

Group Field theory and Tensor Networks: towards a Ryu-Takayanagi formula in full quantum gravity

Goffredo Chirco, Daniele Oriti and Mingyi Zhang

*Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut),
Am Mühlenberg 1, 14476 Golm, Germany*

E-mail: [goffredo.chirco\(AT\)aei.mpg.de](mailto:goffredo.chirco@aei.mpg.de), [daniele.oriti\(AT\)aei.mpg.de](mailto:daniele.oriti@aei.mpg.de),
[mingyi.zhang\(AT\)aei.mpg.de](mailto:mingyi.zhang@aei.mpg.de)

ABSTRACT: We establish a dictionary between group field theory (thus, spin networks and random tensors) states and generalized random tensor networks. Then, we use this dictionary to compute the Rényi entropy of such states and recover the Ryu-Takayanagi formula, in three different cases corresponding to three different truncations/approximations, suggested by the established correspondence.

Contents

1	Introduction	1
2	Group Field theory and Tensor Networks	4
2.1	Group Field Theory	5
2.2	Fourier modes of the group field as tensor fields	7
2.3	Group Field Single Particle States	8
2.4	Many-Body Description and Tensor Network States	9
2.5	Link state as a gluing operation	11
2.6	Link function in spin decomposition	13
2.7	Random Tensor Network States	14
2.8	Dictionary	16
3	Ryu-Takayanagi formula for Random Tensor Networks with Gauge Symmetry	18
3.1	Replica Trick and Rényi Entropy	19
3.2	S_2 in RTN with Gauge Symmetry	20
4	Ryu-Takayanagi formula from Group Field Theory and spin networks	26
4.1	Rényi Entropy for GFT states in the group representation as generalised tensor networks	26
4.2	Rényi Entropy from Spin-Networks	39
5	Randomness and Universality	42
6	Conclusions	44
A	Average over the random tensor	44
B	Structure of the matrix of $\mathbb{E}_T(\rho^{\otimes n})$	47
C	From (4.87) to (4.88)	48

1 Introduction

Background independent approaches to quantum gravity suggest a picture of the microstructure of the universe in which continuum spacetime and geometry disappear and are replaced by discrete and non-spatiotemporal entities. Among them, Loop Quantum Gravity (LQG) [1–5], the modern incarnation of the canonical quantization programme for the gravitational field, together with its covariant counterpart (spin foam models), and

Group Field Theory (GFT) [6–9], a closely related formalism sharing the same type of fundamental degrees of freedom, identify this microstructure with (superpositions of) spin networks, which are graphs labeled by group-theoretic data. More precisely, in GFT models of quantum gravity spin network states arise as many-body states in a 2nd quantised context, whose kinematics and dynamics are governed by a quantum field theory over a group manifold with quanta corresponding to *tensor maps* associated to nodes of the spin network graphs. Random combinatorial structures, corresponding both to the elementary building blocks of quantum spacetime and to their interaction processes, become central. The same is true in the related context of random tensor models [10–12], which, for our present purposes can be seen as a simplified version of GFTs, stripped down of the group-theoretic data, leaving only the combinatorial aspects. Indeed, the random tensors can be understood as the GFT fields considered for the special case of a finite group. For a more detailed account of these three quantum gravity formalisms, and for the many results obtained, we refer to the cited literature. In the following, we will provide more precise definitions of their main ingredients.

Tensor networks, in recent years, as powerful quantum information tools in the context of condensed matter and, more generally, quantum many-body systems (including quantum field theory). For recent reviews, see [13, 14]. Also in this case, we will give precise definitions in the following. Here it suffices to say that tensor networks encode the entanglement properties of many-body systems in their combinatorial structure, in which tensors are connected along a network pattern and identify (the coefficients, in a given basis, of the wave function corresponding to) quantum states of the given system. Born as convenient mathematical tools for numerical evaluations of many-body wavefunctions, which become translatable into graphical manipulations, tensor network techniques have found an amazing number of applications: from the classification of exotic phases of quantum matter (e.g. topological order) [15, 16] to new formulation of the non-perturbative renormalization of interacting quantum field theories [17–19], down to realizations of the AdS/CFT correspondence [20–23].

Despite their disparate origin, it should be clear already from our sketchy description that the type of mathematical structures identified by quantum gravity approaches and used in the theory of tensor networks are very similar. And consequently, it is very natural to try to put the two frameworks in more direct contact. This is the main goal of the present article. Indeed, the structural similarity had been noted before [24–27], and also exploited in the context of renormalization of spin foam models treated as lattice gauge theories [28–31]. The last set of works, in particular, has already shown how fruitful tensor network techniques can be for quantum gravity models.

Before we start presenting our results, we want to offer some motivations for our work, both from the quantum gravity perspective and from the tensor network side.

From the quantum gravity point of view, the general motivation is clear. Tensor networks provide a host of tools and results that could find useful application in quantum gravity; in particular they may become central tools in the renormalization analysis of GFT models [32–37], in addition to their mentioned role in the renormalization analysis of spin foams models [38–40]. And such renormalization analyses are, in turn, the main

avenue for solving the crucial problem of the continuum limit in such formalism.

More specifically, tensor networks are very effective in taking into account and controlling the entanglement properties of quantum states in many-body systems. This is exactly the language in which GFT deals with quantum gravity states; moreover, in GFT, the very connectivity of spin network states, encoded in the links of the underlying graphs, is associated with entanglement between the fundamental quanta constituting them (associated to nodes) [41]. One example of this type of application, as we show in this paper, is the computation of entanglement entropy in spin network states and relate LQG with holography, which was also the subject of a number of other works in the LQG/GFT literature [27, 42–55].

Further, the identification of the true (interacting) vacuum state of a quantum gravity theory, in absence of any space-time background or preferred notions of energy, is a difficult matter even at the purely conceptual level, leaving aside the formidable technical challenges. One possible criterion, suited to this context, is to look for states which maximize entanglement, by some measure (e.g. entanglement entropy). In this respect, to reformulate the kinematics and dynamics of GFT and LQG states in terms of tensor networks, and to do the same for their renormalization, seems a promising strategy.

Finally, recent results in the application of tensor networks to AdS/CFT [21–23] suggest that this application would be fruitful even within the conventional perspective of canonical quantum gravity (which includes, at least historically, canonical LQG). From this perspective, in fact, the task of quantum gravity is the construction of the space of quantum states of the gravitational field which satisfy the (quantum counterpart of the) Hamiltonian constraint encoding the dynamics of quantised GR. A number of results in AdS/CFT suggest that a static AdS space-time, which we expect to be one such state, at the quantum level, satisfies the Ryu-Takayanagi (RT) formula [20] for the entanglement entropy, which is very efficiently computed (as we also show in this paper) via random tensor network techniques[23]. One is led to conjecture that this may be a general properties of physically interesting quantum states of the gravitational field, and so far no counterexample to this conjecture has been found. This prompts the search, by the same techniques, for similar states in canonical quantum gravity.

From the perspective of the theory of tensor networks, one general good point of dwelling into the correspondence with quantum gravity states should also be obvious. This identifies a new domain of applications, of truly fundamental nature, for techniques and ideas which have already proven powerful in others. Indeed, we expect that a number of key results obtained via tensor network techniques, most notably holographic mappings and indications of new topological phases in many-body systems, can be reproduced in this new context, with deep implications. In perspective, it is here that one will be able to test the suggestion that quantum information has a truly foundational role to play in our understanding of physical reality.

More practically, a number of techniques have been developed, and many results obtained, concerning the dynamics of GFT and spin-network states, also thanks to the many related developments in the theory of random tensors, and our dictionary proves that the GFT formalism provides a natural definition of the dynamics of random tensor networks.

Specifically, it means that the many results in GFT can help dealing with general (non-Gaussian) probability distributions over random tensor networks, as well as offering new takes on more standard problems, like entropy calculations, in tensor network theory. In fact, we offer some examples of these applications in the following.

In this paper, we do not target the more ambitious objective of a calculation of the RT formula for the entanglement entropy in the full quantum gravity formalism of group field theory. Having established the general dictionary between group field theory states and (generalized) random tensor networks, we content ourselves with reproducing the RT formula in three new cases, but staying close to the standard derivation: for random tensor networks endowed with a gauge invariance property motivated by the quantum gravity setting; for group field theory states corresponding to generalized tensor networks, but only using a group field theory dynamics in the simplest approximation and dealing only with averages over the tensor functions associated to the network nodes, rather than treating the full tensor network as a group field theory observable; for the simple truncation of group field theory states corresponding to spin networks with fixed spin labels. We leave a more complete and comprehensive analysis for forthcoming work.

