J_{m_{K-1}} & 0 \end{bmatrix}, \tag{34}$$

where as before, J_{m_i} denotes the identity matrix of size m_i. We introduce a new permutation matrix $\hat{\sigma}^i\hat{\mathcal{P}}$ at each iteration step i in the computation of (32), where $\hat{\sigma}^i\hat{\mathcal{P}}$ changes in a cyclic manner by a backward block-column shift at each iteration step.

Similar to the causal case, the permutation matrix is periodic with a periodicity K, i.e., $\hat{\sigma}^K\hat{\mathcal{P}} = \hat{\mathcal{P}}$, where K is the period of system (1). Then (32) gets the new form

$$\hat{\mathcal{X}}_\nu = \sum_{i=0}^{\nu-1}(\mathcal{A}^{-1}\mathcal{E})^i\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}(\hat{\sigma}^i\hat{\mathcal{P}})\,(\hat{\sigma}^i\hat{\mathcal{P}})^T\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T((\mathcal{A}^{-1}\mathcal{E})^T)^i = \hat{\mathcal{X}}. \tag{35}$$

Therefore, the Cholesky factor $\hat{\mathcal{R}}_\nu$ has the form

$$\hat{\mathcal{R}}_\nu = [\,\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\hat{\mathcal{P}}, \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}(\hat{\sigma}\hat{\mathcal{P}}), \ldots, (\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}(\hat{\sigma}^{\nu-1}\hat{\mathcal{P}})\,]. \tag{36}$$

It can be verified that each column block in (36) preserves the block diagonal structure analogous to the solution of (30). The computation of this factor is presented in Algorithm 3.

We note that if the index ν is unknown, then Algorithm 3 can be stopped as soon as $\|W_i\|_F \leq \epsilon$ or $\|W_i\|_F/\|\hat{\mathcal{R}}_i\|_F \leq \epsilon$ with the machine precision ϵ. Thus, for systems of low index, the solution of (8) can be obtained with few computations. Also, we have to project W_i onto the image of $\mathcal{Q}_r$ to guarantee that the second equation in (12) (and also in (30)) is satisfied.

Algorithm 3 Generalized Smith method for noncausal PLDALEs

Input: $\mathcal{A}, \mathcal{E}, \mathcal{B}$, spectral projector $\mathcal{Q}_r$, cyclic permutation matrix $\hat{\mathcal{P}}$.
Output: Low-rank factor $\hat{\mathcal{R}}_\nu$ such that $\hat{\mathcal{X}} = \hat{\mathcal{R}}_\nu \hat{\mathcal{R}}_\nu^T$.

1: $W_1 = \mathcal{Q}_r \mathcal{A}^{-1} \mathcal{B}$
2: $Z_1 = W_1 \hat{\mathcal{P}}$
3: $\hat{\mathcal{R}}_1 = Z_1$
4: **for** $i = 2, 3, \ldots, \nu$ **do**
5:     $W_i = \mathcal{A}^{-1} \mathcal{E} W_{i-1}$
6:     $Z_i = W_i(\hat{\sigma}^{i-1} \hat{\mathcal{P}})$
7:     $\hat{\mathcal{R}}_i = [\hat{\mathcal{R}}_{i-1}, \ Z_i]$
8: **end for**

Note that the matrix $\hat{\mathcal{R}}_\nu$ in (36) is not block diagonal but each column block computed at the i-th iteration step in (36) preserves the block diagonal structure analogous to the solution of (30).Therefore, we construct the block diagonal Cholesky factor $\mathrm{diag}(\hat{R}_{1,i}^b, \ldots, \hat{R}_{K-1,i}^b, \hat{R}_{0,i}^b)$ of $\hat{\mathcal{X}}_i$ which is obtained from $\hat{\mathcal{R}}_i$ by additional block permutation at each iteration step i, $i = 2, 3, \ldots, \nu$. Note that $\hat{R}_{k,i}^b$ collects all the iterative counterparts of the computed $\hat{R}_k$ at the i-th iteration steps for $k = 0, 1, \ldots, K-1$, and $\hat{X}_k = \hat{R}_k \hat{R}_k^T \approx \hat{R}_{k,i}^b (\hat{R}_{k,i}^b)^T$.

