Subsystem eigenstate thermalization hypothesis for entanglement entropy in CFT

Song He[1,2*], Feng-Li Lin[3†] and Jia-ju Zhang[4‡]

[1] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute),*
Am Mühlenberg 1, 14476 Golm, Germany

[2] *CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences,*
55 Zhong Guan Cun East Road, Beijing 100190, China

[3] *Department of Physics, National Taiwan Normal University, Taipei 11677, Taiwan*

[4] *Dipartimento di Fisica, Universitá degli Studi di Milano-Bicocca, Piazza della Scienza 3, I-20126 Milano, Italy*

Abstract

In order to investigate subsystem eigenstate thermalization hypothesis (ETH) for two-dimensional large central charge conformal field theory, we evaluate the single-interval Rényi entropy and entanglement entropy for a heavy primary state in short interval expansion. We verify the results of Rényi entropy by three different replica methods. We find nontrivial results at the eighth order of short interval expansion, which include an infinite number of higher order terms in the large central charge expansion. We then evaluate the relative entropy of the reduced density matrices to measure the difference between the heavy primary state and thermal state, and find that the aforementioned nontrivial eighth order results make the relative entropy unsuppressed in the large central charge limit. By Fannes-Audenaert inequality, these results yield a lower bound on trace distance of the excited state and thermal state reduced density matrices, which is crucial in checking the validity of subsystem ETH. We find that whether the subsystem ETH is violated depends on how the effective dimension of the reduced density matrix scales with the large central charge. If the effective dimension is strictly infinite, then it yields no useful information for checking the validity of subsystem ETH. If the effective dimension scales exponentially with the large central charge, the trace distance is at most power suppressed, and subsystem ETH would be violated, while the local ETH remains intact. As a byproduct we also calculate the relative entropy and distance to measure the difference between the reduced density matrices of two different heavy primary states.

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*hesong17@gmail.com
†fengli.lin@gmail.com
‡jiaju.zhang@unimib.it

Contents

1 Introduction

According to eigenstate thermalization hypothesis (ETH) [1,2], a highly excited state of a chaotic system behaves like a high temperature thermal state. More precisely, it states that (i) the diagonal matrix element $A_{\alpha\alpha}$ of a few-body operator A with respect to the energy eigenstate α change slowly with the state in a way of suppression by the exponential of the system size; (ii) the off-diagonal element $A_{\alpha\beta}$ is much smaller than the diagonal element by the factor of exponential of the system size. This will then yield that the expectation value of the few-body observable in a generic state $|\phi\rangle$ behaves like the ones in the thermal state with inverse temperature β

$$\langle A \rangle_\phi - \langle A \rangle_\beta \sim \mathrm{e}^{-\mathcal{O}(S(E))}, \tag{1.1}$$

where $S(E)$ is the system entropy for the state ϕ of energy E.

Recently, a new scheme called subsystem ETH has been proposed in [3,4], in contrast to the old one which is then called local ETH. The subsystem ETH states that the reduced density matrices of a subregion A for high energy primary eigenstates are universal to the reduced density matrix for thermal state up to exponential

suppression by order of the system entropy, i.e.

$$t(\rho_{A,\phi}, \rho_{A,\beta}) \sim e^{-\mathcal{O}(S(E))} \tag{1.2}$$

where $t(\rho_{A,\phi}, \rho_{A,\beta})$ denotes the trace distance between $\rho_{A,\phi}$ and $\rho_{A,\beta}$. As the subsystem ETH is a statement regarding the reduced density matrices, their derived quantities such as correlation functions, entanglement entropy and Rényi entropy should also satisfy some sort of the subsystem ETH. In this sense, the subsystem ETH is strongest form of ETH, i.e., stronger than local ETH.

For a conformal field theory (CFT) there are infinite degrees of freedom, and nonlocal quantities like entanglement entropy and Rényi entropy do not necessarily be exponentially suppressed [3, 4]. In fact the primary excited state Rényi entropy in two-dimensional (2D) CFT is not exponentially suppressed in the large central charge limit [3, 5]. In this case, it is possible to relax the subsystem ETH (1.2) in a way such that each component of the reduced density matrices are different exponentially

$$\rho_{A,\phi} - \rho_{A,\beta} \sim e^{-\mathcal{O}(S(E))}. \tag{1.3}$$

We call it weak subsystem ETH. In case of finite-dimensional models the weak subsystem ETH is equivalent to the subsystem ETH, but it is not necessarily true for a CFT. It can be seen to be weaker than (1.2) in the diagonal basis of $\rho_{A,\phi} - \rho_{A,\beta}$, and a CFT satisfies the weak subsystem ETH (1.3) can possibly violate the subsystem ETH (1.2). A few-body operator probes only a finite number of degrees of freedom, and the local ETH (1.1) can be derived from (1.3). The weak subsystem ETH is stronger than local ETH.

In this paper we investigate the validity of the subsystem ETH in a 2D large central charge CFT. In this case, the worldsheet description of an excited state $|\phi\rangle$ for a CFT living on a circle of size L corresponds to an infinitely long cylinder of spatial period L capped by an operator ϕ at each end. On the other hand, the thermal state of a CFT living on a circle with temperature T has its worldsheet description as a torus with temporal circle of size $\beta = 1/T$. In the high temperature limit with $L \gg \beta$, the torus is approximated by a horizontal cylinder. Naively the vertical and horizontal cylinders should be related by Wick rotation and can be compared after taking care of the capped states. This is indeed what have been done in [6, 7] by comparing the two-point functions two light operators of large c CFT, and in [8, 9] for the single-interval entanglement entropy. These comparisons all show that the subsystem ETH holds. However, in [3, 5] the one-interval Rényi entropy for small interval of size $\ell \ll L$ are compared by ℓ expansion up to order ℓ^6 and it was found that one cannot find a universal relation between β and L to match the excited-state Rényi entropy with the thermal one in the series expansion of ℓ.

Moreover, in the context of AdS/CFT correspondence [10–13], a large N CFT is dual to the AdS gravity of large AdS radius, and so the subsystem ETH implies that the backreacted geometry by the massive bulk field is approximately equivalent to the black hole geometry for the subregion observer. Especially, for AdS$_3$/CFT$_2$ the CFT has infinite dimensional conformal symmetries as the asymptotic symmetries of AdS space [14], along with ETH it could imply that the infinite varieties of Bañados geometries [15] dual to the excited CFT states are universally close to the BTZ black hole [16]. Thus, the study of ETH could help us to understand the black hole formation in AdS$_3$ space and the nature of the microstates of BTZ black hole.

As discussed in [3], the validity of subsystem ETH depends on how the operator product expansion (OPE) coefficients scale with the conformal dimension of the eigen-energy operator in the thermodynamic limit. This

means that the subsystem ETH could be violated for some circumstances. In this paper we continue to investigate the validity of ETH for a 2D large c CFT by more extensive calculations, and indeed find the surprising results. According to Cardy's formula [17] the thermal entropy is proportional to the central charge c, and so we just focus on how various quantities behave in large c limit. We calculate the entanglement entropy and Rényi entropy up to order ℓ^8 in the small ℓ expansion. We then find that there appear subleading corrections of $1/c$ expansion at the order ℓ^8. Because the appearance of these subleading corrections at order ℓ^8 is quite unexpected, we solidify the results by adopt three different method to calculate them. These methods are (i) the OPE of twist operators [18–21] on cylinder; (ii) OPE of twist operators on complex plane; and (iii) $2n$-point correlation function on complex plane [22–28]. By all three methods we all get the same results. Moreover, we turn the comparison of the entanglement entropy into the relative entropy between reduced density matrices for excited state and thermal state by the modular Hamiltonian argument in [3]. Then, the above discrepancy yields that the relative entropy is of order c^0.

Based on the above results, we use the Fannes-Audenaert inequality [29, 30] and Pinsker's inequality by relating the trace distance to entanglement entropy or relative entropy, to argue how the trace distance of the reduced density matrices of the excited and thermal states scales with the large central charge. According to [3,4], the reduced density matrix is of infinite dimension such that the Fannes-Audenaert inequality is trivially satisfied, and yields no constraint on the trace distance for subsystem ETH. We propose that there may exist a large but finite effective dimension for the reduced density matrix that satisfies the Fannes-Audenaert inequality. If the effective dimension scales exponentially with the large central charge, the trace distance would be at most power suppressed, causing subsystem ETH to be violated despite that the weak subsystem ETH remains intact.

Finally, using the replica method based on evaluating the multi-point function on a complex plane, as a byproduct of this project we explicitly calculate the relative entropy and distance to measure the difference between the reduced density matrices of two different heavy primary states. This result could be useful for future investigation on the related topics.

The rest of the paper is organized as follows. In section 2 we briefly review the known useful results about Rényi entropy, entanglement entropy and relative entropy, and also evaluate the relative entropy between chiral vertex state and thermal state in 2D free massless scalar theory. In section 3 we calculate the excited state Rényi entropy in three different replica methods in the short interval expansion. Using these results, in section 4 we check the subsystem ETH. We find that the validity of the subsystem ETH depend only on whether the large but finite effective dimension of the reduced density matrix exists and if yes how it scales with the large central charge. In section 5 we evaluate the relative entropy and distance of the reduced density matrices to measure the distance between two two heavy primary states. Finally, we conclude our paper in section 6. Besides, we relegate some technical details in three appendices. In appendix A we give some details of the vacuum conformal family. In appendix B we give the results of OPE of twist operators, including both the review of the formulism and some new calculations. In appendix C we list some useful summation formulas.

2 Relative entropy and ETH

In this section we will briefly review the basics of relative entropy and then ETH. In the next section, the relative entropy between the reduced density matrices of heavy state and thermal state will be evaluated for large central charge 2D CFT to check the validity of ETH. We will end this section by calculating the relative entropy of a toy example CFT, namely the 2D massless scalar.

2.1 Relative entropy

Given a quantum state of a system denoted by the density matrix ρ, then the reduced density matrix on some region A is given by

$$\rho_A = \mathrm{tr}_{A^c} \rho. \tag{2.1}$$

where A^c is the complement of A. One can then define the Rényi entropy

$$S_{A,n} = -\frac{1}{n-1} \log \mathrm{tr}_A \rho_A^n, \tag{2.2}$$

and the entanglement entropy

$$S_A = -\mathrm{tr}_A(\rho_A \log \rho_A), \tag{2.3}$$

which is formally equivalent to taking $n \to 1$ limit of the Rényi entropy.

In this work we will focus on 2D CFT of central charge c, and the Rényi entropy for a region A of size ℓ, i.e., $A = [-\ell/2, \ell/2]$, for a vacuum state is known [18]

$$S_{n,L} = \frac{c(n+1)}{12n} \log\left(\frac{L}{\pi\epsilon} \sin\frac{\pi\ell}{L}\right), \tag{2.4}$$

where L is the size of the spatial circle on which the CFT lives. Similarly, the Rényi entropy of a thermal state with temperature $1/\beta$ for a CFT living on a infinite straight line is

$$S_{n,\beta} = \frac{c(n+1)}{12n} \log\left(\frac{\beta}{\pi\epsilon} \sinh\frac{\pi\ell}{\beta}\right). \tag{2.5}$$

Taking $n \to 1$ limit, one can get the corresponding entanglement entropy straightforwardly. For simplicity, in this paper we only consider the contributions from the holomorphic sector of CFT, and the anti-holomorphic sector can be just added for completeness without complication. Also we will not consider the subtlety due to the boundary conditions imposed on the entangling surface [31, 32].