The paper is organized as follows. In the next section, we summarize the basic elements of spin network states and of their embedding in the GFT formalism, as well as the definition of tensor networks. Having done so, we show the precise correspondence between GFT states and tensor networks, showing how the first generalizes and provides a Fock space setting for the second. In the following section, as a first application of this correspondence, we compute the 2nd Rényi entropy for random tensor networks and derive the RT formula for them, in the case of tensors endowed with a gauge symmetry. This case is immediately suggested by the GFT framework, since this type of gauge symmetry plays a crucial role in GFT models of (lattice) gauge theory and quantum gravity. In section 4 we derive the N th Rényi entropy using GFT techniques, in the group representation and for a generalized tensor network, but without taking advantage of the full GFT formalism; next, we compute the same Rényi entropy and derive the RT formula from a purely spin-network perspective, see as a truncation of more general GFT states. This is meant to be a clear example of how the same problem can be fruitfully approached from both sides of the correspondence. Finally, in the last section, we discuss one key universality result from the theory of random tensors, which extends straightforwardly to GFTs, and which could have direct impact on the applications of random tensor networks. We end up with a summary of our results.

2 Group Field theory and Tensor Networks

A d -dimensional GFT is a combinatorially non-local field theory living on (d copies of) a group manifold [6–9]. Due to the defining combinatorial structure, the Feynman diagrams \mathcal{F} of the theory are dual to cellular complexes, and the perturbative expansion of the quantum dynamics defines a sum over random lattices of (a priori) arbitrary topology. A similar lattice interpretation can be given to the quantum states of the theory. For GFT models

where appropriate group theoretic data are used and specific properties are imposed on the states and quantum amplitudes, the same lattice structures can be understood in terms of simplicial geometries. The associated many-body description of such lattice states can be given in terms of a tensor network decomposition. The corresponding (generalized) tensor networks are thus provided with a field theoretic formulation and a quantum dynamics (and, in specific models, with additional symmetries). In this section, after a brief introduction to the GFT formalism, we detail this correspondence between GFT states and (generalized) tensor networks.

2.1 Group Field Theory

Let G denote an arbitrary semi-simple Lie group; in the following, we assume for simplicity that G is compact, but the framework can easily be generalized to the non-compact case. A *group field* φ is a complex function defined on a number of copies of the group manifold G :

$$\begin{aligned} \varphi : G^d &\rightarrow \mathbb{C} \\ g_i &\mapsto \varphi(g_i) \end{aligned} \tag{2.1}$$

where we use the shorthand notation g_i for the set of d group elements $\{g_1, g_2, \dots, g_d\}$.

The GFT field can be also seen as an infinite-dimensional tensor, transforming under the action of some (unitary) group $U^{\times d}$, as:

$$\varphi(g_1, \dots, g_d) \rightarrow \int [dg_i] U(g'_1, g_1) \cdots U(g'_d, g_d) \varphi(g_1, \dots, g_d),$$

and

$$\begin{aligned} \varphi^*(g_1, \dots, g_d) &\rightarrow \int [dg_i] U^*(g_1, g'_1) \cdots U^*(g_d, g'_d) \varphi^*(g_1, \dots, g_d) \\ \text{for } \int dg U(g'_i, g_i)^* U(g_i, \tilde{g}_i) &= \delta(g'_i, \tilde{g}_i). \end{aligned} \tag{2.2}$$

This requires the d arguments of the GFT field to be *labeled and ordered*. We will see in the following how one can decompose the same field into finite-dimensional tensors; in this finite-dimensional case, the correspondence with tensor network formalism will be evident, and it will also be evident then in which sense GFTs provide a generalization of it.

The GFT dynamics is defined by an action, at the classical level, and a partition function at the quantum level. The combinatorial structure of the pairing of field arguments in the GFT interactions is part of the definition of a GFT model. An interesting class of models [6–12] is defined by the requirement that the interaction monomials are *tensor invariants*, i.e. that GFT fields are convoluted in such a way as to produce an invariant under the above mentioned (unitary) transformations¹.

¹Such invariants are in one to one correspondence with *colored d-graphs* \mathcal{B} constructed as follows: for each GFT field (resp. its complex conjugate) draw a white (resp. black) node with d outgoing links each labeled by d different colors, then connect all links in such a way that a white (resp. black) node is always connected to a black (resp. white) node and that only links with the same color can be connected.

Another class of GFT models is instead based on the requirement that the Feynman diagrams of the theory are simplicial complexes, which in turn requires the interaction kernels to have the combinatorial structure of d -simplices. This class of models is also the one on which model building for 4d quantum gravity has focused on, producing models whose Feynman amplitudes have the form of simplicial gravity path integrals and spin foam models [6–9], and, more generally, lattice gauge theories. This leads to additional symmetry requirements on the GFT fields and interactions, which will play a crucial role in the following.

In this last case, the GFT action has the general form

$$S_d[\varphi] = \frac{1}{2} \int dg_i dg'_i \varphi(g_i) \mathcal{K}(g_i g'_i{}^{-1}) \varphi(g'_i) + \frac{\lambda}{d+1} \int \prod_{i \neq j=1}^{d+1} dg_{ij} \mathcal{V}(g_{ij} g'_{ji}{}^{-1}) \varphi(g_{1j}) \cdots \varphi(g_{d+1j}), \quad (2.3)$$

where dg_i is an invariant measure on G and we use the notation $\varphi(g_{1j}) = \varphi(g_{12}, \dots, g_{1d+1})$. \mathcal{K} is the kinetic kernel, \mathcal{V} the interaction kernel, λ a coupling constant for the $d+1$ -degree homogeneous interaction. The two kernels satisfy the invariance properties

$$\begin{aligned} \mathcal{K}(h g_i g'_i{}^{-1} h') &= \mathcal{K}(g_i g'_i), \\ \mathcal{V}(h_i g_{ij} g'_{ji}{}^{-1}, h_j{}^{-1}) &= \mathcal{V}(g_{ij} g'_{ji}{}^{-1}) \quad \forall h, h', h_i \in G. \end{aligned} \quad (2.4)$$

This implies that the action is invariant under the gauge transformations $\delta\varphi(g_i) = \tilde{\varphi}(g_i)$, where $\tilde{\varphi}$ is any function satisfying

$$\int_G dh \tilde{\varphi}(hg_1, \dots, hg_d) = 0. \quad (2.5)$$

This symmetry is gauge fixed if one restricts the field φ to satisfy

$$\varphi(hg_i) = \varphi(g_i). \quad (2.6)$$

The action is also invariant under the *global symmetry*

$$\varphi(g_1, \dots, g_d) \rightarrow \varphi(g_1 h, \dots, g_d h). \quad (2.7)$$

GFT's Feynman diagrams define cellular complexes \mathcal{F} weighted by amplitudes assigned to the faces, edges and vertices of the dual two-skeleton of a chosen triangulation of a d dimensional topological spacetime $\mathcal{M}_{\mathcal{F}}$. As mentioned, their Feynman diagram evaluations reproduce the associated amplitudes of a spin foam model, or, in different variables, of a simplicial gravity path integral [56–58], providing a generalisation of the lattice formulation of gravity à la Regge, with an accompanying sum over lattices, generalising matrix models for 2d gravity to any dimension [6–12].