6.2 Cyclic computations of periodic noncausal Cholesky factors

In fact, the iteration (36) implemented in Algorithm 3 not only proves that each column block computed in $\hat{\mathcal{R}}_\nu$ stay block diagonal, it also enables us to rewrite (33) in such a way that one can directly compute the periodic Cholesky factors for different k, $k = 0, 1, \ldots, K-1$. From simple algebraic manipulation of (36), we observe that the periodic matrices E_k, A_k, and B_k appear in a cyclic manner in the computation of the periodic Cholesky factors $\hat{R}_{k,i}^b$ in every iteration step i for different values of k, $k = 0, 1, \ldots, K-1$. Observing these cyclic relations and handling them technically, we can compute the periodic Cholesky factors $\hat{R}_{k,i}^b$, $k = 0, 1, \ldots, K-1$, $i = 1, 2, \ldots$, directly. We represent some of those computations in the following.

For $i = 1$, we get

$$\hat{R}_{0,1} = \hat{R}_{K,1} = Q_r(0) A_0^{-1} B_0,$$
$$\hat{R}_{1,1} = Q_r(1) A_1^{-1} B_1,$$
$$\vdots$$
$$\hat{R}_{K-1,1} = Q_r(K-1) A_{K-1}^{-1} B_{K-1}.$$

For $i = 2$, we get

$$\hat{R}_{0,2} = \hat{R}_{K,2} = A_0^{-1} E_0 Q_r(1) A_1^{-1} B_1 = A_0^{-1} E_0 \hat{R}_{1,1},$$
$$\hat{R}_{1,2} = A_1^{-1} E_1 Q_r(2) A_2^{-1} B_2 = A_1^{-1} E_1 \hat{R}_{2,1},$$
$$\vdots$$
$$\hat{R}_{K-1,2} = A_{K-1}^{-1} E_{K-1} Q_r(0) A_0^{-1} B_0 = A_{K-1}^{-1} E_{K-1} \hat{R}_{0,1}.$$

For $i = 3$, we get

$$\hat{R}_{0,3} = \hat{R}_{K,3} = A_0^{-1} E_0 A_1^{-1} E_1 Q_r(2) A_2^{-1} B_2 = A_0^{-1} E_0 \hat{R}_{1,2},$$
$$\hat{R}_{1,3} = A_1^{-1} E_1 A_2^{-1} E_2 Q_r(3) A_3^{-1} B_3 = A_1^{-1} E_1 \hat{R}_{2,2},$$
$$\vdots$$
$$\hat{R}_{K-1,3} = A_{K-1}^{-1} E_{K-1} A_0^{-1} E_0 Q_r(1) A_1^{-1} B_1 = A_{K-1}^{-1} E_{K-1} R_{0,2},$$

and so on.

Algorithm 4 Generalized Smith method for noncausal PLDALEs.

Input: (E_k, A_k, B_k), spectral projectors $Q_r(k)$ for $k = 0, 1, \ldots, K - 1$.
Output: Low-rank periodic Cholesky factor $\hat{R}_k$ such that $\hat{X}_k = \hat{R}_k \hat{R}_k^T$.

1: **for** $k = 0 : K - 1$ **do**
2:     $\hat{R}_{k,1} = Q_r(k) A_k^{-1} B_k$ % note that $\hat{R}_{0,1} = \hat{R}_{K,1}$
3:     $\hat{R}_{k,1}^b = \hat{R}_{k,1}$
4: **end for**

5: **for** $i = 2 : \nu$ **do**
6:     **for** $k = 0 : K - 1$ **do**
7:         $\hat{R}_{k,i} = A_k^{-1} E_k \hat{R}_{k+1,i-1}$
8:         $\hat{R}_{k,i}^b = [\hat{R}_{k,i-1}^b \quad \hat{R}_{k,i}]$
9:         $\hat{R}_k = \mathrm{RRQR}(\hat{R}_{k,i}^b, \tau)$
10:     **end for**
11: **end for**