For subsystem ETH it is to compare the reduced density matrices between a heavy state and the excited state, and in this paper we consider the Rényi entropy difference, the entanglement entropy difference, and the relative entropy, as well as the trace distance. The relative entropy is defined as

$$S(\rho_A' \| \rho_A) = \mathrm{tr}_A \rho_A' \log \rho_A' - \mathrm{tr}_A \rho_A' \log \rho_A. \tag{2.6}$$

where ρ_A and ρ_A' are the reduced density matrixes over region A for state ρ and ρ', respectively. Note that the relative entropy is not symmetric for its two arguments, i.e., $S(\rho_A' \| \rho_A) \neq S(\rho_A \| \rho_A')$. One is then motivated to symmetrize it and define the "distance"

$$d(\rho_A', \rho_A) = S(\rho_A' \| \rho_A) + S(\rho_A \| \rho_A'). \tag{2.7}$$

One can also express the relative entropy as follows:

$$S(\rho'_A \| \rho_A) = \langle H_A \rangle_{\rho'} - \langle H_A \rangle_{\rho} - S'_A + S_A. \tag{2.8}$$

where the modular Hamiltonian H_A is defined by

$$H_A \equiv -\log \rho_A. \tag{2.9}$$

The modular Hamiltonian is in general quite nonlocal and known only for some special cases [25,32–34]. One of these cases fitted for our study in this paper is just the case considered for the Rényi entropy (2.5) of a thermal state, and the modular Hamiltonian is given by [34]

$$H_{A,\beta} = -\frac{\beta}{\pi} \int_{-\ell/2}^{\ell/2} dx \; \frac{\sinh \frac{\pi(\ell-2x)}{2\beta} \sinh \frac{\pi(\ell+2x)}{2\beta}}{\sinh \frac{\pi\ell}{\beta}} \; T(x) \tag{2.10}$$

where $T(x)$ is the holomorphic sector stress tensor of the 2D CFT.

In this paper we will check the subsystem ETH for a normalized highly and globally excited state[1] created by a (holomorphic) primary operator ϕ of conformal weight $h_\phi = c\epsilon_\phi$ acting on the vacuum, i.e., $|\phi\rangle = \phi(0)|0\rangle$. The first step to proceed the comparison for checking ETH is to make sure the excited state and the thermal state have the same energy, and this then requires

$$\langle \phi|T|\phi \rangle_L = \langle T \rangle_\beta. \tag{2.11}$$

The right hand side of the above equation is just the Casimir energy of the horizontal worldsheet cylinder

$$\langle T \rangle_\beta = -\frac{\pi^2 c}{6\beta^2}. \tag{2.12}$$

and the left hand side is given by (A.16). Thus, (2.11) yields a relation between the inverse temperature β and the conformal weight h_ϕ (or ϵ_ϕ) [6,7]

$$\beta = \frac{L}{\sqrt{24\epsilon_\phi - 1}}. \tag{2.13}$$

Moreover, the relation (2.11) and (2.10) ensure $\langle H_{A,\beta} \rangle_\phi = \langle H_{A,\beta} \rangle_\beta$ so that [3]

$$S(\rho_{A,\phi} \| \rho_{A,\beta}) = -S_{A,\phi} + S_{A,\beta}. \tag{2.14}$$

2.2 ETH

ETH states that a highly excited state of a chaotic system behaves thermally. One way to formulate this is to compare the expectation values of few-body operators for high energy eigenstate and the thermal state, as explicitly formulated in (1.1). This is called the local ETH in [4] in contrast to a stronger statement called subsystem ETH proposed therein, which is formulated as in (1.2) by comparing the reduced density matrices, i.e., requiring that trace distance between the two reduced states should be exponentially suppressed by the system entropy. The trace distance for two reduced density matrices ρ'_A, ρ_A is defined as

$$t(\rho'_A, \rho_A) = \frac{1}{2} \text{tr}_A |\rho'_A - \rho_A|, \tag{2.15}$$

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[1]In this paper, we mainly focus on global excited states which are quite different from so called local excited states studied in [35–38].

and by definition $0 \le t(\rho'_A, \rho_A) \le 1$.

As the subsystem ETH is stronger than local ETH, it could be violated for the system of infinite number of degrees of freedom. However, we do not directly calculate the trace distance but the Rényi entropies and entanglement entropies for both heavy state and thermal state. After doing this, we can then use some inequalities to constrain the trace distance with the difference of the Rényi entropies or relative entropy, thus check the validity of ETH.

Here are three such kinds of inequality. First, the Fannes-Audenaert inequality [29, 30] relating the difference of entanglement entropy, $\Delta S_A := S_{A,\phi} - S_{A,\beta}$ to the trace distance $t := t(\rho_{A,\phi}, \rho_{A,\beta})$ as follows:

$$|\Delta S_A| \le t \log(d-1) + h, \tag{2.16}$$

with $h = -t \log t - (1-t) \log(1-t)$ and d being the dimension of Hilbert space $\mathcal{H}_A$ for the effective degrees of freedom in subsystem A. On the other hand, there is the Audenaert inequality for the Rényi entropy of order $0 < n < 1$ [30]

$$|\Delta S_n| \le \frac{1}{1-n} \log[(1-t)^n + (d-1)^{1-n} t^n], \tag{2.17}$$

with $\Delta S_n := S_{n,\phi} - S_{n,\beta}$. Both the right hand sides of (2.16) and (2.17) are vanishing at $t = 0$, are $\log(d-1)$ at $t = 1$, monotonically increase at $0 < t < 1 - \frac{1}{d}$, monotonically decrease at $1 - \frac{1}{d} < t < 1$, and have a maximal value $\log d$ at $t = 1 - \frac{1}{d}$. Since d is very large, the right hand sides of (2.16) and (2.17) are approximately monotonically increase at $0 < t < 1$. Finally, we also need Pinsker's inequality to give upper bound on trace distance by the square root of relative entropy, i.e.

$$t \le \sqrt{\frac{1}{2} S(\rho_{A,\phi} \| \rho_{A,\beta})}. \tag{2.18}$$

By using (2.16) we see that $|\Delta S_A|$ gives the tight lower-bound on the trace distance if the d is finite, thus the validity of subsystem ETH can be pin down by the scaling behavior of $|\Delta S_A|$ with respect to the system entropy. This is no longer true if d is infinite as one would expect for generic quantum field theories, then both (2.16) and (2.17) are trivially satisfied and can tell no information about the trace distance [3, 4]. However, it is a subtle issue to find out how the effective dimension d of the reduced density matrix scale with the large c and if it is finite once a UV cutoff is introduced. It is possible that d is not strictly infinite but scale exponentially with c for a large c CFT so that the inequalities (2.16) and (2.17) are satisfied and the subsystem ETH is violated. We will discuss in more details in section 4.

2.3 A toy example

We now apply the above formulas to a toy 2D CFT, the massless free scalar. This CFT has central charge $c = 1$ so that it makes less sense to check subsystem ETH. Despite that, we will still calculate the relative entropy between excited state and thermal state, and the result can be compared to the large c ones obtained later.

Let the massless scalar denoted by φ, and from it we can construct the chiral vertex operator [39]

$$V_\alpha(z) = e^{i\alpha\varphi(z)}, \tag{2.19}$$

with conformal weight

$$h_\alpha = \frac{\alpha^2}{2}. \tag{2.20}$$

Choosing $\alpha \gg 1$ we can create the highly excited state as follows:

$$|V_\alpha\rangle = V_\alpha(0)|0\rangle. \tag{2.21}$$

The Rényi entropy for the state $|V_\alpha\rangle$ was calculated before in [23,24,28], and the result is the same as (2.4) for the vacuum state, no matter what the value α is. Thus, the relative entropy (2.14) can be obtained straightforwardly and the result is

$$S(\rho_{A,\alpha}\|\rho_{A,\beta}) = \frac{1}{6}\log\frac{\beta\sinh\frac{\pi\ell}{\beta}}{L\sin\frac{\pi\ell}{L}}. \tag{2.22}$$

In Fig. 1 we have plotted the results as a function of ℓ/L for various β/L. We see that the relative entropy is overall larger for heavier excited state. Note that α appears in (2.22) implicitly through

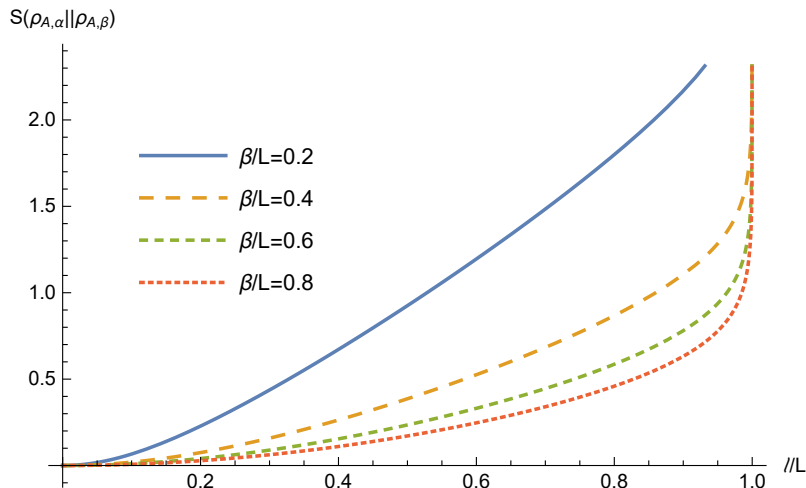$$\beta = \frac{L}{\sqrt{12\alpha^2 - 1}}. \tag{2.23}$$

Figure 1: Relative entropy (2.22) as a function of ℓ/L in comparing the reduced density matrices for chiral primary state and thermal state of 2D massless scalar. Note that the higher the temperature is, the larger the relative entropy becomes.

3   Excited state Rényi entropy

We now consider the 2D CFT with large central charge, which can be also thought as dual CFT of AdS$_3$. We aim to calculate the Rényi entropy $S_{n,\phi}$ for a highly excited state $|\phi\rangle$, i.e., the conformal weight h_ϕ is order c for short interval $\ell \ll L$ so that we can obtain the results with three different methods based on short interval expansion up to order $(\ell/L)^8$.

The first method is to use OPE of twist operators on the cylinder to evaluate the excited state Rényi entropy [18–21]. We have used this method in [5] to get the result up to order $(\ell/L)^6$ and find that the subsystem ETH is violated for $n \neq 1$ but holds for $n = 1$, i.e., the entanglement entropy. In this paper we calculate up to one more order and find nontrivial violation of subsystem ETH at the new order. For consistency check we also use the other two methods to calculate and obtain the same result. The second method is to use OPE of the twist operator on complex plane to evaluate the Rényi entropy, and the third one is to use the multi-point correlation functions on complex plane [22–28]. As in [5], we focus on the contributions of the

holomorphic sector of the vacuum conformal family. Some details of the vacuum conformal family are collected in appendix A.

3.1 Method of twist operators on cylinder

By the replica trick for evaluating the single-interval Rényi entropy, we get the one-fold CFT on an n-fold cylinder, or equivalently an n-fold CFT, which we call CFT^n on the one-fold cylinder. The boundary conditions of the CFT^n on cylinder can be replaced by twist operators [18]. Thus, the partition function of CFT^n on cylinder capped by state $|\phi\rangle$ can be expressed as the two-point function of twist operators, i.e.

$$\text{tr}_A \rho_{A,\phi}^n = \langle \Phi | \sigma(\ell/2) \tilde{\sigma}(-\ell/2) | \Phi \rangle_{\text{cyl}}, \tag{3.1}$$

with the definition $\Phi \equiv \prod_{j=0}^{n-1} \phi_j$ and the index j marking different replicas.

Formally and practically, we can use the OPE of the twist operators to turn the above partition function into a series expansion, and the formal series expansion for the excited state Rényi entropy is

$$S_{n,\phi} = \frac{c(n+1)}{12n} \log \frac{\ell}{\epsilon} - \frac{1}{n-1} \log \left(\sum_K d_K \ell^{h_K} \langle \Phi_K \rangle_\Phi \right). \tag{3.2}$$

The details about the OPE of twist operators [19–21] is reviewed in appendix B. In arriving the above, we have used (B.2) and the fact that $\langle \Phi_K \rangle_\Phi \equiv \langle \Phi | \Phi_K | \Phi \rangle_{\text{cyl}}$ is a constant.