Let us give some more detail on the construction, to clarify the above points. A specific theory, with a specific related Feynman cellular complex, is completely defined by

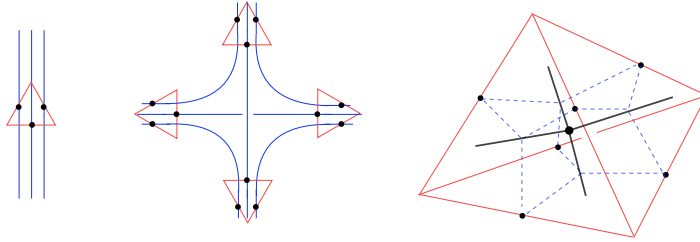


Figure 1. Correspondence between Feynman diagram and triangulation: Each strand of the graph forms a closed loop which can be interpreted as the boundary of a 2d disk. These data are enough to reconstruct a topological 2d complex \mathcal{F} , the vertices and edges of this complex correspond to vertices and edges of the Feynman graph, the boundary of the faces of \mathcal{F} correspond to the strands of the Feynman graph.

the choice of the kernels. Lets consider the simplest case, consisting in the choice

$$\mathcal{K}(g_i, g'_i) = \int_G dh \prod_i \delta(g_i g'_i{}^{-1} h), \quad (2.8)$$

$$\mathcal{V}(g_{ij} g'_{ji}{}^{-1}) = \int_G \prod_i dh_i \prod_{i < j} \delta(h_i g_{ij} g'_{ji}{}^{-1}, h_j^{-1}) \quad (2.9)$$

where $\delta(\cdot)$ is the delta function on G and the integrals ensure the gauge invariance defined in (2.5), and let us restrict to the case of dimension $d = 3$. To keep track of the combinatorics of field arguments in the kernels, it is useful to represent the Feynman diagram as a *stranded graph*. The field φ has three arguments, so each edge of a Feynman diagram comprises three strands running parallel to it. Four edges meet at each vertex and the form of the interaction \mathcal{V} in (2.9) forces the strands to recombine as in Figure 1.

The three strands running along the edges can be understood to be dual to a triangle and the propagator \mathcal{K} gives a prescription for the *gluing* of two triangles. At the vertex, four triangles meet and their gluing via \mathcal{V} form a tetrahedron. With this interpretation the Feynman diagram of a GFT is clearly dual to a triangulated 3d simplicial complex (which will be generically a singular pseudo-manifold) and this is true in any dimension [59–61].

The quantum states of the theory can be given a similar combinatorial characterization in terms of graphs and dual cellular complexes, as it should be already intuitive in the above example, in which GFT fields themselves are associated to triangles. We will no detail this aspect of the formalism.

2.2 Fourier modes of the group field as tensor fields

As a function on a group G , the field φ can be decomposed in terms of unitary irreducible representations (ρ, V_ρ) of G using the Peter-Weyl theorem, $L^2(G) \simeq \bigoplus_\rho V_\rho \otimes V^{*\rho}$, giving

$$\varphi(g) = \sum_{\rho} d_{\rho} \text{Tr}[\hat{\varphi}_{ab} \rho^{ab}(g)] \quad (2.10)$$

Here, $d_{\rho} \in \mathbb{N}$ is the dimension of the representation $\rho : G \rightarrow \text{Aut}(V_{\rho})$, the indices $a, b = 1, \dots, d_{\rho}$ are matrix indices associated to the matrix $\rho(g)$ representing the group element g , and $\hat{\varphi}_{\rho} \in V^{\rho} \otimes V^{*\rho} \simeq \text{End}(V_{\rho})$ is the matrix Fourier coefficient of the function φ . In other words, each $\hat{\varphi}_{\rho}$ is a rank $d_{\rho} = N$ matrix.

Let us consider, as a specific example, the same decomposition for the case of $d = 3$, with $G = SU(2)$. The unitary irreps of $SU(2)$, \mathbb{V}^j , are labeled by the spin $j \in \mathbb{N}/2$. Using the right invariance property of the field, one obtains the following decomposition

$$\varphi(g_1, g_2, g_3) = \sum_{\{j\}} \text{Tr} \left[\varphi_{m_1, m_2, m_3}^{\{j\}} \left(\prod_i \sqrt{d_{j_i}} D_{m_i, n_i}^{j_i}(g_i) \right) \bar{i}_{n_1, n_2, n_3}^{\{j\}} \right] \quad (2.11)$$

where d_j is the dimension, $D^j(g) \in \text{End}(\mathbb{V}^j)$ the group matrix element and $i_{n_1, n_2, n_3}^{\{j\}} \in \text{Hom}_G(\mathbb{V}^{j_1} \otimes \mathbb{V}^{j_2} \otimes \mathbb{V}^{j_3}, \mathbb{C})$ is the three-valent intertwiner operator (related to the Clebsch-Gordan map $\Psi_{j_1 j_2}^{j_3} : \mathbb{V}^{j_1} \otimes \mathbb{V}^{j_2} \rightarrow \mathbb{V}^{j_3}$). We used the shorthand notation $\{j\}$ for the set of spin labels (j_1, j_2, j_3) .

The fields $\varphi_{m_1, m_2, m_3}^{\{j\}}$ result from the contraction of the Fourier transformed GFT fields $\hat{\varphi}_{\{j\}}$ with the intertwiner tensor imposing the gauge symmetry at the vertex. ²

$$\varphi_{\{m\}}^{\{j\}} = \sum_{\{k\}} \hat{\varphi}_{\{m\}; \{k\}}^{\{j\}} i_{\{j\}}^{\{k\}} \prod_i \sqrt{d_{j_i}}. \quad (2.12)$$

The Fourier transformed fields depend on the (discrete) representation space of the Lie group in question. Thus, generically Fourier transformed GFT fields are tensors of some rank d , $\varphi_{\{m_j\}}$ with *discrete* indices $\vec{m}_j = \{m_1, \dots, m_d\}$. ³

In (2.11), such tensors are contracted with the *spin network basis* tensors

$$S_{\{m\}}^{\{j\}} = \left(\prod_i \sqrt{d_{j_i}} D_{m_i, n_i}^{j_i}(g_i) \right) \bar{i}_{\{n\}}^{\{j\}}, \quad (2.13)$$

encoding the properties of the vertex of the spin network graph dual to the $(d-1)$ -dimensional triangulation that can be associated to the GFT states.

2.3 Group Field Single Particle States

Functions $\varphi(g_i)$ can also be understood as *single particle* wave functions for quanta corresponding to single open vertices of a spin network graph (in fact, they also label coherent

²This is the standard factorization of a symmetric tensor into a degeneracy tensor with all the degrees of freedom and a structural tensors (the Clebsch-Gordan coefficients) completely determined by the symmetry group G (Wigner-Eckart theorem) [25].

³To regularize some quantities, especially at the dynamical level, it may be necessary to impose a (large) cut-off N in the range of the representation indices.

states of the GFT field operator, which define the simplest condensate states of the theory [55, 62, 63]).

Let us define these ‘single-particle’ quantum states as

$$|\varphi\rangle = \int_{G^d} dg_i \varphi(g_i) |g_i\rangle \quad (2.14)$$

where $dg_i \equiv dg_1 dg_2 \dots dg_d$ is the Haar measure on the group manifold G^d , invariant under the gauge transformation, and the vectors $|g_1\rangle \dots |g_d\rangle$ provide a basis on the respective infinite dimensional spaces $\mathbb{H} \simeq L^2[G]$.

The single particle state $|\varphi\rangle$ is then defined in $\mathbb{H}^{\otimes d}$. Moreover we require $|\varphi\rangle$ to be normalized (this is of course not the case for the classical GFT fields or the GFT condensate wavefunctions):

$$\langle\varphi|\varphi\rangle = \int dg_i \overline{\varphi(g_i)} \varphi(g_i) = 1. \quad (2.15)$$

Considering the case of $G = SU(2)$, we can decompose the basis $|g\rangle$ into the unitary irreducible representation of $SU(2)$ as

$$|g\rangle \equiv \sum_{j,m,n} \sqrt{d_j} \overline{D_{mn}^j(g)} |j, n, m^\dagger\rangle \quad (2.16)$$

and vice versa

$$|j, n, m^\dagger\rangle = \int_{SU(2)} dg \sqrt{d_j} D_{mn}^j(g) |g\rangle. \quad (2.17)$$

In particular, the tensor decomposition given in (2.11) holds at the quantum level, hence defining the quantum fields $\varphi_{m_1, m_2, m_3}^{\{j\}}$ as actual *tensors states*.

Tensors in (2.13) defines the $SU(2)$ -invariant single vertex spin network wave functions (in group representation)

$$\psi_\chi(g_i) = \langle\chi|g_i\rangle = \left(\prod_i \sqrt{d_{j_i}} D_{m_i, n_i}^{j_i}(g_i) \right) \overline{i_{\{n\}}^{\{j\}}}, \quad (2.18)$$

The basis vector $|\chi\rangle = |j, m, i\rangle$ denotes the standard $SU(2)$ spin network basis (labelled by spins and angular momentum projections associated to their d open edges, and intertwiner quantum numbers).