The whole computation is summarized in Algorithm 4. Similar to the causal case, we consider the periodicity of the coefficient matrices and that of the projectors in Algorithm 4. Here $Q_r(K) = Q_r(0)$, $E_K = E_0$, $E_{K-1} = E_{-1}$, $E_{K+k} = E_k$, etc. Obviously, $\hat{X}_K = \hat{X}_0 = \hat{R}_K \hat{R}_K^T = \hat{R}_0 \hat{R}_0^T$. In Algorithm 4, $\hat{R}_{k,i}^b$ means the computed $\hat{R}_k$ at the i-th iteration step. Also note that $\hat{R}_{k,i}^b$ collocates all these iterative counterparts for an individual k, where $k = 0, 1, \ldots, K - 1$. That means for $k = 0$, we compute

$$\hat{R}_{0,i}^b = [\hat{R}_{0,1}, \hat{R}_{0,2}, \ldots, \hat{R}_{0,i}],$$

and similarly the others.

For unknown index ν, we propose the RRQR decomposition [8] of $\hat{R}_{k,i}^b$ with tolerance τ to truncate redundant columns in the iterations of Algorithm 4. In that case, we assume that after the i-th iteration we have the exact computation of the periodic Cholesky factors $\hat{R}_k$ satisfying relation $\|\hat{R}_{k,i}\|_F \leq \epsilon$ or $\|\hat{R}_{k,i}\|_F / \|\hat{R}_k\|_F \leq \epsilon$, where ϵ is the machine precision, and then, $\hat{X}_k = \hat{R}_k \hat{R}_k^T$ are the periodic solutions of (8) for $k = 0, 1, \ldots, K - 1$. We also need to project $\hat{R}_{k,i}$ onto the image of Q_r to guarantee that the second equation in (8) is satisfied.

Remark 3 As the A_k can be singular in the discrete-time case, Algorithm 4 is restricted in that case. We restrict our model reduction approach for systems where A_k, for $k = 0, 1, \ldots, K - 1$, are nonsingular.

Remark 4 The causal and noncausal observabiliy Gramians of the periodic descriptor system (1) can also be determined from the corresponding PLDALEs that are dual to the PLDALE (11) and (12), see [1] for details. Applying Algorithm 1 and Algorithm 3 (and also their corresponding cyclic reformulations Algorithm 2 and Algorithm 4) to these equations, we find, respectively, the low-rank Cholesky factors L_k of the causal observability Gramians $Y_k \approx L_k L_k^T$ and the Cholesky factor $\tilde{Y}_k$ of the noncausal observability Gramians $\hat{Y}_k = \hat{L}_k \hat{L}_k^T$.

7 Application to model order reduction

Model order reduction (MOR) is an approach, where a large dynamical system is approximated by a reduced-order model which can be fast and efficiently simulated and which has nearly the same response characteristics as compared to the original large model. Balanced truncation for periodic standard discrete-time system and periodic descriptor systems has been considered in [7,30], and [6], respectively. For the periodic descriptor system (1), a reduced-order model of dimension $\bar{\mathbf{r}} = (r_0, r_1, \ldots, r_{K-1})$ has the form

$$\tilde{E}_k \tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{B}_k u_k, \quad \tilde{y}_k = \tilde{C}_k \tilde{x}_k, \qquad k \in \mathbb{Z}, \tag{37}$$

where $\tilde{E}_k \in \mathbb{R}^{\gamma_k \times r_{k+1}}$, $\tilde{A}_k \in \mathbb{R}^{\gamma_k \times r_k}$, $\tilde{B}_k \in \mathbb{R}^{\gamma_k \times m_k}$, $\tilde{C}_k \in \mathbb{R}^{q_k \times r_k}$ are K-periodic matrices, $\sum_{k=0}^{K-1} \gamma_k = \sum_{k=0}^{K-1} r_k = \mathbf{r}$, and $\mathbf{r} \ll \mathbf{n}$. It is also important that the reduced-order model preserves physical properties of the original system such as regularity and stability, and that the approximation error is small.