Further using the properties for the vacuum conformal family and its OPE in appendix A and B, i.e., specifically (A.16), (B.9), (B.10) and (B.14), we can obtain the explicit result of the short interval expansion up to order $(\ell/L)^8$ as follows:

$$
\begin{aligned}
S_{n,\phi} = {} & \frac{c(n+1)}{12n} \log \frac{\ell}{\epsilon} + \frac{\pi^2 c(n+1)(24\epsilon_\phi - 1)\ell^2}{72nL^2} - \frac{\pi^4 c(n+1)\left[48(n^2+11)24\epsilon_\phi^2 - 24(n^2+1)\epsilon_\phi + n^2\right]\ell^4}{2160n^3 L^4} \\
& - \frac{\pi^6 c(n+1)\left[96(n^2-4)(n^2+47)\epsilon_\phi^3 + 36(2n^4+9n^2+37)\epsilon_\phi^2 - 24(n^4+n^2+1)\epsilon_\phi + n^4\right]\ell^6}{34020n^5 L^6} \\
& + \frac{\pi^8 c(n+1)\ell^8}{453600(5c+22)n^7 L^8} \Big\{ c\big[64(13n^6 - 1647n^4 + 33927n^2 - 58213)\epsilon_\phi^4 \\
& - 64(n^2+11)(13n^4 + 160n^2 - 533)\epsilon_\phi^3 - 48(9n^6 + 29n^4 + 71n^2 + 251)\epsilon_\phi^2 \\
& + 120(n^2+1)(n^4+1)\epsilon_\phi - 5n^6\big] - 5632(n^2-4)(n^2-9)(n^2+119)\epsilon_\phi^4 \\
& - 2816(n^2-4)(n^2+11)(n^2+19)\epsilon_\phi^3 - 128(15n^6 + 50n^4 + 134n^2 + 539)\epsilon_\phi^2 \\
& + 528(n^2+1)(n^4+1)\epsilon_\phi - 22n^6\Big\} + O((\ell/L)^9). \tag{3.3}
\end{aligned}
$$

Note that the result up to order $(\ell/L)^6$ is just proportional to c, and agrees with the result obtained previously in [3, 5]. At the order $(\ell/L)^8$, however, novel property appears. There appears a nontrivial $5c + 22$ factor in the overall denominator, which yields infinite number of higher order subleading terms in the $1/c$ expansion for large c. These subleading terms come from the contributions of the quasiprimary operator $\mathcal{A}$ defined by (A.1) at level four of the vacuum family. We will obtain the same result for other two methods in subsection 3.2 and subsection 3.3.

3.2 Method of twist operators on complex plane

In the second method we map CFT^n on cylinder into CFT^n on complex plane, as shown in figure 2. The cylinder with coordinate w is mapped to a complex plane with coordinate z by a conformal transformation $z = e^{2\pi i w/L}$.
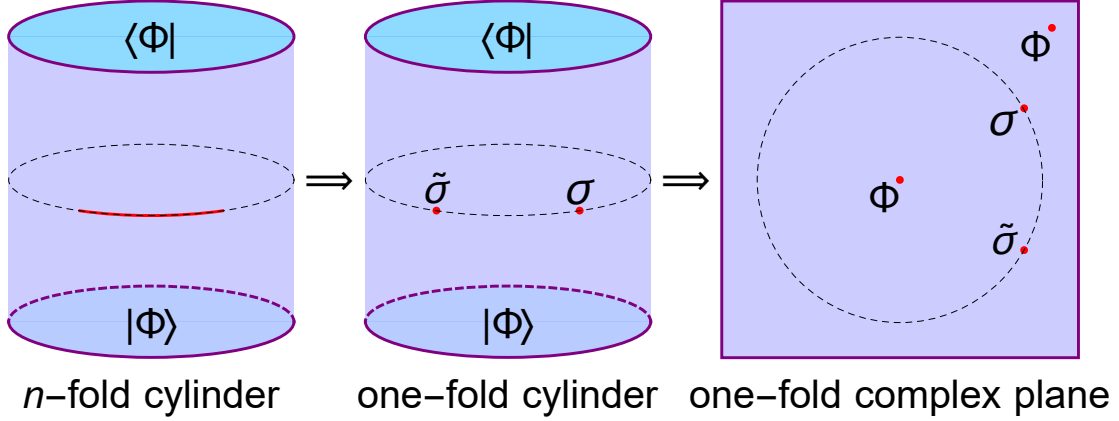
Figure 2: This figure illustrates how our first two methods of replica trick are related. The one-fold CFT on a one-fold CFT on an n-fold cylinder is equivalent to a n-fold CFT on a one-fold cylinder. The boundary conditions of the n-fold CFT can be replaced by the insertion of a pair of twist operators [18]. The cylinder with twist operators can be mapped to a complex plane with twist operators.

The partition function then becomes a four-point function on complex plane

$$\text{tr}_A\rho_{A,\phi}^n = \left(\frac{2\pi\text{i}}{L}\right)^{2h_\sigma}\langle\Phi(\infty)\sigma(\text{e}^{\pi\text{i}\ell/L})\tilde\sigma(\text{e}^{-\pi\text{i}\ell/L})\Phi(0)\rangle_\mathcal{C}. \tag{3.4}$$

Using (B.2) for the OPE of twist operators on complex plane, we get the excited state Rényi entropy

$$S_{n,\phi} = \frac{c(n+1)}{12n}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right) - \frac{1}{n-1}\Big[\sum_K d_K C_{\Phi\Phi K}(1-\text{e}^{2\pi\text{i}\ell/L})^{h_K}\,_2F_1(h_K,h_K;2h_K;1-\text{e}^{2\pi\text{i}\ell/L})\Big]. \tag{3.5}$$

Using (A.13), (B.5), (B.6), (B.7), (B.8), (B.14), (B.15) we can reproduce (3.3).

3.3 Method of multi-point function on complex plane

In the third method we use the formulism of multi-point function on complex plane, see [22–28]. The idea is illustrated in Fig.3. Using the state/operator correspondence, we map the partition function on the capped n-fold cylinder into the two-point function on the n-fold complex plane $\mathcal{C}^n$, i.e., formally

$$\frac{\text{tr}_A\rho_{A,\phi}^n}{\text{tr}_A\rho_{A,0}^n} = \langle\Phi(\infty)\Phi(0)\rangle_{\mathcal{C}^n}. \tag{3.6}$$

We then map each copy of complex plan into a wedge of deficit angle $2\pi/n$ by the following conformal transformation

$$f(z) = \left(\frac{z - \text{e}^{\pi\text{i}\ell/L}}{z - \text{e}^{-\pi\text{i}\ell/L}}\right)^{1/n}. \tag{3.7}$$

The two boundaries of each wedge correspond to the intervals just right above or below the interval A. Gluing all the n wedges along the boundaries, we then obtain the one-fold complex plane $\mathcal{C}$ so that the above two-point function on $\mathcal{C}^n$ becomes a $2n$-point function on a one-fold complex plane $\mathcal{C}$, i.e.

$$\langle\Phi(\infty)\Phi(0)\rangle_{\mathcal{C}^n} = \left(\frac{2\text{i}}{n}\sin\frac{\pi\ell}{L}\right)^{2nh_\phi}\Big\langle\prod_{j=0}^{n-1}\Big(\text{e}^{2\pi\text{i}(\frac{\ell}{nL}+\frac{2j}{n})}\phi(\text{e}^{2\pi\text{i}(\frac{\ell}{nL}+\frac{j}{n})})\phi(\text{e}^{2\pi\text{i}\frac{j}{n}})\Big)\Big\rangle_\mathcal{C}. \tag{3.8}$$

Based on the above, it is straightforward to see that

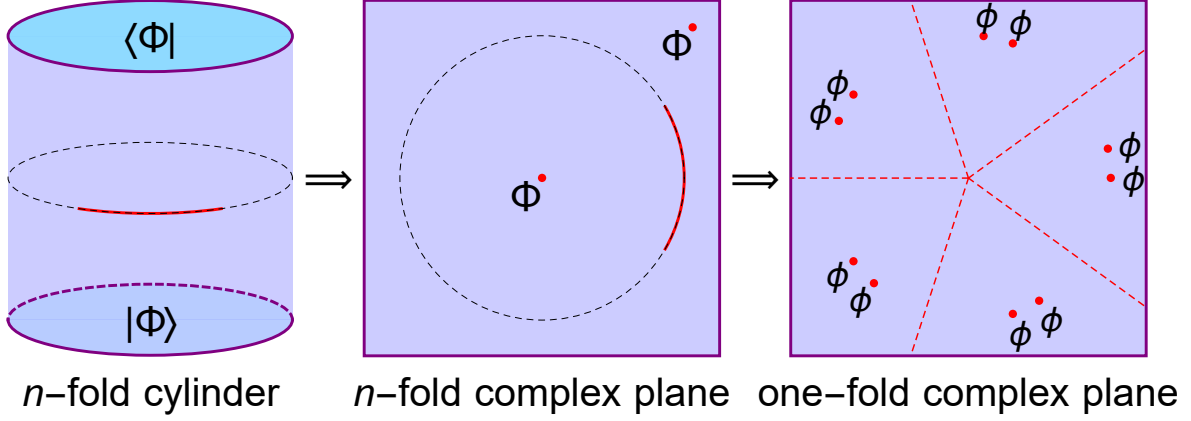$$S_{n,\phi}(\ell) = S_{n,\phi}(L-\ell), \tag{3.9}$$

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Figure 3: This figure illustrates the replica method of multi-point function on complex plane [22–28]. Firstly, one has the one-fold CFT on an n-fold cylinder in excited state $|\Phi\rangle$. Then, by state/operator correspondence one gets a two-point function on an n-fold complex plane $\mathcal{C}^n$. Lastly, by a conformal transformation one gets a $2n$-point function on a one-fold complex plane. In the last part of the figure we use $n = 5$ as an example.

which is expected for a pure state.

Formally, the OPE of a primary operator with itself is given [40]

$$\phi(z)\phi(w) = \frac{1}{(z-w)^{2h_\phi}}\mathcal{F}_\phi(z,w). \tag{3.10}$$

with

$$\mathcal{F}_\phi(z,w) = \sum_{\mathcal{X}} \frac{C_{\phi\phi\mathcal{X}}}{\alpha_{\mathcal{X}}} \sum_{r=0}^{\infty} \frac{a_{\mathcal{X}}^r}{r!}(z-w)^{h_{\mathcal{X}}+r}\partial^r\mathcal{X}(w), \quad a_{\mathcal{X}}^r = \frac{C_{h_{\mathcal{X}}+r-1}^r}{C_{2h_{\mathcal{X}}+r-1}^r}, \tag{3.11}$$

where the summation runs over all the holomorphic quasiprimary operators $\{\mathcal{X}\}$ with each $\mathcal{X}$ being of conformal weight $h_{\mathcal{X}}$, and C_x^y denotes the binomial coefficient.

In a unitary CFT, the operator with the lowest conformal weight is the identity operator, and so in $z \to w$ limit

$$\mathcal{F}_\phi(z,w) = 1 + \cdots. \tag{3.12}$$

Putting (3.10) in (3.8) and using [18]

$$\text{tr}_A\rho_{A,0}^n = \left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right)^{-2h_\sigma}, \tag{3.13}$$

we get the excited state Rényi entropy

$$S_{n,\phi} = \frac{c(n+1)}{12n}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right) - \frac{2nh_\phi}{n-1}\log\frac{\sin\frac{\pi\ell}{L}}{n\sin\frac{\pi\ell}{nL}} - \frac{1}{n-1}\log\left\langle\prod_{j=0}^{n-1}\mathcal{F}_\phi\left(e^{2\pi i(\frac{\ell}{nL}+\frac{j}{n})}, e^{2\pi i\frac{j}{n}}\right)\right\rangle_{\mathcal{C}}. \tag{3.14}$$

We now perform the short-interval expansion for (3.14) up to order $(\ell/L)^8$ by considering only the contributions from the vacuum conformal family, i.e., including its descendants up to level eight, see appendix A. For $n = 2$ there is a compact formula

$$S_{2,\phi} = \frac{c}{8}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right) - 4h_\phi\log\frac{\sin\frac{\pi\ell}{L}}{2\sin\frac{\pi\ell}{2L}} - \log\left[\sum_\psi\frac{C_{\phi\phi\psi}^2}{\alpha_\psi}\left(\sin\frac{\pi\ell}{2L}\right)^{2h_\psi}{}_2F_1\left(h_\psi, h_\psi; 2h_\psi; \sin^2\frac{\pi\ell}{2L}\right)\right]. \tag{3.15}$$