2.4 Many-Body Description and Tensor Network States

We now describe the quantum states of the formalism, emphasizing their many-body structure, following [64].

Consider a d -valent graph formed by V disconnected components, each corresponding to a single *gauge invariant* d -valent vertex and d 1-valent vertices, thus having d edges.⁴ We refer to this type of disconnected components as *open* spin network vertices.

⁴One could work instead with the larger Hilbert spaces of non-gauge invariant states $L^2[G^{d \times V}]$ without imposing any gauge symmetry at the vertices of spin network graphs, and consider this condition as part of the dynamics. The above construction would proceed identically, with the same final result, but with the basis of single-vertex states now given by the above functions without the contraction of representation function with a G -intertwiner.

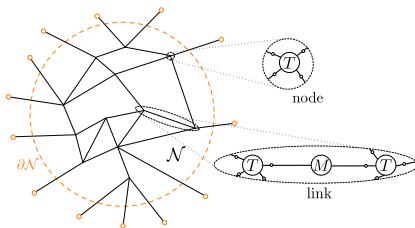


Figure 2. A tensor network Γ is a set of tensors whose indices are contracted according to a network pattern. A network pattern can be always represented as a *graph*, given by a set of nodes (n) and links (ℓ) connecting nodes. A link is called an *internal* link when it connects two *different* nodes; while it is called a *boundary* link when it connects only one node. The number of links that connect to a node is called the valence of the node..

To such a graph we can associate a generic wavefunction given by a function of $d \times V$ group elements,

$$\Phi(g_a^i) = \Phi(g_1^1, \dots, g_1^d, g_2^1, \dots, g_2^d, \dots, g_V^1, \dots, g_V^d) \quad (2.19)$$

defined on the group space $G^{d \times V} / G^V$ (V copies of G^d , quotiented by the isotropy group of the single particle function $\varphi_{(v)}(g_i)$ at the each vertex); here the index a runs over the set of vertices, while the index i still runs over the links attached to each vertex).

These functions are exactly like many-particles wave functions for point particles living on the group manifold G^d , and having as classical phase space $(\mathcal{T}^*G)^d$ (the classical phase space of a single open spin network vertex or polyhedron).

Accordingly, a state $|\Phi\rangle \in \mathbb{H}_V \simeq L^2[G^{d \times V} / G^V]$ can be conveniently decomposed into products of single-particle (single-vertex) states,

$$\Phi(g_i^a) = \langle g_i^a | \Phi \rangle = \sum_{\chi_i, i=1 \dots V} \varphi^{\chi_1 \dots \chi_V} \psi_{\chi_1}(g_i) \cdots \psi_{\chi_V}(g_i) \quad (2.20)$$

While the above decomposition is completely general, a special class of states can be constructed in direct association with a graph or network Γ . The association works as follows. Start from the d -valent graph with V disconnected components (open spin network vertices) to which a generic V -body state of the theory is associated. A partially connected d -valent graph can be constructed by choosing at least one edge i in a vertex a and *gluing* it to one edge j of the vertex b , i.e. joining the two edges along their 1-valent vertices. The final graph will be fully connected if all edges have been glued. Each pair of glued edges $\{ai, bj\}$ will identify a *link* L of the resulting (partially) connected graph. In the spin representation, i.e. in terms of the basis of functions $\psi_{\chi_1}(g_i) \cdots \psi_{\chi_V}(g_i)$, the gluing is implemented by the identification of the spin labels j_i^a and j_j^b associated to the two edges being glued and by the contraction of the corresponding vector indices m_i^a and m_j^b . In other words, the corresponding wave functions for closed graphs can be decomposed in a basis of *closed spin network wave functions*, obtained from the general product basis by means of the same contractions:

$$\Phi_\Gamma(g_i^a) = \langle g_i^a | \Phi_\Gamma \rangle = \sum_{\chi_a, a=1 \dots V} \Phi_\Gamma^{j_1^1 \dots j_V^V} \left[\left(\prod_{L \in \Gamma} \delta_{j_i^a, j_j^b} \delta_{m_i^a, m_j^b} \right) \psi_{\chi_1}(g_i) \cdots \psi_{\chi_V}(g_i) \right] \quad (2.21)$$

where the coefficients of the wave function can in turn be understood as the resulting of considering generic coefficients $\varphi^{\chi_1 \dots \chi_V}$ and contracting them with some choice of functions

$$M_{n_i^a n_j^b}^{j_i^a j_j^b} \delta_{j_i^a, j_j^b}:$$

$$\Phi_{\Gamma}^{j_1^1 \dots j_1^V} = \varphi^{\chi_1 \dots \chi_V} \left(\prod_{L \in \Gamma} \delta_{j_i^a, j_j^b} M_{n_i^a, n_j^b}^{j_i^a, j_j^b} \right), \quad (2.22)$$

where the contraction is left implicit.

For fixed $\{j\}$, each resulting contraction scheme of tensors (each identified by a set of labels χ) defines a *tensor network state*.

In the group representation, the gluing amounts to considering wave functions with a specific symmetry under simultaneous group translation of the arguments associated to the edges being glued:

$$\Phi_{\Gamma}(g_a^i) = \Phi_{\Gamma}(g_1^1, \dots, g_1^d h_{1V}^{d1}, g_2^1, \dots, g_2^d, \dots, g_V^1 h_{1V}^{d1}, \dots, g_V^d) \quad . \quad (2.23)$$

In the end, given a tensor network with graph Γ , the $\Phi^{j_1^1 \dots j_1^V}$ defined above will contain all the information about the combinatorics of the quantum geometry state.

A further special case corresponds to those states for which the coefficients $\varphi^{\chi_1 \dots \chi_V}$ themselves have a product form, i.e. can be decomposed in terms of tensors. In this case, as it is for the spin network wave functions, the coefficients $\Phi^{j_1^1 \dots j_1^V}$ can be obtained as a tensor trace

$$\Phi^{j_1^1 \dots j_1^V} = \text{Tr} \left[\bigotimes_L M \bigotimes_v \varphi_{\{m\}}^{\{j\}(v)} \right], \quad (2.24)$$

again, in the case of fully connected graphs Γ (otherwise, some angular momentum labels will remain on the left had side, corresponding to the edges that have not been glued). In lattice theory, we would say that the network Γ (fixed $\{j\}$) provides a tensor network decomposition of the tensor state $\Phi^{j_1^1 \dots j_1^V}$.

The *equivalence* of a special class of GFT states with the lattice tensor network states, and the sense in which GFT states generalise them, can be further elucidated by the following example.

2.5 Link state as a gluing operation

A tensor \hat{T} is a multidimensional array of complex numbers $\hat{T}_{\lambda_1, \dots, \lambda_d} \in \mathbb{C}$. The rank of tensor \hat{T} is the number d of indices. The size of an index λ , denoted $d_{|\lambda|}$, is the number of values that the index $\lambda \in \mathbb{N}$ takes [65].

Analogously, at the quantum level, to each leg of the tensor one associates a Hermitian inner product space \mathbb{H}_D , with dimension D given by the size of the indices $\lambda \in \{1, 2, \dots, d_{|\lambda|} = D\}$. Given an orthonormal basis $|\lambda_n\rangle$, $n = 1, \dots, |D|$ in \mathbb{H}_D , a covariant tensor of rank d is a multilinear form on the Hilbert space of the vertex $T: \mathbb{H}_n \equiv \mathbb{H}_D^{\otimes d} \rightarrow \mathbb{C}$.