Consider that the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable, and the Cholesky factors of the causal and noncausal Gramians satisfy

$$\begin{aligned} X_k &= R_k R_k^T, & Y_k &= L_k L_k^T, \\ \hat{X}_k &= \hat{R}_k \hat{R}_k^T, & \hat{Y}_k &= \hat{L}_k \hat{L}_k^T. \end{aligned}$$

Then the *causal* and *noncausal Hankel singular values* of the periodic descriptor system (1) are defined as

$$\sigma_{k,j} = \zeta_j(L_k^T E_{k-1} R_k), \quad \theta_{k,j} = \zeta_j(\hat{L}_{k+1}^T A_k \hat{R}_k),$$

respectively, where $\zeta_j(.)$ denotes the singular values of the corresponding product matrices. For a balanced system, truncation of states related to the small causal Hankel singular values does not change system properties essentially. Unfortunately, truncation of small non-zero noncausal Hankel singular values may lead the system to become unstable [9].

Let us consider the singular value decompositions of the product matrices

$$
\begin{aligned}
L_k^T E_{k-1} R_k &= [U_{k,1}, U_{k,2}] \begin{bmatrix} \Sigma_{k,1} & \\ & \Sigma_{k,2} \end{bmatrix} [V_{k,1}, V_{k,2}]^T, \\
\hat{L}_{k+1}^T A_k \hat{R}_k &= U_{k,3} \Theta_k V_{k,3}^T,
\end{aligned}
\tag{38}
$$

where $[U_{k,1}, U_{k,2}]$, $[V_{k,1}, V_{k,2}]$, $U_{k,3}$ and $V_{k,3}$ are orthogonal,

$$
\Sigma_{k,1} = \mathrm{diag}(\sigma_{k,1}, \ldots, \sigma_{k,r_k^f}), \quad \Sigma_{k,2} = \mathrm{diag}(\sigma_{k,r_k^f+1}, \ldots, \sigma_{k,n_k^f}),
$$

with $\sigma_{k,1} \geq \cdots \geq \sigma_{k,r_k^f} > \sigma_{k,r_k^f+1} \geq \ldots \geq \sigma_{k,n_k^f}$, and $\Theta_k = \mathrm{diag}(\theta_{k,1}, \ldots, \theta_{k,r_k^\infty})$ is nonsingular for $k = 0, 1, \ldots, K-1$.

Defining the projection matrices [6] as

$$
\begin{aligned}
S_{k,r} &= [\, L_{k+1} U_{k+1,1} \Sigma_{k+1,1}^{-1/2}, \;\; \hat{L}_{k+1} U_{k,3} \Theta_k^{-1/2} \,] \in \mathbb{R}^{\mu_k \times \gamma_{k+1}}, \\
T_{k,r} &= [\, R_k V_{k,1} \Sigma_{k,1}^{-1/2}, \;\;\; \hat{R}_k V_{k,3} \Theta_k^{-1/2} \,] \in \mathbb{R}^{n_k \times r_k},
\end{aligned}
$$

with $r_k = r_k^f + r_k^\infty$ and $\gamma_{k+1} = r_{k+1}^f + r_k^\infty$, we compute the reduced-order system (37) as

$$
\tilde{E}_k = S_{k,r}^T E_k T_{k+1,r}, \; \tilde{A}_k = S_{k,r}^T A_k T_{k,r}, \; \tilde{B}_k = S_{k,r}^T B_k, \; \tilde{C}_k = C_k T_{k,r}, \tag{39}
$$

Let $\tilde{\mathcal{H}}(z)$ be the transfer functions of the reduced-order lifted system formed from the reduced-order matrices in (39). Then the reduced-order model satisfies the following $\mathbb{H}_\infty$-norm error bound

$$
\begin{aligned}
\|\mathcal{H} - \tilde{\mathcal{H}}\|_{\mathbb{H}_\infty} &= \sup_{\omega \in [0,2\pi]} \|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2 \\
&\leq 2\,\mathrm{trace}\,(\mathrm{diag}(\Sigma_{0,2}, \ldots, \Sigma_{K-1,2})),
\end{aligned}
\tag{40}
$$

where $\mathcal{H}(z)$ is defined in (4), $\|.\|_2$ denotes the spectral matrix norm, and $\Sigma_{k,2}$, $k = 0, 1, \ldots, K-1$, contains the truncated causal Hankel singular values [9, 2].