11

Using the details in the appendix A, we can obtain the explicit result as follows[2]

$$
\begin{aligned}
S_{2,\phi} = {} & \frac{c}{8}\log\frac{\ell}{\epsilon} + \frac{\pi^2 c(24\epsilon_\phi - 1)\ell^2}{48L^2} - \frac{\pi^4 c(180\epsilon_\phi^2 - 30\epsilon_\phi + 1)\ell^4}{1440L^4} - \frac{\pi^6 c(945\epsilon_\phi^2 - 126\epsilon_\phi + 4)\ell^6}{90720L^6} \\
& + \frac{\pi^8 c[5c(83160\epsilon_\phi^4 - 7560\epsilon_\phi^3 - 1890\epsilon_\phi^2 + 255\epsilon_\phi - 8) - 2(22680\epsilon_\phi^2 - 2805\epsilon_\phi + 88)]\ell^8}{2419200(5c + 22)L^8} \\
& + \frac{\pi^{10} c[5c(1372140\epsilon_\phi^4 - 124740\epsilon_\phi^3 - 14355\epsilon_\phi^2 + 2046\epsilon_\phi - 64) - 44(8595\epsilon_\phi^2 - 1023\epsilon_\phi + 32)]\ell^{10}}{239500800(5c + 22)L^{10}} \\
& + O((\ell/L)^{12}).
\end{aligned}
\tag{3.16}
$$

To perform the short-interval expansion Rényi entropy of general rank n, i.e., (3.14), we have to calculate a series of j-point correlation functions with $j = 1, 2, \cdots, n/2$. To order ℓ^8 the number of these multi-point correlation functions can be counted in each order by the following:

$$
\begin{aligned}
\prod_{k=2}^{\infty} \frac{1}{(1-x^k)^n} = {} & 1 + nx^2 + nx^3 + \frac{n(n+3)}{2}x^4 + n(n+1)x^5 + \frac{n(n+1)(n+11)}{6}x^6 \\
& + \frac{n(n^2+5n+2)}{2}x^7 + \frac{n(n+3)(n^2+27n+14)}{24}x^8 + O(x^9).
\end{aligned}
\tag{3.17}
$$

These multi-point correlation functions are listed in table 1, and we note that many of them are trivially vanishing. Putting the results of appendix A in (3.14), we reproduce the Rényi entropy (3.3).

4 Check subsystem ETH

We now can use the result in the previous section to check the subsystem ETH for the 2D large c CFT. We first take the $n \to 1$ limit of the excited state Rényi entropy (3.3) and the thermal one (2.5) up to order $(\ell/L)^8$ to get the corresponding entanglement entropy. We then have

$$
\begin{aligned}
S_{A,\phi} = {} & \frac{c}{6}\log\frac{\ell}{\epsilon} + \frac{\pi^2 c(24\epsilon_\phi - 1)\ell^2}{36L^2} - \frac{\pi^4 c(24\epsilon_\phi - 1)^2\ell^4}{1080L^4} + \frac{\pi^6 c(24\epsilon_\phi - 1)^3\ell^6}{17010L^6} - \frac{\pi^8 c\ell^8}{226800(5c+22)L^8}\big[5c(24\epsilon_\phi - 1)^4 \\
& + 2(8110080\epsilon_\phi^4 - 1013760\epsilon_\phi^3 + 47232\epsilon_\phi^2 - 1056\epsilon_\phi + 11)\big] + O((\ell/L)^9),
\end{aligned}
\tag{4.1}
$$

and

$$
S_{A,\beta} = \frac{c}{6}\log\frac{\ell}{\epsilon} + \frac{\pi^2 c\ell^2}{36\beta^2} - \frac{\pi^4 c\ell^4}{1080\beta^4} + \frac{\pi^6 c\ell^6}{17010\beta^6} - \frac{\pi^8 c\ell^8}{226800\beta^8} + O((\ell/\beta)^{10}).
\tag{4.2}
$$

It is straightforward to see that (4.1) fails to match with (4.2) at order $(\ell/L)^8$ under the identification of inverse temperature and conformal weight by the relation (2.13), and the discrepancy is

$$
S_{A,\beta} - S_{A,\phi} = \frac{128\pi^8 c\epsilon_\phi^2(22\epsilon_\phi - 1)^2\ell^8}{1575(5c+22)L^8} + O((\ell/L)^9).
\tag{4.3}
$$

From (2.14) we know that this is nothing but the relative entropy $S(\rho_{A,\phi}\|\rho_{A,\beta})$. Note that this discrepancy is of order c^0 in the large c limit.

Based on the result (4.3), we then use the inequalities (2.16), (2.17), and (2.18) to estimate the order of the trace distance in large c limit and check the validity of ETH. We have obtained the Rényi entropy, entanglement

---

[2]Since $C_{\phi\phi\mathcal{X}} = 0$ for a bosonic operator $\mathcal{X}$ with odd integer conformal weight, using results in the appendix A we can get the result up to order $(\ell/L)^{19}$. However higher order results are too complicated to be revealing. We just write down the result up to order $(\ell/L)^{11}$.

order	multi-point function	? ? ? ? ?	number	number
0	1	$\checkmark\times\times\times\times$	1	1
2	T	$\times\checkmark\times\times\times$	n	n
3	∂T	$\times\checkmark\times$	n	n
4	$\mathcal{A}$	$\times\times\checkmark$	n	$\frac{n(n+3)}{2}$
	TT	$\checkmark\checkmark\checkmark$	$\frac{n(n-1)}{2}$	
	$\partial^2 T$	$\times\checkmark\times$	n	
5	$\partial^3 T$	$\times\checkmark$	n	$n(n+1)$
	$\partial\mathcal{A}$	$\times\times$	n	
	$T\partial T$	$\checkmark\checkmark$	$n(n-1)$	
6	$\mathcal{B}, \mathcal{D}$	$\times\times$	$2n$	$\frac{n(n+1)(n+11)}{6}$
	$T\mathcal{A}$	$\times\checkmark$	$n(n-1)$	
	TTT	$\checkmark\checkmark$	$\frac{n(n-1)(n-2)}{6}$	
	$\partial^4 T$	$\times\checkmark$	n	
	$\partial^2 \mathcal{A}$	$\times\times$	n	
	$T\partial^2 T, \partial T\partial T$	$\checkmark\checkmark$	$\frac{3n(n-1)}{2}$	
7	$\partial^5 T, \partial^3\mathcal{A}, \partial\mathcal{B}, \partial\mathcal{D}$	$\times$	$4n$	$\frac{n(n^2+5n+2)}{2}$
	$T\partial^3 T, \partial T\partial^2 T$	$\checkmark$	$2n(n-1)$	
	$T\partial\mathcal{A}, \partial T\mathcal{A}$	$\times$	$2n(n-1)$	
	$TT\partial T$	$\checkmark$	$\frac{n(n-1)(n-2)}{2}$	
8	$\mathcal{E}, \mathcal{H}, \mathcal{I}$	$\times$	$3n$	$\frac{n(n+3)(n^2+27n+14)}{24}$
	$T\mathcal{B}, T\mathcal{D}$	$\times$	$2n(n-1)$	
	$\mathcal{A}\mathcal{A}$	$\checkmark$	$\frac{n(n-1)}{2}$	
	$TT\mathcal{A}$	$\checkmark$	$\frac{n(n-1)(n-2)}{2}$	
	$TTTT$	$\checkmark$	$\frac{n(n-1)(n-2)(n-4)}{24}$	
	$\partial^6 T, \partial^4\mathcal{A}, \partial^2\mathcal{B}, \partial^2\mathcal{D}$	$\times$	$4n$	
	$T\partial^4 T, \partial T\partial^3 T, \partial^2 T\partial^2 T$	$\checkmark$	$\frac{5n(n-1)}{2}$	
	$T\partial^2\mathcal{A}, \partial T\partial\mathcal{A}, \partial^2 T\mathcal{A}$	$\times$	$3n(n-1)$	
	$TT\partial^2 T, T\partial T\partial T$	$\checkmark$	$n(n-1)(n-2)$	

Table 1: All the multi-point functions we have to consider in obtaining the excited state Rényi entropy up to order $(\ell/L)^8$. In the 1st column it is the order from which each multi-point function starts to contribute. In the 2nd column listed are the multi-point functions, and for simplicity we have omitted the correlation function symbol $\langle\cdots\rangle_{\mathcal{C}}$ and the positions of the operators, which should be $\mathrm{e}^{2\pi\mathrm{i}j_1/n}, \mathrm{e}^{2\pi\mathrm{i}j_2/n}, \cdots$. The j's take values from $0, 1, \cdots, n-1$ with constraints that can be figured out easily. For example, $TT\mathcal{A}$ denotes the three-point functions on complex plane $\langle T(\mathrm{e}^{2\pi\mathrm{i}j_1/n})T(\mathrm{e}^{2\pi\mathrm{i}j_2/n})\mathcal{A}(\mathrm{e}^{2\pi\mathrm{i}j_3/n})\rangle_{\mathcal{C}}$ with $0 \leq j_{1,2,3} \leq n-1$ and the constraints $j_1 < j_2, j_1 \neq j_3, j_2 \neq j_3$. In the 3rd column we mark the answers to several questions. For the 1st question we mark $\checkmark$ if the multi-point functions are non-vanishing and we $\times$ if they are vanishing. The 2nd, 3rd, 4th and 5th questions are for the calculation of (5.2) in section 5, and they are about whether the multi-point functions are non-vanishing or vanishing after the inversion, respectively, of the operators T, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$. In the 4th and 5th columns are the numbers of the multi-point functions as counted by (3.17). Note that the table is similar to but different from table 2 which counts the CFT^n quasiprimary operators.

entropy and relative entropy firstly in expansion of small ℓ, and then in expansion of large c. Focusing on the order of large c, we have

$$\Delta S_A \sim \mathcal{O}(c^0),$$
$$\Delta S_n \sim \mathcal{O}(c),$$
$$S(\rho_{A,\phi}\|\rho_{A,\beta}) \sim \mathcal{O}(c^0), \tag{4.4}$$

and we assume that these orders still apply when $0 < \ell/L < 1$ is neither too small nor too large. From (2.16), (2.17), and (2.18) we get respectively

$$t \geq \frac{\mathcal{O}(c^0)}{\log d},$$
$$d \geq \mathrm{e}^{\mathcal{O}(c)},$$
$$t \leq \mathcal{O}(c^0). \tag{4.5}$$

From the first inequality of (4.5), the lower bound of the trace distance t, which is crucial for the validity of subsystem ETH, depends on if the effective dimension d of the subsystem A is strictly infinite or how it scales with c. It is a subtle issue to determine d for generic CFTs. We will raise this as an interesting issue for further study, but now consider some interesting scenarios.[3]

If d is strictly infinite, from (4.5) we get

$$0 \leq t \leq \mathcal{O}(c^0). \tag{4.6}$$

This yield no useful information to check the validity of subsystem ETH.

It is possible there exists a large but finite effective dimension d of the subsystem A that satisfies (2.16), (2.17) and so satisfies (4.5). The Cardy's formula [17] and Boltzmann's entropy formula $\Omega(E) \sim \mathrm{e}^{S(E)}$ state that the number of states at a specific high energy is $\mathrm{e}^{\mathcal{O}(c)}$. It is plausible that for both the reduced density matrices $\rho_{A,\phi}$ and $\rho_{A,\beta}$ only $\mathrm{e}^{\mathcal{O}(c)}$ components are nontrivial and other components are even smaller than exponential suppression. Then we get the tentative result

$$d \sim \mathrm{e}^{\mathcal{O}(c)}. \tag{4.7}$$

If this is true, from (4.5) we get

$$\mathcal{O}(c^{-1}) \leq t \leq \mathcal{O}(c^0). \tag{4.8}$$

This shows the violation of subsystem ETH (1.2), while the weak subsystem ETH (1.3) remains intact.

5   Relative entropy and distance between primary states

In this section we present some byproducts of this paper obtained by using the same method as the one in subsection 3.3 [22–28], which has also been used to calculate the relative entropy [24, 26–28]. We will calculate the relative entropy $S(\rho_{A,\phi}\|\rho_{A,\psi})$, the 2nd relative distance $d_2(\rho_{A,\phi}, \rho_{A,\psi})$, and the Schatten 2-distance $\|\rho_{A,\phi} - \rho_{A,\psi}\|_2$ between the reduced density matrices of two primary states $|\phi\rangle$ and $|\psi\rangle$ in the short interval expansion, where ψ is similar to ϕ and is the primary field of conformal weight $h_\psi = c\epsilon_\psi$.