We denote its components in the canonical dual tensor product basis by

$$\hat{T}_{\lambda_1 \dots \lambda_d} \equiv T(\lambda_{n_1}, \dots, \lambda_{n_d}), \quad (2.25)$$

hence the tensor state

$$|T\rangle = \sum_{\lambda_1, \dots, \lambda_d} \hat{T}_{\lambda_1 \dots \lambda_d} |\lambda_1\rangle \otimes \dots \otimes |\lambda_d\rangle \quad (2.26)$$

A tensor network is generally given by a set of d -valent vertices v , corresponding to rank d tensors. In particular, a state corresponding to a set of unconnected vertices is written as a tensor product of individual vertex states

$$|\mathcal{T}_{\mathcal{N}}\rangle \equiv \bigotimes_n |T_n\rangle \quad (2.27)$$

Individual vertex states are *glued* by links. To each end of a link we associate a Hilbert space \mathbb{H}_D . The Hilbert space of the link ℓ is then $\mathbb{H}_\ell = \mathbb{H}_D^{\otimes 2}$ and a link state can be written as

$$|M\rangle = M_{\lambda_1 \lambda_2} |\lambda_1\rangle \otimes |\lambda_2\rangle \quad (2.28)$$

where we choose to take the link states $|M\rangle$ to be generically entangled.⁵ In general, the entanglement of the links will encode the information on the connectivity of the graph. Two nodes are connected if their corresponding states contract with a link state,

$$\hat{\mathcal{T}}_{12} \equiv \langle M | |T_1\rangle |T_2\rangle = T_{\lambda_1 \dots \lambda_a \dots \lambda_v}^{(1)} \bar{M}_{\lambda_a \lambda_b} T_{\lambda'_1 \dots \lambda_b \dots \lambda'_u}^{(2)} \bigotimes_{i \neq a}^v |\lambda_i\rangle \otimes \bigotimes_{j \neq b}^u |\lambda'_j\rangle \quad (2.30)$$

Notice that if $|M\rangle$ was a non-entangled state, the connection would be trivial, i.e. the two nodes would be practically disconnected and the corresponding state could be written as a tensor product of two states,

$$\begin{aligned} \hat{\mathcal{T}}_{12} &= T_{\lambda_a \lambda_1 \dots \lambda_v}^{(1)} \bar{A}_{\lambda_a} \bigotimes_{i=1}^v |\lambda_i\rangle \otimes \bar{B}_{\lambda_b} T_{\lambda_b \lambda'_1 \dots \lambda'_u}^{(2)} \bigotimes_{j=1}^u |\lambda'_j\rangle \\ &= |T'_1\rangle \otimes |T'_2\rangle \end{aligned} \quad (2.31)$$

Then given a network \mathcal{N} with N nodes and L links, the corresponding state is

$$|\Psi_{\mathcal{N}}\rangle \equiv \bigotimes_{\ell}^L \langle M_{\ell} | \bigotimes_n^N |T_n\rangle \quad (2.32)$$

Because all links are contracted with nodes, $|\Psi_{\mathcal{N}}\rangle$ is then in the Hilbert space associated to the boundary links of the network, which is denoted as $\mathbb{H}_{\partial\mathcal{N}}$. $|\Psi_{\mathcal{N}}\rangle$ is a state in $\mathbb{H}_{\partial\mathcal{N}}$.

⁵One can observe it by defining a density matrix $\rho_M \equiv |M\rangle \langle M|$ and tracing out one of the Hilbert space, without losing generality, tracing out \mathbb{H}_D of $|\lambda_2\rangle$, then computing the von Neumann entropy of the reduced density matrix $\rho_1 \equiv \text{Tr}_2 \rho_M = M^\dagger M$. The entropy $S = -\text{Tr} \rho_1 \ln \rho_1$ is non-zero unless $M_{\lambda_1 \lambda_2}$ can split as $M_{\lambda_1 \lambda_2} = A_{\lambda_1} B_{\lambda_2}$. For simplicity, in the next sections we will often assume that the link state is maximally entangled, i.e.

$$|M\rangle = \frac{1}{\sqrt{D}} \delta_{\lambda_1 \lambda_2} |\lambda_1\rangle \otimes |\lambda_2\rangle. \quad (2.29)$$

The above structure can be identified also for the special GFT states mentioned at the end of the previous subsection, which are formed by generalised $L^2(G^d)$ functions associated to the nodes of the network. In this case, the analogous of the generic link state in (2.28), which is also the group counterpart of the gluing operators associated in the spin representation to the matrices M , can be defined in as the convolution functional

$$\langle M_{g_\ell} | \equiv \int dg_1 dg_2 M(g_1^\dagger g_\ell g_2) \langle g_1 | \otimes \langle g_2 | \in \mathbb{H}^{*\otimes 2}, \quad (2.33)$$

where the functions $M(g)$ are assumed to be invariant under conjugation $M(g) = M(hgh^{-1})$. When a link ℓ connects two nodes, say a and b , the corresponding state $\langle M_{g_\ell} |$ contracts with states $|\varphi_a\rangle$ and $|\varphi_b\rangle$

$$\langle M_{g_\ell} | |\varphi_a\rangle |\varphi_b\rangle = \int dg_1 dg_2 dg_i^a dg_i^b M(g_1^\dagger g_\ell g_2) \varphi_a(g_1, g_i^a) \varphi_b(g_2, g_i^b) |g_i^a\rangle |g_i^b\rangle \quad (2.34)$$

where we have singled out, among the arguments of the vertex wave functions φ the ones affected by the gluing operation. In these terms, the open d -valent tensor network graph Γ with V vertices, can be written as

$$|\Phi_\Gamma^{g_\ell}\rangle \equiv \bigotimes_{\ell \in \Gamma} \langle M_{g_\ell} | \bigotimes_n^V |\varphi_n\rangle = \int dg_\partial \Phi_\Gamma(g_\ell, g_\partial) |g_\partial\rangle \quad (2.35)$$

where the $\{g_\partial\}$ denote the group elements on the open links.

The role of the link state in tensor network, thus, is naturally generalised by the convolution function, defined for the group field variables. This is due to the fact that the group fields $\varphi(g_i)$ on G^d can be interpreted as rank d tensors, with indices spanning the group space G , and associated Hilbert space (for each index) being $L^2(G)$.⁶ The multiparticle state given in (2.23) can then be interpreted as a tensor state with indices g_∂ and rank given by the number of open links of the spin network graph.

2.6 Link function in spin decomposition

As showed in 2.4, many-body state can also be decomposed into spin representations. Suppose $M(g_1^\dagger g_\ell g_2)$ can be written as

$$M(g_1^\dagger g_\ell g_2) = \sum_{jmn} d_j M_{mn}^j D_{mn}^j(g_1^\dagger g_\ell g_2) \quad (2.38)$$

⁶The case of ordinary, finite-dimensional tensors is obtained if we pass from a Lie group to a discrete group. Let us consider, as a basic example, the case of a field defined on the discrete n th cyclic group \mathbb{Z}_n . Given the nonempty set

$$X = \{\vec{\lambda} | \vec{\lambda} = (\lambda_1, \dots, \lambda_d), \lambda_k \in \mathbb{Z}_n\}. \quad (2.36)$$

the field $\varphi : X \rightarrow \mathbb{R}$ (or \mathbb{C}) is a real or complex valued function on X and we indicate by

$$\varphi_{\vec{\lambda}} \equiv \varphi(\vec{\lambda}). \quad (2.37)$$

the value of φ on the set of d elements $\vec{\lambda}$. The function $\varphi_{\vec{\lambda}}$ can be interpreted as a tensor with d discrete indices $\varphi_{\lambda_1, \dots, \lambda_d}$, where $\lambda \in \{1, 2, \dots, |\dim(\mathbb{Z}_n)|\}$.

Then, as a simple example, the state $\langle M_{g_\ell} | |\varphi_a\rangle |\varphi_b\rangle$ can be written in terms of $\varphi_{\mathbf{kn}}^{\mathbf{j}}$, $i_{\mathbf{m}}$ and M_{mn}^j as⁷

$$\begin{aligned}
\langle M_{g_\ell} | |\varphi_a\rangle |\varphi_b\rangle &= \int dg_1 dg_2 dg_i^a dg_i^b M(g_1^\dagger g_\ell g_2) \varphi_a(g_1, g_i^a) \varphi_b(g_2, g_i^b) |g_i^a\rangle |g_i^b\rangle \\
&= \sum_{jmnklpq} \sum_{i_a i_b} \overline{[i_a]_{\mathbf{p}}} [\varphi_a]_{\mathbf{p}\mathbf{n}_a}^{\mathbf{j}i_a} M_{nm}^j \overline{[i_b]_{\mathbf{q}}} [\varphi_b]_{\mathbf{q}\mathbf{m}_b}^{\mathbf{j}i_b} (-)^m |\mathbf{j}_b, \mathbf{m}_b\rangle |\mathbf{j}_a, \mathbf{n}_a\rangle [i_a]_{\mathbf{k}_a k} D_{kl}^j(g_\ell) [i_b]_{\mathbf{l}_b(-l)} \times \\
&\times (-)^l |\mathbf{j}_a, \mathbf{k}_a^\dagger\rangle |\mathbf{j}_b, \mathbf{l}_b^\dagger\rangle
\end{aligned} \tag{2.39}$$

Graphically, the last line can be presented as

$$\langle M_{g_\ell} | |\varphi_a\rangle |\varphi_b\rangle = \sum_{jmnkl} \sum_{i_a i_b} \tag{2.40}$$

From the graphic equation, one can immediately observe that the upper part is an open tensor network $|\Phi^{\mathbf{j}i}\rangle$, given by the tensor trace of a collection of tensors

$$\phi_{\mathbf{m}}^{\mathbf{j}i} \equiv \sum_{\mathbf{n}} \overline{i_{\mathbf{n}}} \varphi_{\mathbf{nm}}^{\mathbf{j}} \tag{2.41}$$

for each node and matrices M_{mn}^j for each link.