8 Results

Example 1 We consider first an artificial periodic discrete-time descriptor system on index-1 from [2, Example 1], which is reformulated from its original model in [6, Example 1]. In this reformulation, the periodic descriptor system has $\mu_k = n_k = 404$, $m_k = 2$ and $p_k = 3$ for the periodicity $K = 10$, i.e., $k = 0, 1, \ldots, 9$. The spectral projectors $P_l(k)$ and $P_r(k)$ are computed using the Kronecker-like forms of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ proposed
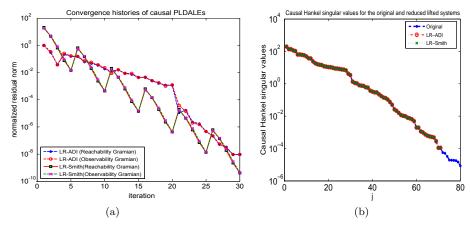
Fig. 1 (a) Normalized residual norms for the causal reachability and observability lifted projected Lyapunov equations; (b) Causal Hankel singular values for original and reduced-order lifted systems.

in [28,6]. The set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ is periodic stable with causal eigenvalues, $n_k^f = 400$ and noncausal eigenvalues, $n_k^\infty = 4$ for $k = 0, 1, \ldots, 9$. The original lifted system has order $\mathbf{n} = 4040$. Since $n_k^f = 400$ for $k = 0, 1, \ldots, 9$, computing the $\bar{E}_k$ is straightforward using relation (15).

We solve the causal and noncausal lifted projected Lyapunov equations using Algorithm 2 and Algorithm 4, respectively. Since, Algorithm 2 is the cyclic reformulation of Algorithm 1, we compute the normalized residual norms at each Smith iteration step for the reachability and observability type of the causal lifted Lyapunov equations using relation (27). We apply the RRQR decomposition at every fifth iteration step, since it is computationally expensive. Experiments confirm that computing it every $\ell = 5$ steps often provides a good efficiency balance of computational cost and rank growth compared to other choices of ℓ. To illustrate the efficiencies of the proposed algorithms, we compare the results with the corresponding ADI (Alternating Direction Implicit) computation of [2]. Fig. 1(a) shows the decay of the residual norms computed at each Smith iteration step.

This iteration is stopped as soon as the normalized Lyapunov residual reaches the tolerance $tol = 10^{-10}$. We approximate system (1) by a reduced-order model obtained by truncating the states corresponding to the small causal Hankel singular values satisfying $\sigma_{k,j} < 10^{-4}$. The largest 80 causal Hankel singular values of the original lifted system, and the approximate 71 causal Hankel singular values for the reduced-order lifted system are shown in Fig. 1(b).
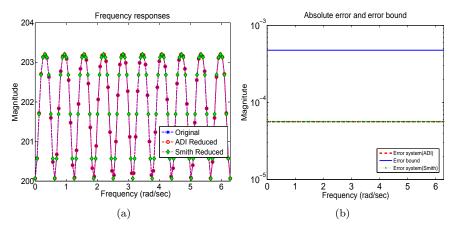
For different subsystems, the numbers of the computed non-zero noncausal Hankel singular values are identical and given by $r_k^\infty = 2$ for $k = 0, 1, \ldots, 9$. The computed reduced-order model has subsystems of orders $(9, 9, 9, 9, 9, 9, 9, 9, 10, 9)$. Note that stability is preserved in the reduced-order system.