---

[3]In [4], instead a similar proposal to our first inequality of (4.5) for subsystem ETH, i.e., $t \sim d\,\mathrm{e}^{-\frac{1}{2}S(E)}$ is used as the definition for the effective definition d.

5.1 Relative entropy

To calculate the relative entropy $S(\rho_{A,\phi}\|\rho_{A,\psi})$, we first need to calculate the "Rényi" relative entropy

$$S_n(\rho_{A,\phi}\|\rho_{A,\psi}) = \frac{1}{n-1}\big(\log \operatorname{tr}_A \rho_{A,\phi}^n - \log \operatorname{tr}_A(\rho_{A,\phi}\rho_{A,0}^{n-1})\big). \tag{5.1}$$

and then take $n \to 1$ limit.

We have already calculated $\operatorname{tr}_A \rho_{A,\phi}^n$ as illustrated in figure 3 of in subsection 3.3, and this inspires us to calculate $\operatorname{tr}_A(\rho_{A,\phi}\rho_{A,\psi}^{n-1})$ as illustrated in figure 4. Similar to the manipulation in subsection 3.3, in the end we can obtain the formal result

$$\frac{\operatorname{tr}_A(\rho_{A,\phi}\rho_{A,\psi}^{n-1})}{\operatorname{tr}_A\rho_{A,0}^n} = \langle \Psi_\phi(\infty)\Psi_\phi(0)\rangle_{\mathcal{C}^n} = \left(\frac{\sin\frac{\pi\ell}{L}}{n\sin\frac{\pi\ell}{nL}}\right)^{2(h_\phi+(n-1)h_\psi)} \left\langle \mathcal{F}_\phi\big(e^{2\pi i\frac{\ell}{nL}},1\big)\prod_{j=1}^{n-1}\mathcal{F}_\psi\big(e^{2\pi i(\frac{\ell}{nL}+\frac{j}{n})},e^{2\pi i\frac{j}{n}}\big)\right\rangle, \tag{5.2}$$

Here $\Psi_\phi \equiv \phi_0\prod_{j=1}^{n-1}\psi_j$, with ϕ existing in one copy and ψ existing in the other $n-1$ copies. The explicit result up to order $(\ell/L)^8$ is

$$
\begin{aligned}
S_n(\rho_{A,\phi}\|\rho_{A,\psi}) =\ & \frac{\pi^4 c(\epsilon_\phi - \epsilon_\psi)(n+1)(n^2+11)\ell^4}{45n^4 L^4}\big(n\epsilon_\phi + (n-2)\epsilon_\psi\big)\\
&+ \frac{\pi^6 c(\epsilon_\phi - \epsilon_\psi)(n+1)\ell^6}{2835 L^6 n^6}\big(8n(n^2-4)(n^2+47)\epsilon_\phi^2 + 8(n-3)(n^2-4)(n^2+47)\epsilon_\psi^2\\
&+ 8n(n^2-4)(n^2+47)\epsilon_\phi\epsilon_\psi + 3n(2n^4+9n^2+37)\epsilon_\phi + 3(n-2)(2n^4+9n^2+37)\epsilon_\psi\big)\\
&- \frac{\pi^8 c(\epsilon_\phi - \epsilon_\psi)(n+1)\ell^8}{28350(5c+22)n^8 L^8}\big[c\big(4n(13n^6 - 1647n^4 + 33927n^2 - 58213)\epsilon_\phi^3\\
&+ 4(n-2)(13n^6 + 40n^5 - 1567n^4 + 4400n^3 + 42727n^2 - 42840n - 143893)\epsilon_\psi^3\\
&+ 4n(13n^6 - 1647n^4 + 33927n^2 - 58213)\epsilon_\phi^2\epsilon_\psi\\
&+ 4(n-2)(13n^6 - 40n^5 - 1727n^4 - 4400n^3 + 25127n^2 + 42840n + 27467)\epsilon_\phi\epsilon_\psi^2\\
&- 4n(n^2+11)(13n^4 + 160n^2 - 533)\epsilon_\phi^2\\
&- 4(n-2)(n^2+11)(13n^4 - 10n^3 + 140n^2 - 190n - 913)\epsilon_\psi^2\\
&- 4(n^2+11)(13n^5 - 3n^4 + 160n^3 - 10n^2 - 533n - 227)\epsilon_\phi\epsilon_\psi\\
&- 3n(9n^6 + 29n^4 + 71n^2 + 251)\epsilon_\phi - 3(n-2)(9n^6 + 29n^4 + 71n^2 + 251)\epsilon_\psi\big)\\
&- 352n(n^2-9)(n^2-4)(n^2+119)\epsilon_\phi^3 - 352(n-4)(n^2-9)(n^2-4)(n^2+119)\epsilon_\psi^3\\
&- 352n(n^2-9)(n^2-4)(n^2+119)\epsilon_\phi^2\epsilon_\psi - 352n(n^2-9)(n^2-4)(n^2+119)\epsilon_\phi\epsilon_\psi^2\\
&- 176n(n^2-4)(n^2+11)(n^2+19)\epsilon_\phi^2 - 176(n-3)(n^2-4)(n^2+11)(n^2+19)\epsilon_\psi^2\\
&- 176n(n^2-4)(n^2+11)(n^2+19)\epsilon_\phi\epsilon_\psi - 8n(15n^6 + 50n^4 + 134n^2 + 539)\epsilon_\phi\\
&- 8(n-2)(15n^6 + 50n^4 + 134n^2 + 539)\epsilon_\psi\big] + O((\ell/L)^9).
\end{aligned}
\tag{5.3}
$$

By taking $n \to 1$ limit we get

$$
\begin{aligned}
S(\rho_{A,\phi}\|\rho_{A,\psi}) =\ & \frac{8\pi^4 c(\epsilon_\phi - \epsilon_\psi)^2\ell^4}{15 L^4} - \frac{32\pi^6 c(\epsilon_\phi - \epsilon_\psi)^2\big(8(\epsilon_\phi + 2\epsilon_\psi) - 1\big)\ell^6}{315 L^6}\\
&+ \frac{8\pi^8 c(\epsilon_\phi - \epsilon_\psi)^2\ell^8}{1575(5c+22)L^8}\big(5c(288\epsilon_\phi^2 + 1568\epsilon_\psi^2 + 576\epsilon_\phi\epsilon_\psi - 48\epsilon_\phi - 128\epsilon_\psi + 3)\\
&+ 2(7040\epsilon_\phi^2 + 21120\epsilon_\psi^2 + 14080\epsilon_\phi\epsilon_\psi - 880\epsilon_\phi - 1760\epsilon_\psi + 41)\big) + O((\ell/L)^9).
\end{aligned}
\tag{5.4}
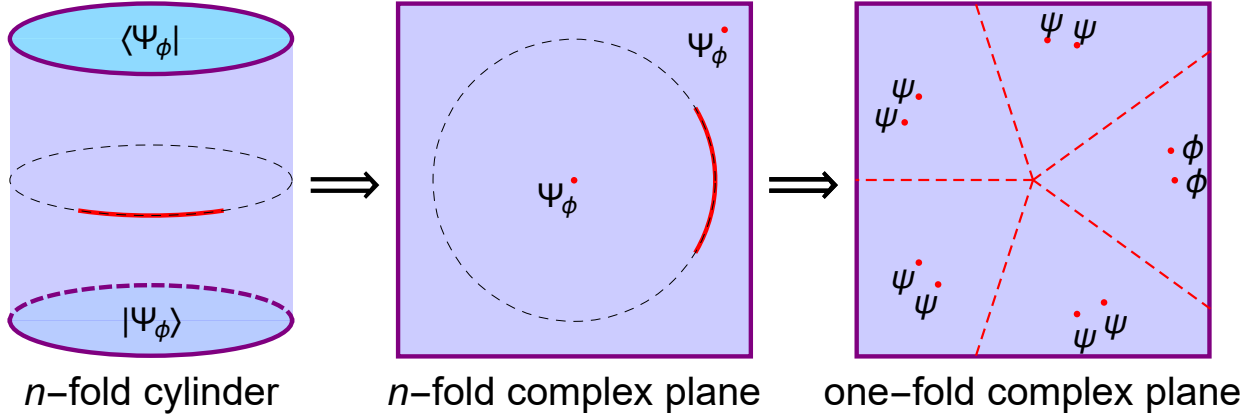$$

To order ℓ^6 the result is in accord with [26, 27].

Figure 4: The calculation of $\mathrm{tr}_A(\rho_{A,\phi}\rho_{A,\psi}^{n-1})$. Here $\Psi_\phi \equiv \phi_0 \prod_{j=1}^{n-1} \psi_j$, with ϕ existing in one copy and ψ existing in the other $n-1$ copies.

Using the above result we can obtain $S(\rho_{A,\psi}\|\rho_{A,\phi})$ by swapping ϵ_ϕ and ϵ_ψ in (5.4). After that we can get the symmetric distance

$$
d(\rho_{A,\phi}, \rho_{A,\psi}) = \frac{16\pi^4 c(\epsilon_\phi - \epsilon_\psi)^2 \ell^4}{15 L^4} - \frac{64\pi^6 c(\epsilon_\phi - \epsilon_\psi)^2(12(\epsilon_\phi + \epsilon_\psi) - 1)\ell^6}{315 L^6}
$$
$$
+ \frac{16\pi^8 c(\epsilon_\phi - \epsilon_\psi)^2 \ell^8}{1575(5c + 22)L^8} \big[5c(928(\epsilon_\phi^2 + \epsilon_\psi^2) + 576\epsilon_\phi\epsilon_\psi - 88(\epsilon_\phi + \epsilon_\psi) + 3)
$$
$$
+ 2\big(14080(\epsilon_\phi^2 + \epsilon_\psi^2) + 14080\epsilon_\phi\epsilon_\psi - 1320(\epsilon_\phi + \epsilon_\psi) + 41\big)\big] + O((\ell/L)^9). \tag{5.5}
$$

Note that if we take $\epsilon_\psi = 0$, we can obtain $S(\rho_{A,\phi}\|\rho_{A,0})$, $S(\rho_{A,0}\|\rho_{A,\phi})$ and $d(\rho_{A,\phi}, \rho_{A,0})$ which compare the excited state $|\phi\rangle$ and the CFT vacuum state $|0\rangle$. Moreover, all the above results show nontrivial subleading $1/c$ corrections at the order $(\ell/L)^8$.

5.2 Relative distances

The "Rényi" relative entropy for $n \neq 1$ is not positive definite so that it cannot used as the measure for the distance between two quantum states, similarly for the symmetrized quantity like

$$
d_n(\rho, \rho') = S_n(\rho\|\rho') + S_n(\rho'\|\rho), \tag{5.6}
$$

except for $n = 1$ as done in the previous subsection. However, it turns out the 2nd Rényi distance $d_2(\rho, \rho')$ is positive definite because it can be rewritten as

$$
d_2(\rho, \rho') = \log \frac{\mathrm{tr}\rho^2 \mathrm{tr}\rho'^2}{[\mathrm{tr}(\rho\rho')]^2}. \tag{5.7}
$$

Thus, $d_2(\rho, \rho')$ can be used as a distance measure between two quantum states. Note that one requires $\mathrm{tr}(\rho\rho') \neq 0$ for the 2nd Rényi distance $d_2(\rho, \rho')$ to be well-defined. Also $d_2(\rho, \rho')$ may be vanishing for two different density matrices that can be diagonalized simultaneously.