2.7 Random Tensor Network States

A random tensor network (RTN) is a tensor network in which tensors $T_{\lambda_1 \dots \lambda_v}$ on each node are unit vectors chosen independently at random from their respective Hilbert spaces. A tensor state $|T\rangle$, defined in the Hilbert space $\mathbb{H}_n = \mathbb{H}_D^{\otimes v}$, contains D^v complex components. These components define a complex vector T_μ in a D^v dimensional vector space \mathbb{H}_T . We choose a gauge such that numerically $T_\mu \equiv T_{\lambda_1 \dots \lambda_v}$,

$$\mu = \sum_{a=1}^v \lambda_a D^{v-a} = 0, 1, \dots, D^v - 1. \tag{2.42}$$

Since \mathbb{H}_T is isomorphic to \mathbb{H}_n , one can give an inner product between two vectors in \mathbb{H}_T as $\overline{T}_\mu T'_\mu$. Moreover, $|T\rangle \in \mathbb{H}_n$ being normalized, one has as well

$$\langle T|T\rangle = \overline{T}_{\lambda_1 \dots \lambda_v} T_{\lambda_1 \dots \lambda_v} = \overline{T}_\mu T_\mu = 1 \tag{2.43}$$

⁷Notice that we are introducing the bold font for vectorial quantities, in order to shorten the notation in spin representation.

The Hilbert space \mathbb{H}_T is the fundamental representation of $U(D^v)$ group.

Given an arbitrary reference state T_μ^0 , a group element $U \in U(D^v)$ will transform T_μ^0 to a new vector $T_\mu^U \equiv (UT^0)_\mu$. A random tensor $T_{\lambda_1 \dots \lambda_v}$ corresponds then to a random choice of the group element $U \in U(D^v)$ defining T_μ^U .⁸ In a random network, a group element U_n is independently chosen for each node of the network.

Let us consider now a tensor $T_{\lambda_1 \dots \lambda_v}$ which satisfies the following symmetry

$$T_{\lambda_1 \dots \lambda_v} = T_{[\lambda_1 + \ell]_D \dots [\lambda_v + \ell]_D}, \quad \forall \ell \in \mathbb{Z} \quad (2.44)$$

The square bracket $[\dots]_D$ denotes the modular arithmetic: for all $k \in \mathbb{Z}$ and $D \in \mathbb{Z}^+$

$$[k]_D = k \bmod D, \quad [k]_D \in \mathbb{Z}/D \quad (2.45)$$

where \mathbb{Z}/D is the set of integers modulo D . Such a tensor state can be considered as a particular case of a group field tensor, with indices taking values on a finite group space.⁹

For all $k_1, k_2 \in \mathbb{Z}$ and $D \in \mathbb{Z}^+$, it satisfies the following rules:

$$\begin{aligned} [k_1] + [k_2]_D &= [k_1 + k_2]_D, \\ [k_1] + [k_2]_D &= [k_1 + k_2]_D, \quad [D]_D = 0 \end{aligned} \quad (2.46)$$

In the case with symmetry (2.44), only D^{v-1} components are independent. Because of the gauge choice (2.42), T_μ can be written as a direct sum of D vectors

$$T_\mu \equiv \begin{pmatrix} T_{\mu_0} \\ T_{\mu_1} \\ \vdots \\ T_{\mu_{D-1}} \end{pmatrix}, \mu_i = 0, 1, \dots, D^{v-1} - 1, \quad (2.47)$$

$\forall i = 0, 1, \dots, D - 1$, in which the components in T_{μ_i} are independent from each other. However one can observe that if we use the gauge choice (2.42)

$$T_{\mu_i} \neq T_{\mu_j}, \quad \forall i \neq j \quad (2.48)$$

In order to avoid this inconsistency, we choose a new gauge $T_\mu = \bigoplus_{i=0}^{D-1} T_{\mu_i} \equiv T_{\lambda_1 \dots \lambda_v}$, with

$$\mu = \lambda_1 D^{v-1} + \sum_{a=2}^v [\lambda_a - \lambda_1]_D D^{v-a} \quad (2.49)$$

In the new gauge, numerically the components satisfy

$$T_{\mu_i} = T_{\mu_j}, \quad \forall i, j \quad (2.50)$$

and in direct summed representation, for a given component T_μ

$$T_\mu = T_{[\mu + \ell D^{v-1}]_{D^v}}, \quad \forall \ell \in \mathbb{Z} \quad (2.51)$$

⁸The random average of an arbitrary function $f(|V_x\rangle)$ of the state $|V_x\rangle$ is equivalent to an integration over U according to the Haar probability measure $\int dU f(U|0_x\rangle)$, with normalization $\int dU = 1$.

⁹In a tensor field theory the gauge symmetry characterising the group field function, as defined in (2.6), can also be imposed as a ‘‘dynamical’’ feature via the choice of a specific kernel in the action.

From now on, for simplicity, we denote the tensor with the symmetry (2.44) with two indices: μ and $i = \lambda_1 \in \mathbb{Z}/D$ as $T_{\mu_i} = T_{\lambda_1 \dots \lambda_v}$, with

$$\mu_i = \sum_{a=2}^v [\lambda_a - \lambda_1]_D D^{v-a} \in \mathbb{Z}/D^{v-1} \quad (2.52)$$

For a given i , the vector T_{μ_i} is lying on a D^{v-1} dimensional space, which is a fundamental representation space of the group $U(D^{v-1})$. Because $\bar{T}_\mu T_\mu = 1$ and (2.44), T_{μ_i} is also normalized

$$\bar{T}_{\mu_i} T_{\mu_i} = D^{-1}, \quad \forall i = 0, 1, \dots, D-1 \quad (2.53)$$

Then in this case the tensor $T_{\lambda_1 \dots \lambda_v}$ is at random means that with respect to the same T_0^μ , the group element $U_n \in U(D^{v-1})$ is randomly chosen for each node.

The random character plays a central role in the study of the entanglement area laws in tensor networks. Indeed, random pure states are nearly maximally entangled states [66], hence can be used as a toy model of a thermal state [67, 68]. This in particular implies that the computation of typical entropies and other quantities of interest for these states can be mapped to the evaluation of partition functions of classical statistical models as showed in [23].

The interpretation of group fields as tensors provides an actual generalisation of the tensor network decomposition techniques in terms of random tensor fields. Moreover, a given GFT model provide the same generalised tensor network states with a dynamical description, each model corresponding to a specific probability measure $\frac{1}{Z} d\nu(\varphi)$ for the field φ , defined through its partition function Z and momenta (correlations).

2.8 Dictionary

We summarize the established dictionary between group field theory states and generalized random tensor networks in terms of two synthetic tables.