Fig. 2 (a) The frequency responses of the original and the reduced-order lifted systems; (b) absolute error and error bound.

Fig. 2(a) shows the norms of the frequency responses $\mathcal{H}(e^{i\omega})$ and $\tilde{\mathcal{H}}(e^{i\omega})$ of the original and reduced-order lifted systems for a frequency range $[0, 2\pi]$. We observe a good match of the system norms. Finally, in Fig. 2(b), we display the absolute error $\|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2$ and the error bound given in (40).

Example 2 As a second model problem we consider here an artificial continuous-time model from Section 4.3 of [26], where a spring-damper model is considered as an artificial model of piezo-mechanical systems. We consider $n = 500, l = 100, \text{nin} = 2, \text{nout} = 3$ for our model problem, and hence the dimension of the continuous-time model is $2n + l = 1100$. The formulated continuous-time model is converted to a discrete-time model by an Euler discretization method [20]. We then change the damping matrix periodically by introducing some periodic coefficients inside it. As a result, the model is time-varying and periodic. The details of this periodic model formulation are given in the appendix.

For the resulting periodic model, we have $n_k = 1100$, $m_k = 2$, $p_k = 3$, and a period $K = 10$. The periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are periodic stable with $n_k^f = 1000$ and $n_k^\infty = 100$ for every $k = 0, 1, \ldots, 9$. The resulting periodic system is of index 1, and the original lifted system has order $\mathbf{n} = 11000$. The sparsity pattern of the periodic pair at $k = 0$ is shown in Fig. 3.

In Fig. 4(a), we present the largest 260 causal Hankel singular values computed by the proposed Smith method in Algorithm 2. We approximate system (1) to the tolerance 10^{-4} and truncate the states corresponding to the smallest 200 causal Hankel singular values. The system has 20 noncausal Hankel singular values which are positive, but very small. The values of these noncausal Hankel singular values lie in the range of $[10^{-13}, 10^{-15}]$, and they are shown in Fig. 4(b). It is to be mentioned that our model problem is of index 1, and hence we need only one iteration to compute the noncausal Cholesky factor for noncausal PLDALEs using Equation (36).
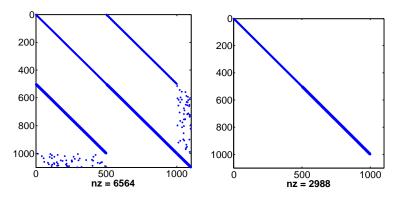
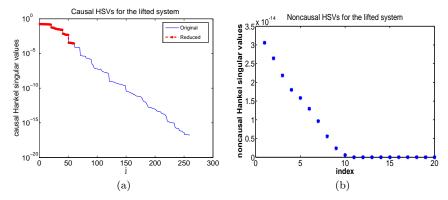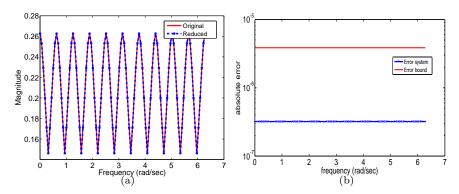**Fig. 3** Sparsity patterns of A_0 (left) and E_0 (right).

Fig. 4 (a) Causal Hankel singular values for original and reduced-order lifted systems. (b) Noncausal Hankel singular values for original and reduced-order lifted systems.

The computed reduced-order model has subsystems of order $\bar{\mathbf{r}} = (9, 8, 8, 7, 8, 9, 7, 8, 8, 8)$, and $\mathbf{r} = 80$. In Fig. 5(a), we show the norms of the frequency responses $\mathcal{H}$ and $\tilde{\mathcal{H}}$ for a frequency range $[0, 2\pi]$. We also display the absolute error $\|\mathcal{H}(e^{i\omega}) - \tilde{\mathcal{H}}(e^{i\omega})\|_2$ and the error bound in Fig. 5(b). One can see that the error bound is tight in this example.