More general, one can also use Schatten n-norm to measure the distance between two quantum states, i.e., called Schatten n-distance

$$
\|\rho - \rho'\|_n = [\mathrm{tr}(|\rho - \rho'|^n)]^{1/n}, \tag{5.8}
$$

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For $n = 1$ it is just the trace distance, and for $n = 2$ we have

$$\|\rho - \rho'\|_2 = [\mathrm{tr}\rho^2 + \mathrm{tr}\rho'^2 - 2\mathrm{tr}(\rho\rho')]^{1/2}, \tag{5.9}$$

Below we will calculate both $d_2(\rho_{A,\phi}, \rho_{A,\psi})$ and $\|\rho_{A,\phi} - \rho_{A,\psi}\|_2$ by following the similar trick used in the previous subsection. In fact, the ingredients needed to carry out the calculations such as $\frac{\mathrm{tr}\rho_{A,\phi}^2}{\mathrm{tr}\rho_{A,0}^2}$ and $\frac{\mathrm{tr}(\rho_{A,\phi}\rho_{A,\psi})}{\mathrm{tr}\rho_{A,0}^2}$, have all been done already in previous sections. Packing them up, we then obtain the formal results as follows:

$$
d_2(\rho_{A,\phi}, \rho_{A,\psi}) = \log \left\{ \left[\sum_{\mathcal{X}} \frac{C_{\phi\phi\mathcal{X}}^2}{\alpha_{\mathcal{X}}} \left(\sin \frac{\pi\ell}{2L} \right)^{2h_{\mathcal{X}}} {}_2F_1(h_{\mathcal{X}}, h_{\mathcal{X}}; 2h_{\mathcal{X}}; \sin^2 \frac{\pi\ell}{2L}) \right] \right.
$$
$$
\times \left[\sum_{\mathcal{X}} \frac{C_{\psi\psi\mathcal{X}}^2}{\alpha_{\mathcal{X}}} \left(\sin \frac{\pi\ell}{2L} \right)^{2h_{\mathcal{X}}} {}_2F_1(h_{\mathcal{X}}, h_{\mathcal{X}}; 2h_{\mathcal{X}}; \sin^2 \frac{\pi\ell}{2L}) \right]
$$
$$
\left. \div \left[\sum_{\mathcal{X}} \frac{C_{\phi\phi\mathcal{X}} C_{\psi\psi\mathcal{X}}}{\alpha_{\mathcal{X}}} \left(\sin \frac{\pi\ell}{2L} \right)^{2h_{\mathcal{X}}} {}_2F_1(h_{\mathcal{X}}, h_{\mathcal{X}}; 2h_{\mathcal{X}}; \sin^2 \frac{\pi\ell}{2L}) \right]^2 \right\}, \tag{5.10}
$$

$$
\|\rho_{A,\phi} - \rho_{A,\psi}\|_2 = \left(\frac{L}{\pi\epsilon} \sin \frac{\pi\ell}{L} \right)^{-c/16} \left\{ \left(\frac{\sin \frac{\pi\ell}{L}}{2 \sin \frac{\pi\ell}{2L}} \right)^{4h_{\phi}} \left[\sum_{\mathcal{X}} \frac{C_{\phi\phi\mathcal{X}}^2}{\alpha_{\mathcal{X}}} \left(\sin \frac{\pi\ell}{2L} \right)^{2h_{\mathcal{X}}} {}_2F_1(h_{\mathcal{X}}, h_{\mathcal{X}}; 2h_{\mathcal{X}}; \sin^2 \frac{\pi\ell}{2L}) \right] \right.
$$
$$
+ \left(\frac{\sin \frac{\pi\ell}{L}}{2 \sin \frac{\pi\ell}{2L}} \right)^{4h_{\psi}} \left[\sum_{\mathcal{X}} \frac{C_{\psi\psi\mathcal{X}}^2}{\alpha_{\mathcal{X}}} \left(\sin \frac{\pi\ell}{2L} \right)^{2h_{\mathcal{X}}} {}_2F_1(h_{\mathcal{X}}, h_{\mathcal{X}}; 2h_{\mathcal{X}}; \sin^2 \frac{\pi\ell}{2L}) \right] \tag{5.11}
$$
$$
\left. - 2 \left(\frac{\sin \frac{\pi\ell}{L}}{2 \sin \frac{\pi\ell}{2L}} \right)^{2(h_{\phi}+h_{\psi})} \left[\sum_{\mathcal{X}} \frac{C_{\phi\phi\mathcal{X}} C_{\psi\psi\mathcal{X}}}{\alpha_{\mathcal{X}}} \left(\sin \frac{\pi\ell}{2L} \right)^{2h_{\mathcal{X}}} {}_2F_1(h_{\mathcal{X}}, h_{\mathcal{X}}; 2h_{\mathcal{X}}; \sin^2 \frac{\pi\ell}{2L}) \right] \right\}^{1/2}.
$$

Then, in the short interval expansion we obtain the explicit results as follows:

$$
d_2(\rho_{A,\phi}, \rho_{A,\psi}) = \frac{\pi^4 c (\epsilon_{\phi} - \epsilon_{\psi})^2 \ell^4}{8L^4} \left[1 + \frac{\pi^2 \ell^2}{12L^2} + \frac{\pi^4 (5c + 24 - 20c(\epsilon_{\phi} + \epsilon_{\psi})(11\epsilon_{\phi} + 11\epsilon_{\psi} - 1))\ell^4}{160(5c + 22)L^4} \right.
$$
$$
\left. + \frac{\pi^6 (145c + 764 - 1260c(\epsilon_{\phi} + \epsilon_{\psi})(11\epsilon_{\phi} + 11\epsilon_{\psi} - 1))\ell^6}{60480(5c + 22)L^6} + O((\ell/L)^8) \right], \tag{5.12}
$$
$$
\|\rho_{A,\phi} - \rho_{A,\psi}\|_2 = \frac{\pi^2 \sqrt{c(c+2)} (\epsilon_{\phi} - \epsilon_{\psi})^2 \ell^2}{4L^2} \left(\frac{\ell}{\epsilon} \right)^{-c/16} \left[1 + \frac{\pi^2 (c+4)(c+2-12c(\epsilon_{\phi}+\epsilon_{\psi}))\ell^2}{96(c+2)L^2} + O((\ell/L)^4) \right].
$$

We see that both distances have nontrivial large c corrections though with different structures.

6 Conclusion and discussion

ETH is a fundamental issue in quantum thermodynamics and its validity for various situation should be scrutinized. An interesting version of ETH was proposed very recently in [3, 4], the so-called subsystem ETH which requires the difference between high energy state and thermal state over all a local region should be exponentially suppressed by the entropy of the total system. To be precise, the trace distance of the reduced density matrices should be exponentially suppressed. The subsystem ETH can be relaxed in a weak form that only requires each competent of reduced density matrix is different by an exponentially suppressed term.

In this paper we would like to check the validity of subsystem ETH for 2D large c CFT. We evaluate the Rényi entropy, entanglement entropy, and relative entropy of the reduced density matrices to measure the difference between thermal state and heavy primary state. We use these results and Fannes-Audenaert inequality to get the lower bound for the trace distance in large c limit, and find

$$t \geq \frac{\mathcal{O}(c^0)}{\log d}. \tag{6.1}$$

The bound then depends on how the effective dimension d of the subsystem scales with c, which is subtle to determine. Instead of using the relation (6.1) as a definition of d, see, e.g., [4], we treat it as an open issue and consider the possible interesting scenarios. One of these is that d satisfies Fannes-Audenaert inequality and at the same time yields nontrivial lower bound of t. It is plausible that

$$d \sim \mathrm{e}^{\mathcal{O}(c)}. \tag{6.2}$$

If this is true the trace distance would be at most power suppressed and the subsystem ETH would be violated for the 2D large central CFT.

We have to say we do not have a concrete proof that the result (6.2) is correct. The validity of the subsystem ETH really depends only on whether the large but finite effective dimension exists and if yes how it scales with the large central charge. It is an open question, and it is possible one has to calculate the trace distance explicitly to find the answer.

As the 2D large c CFT is holographically dual to the AdS$_3$ gravity theory, it would be interesting to understand the infinite subleading terms of large central charge to the Rényi entropy and entanglement entropy. It is also interesting to understand the bulk story about the possible subtle violation of subsystem ETH. We hope to revisit this problem in the future.

Acknowledgments

A   Some details of vacuum conformal family

We list the holomorphic quasiprimary operators in vacuum conformal family to level 8. In level 2, we have the quasiprimary operator T, with the usual normalization $\alpha_T = \frac{c}{2}$. In level 4, we have

$$\mathcal{A} = (TT) - \frac{3}{10}\partial^2 T, \quad \alpha_{\mathcal{A}} = \frac{c(5c+22)}{10}. \tag{A.1}$$

In level 6, we have the orthogonalized quasiprimary operators

$$\mathcal{B} = (\partial T \partial T) - \frac{4}{5}(\partial^2 TT) - \frac{1}{42}\partial^4 T, \quad \mathcal{D} = \mathcal{C} + \frac{93}{70c+29}\mathcal{B}, \tag{A.2}$$

with the definition

$$\mathcal{C} = (T(TT)) - \frac{9}{10}(\partial^2 TT) - \frac{1}{28}\partial^4 T, \tag{A.3}$$

and the normalization factors

$$\alpha_{\mathcal{B}} = \frac{36c(70c+29)}{175}, \quad \alpha_{\mathcal{D}} = \frac{3c(2c-1)(5c+22)(7c+68)}{4(70c+29)}. \tag{A.4}$$

In level 8 we have the orthogonalized quasiprimary operators

$$\mathcal{E} = (\partial^2 T \partial^2 T) - \frac{10}{9}(\partial^3 T \partial T) + \frac{10}{63}(\partial^4 T T) - \frac{1}{324}\partial^6 T,$$

$$\mathcal{H} = \mathcal{F} + \frac{9(140c+83)}{50(105c+11)}\mathcal{E}, \tag{A.5}$$

$$\mathcal{I} = \mathcal{G} + \frac{81(35c-51)}{100(105c+11)}\mathcal{E} + \frac{12(465c-127)}{5c(210c+661)-251}\mathcal{H},$$

with the definitions

$$\mathcal{F} = (\partial T (\partial T T)) - \frac{4}{5}(\partial^2 T (T T)) + \frac{2}{15}(\partial^3 T \partial T) - \frac{3}{70}(\partial^4 T T), \tag{A.6}$$

$$\mathcal{G} = (T(T(TT))) - \frac{9}{5}(\partial^2 T (TT)) + \frac{3}{10}(\partial^3 T \partial T) - \frac{13}{70}(\partial^4 T T) - \frac{1}{240}\partial^6 T,$$

and the normalization factors

$$\alpha_{\mathcal{E}} = \frac{22880c(105c+11)}{1323},$$

$$\alpha_{\mathcal{H}} = \frac{26c(5c+22)(5c(210c+661)-251)}{125(105c+11)}, \tag{A.7}$$

$$\alpha_{\mathcal{I}} = \frac{3c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)}{2(5c(210c+661)-251)}.$$

In this paper we need the structure constants

$$C_{TTT} = c, \quad C_{TT\mathcal{A}} = \frac{c(5c+22)}{10}, \tag{A.8}$$

and the four-point function on complex plane

$$\langle T(z_1)T(z_2)T(z_3)T(z_4)\rangle_{\mathcal{C}} = \frac{c^2}{4}\left(\frac{1}{(z_{12}z_{34})^4} + \frac{1}{(z_{13}z_{24})^4} + \frac{1}{(z_{14}z_{23})^4}\right) + c\frac{(z_{12}z_{34})^2 + (z_{13}z_{24})^2 + (z_{14}z_{23})^2}{(z_{12}z_{34}z_{13}z_{24}z_{14}z_{23})^2}, \tag{A.9}$$

with the definitions $z_{ij} \equiv z_i - z_j$.