The correspondence between group field and tensor network description is resumed in Table A:

Table A	Group Fields	Tensors	
classical	$\varphi : G^d \rightarrow \mathbb{C}$ $g_i \mapsto \varphi(g_i)$ $\varphi(\vec{g}) \equiv \varphi(g_1, g_2, \dots, g_d)$ $g_i \in G$ generic group	$T : X \rightarrow \mathbb{C}$ $X = \{\vec{\lambda} \vec{\lambda} = (\lambda_1, \dots, \lambda_d)\}$ $T_{\lambda_1 \lambda_2 \dots \lambda_d} \equiv T(\vec{\lambda})$ e.g. $\lambda_i \in \mathbb{Z}_n$, n th cyclic group	
gauge sym	$\varphi(h\vec{g}) = \varphi(\vec{g}_i)$	our case $T(\vec{\lambda} + \vec{\ell}) = T(\vec{\lambda})$ $\vec{\ell} \equiv (\ell, \dots, \ell), \ell \in \mathbb{Z}$	
quantum	$ \vec{g}\rangle \in \mathbb{H}^{\otimes d} \simeq L^2[G^d]$	$ \lambda_i\rangle, i = 1, \dots, d _{\lambda} = D$ in \mathbb{H}_D	
one particle state	$ \varphi\rangle = \varphi(\vec{g}) \vec{g}\rangle$	$ T_n\rangle = T_{\lambda_1 \dots \lambda_d} \vec{\lambda}\rangle \in \mathbb{H}_n = \mathbb{H}_D^{\otimes d}$	tensor state
gluing functional	$\langle M_{g_\ell} =$ $\int dg_1 dg_2 M(g_1^\dagger g_\ell g_2) \langle g_1 \langle g_2 $ $\in \mathbb{H}^{*\otimes 2}$	$ M\rangle = M_{\lambda_1 \lambda_2} \lambda_1\rangle \otimes \lambda_2\rangle \in$ $\mathbb{H}_\ell = \mathbb{H}_D^{\otimes 2}$	link state
multiparticle state	$ \Phi_\Gamma\rangle \in \mathbb{H}_V \simeq L^2[G^{d \times V} / G^V]$	$ \Psi_{\mathcal{N}}\rangle$	tensor network state
product state convolution	$ \Phi_\Gamma^{g_\ell}\rangle \equiv \bigotimes_{\ell \in \Gamma} \langle M_{g_\ell} \bigotimes_n^V \varphi_n\rangle$ $= \int dg_\partial \Phi_\Gamma(g_\ell, g_\partial) g_\partial\rangle$	$ \Psi_{\mathcal{N}}\rangle \equiv \bigotimes_\ell^L \langle M_\ell \bigotimes_n^N T_n\rangle \in \mathbb{H}_{\partial \mathcal{N}}$	tensor network decomposition
randomness	$\frac{1}{Z} d\nu(\varphi)$ field theory probability measure	$T_\mu^U \equiv (UT^0)_\mu$ $T_\mu^0 \equiv T_{\lambda_1 \dots \lambda_d}^0 \in \mathbb{H}_T,$ $U \in \text{U}(\dim(\mathbb{H}_T))$	random tensor state

The generalisation of tensor networks in terms of group fields states is evident in the spin- j decomposition of the latter $\varphi(g_i) = \sum_j \text{Tr}[\varphi_{\{m\}}^j \left(\prod_i \sqrt{d_{j_i}} D_{m_i, n_i}^{j_i}(g_i) \right) \bar{i}_{\{n\}}^j]$.

Once we turn off the sum over all possible js , fix the representation labels and ask them to be equal, generically Fourier transformed GFT fields $\varphi_{\{m\}}^j$, are tensors of single rank d , with *discrete* indices $m_i = \{m_1, \dots, m_d\}$ spanning a finite dimensional space. The

equivalence is resumed in table B:

Table B	GFT network	Spin Tensor Network	Tensor Network
node	$\varphi(\vec{g})$ $\equiv \varphi(g_1, g_2, g_3, g_4)$	$\varphi_{\{m\}}^{\mathbf{j}}$ $\propto \sum_{\{k\}} \hat{\varphi}_{\{m\}\{k\}}^{\mathbf{j}} i^{\mathbf{j}\{k\}}$	$T_{\{\mu\}}$
link	$M(g_1^\dagger g_\ell g_2)$	M_{mn}^j	$M_{\lambda_1 \lambda_2}$
sym	$\varphi(h\vec{g}) = \varphi(\vec{g})$	$\prod_s^v D_{m_s m'_s}^j(g) i_{m'_1 \dots m'_v}^i$ $= i_{m_1 \dots m_v}^i$	$\prod_s^v U_{\mu_s \mu'_s} T_{\mu'_1 \dots \mu'_v} =$ $T_{\mu_1 \dots \mu_v}$
state	$ \Phi_\Gamma^{g_\ell}\rangle \equiv$ $\otimes_\ell \langle M_{g_\ell} \otimes_n \psi_n\rangle$	$ \Psi_\Gamma^{\mathbf{j}i}\rangle \equiv$ $\otimes_\ell \langle M^{j_\ell} \otimes_n \phi_n^{\mathbf{j}i_n}\rangle$	$ \Psi_{\mathcal{N}}\rangle \equiv$ $\otimes_\ell^L \langle M_\ell \otimes_n^N T_n\rangle$
indices	$g_i \in G$, $ g_i\rangle \in \mathbb{H} \simeq L^2[G]$	$m_i \in \mathbb{H}_j$, SU(2) spin- j irrep.	$\mu_i \in \mathbb{Z}_n$, n th cyclic group
dim	∞	$\dim \mathbb{H}_j = 2j + 1$	$\dim \mathbb{Z}_n = n$

In the following sections, with the longer-term goal of a full understanding and computation of the RT formula in the field-theoretic GFT context, we are going to use the inputs provided by the established dictionary between GFT states and (random) tensor networks to reproduce the RT formula in three different cases corresponding to three different truncations/approximations, suggested by the established correspondence. In the next section, we derive a RT formula by calculating the 2nd Rényi entropy, reproducing the original argument given in [23] for the case of a random tensor network with additional gauge symmetry described above. Then, we further generalize the approach by means of the GFT formalism and spin network techniques, as further steps towards the calculation of the RT formula within a complete quantum gravity setting. We expect the random character at the core of the original derivation to be naturally captured by our field theoretic generalization. In particular, the correspondence will allow us to use the standard path integral formalism to evaluate the expectation values of entropies and other tensor observables.

3 Ryu-Takayanagi formula for Random Tensor Networks with Gauge Symmetry

The Ryu-Takayanagi formula[20], originally derived in the context of the gauge gravity duality, for continuum fields on a smooth background, shows that the entanglement entropy in $d+1$ dimensional conformal field theories can be obtained from the area of d -dimensional minimal surfaces in AdS_{d+2} . This entropy-area relation is recognised to be of fundamental importance for at least three key reasons. First, it suggests a convenient approach to

the calculation of entanglement entropy of quantum fields (a notoriously difficult problem) via the calculation of geometric quantities which is, most often, much easier. Second, it supports a general relation between entanglement and geometry, in turn leading to the suggestion that the whole of spacetime geometry can be understood as *emergent* from (quantum information-theoretic) properties of non-spatiotemporal quantum building blocks. Of course, this last suggestion has a life on its own and it has been brought forward in many different contexts [63, 69–76]. Third, it provides a concrete and general indication of the holographic character of gravity, which goes beyond the AdS/CFT gauge gravity duality framework.

In this section, we derive the RT formula by performing the calculation of the 2nd Rényi entropy of the random tensor network. A generalized calculation on the N th Rényi entropy will be performed in next section in the context of GFT and spin-networks.

3.1 Replica Trick and Rényi Entropy

We start by recalling some general tools for the computation of the entanglement entropy between two subsystems of a given quantum system.

The entanglement entropy between two subsystems is defined as the von Neumann entropy computed from the reduced density matrix of one subsystem.

Suppose a system S is divided into two regions A and B . The reduced density matrix of region A is defined by tracing out the Hilbert space in part B from the density matrix ρ of the system S :

$$\rho_A = \text{Tr}_B(\rho) \quad , \quad (3.1)$$

and the entanglement entropy between A and B is given by the von Neumann entropy

$$S_{\text{EE}}(A) = -\text{Tr}[\widehat{\rho}_A \ln \widehat{\rho}_A], \quad (3.2)$$

where

$$\widehat{\rho}_A \equiv \frac{\rho_A}{\text{Tr}\rho} \quad (3.3)$$

is the normalized reduced density matrix.