9 Conclusions

We discussed the structure preserving Smith iterations to compute the low-rank factors for the solutions of large sparse projected periodic discrete-time algebraic Lyapunov equations exploiting the reflexive generalized inverses of the periodic matrix coefficients associated with the periodic descriptor system. These low-rank factors are used in a balanced truncation model reduction approach to find a reduced-order model for periodic discrete-time descriptor systems. The proposed model reduction method delivers a reduced-order model that preserves the regularity and stability properties of the original system.

Fig. 5 The frequency responses and error bounds of the original and the reduced-order lifted systems.

An important advantage of our computational approach is that one can directly compute the reflexive generalized inverses of the periodic descriptor system, without explicitly manipulating the lifted representations. Beside this, the proposed Smith iterations preserve the cyclic block diagonal structures at all iteration steps which is the main challenging task in periodic iterative computations.

The major drawback of our proposed method is that it requires the periodic projectors explicitly for the computations of periodic reflexive generalized inverses which is numerically expensive and not a wise approach for higher index systems. Hence we restrict the proposed model reduction approach for systems of index-1.

Appendix

The MATLAB codes of the periodic model formulation used in the second model problem. The following codes will construct the periodic matrices E_k, A_k, B_k, and C_k for $k = 0, 1, \ldots, 9$. Note that E_k are constant matrices in the following construction.

```
--------------------------------------------------------
This is an index 1 problem
--------------------------------------------------------
n=500;
l=100;
nin=3;  % number of input
nout=2; % number of output
K=10;   %period
den=0.001;
I=speye(n);
M=.5*I+spdiags(-0.2*ones(n,1),2,n,n)+spdiags(-0.2*ones(n,1),-2,n,n)+
  spdiags(0.2*ones(n,1),4,n,n)+spdiags(0.2*ones(n,1),-4,n,n);

K_uu=spdiags(5*ones(n,1),0,n,n)+spdiags(-1*ones(n,1),2,n,n)+
     spdiags(-1*ones(n,1),-2,n,n)+ spdiags(2*ones(n,1),4,n,n)+spdiags(2*ones(n,1),-4,n,n);

D=cell(1,K);
for i=1:K
    mu(i)=0.05+.01*i;
    nu(i)=.8+.01*i;
    D{i}=mu(i)*M+nu(i)*K_uu;
end

K_pp=spdiags(-5*ones(l,1),0,l,l)+spdiags(ones(l,1),2,l,l)+spdiags(ones(l,1),-2,l,l)+
     spdiags(-2*ones(n,1),4,l,l)+spdiags(-2*ones(n,1),-4,l,l);
K_up=sprand(n,l,den);

          % Transformation to first order system %

E1=[I spalloc(n,n,0);spalloc(n,n,0) M];
J1=cell(1,K);
for i=1:K
    J1{i}=[spalloc(n,n,0) I;-K_uu -D{i}];
end

J2=[spalloc(n,l,0);-K_up];
J3=[-K_up' spalloc(l,n,0) ];
J4=-K_pp;
clear M K_uu D K_pp K_up
```

```
B11=[spalloc(n,nin,0);spdiags(ones(n,1),0,n,nin)];
B2=spdiags(zeros(l,1),0,l,nin);
C11=[spdiags(ones(n,1),0,nout,n)  spalloc(nout,n,0)];
C22=spdiags(zeros(l,1),0,nout,l);

E= [E1 spalloc(size(J2,1),size(J2,2),0);
   spalloc(size(J3,1),size(J3,2),0)
   spalloc(size(J4,1),size(J4,2),0)];  % note E{i}=E; for i=0,1,...,9.

      % Set the periodic matrices %

Ad=cell(1,K);B=cell(1,K); C=cell(1,K);

for i=1:K
    Ad{i}=[J1{i} J2;J3 J4];
    B1{i}=B11*cos(i);
    B{i}=[B1{i}; B2];
    C1{i}=C11*sin(i);
    C{i}=[C1{i} C22];
end

A=cell(1,K);
for i=1:K
    A{i}=(0.6*E-.015*Ad{i});
end
```

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