Under a general coordinate transformation $z \to f(z)$, we have the transformation rules

$$T(z) = f'^2 T(f) + \frac{c}{12}s, \quad \mathcal{A}(z) = f'^4 \mathcal{A}(f) + \frac{5c+22}{30}s\left(f'^2 T(f) + \frac{c}{24}s\right),$$

$$\mathcal{B}(z) = f'^6 \mathcal{B}(f) - \frac{8}{5}sf'^4 \mathcal{A}(f) - \frac{70c+29}{1050}sf'^4 \partial^2 T(f) + \frac{70c+29}{420}f'^2(f's' - 2f''s)\partial T(f)$$

$$- \frac{1}{1050}\left(28(5c+22)f'^2 s^2 + (70c+29)(f'^2 s'' - 5f'f''s' + 5f''^2 s)\right)T(f)$$

$$- \frac{c}{50400}\left(744s^3 + (70c+29)(4ss'' - 5s'^2)\right),$$

$$\mathcal{D}(z) = f'^6 \mathcal{D}(f) + \frac{(2c-1)(7c+68)}{70c+29}s\left(\frac{5}{4}f'^4 \mathcal{A}(f) + \frac{5c+22}{48}s\left(f'^2 T(f) + \frac{c}{36}s\right)\right),$$

$$\mathcal{E}(z) = f'^8 \mathcal{E}(f) - \frac{4510}{567}sf'^6 \mathcal{B}(f) + \frac{50}{567}sf'^6 \partial^2 \mathcal{A}(f) - \frac{25}{63}f'^4(f's' - 2sf'')\partial \mathcal{A}(f)$$

$$+ \frac{4}{63}f'^2(25sf''^2 + 5f'^2 s'' + 98s^2 f'^2 - 25f'f''s')\mathcal{A}(f) + \frac{105c+11}{7938}sf'^6 \partial^4 T(f)$$

$$- \frac{105c+11}{1134}f'^4(f's' - 2sf'')\partial^3 T(f) + \frac{1}{5670}f'^2\left(9(105c+11)(f'^2 s'' + 5sf''^2 - 5f'f''s')\right)$$

$$+ 10(120c+77)s^2 f'^2\right)\partial^2 T(f) - \frac{1}{2268}\left((105c+11)(2s^{(3)}f'^3 - 30sf''^3 - 18f'^2 f''s'' + 45f'f''^2 s')\right)$$

$$+ 10(120c+77)sf'^2(f's' - 2sf'')\right)\partial T(f) + \frac{1}{79380f'^2}\left(8(3570c+2629)sf'^4 s''\right)$$

$$- 5(2940c + 2563)f'^4 s'^2 + 12(1225c + 9449)s^3 f'^4 + 700(120c + 77)sf'^2 f''(sf'' - f's')$$

$$+ 5(105c + 11)(105sf''^4 + 2s^{(4)}f'^4 - 28s^{(3)}f'^3 f'' + 126f'^2 f''^2 s'' - 210f'f''^3 s'))T(f)$$

$$+ \frac{c}{952560}\big((105c + 11)(10ss^{(4)} + 63s''^2 - 70s^{(3)}s') + 451s(20ss'' - 25s'^2 + 52s^3)\big), \qquad \text{(A.10)}$$

$$\mathcal{H}(z) = f'^8 \mathcal{H}(f) - \frac{8}{5}sf'^6 \mathcal{D}(f) + \frac{91(5c + 22)(5c(210c + 661) - 251)}{540(70c + 29)(105c + 11)}sf'^6 \mathcal{B}(f)$$

$$- \frac{5c(210c + 661) - 251}{540(105c + 11)}sf'^6 \partial^2 \mathcal{A}(f) + \frac{5c(210c + 661) - 251}{120(105c + 11)}f'^4(f's' - 2sf'')\partial \mathcal{A}(f)$$

$$+ \frac{1}{150(105c + 11)}f'^2\big((5c(210c + 661) - 251)(5f'f''s' - 5sf''^2 - f'^2 s'')$$

$$- 8(25c(21c + 187) - 951)s^2 f'^2\big)\mathcal{A}(f) - \frac{(5c + 22)(5c(210c + 661) - 251)}{9000(105c + 11)}s^2 f'^4 \partial^2 T(f)$$

$$+ \frac{(5c + 22)(5c(210c + 661) - 251)}{3600(105c + 11)}sf'^2(f's' - 2sf'')\partial T(f)$$

$$+ \frac{5c + 22}{108000(105c + 11)}\big(3(5c(210c + 661) - 251)(-20s^2 f''^2 - 8sf'^2 s'' + 5f'^2 s'^2 + 20sf'f''s')$$

$$- 8(15c(210c + 2273) - 7357)s^3 f'^2)T(f) - \frac{c(5c + 22)s}{1296000(105c + 11)}(104(465c - 127)s^3$$

$$+ 3(5c(210c + 661) - 251)(4ss'' - 5s'^2)),$$

$$\mathcal{I}(z) = f'^8 \mathcal{I}(f) + \frac{(3c + 46)(5c + 3)(70c + 29)}{3(5c(210c + 661) - 251)}s\Big(\mathcal{D}(f)f'^6 + \frac{5(2c - 1)(7c + 68)}{8(70c + 29)}s\Big(f'^4 \mathcal{A}(f)$$

$$+ \frac{5c + 22}{90}s\Big(f'^2 T(f) + \frac{c}{48}s\Big)\Big)\Big).$$

In the above equations, we have the definition of Schwarzian derivative

$$s(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \qquad \text{(A.11)}$$

and the shorthand notations

$$f' \equiv f'(z), \quad f'' \equiv f''(z), \quad s \equiv s(z), \quad s' \equiv s'(z), \quad s'' \equiv s''(z), \quad s^{(3)} \equiv s^{(3)}(z), \quad \cdots. \qquad \text{(A.12)}$$

For a general primary operator ϕ with conformal weight h_ϕ and normalization factor $\alpha_\phi = 1$, we have the structure constants

$$C_{\phi\phi T} = h_\phi, \quad C_{\phi\phi\mathcal{A}} = \frac{h_\phi(5h_\phi + 1)}{5}, \quad C_{\phi\phi\mathcal{B}} = -\frac{2h_\phi(14h_\phi + 1)}{35},$$

$$C_{\phi\phi\mathcal{D}} = \frac{h_\phi[(70c + 29)h_\phi^2 + (42c - 57)h_\phi + (8c - 2)]}{70c + 29}, \quad C_{\phi\phi\mathcal{E}} = \frac{4h_\phi(27h_\phi + 1)}{63},$$

$$C_{\phi\phi\mathcal{H}} = -\frac{2h_\phi\big(10(105c + 11)h_\phi^2 + (435c - 218)h_\phi + 55c - 4\big)}{25(105c + 11)}, \qquad \text{(A.13)}$$

$$C_{\phi\phi\mathcal{I}} = \frac{h_\phi}{5c(210c + 661) - 251}\big((5c(210c + 661) - 251)h_\phi^3 + 6(c(210c - 83) + 153)h_\phi^2$$

$$+ (c(606c - 701) - 829)h_\phi + 6c(18c + 13) - 6\big).$$

For a general holomorphic quasiprimary operator $\mathcal{X}$, the non-vanishing of $C_{\phi\phi\mathcal{X}}$ require that $\mathcal{X}$ is bosonic and its conformal dimension $h_\mathcal{X}$ is an even integer, and this leads to $C_{\phi\phi\mathcal{X}} = C_{\phi\mathcal{X}\phi}$. We have the three-point function on complex plane

$$\langle\phi(\infty)\partial^r \mathcal{X}(z)\phi(0)\rangle_\mathcal{C} = \frac{(-)^r(h_\mathcal{X} + r - 1)!}{(h_\mathcal{X} - 1)!}\frac{C_{\phi\phi\mathcal{X}}}{z^{h_\mathcal{X} + r}}. \qquad \text{(A.14)}$$

For a general operator $\mathcal{X}$, we denote its expectation value on a cylinder with spatial period L in excited state $|\phi\rangle$ as $\langle\mathcal{X}\rangle_\phi = \langle\phi|\mathcal{X}|\phi\rangle_{\text{cyl}}$. From translation symmetry in both directions of the cylinder, we know that $\langle\mathcal{X}\rangle_\phi$ is a constant. So for $r \in \mathbb{Z}$ and $r > 0$, we have

$$\langle\partial^r\mathcal{X}\rangle_\phi = 0. \tag{A.15}$$

By mapping the cylinder to a complex plane, using (A.10) and (A.13) we get the expectation values

$$\langle T\rangle_\phi = \frac{\pi^2(c-24h_\phi)}{6L^2}, \quad \langle\mathcal{A}\rangle_\phi = \frac{\pi^4(c(5c+22)-240(c+2)h_\phi+2880h_\phi^2)}{180L^4}, \quad \langle\mathcal{B}\rangle_\phi = -\frac{2\pi^6(31c-504h_\phi)}{525L^6},$$

$$\langle\mathcal{D}\rangle_\phi = \frac{\pi^6}{216(70c+29)L^6}\big(c(2c-1)(5c+22)(7c+68)-72(70c^3+617c^2+938c-248)h_\phi$$
$$+1728(c+4)(70c+29)h_\phi^2-13824(70c+29)h_\phi^3\big),$$

$$\langle\mathcal{E}\rangle_\phi = \frac{572\pi^8(41c-480h_\phi)}{59535L^8}, \tag{A.16}$$

$$\langle\mathcal{H}\rangle_\phi = -\frac{13\pi^8}{10125(105c+11)L^8}\big(c(5c+22)(465c-127)-480(195c^2+479c-44)h_\phi+8640(105c+11)h_\phi^2\big),$$

$$\langle\mathcal{I}\rangle_\phi = \frac{\pi^8}{1296(1050c^2+3305c-251)L^8}\big(c(2c-1)(3c+46)(5c+3)(5c+22)(7c+68)$$
$$-96(1050c^5+23465c^4+153901c^3+274132c^2+22388c-6864)h_\phi$$
$$+3456(1050c^4+16325c^3+69963c^2+65686c-648)h_\phi^2$$
$$-55296(c+6)(1050c^2+3305c-251)h_\phi^3+331776(1050c^2+3305c-251)h_\phi^4\big).$$

B OPE of twist operators

We review OPE of twist operators in the n-fold CFT that is denoted as CFT^n [19–21, 41]. We also define and calculate $C_{\Phi\Phi K}$ that would be useful to subsection 3.2, $\langle\Phi_K\rangle_\Phi$ that would be useful to subsection 3.1, as well as b_K that would be useful to both subsection 3.1 and subsection 3.2. Note that in this paper we only consider contributions of the holomorphic sector. The twist operators σ and $\tilde{\sigma}$ are primary operators with conformal weights [18]

$$h_\sigma = h_{\tilde{\sigma}} = \frac{c(n^2-1)}{24n}. \tag{B.1}$$

We have the OPE of twist operators [19–21]

$$\sigma(z)\tilde{\sigma}(w) = \frac{c_n}{(z-w)^{2h_\sigma}}\sum_K d_K \sum_{r=0}^{\infty}\frac{a_K^r}{r!}(z-w)^{h_K+r}\partial^r\Phi_K(w). \tag{B.2}$$

Here c_n is the normalization factor. The summation K is over each orthogonalized holomorphic quasiprimary operator Φ_K in CFT^n, and h_K is the conformal weight of Φ_K. We have definition

$$a_K^r \equiv \frac{C_{h_K+r-1}^r}{C_{2h_K+r-1}^r}, \tag{B.3}$$

with C_x^y denoting the binomial coefficient. To level 8, the CFT^n holomorphic quasiprimary operators has been constructed in [41], and we just list them in table 2. The normalization factors α_K and OPE coefficients d_K for all these quasiprimary operators can also be found in [41].

From a holomorphic primary operator ϕ with normalization $\alpha_\phi = 1$ in the original CFT, we can define the CFT^n primary operator

$$\Phi = \prod_{j=0}^{n-1}\phi_j. \tag{B.4}$$

level	operator	level	operator	level	operator	level	operator
0	1		$\mathcal{B}, \mathcal{D}$	7	$\mathcal{M}$		$\mathcal{O}$
2	T		$T\mathcal{A}$		$T\mathcal{J}, \mathcal{N}$		$\mathcal{P}$
4	$\mathcal{A}$	6	$\mathcal{K}$		$\mathcal{E}, \mathcal{H}, \mathcal{I}$	8	$TT\mathcal{A}$
	TT		TTT	8	$T\mathcal{B}, T\mathcal{D}$		$T\mathcal{K}, \mathcal{Q}, \mathcal{R}$
5	$\mathcal{J}$	7	$\mathcal{L}$		$\mathcal{A}\mathcal{A}$		$TTTT$

Table 2: To level 8, the CFT^n holomorphic quasiprimary operators. We have omitted the replica indices and their constraints. The definitions of $\mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}$ can be found in [41], and the normalization factors α_K and OPE coefficients d_K for all these quasiprimary operators can also be found therein.

In subsection 3.2, we need structure constant $C_{\Phi\Phi K}$ for the quasiprimary operators Φ_K in table 2. The results can be written in terms of (A.13). First of all it is easy to see

$$
\begin{aligned}
& C_{\Phi\Phi T} = C_{\phi\phi T}, \quad C_{\Phi\Phi\mathcal{A}} = C_{\phi\phi\mathcal{A}}, \quad C_{\Phi\Phi\mathcal{B}} = C_{\phi\phi\mathcal{B}}, \quad C_{\Phi\Phi\mathcal{D}} = C_{\phi\phi\mathcal{D}}, \\
& C_{\Phi\Phi\mathcal{E}} = C_{\phi\phi\mathcal{E}}, \quad C_{\Phi\Phi\mathcal{H}} = C_{\phi\phi\mathcal{H}}, \quad C_{\Phi\Phi\mathcal{I}} = C_{\phi\phi\mathcal{I}}, \quad C_{\Phi\Phi TT} = C_{\phi\phi T}^2, \\
& C_{\Phi\Phi T\mathcal{A}} = C_{\phi\phi T}C_{\phi\phi\mathcal{A}}, \quad C_{\Phi\Phi T\mathcal{B}} = C_{\phi\phi T}C_{\phi\phi\mathcal{B}}, \quad C_{\Phi\Phi T\mathcal{D}} = C_{\phi\phi T}C_{\phi\phi\mathcal{D}}, \\
& C_{\Phi\Phi\mathcal{A}\mathcal{A}} = C_{\phi\phi\mathcal{A}}^2, \quad C_{\Phi\Phi TTT} = C_{\phi\phi T}^3, \quad C_{\Phi\Phi TT\mathcal{A}} = C_{\phi\phi T}^2 C_{\phi\phi\mathcal{A}}, \quad C_{\Phi\Phi TTTT} = C_{\phi\phi T}^4.
\end{aligned} \tag{B.5}
$$

There are vanishing structure constants

$$
C_{\Phi\Phi\mathcal{J}} = C_{\Phi\Phi\mathcal{L}} = C_{\Phi\Phi\mathcal{M}} = C_{\Phi\Phi T\mathcal{J}} = C_{\Phi\Phi\mathcal{N}} = 0. \tag{B.6}
$$

For $\mathcal{K}, \mathcal{O},$ and $\mathcal{P}$ we have

$$
C_{\Phi\Phi\mathcal{K}} = -\frac{4}{5}C_{\phi\phi T}^2, \quad C_{\Phi\Phi\mathcal{O}} = -\frac{56}{45}C_{\phi\phi T}C_{\phi\phi\mathcal{A}}, \quad C_{\Phi\Phi\mathcal{P}} = \frac{12}{7}C_{\phi\phi T}^2. \tag{B.7}
$$

Finally we have

$$
C_{\Phi\Phi T\mathcal{K}} = C_{\phi\phi T}C_{\phi\phi\mathcal{K}}, \quad C_{\Phi\Phi\mathcal{Q}} = \frac{7}{9}C_{\Phi\Phi T\mathcal{K}}, \quad C_{\Phi\Phi\mathcal{R}} = \frac{7}{11}C_{\Phi\Phi T\mathcal{K}}. \tag{B.8}
$$

It is easy to get $\langle\Phi_K\rangle_\Phi$ that appear in (3.2) in terms of (A.16)

$$
\begin{aligned}
& \langle T\rangle_\Phi = \langle T\rangle_\phi, \quad \langle\mathcal{A}\rangle_\Phi = \langle\mathcal{A}\rangle_\phi, \quad \langle\mathcal{B}\rangle_\Phi = \langle\mathcal{B}\rangle_\phi, \quad \langle\mathcal{D}\rangle_\Phi = \langle\mathcal{D}\rangle_\phi, \quad \langle\mathcal{E}\rangle_\Phi = \langle\mathcal{E}\rangle_\phi, \quad \langle\mathcal{H}\rangle_\Phi = \langle\mathcal{H}\rangle_\phi, \\
& \langle\mathcal{I}\rangle_\Phi = \langle\mathcal{I}\rangle_\phi, \quad \langle TT\rangle_\Phi = \langle T\rangle_\phi^2, \quad \langle T\mathcal{A}\rangle_\Phi = \langle T\rangle_\phi\langle\mathcal{A}\rangle_\phi, \quad \langle T\mathcal{B}\rangle_\Phi = \langle T\rangle_\phi\langle\mathcal{B}\rangle_\phi, \quad \langle T\mathcal{D}\rangle_\Phi = \langle T\rangle_\phi\langle\mathcal{D}\rangle_\phi, \\
& \langle\mathcal{A}\mathcal{A}\rangle_\Phi = \langle\mathcal{A}\rangle_\phi^2, \quad \langle TTT\rangle_\Phi = \langle T\rangle_\phi^3, \quad \langle TT\mathcal{A}\rangle_\Phi = \langle T\rangle_\phi^2\langle\mathcal{A}\rangle_\phi, \quad \langle TTTT\rangle_\Phi = \langle T\rangle_\phi^4.
\end{aligned} \tag{B.9}
$$

Because of (A.15) we have the vanishing results

$$
\langle\mathcal{J}\rangle_\Phi = \langle\mathcal{K}\rangle_\Phi = \langle\mathcal{L}\rangle_\Phi = \langle\mathcal{M}\rangle_\Phi = \langle\mathcal{N}\rangle_\Phi = \langle\mathcal{O}\rangle_\Phi = \langle\mathcal{P}\rangle_\Phi = \langle\mathcal{Q}\rangle_\Phi = \langle\mathcal{R}\rangle_\Phi = \langle T\mathcal{J}\rangle_\Phi = \langle T\mathcal{K}\rangle_\Phi = 0. \tag{B.10}
$$

From OPE coefficient d_K for quasiprimary operators in table 2 we may define b_K by summing over the indices of Φ_K

$$
b_K = \sum_{j_1,\cdots} d_K^{j_1\cdots}. \tag{B.11}
$$

For examples, in table 2 T denotes operators T_j with $0 \leq j \leq n-1$, and $TT\mathcal{A}$ denotes operators $T_{j_1}T_{j_2}\mathcal{A}_{j_3}$ with $0 \leq j_{1,2,3} \leq n-1$ and the constraints $j_1 < j_2$, $j_1 \neq j_3$, $j_2 \neq j_3$, and so we have

$$b_T = \sum_{j=0}^{n-1} d_T = nd_T, \tag{B.12}$$

and

$$b_{TT\mathcal{A}} = \sum_{0 \leq j_{1,2,3} \leq n-1} d_{TT\mathcal{A}}^{j_1 j_2 j_3} \text{ with constraints } j_1 < j_2, \; j_1 \neq j_3, \; j_2 \neq j_3. \tag{B.13}$$

Using the results of d_K and the summation formulas in [41] we get the b_K we need. In subsection 3.1 we need

$$b_T = \frac{n^2-1}{12n}, \quad b_{\mathcal{A}} = \frac{(n^2-1)^2}{288n^3}, \quad b_{\mathcal{B}} = -\frac{(n^2-1)^2(70cn^2+122n^2-93)}{10368(70c+29)n^5},$$

$$b_{\mathcal{D}} = \frac{(n^2-1)^3}{10368n^5}, \quad b_{\mathcal{E}} = \frac{(n^2-1)^2(11340cn^4+11561n^4-16236n^2+5863)}{65894400(105c+11)n^7},$$

$$b_{\mathcal{H}} = -\frac{(n^2-1)^3(3150c^2n^2+c(15960n^2-6045)-2404n^2+1651)}{539136(5c(210c+661)-251)n^7}, \quad b_{\mathcal{I}} = \frac{(n^2-1)^4}{497664n^7},$$

$$b_{TT} = \frac{(n^2-1)(5c(n+1)(n-1)^2+2n^2+22)}{1440cn^3}, \quad b_{T\mathcal{A}} = \frac{(n^2-1)^2(5c(n+1)(n-1)^2+4n^2+44)}{17280cn^5},$$

$$b_{T\mathcal{B}} = -\frac{(n^2-1)^2}{13063680c(70c+29)n^7}\big(7350n^2c^2(n-1)^2(n+1)$$
$$+ 35c(366n^5-238n^4-645n^3+2369n^2+279n-403)+2(6787n^4+71089n^2-65348)\big),$$

$$b_{T\mathcal{D}} = \frac{(n^2-1)^3(5c(n+1)(n-1)^2+6n^2+66)}{622080cn^7}, \tag{B.14}$$

$$b_{\mathcal{AA}} = \frac{1}{5806080c(5c+22)n^7}(175c^2(n+1)^4(n-1)^5$$
$$+ 70c(n^2-1)^3(11n^3-7n^2-11n+55)+8(n^2-1)(n^2+11)(157n^4-298n^2+381)),$$

$$b_{TTT} = \frac{(n-2)(n^2-1)}{362880c^2n^5}\big(35c^2(n+1)^2(n-1)^3+42c(n^4+10n^2-11)-16(n+2)(n^2+47)\big),$$

$$b_{TT\mathcal{A}} = \frac{(n-2)(n^2-1)}{14515200c^2n^7}\big(175c^2(n+1)^3(n-1)^4+350c(n^2-1)^2(n^2+11)$$
$$- 128(n+2)(n^4+50n^2-111)\big),$$

$$b_{TTTT} = \frac{(n-3)(n-2)(n^2-1)}{87091200c^3n^7}\big(175c^3(n+1)^3(n-1)^4+420c^2(n^2-1)^2(n^2+11)$$
$$- 4c(59n^5+121n^4+3170n^3+6550n^2-6829n-11711)+192(n+2)(n+3)(n^2+119)\big).$$

In subsection 3.2, we further need

$$b_{\mathcal{K}} = -\frac{(n^2-1)(70c(n-1)^2(n+1)n^2-2n^4+215n^2-93)}{725760cn^5},$$

$$b_{\mathcal{O}} = -\frac{(n^2-1)^2(210c(n-1)^2(n+1)n^2+38n^4+1445n^2-403)}{37739520cn^7},$$

$$b_{\mathcal{P}} = \frac{(n^2-1)(11340c(n-1)^2(n+1)n^4-1481n^6+27797n^4-22099n^2+5863)}{6918912000cn^7}, \tag{B.15}$$

$$b_{T\mathcal{K}} + \frac{7}{9}b_{\mathcal{Q}} + \frac{7}{11}b_{\mathcal{R}} = -\frac{(n-2)(n^2-1)}{188697600c^2n^7}\big(1050c^2(n+1)^2n^2(n-1)^3$$
$$+ 5c(n^2-1)(122n^4+2369n^2-403)-4(n+2)(81n^4+4600n^2-2041)\big).$$

C Useful summation formulas

Most of the summation formulas that are used in this paper can be found in [41]. There are two other ones

$$\sum_{\neq} \frac{1}{s_{j_1 j_2}^2 s_{j_2 j_3}^4 s_{j_3 j_1}^4} = \frac{4n(n^2 - 4)(n^2 - 1)(n^2 + 19)(n^4 + 19n^2 + 628)}{467775},$$

$$\sum_{\neq} \frac{c_{j_1 j_2} c_{j_1 j_3}}{s_{j_1 j_2}^3 s_{j_1 j_3}^3 s_{j_2 j_3}^4} = \frac{2n(n^2 - 25)(n^2 - 4)(n^2 - 1)(n^4 + 30n^2 + 419)}{467775}, \tag{C.1}$$

and the summations are in the range $0 \leq j_{1,2,3} \leq n - 1$ with the constraints $j_1 \neq j_2$, $j_1 \neq j_3$ and $j_3 \neq j_1$. Here we have used the shorthand $s_{j_1 j_2} = \sin \frac{\pi(j_1 - j_2)}{n}$, $c_{j_1 j_2} = \cos \frac{\pi(j_1 - j_2)}{n}$, et al.

We define the summation of k indices $0 \leq j_{1,2,\cdots,k} \leq n - 1$

$$\sum_{\neq} f(j_1, j_2, \cdots, j_k), \tag{C.2}$$

with the constraints that any two of the indices are not equal and the function $f(j_1, j_2, \cdots, j_k)$ is totally symmetry for the k arguments. First we have

$$\sum_{\neq'} f(0, j_2, \cdots, j_k) = \frac{1}{n} \sum_{\neq} f(j_1, j_2, \cdots, j_k), \tag{C.3}$$

with the summation $\neq'$ of the left-hand side being over $1 \leq j_{2,\cdots,k} \leq n - 1$ and the constraints that any two of the indices are not equal. Then we have

$$\sum_{\neq'} f(j_1, j_2, \cdots, j_k) = \frac{n - k}{n} \sum_{\neq} f(j_1, j_2, \cdots, j_k), \tag{C.4}$$

with the summation of the left-hand side being over $1 \leq j_{1,2,\cdots,k} \leq n - 1$.

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