Because of the difficulties in computing directly the von Neumann entropy, one use the so-called replica trick to approach the problem. Contracting N copies of the reduced density matrix ρ_A and taking the logarithm of the trace of ρ_A^N , one obtains the N th-order Rényi entropy

$$S_N(A) = -\frac{1}{N-1} \ln \text{Tr} \widehat{\rho}_A^N \quad . \quad (3.4)$$

We can also define

$$\begin{aligned} Z_A^{(N)} &\equiv \text{Tr} \rho_A^N = \text{Tr}[\rho^{\otimes N} \mathbb{P}(\pi_A^0; N, d)], \\ Z_0^N &\equiv (\text{Tr} \rho)^N, \end{aligned} \quad (3.5)$$

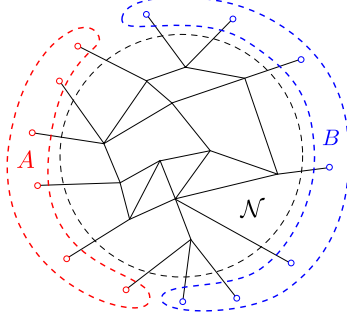


Figure 3. Boundary $\partial\mathcal{N}$ of network \mathcal{N} is divided into two parts A and B .

where $\mathbb{P}(\pi_A^0; N, d)$ is the permutation operator acting on the states in region A ,

$$\mathbb{P}(\pi_A^0; N, d) = \prod_{s=1}^N \delta_{\mu_A^{([s+1]_D)} \mu_A^{(s)}} \quad (3.6)$$

and d is the dimension of the Hilbert space in the same region A .

The replica trick is useful because the Rényi entropy S_N , which is easier to compute, coincides with the von Neumann entropy of region A , and thus with the entanglement entropy between regions A and B , in the limit $N \text{ rightarrow } 1$

$$S_{\text{EE}}(A) = \lim_{N \rightarrow 1} S_N(A) \quad (3.7)$$

3.2 S_2 in RTN with Gauge Symmetry

As the first step, let us calculate the S_2 for a give tensor network state $|\Psi_{\mathcal{N}}\rangle$. The tensor network state $|\Psi_{\mathcal{N}}\rangle$ is given by (2.32), which is in the Hilbert space of $\mathbb{H}_{\partial\mathcal{N}}$. States can be written in terms of index notation.

$$|\Psi_{\mathcal{N}}\rangle \iff \Psi_{\{\lambda_A\}\{\lambda_B\}} \equiv \Psi_{AB} \quad (3.8)$$

$$\bigotimes_n |T_n\rangle \iff \left(\bigotimes_n T_n \right)_{\{\lambda_A\}\{\lambda_B\}\{\lambda_C\}} \equiv \mathcal{T}_{ABC} \quad (3.9)$$

$$\bigotimes_\ell \langle M_\ell| \iff \left(\bigotimes_\ell \bar{M}_\ell \right)_{\{\lambda_C\}} \equiv \bar{\mathcal{M}}_C \quad (3.10)$$

So based on the definition (2.32), the tensor network state is rewritten as

$$\Psi_{AB} = \bar{\mathcal{M}}_C \mathcal{T}_{ABC} \quad (3.11)$$

where we divide the boundary $\partial\mathcal{N}$ into two parts, labeled as A and B .

All links are internal links that contract with nodes. The density matrix corresponding to Ψ_{AB} is

$$\rho_{\bar{A}\bar{A}\bar{B}\bar{B}} = \bar{\Psi}_{\bar{A}\bar{B}} \Psi_{AB} \quad (3.12)$$

Then the S_2 is defined as

$$e^{-S_2} = \frac{Z_A^{(2)}}{Z_0^{(2)}} \quad (3.13)$$

$$Z_A^{(2)} = \text{Tr}[\rho^{\otimes 2} \mathbb{P}(\pi_A^0; 2, d)] \quad (3.14)$$

Because when $n = 2$, the cyclic group \mathcal{S}_n only has two elements: identity $\mathbb{1}$ and swap operator \mathbb{F} , so

$$\mathbb{P}(\pi_A^0; 2, d) \equiv \mathbb{F}^{(A)} \quad (3.15)$$

Then

$$\begin{aligned} Z_A^{(2)} &= \text{Tr}[\rho^{\otimes 2} \mathbb{F}^{(A)}] = \\ &= \rho_{\bar{A}_1 A_1 \bar{B}_1 B_1} \rho_{\bar{A}_2 A_2 \bar{B}_2 B_2} \mathbb{F}_{\bar{A}_1 A_1 \bar{A}_2 A_2}^{(A)} \mathbb{1}_{\bar{B}_1 B_1 \bar{B}_2 B_2}^{(B)} \end{aligned} \quad (3.16)$$

$$\begin{aligned} Z_0^{(2)} &= \text{Tr}[\rho^{\otimes 2}] = \\ &= \rho_{\bar{A}_1 A_1 \bar{B}_1 B_1} \rho_{\bar{A}_2 A_2 \bar{B}_2 B_2} \mathbb{1}_{\bar{A}_1 A_1 \bar{A}_2 A_2}^{(A)} \mathbb{1}_{\bar{B}_1 B_1 \bar{B}_2 B_2}^{(B)} \end{aligned} \quad (3.17)$$

Now let us average the random tensor at each node. As shown in the previous section, the average of the tensor with symmetry (2.44) is given by (B.10) and (B.4)

$$\mathbb{E}_T(\rho_{T_n}^{\otimes 2}) = \frac{1}{D^n D^{v-1}[2]} \sum_{m(1)m(2)} (\mathbb{1}_{m(1)m(2)} + \mathbb{F}_{m(1)m(2)}) \quad (3.18)$$

As defined in (A.4), $D^{v-1}[2] = D^{v-1}(D^{v-1} + 1)$. And let us define the normalization as

$$\mathbb{D}_2 = D^2 D^{v-1}[2] \quad (3.19)$$

For the whole density matrix of tensor with N nodes,

$$\begin{aligned} \mathbb{E}_T(\rho^{\otimes 2}) &= \text{Tr}_C \left[\bigotimes_n \mathbb{E}_T(\rho_{T_n}^{\otimes 2}) \rho_M^{\otimes 2} \right] \\ &= \frac{1}{\mathbb{D}_2^N} \text{Tr}_C \left[\bigotimes_n \sum_{m_n(1)m_n(2)} (\mathbb{1}_{m_n(1)m_n(2)} + \mathbb{F}_{m_n(1)m_n(2)}) \rho_M^{\otimes 2} \right] \\ &= \frac{1}{\mathbb{D}_2^N} \sum_{\{m_n(1)\}\{m_n(2)\}} \text{Tr}_C \left[\bigotimes_n \mathbb{1}_{m_n(1)m_n(2)} \bigotimes_{n'} \mathbb{F}_{m_{n'}(1)m_{n'}(2)} \rho_M^{\otimes 2} \right] \end{aligned} \quad (3.20)$$

For a N -node network, the sum above, naively, is a sum of $(2D^2)^N$ terms, $2D^2$ choices for each node. But we will see later that several of the terms are zero. Consider a link connecting two nodes, there are three classes of situations:

1. $\mathbb{1}_{m(1)m(2)}$ and $\mathbb{1}_{m'(1)m'(2)}$

$$\begin{aligned}
& \text{Tr} \left[\mathbb{1}_{m(1)m(2)} \rho_{M_\ell}^{\otimes 2} \mathbb{1}_{m'(1)m'(2)} \right] \\
&= \delta_{[\lambda(1)-m(1)]_D, \bar{\lambda}(1)} \delta_{[\lambda(2)-m(2)]_D, \bar{\lambda}(2)} \delta_{[\lambda'(1)-m'(1)]_D, \bar{\lambda}'(1)} \times \\
&\quad \times \delta_{[\lambda'(2)-m'(2)]_D, \bar{\lambda}'(2)} \delta_{\lambda(1)\lambda'(1)} \delta_{\bar{\lambda}(1), \bar{\lambda}'(1)} \delta_{\lambda(2)\lambda'(2)} \delta_{\bar{\lambda}(2), \bar{\lambda}'(2)} \\
&= \delta_{[\lambda(1)-m(1)]_D, \bar{\lambda}(1)} \delta_{[\lambda(2)-m(2)]_D, \bar{\lambda}(2)} \delta_{[\lambda(1)-m'(1)]_D, \bar{\lambda}(1)} \\
&\quad \times \delta_{[\lambda(2)-m'(2)]_D, \bar{\lambda}(2)} \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
&= \delta_{[\lambda(1)-m(1)]_D, [\lambda(1)-m'(1)]_D} \delta_{[\lambda(2)-m(2)]_D, [\lambda(2)-m'(2)]_D} \\
&= D^2 \delta_{m(1)m'(1)} \delta_{m(2)m'(2)} \tag{3.22}
\end{aligned}$$

2. $\mathbb{F}_{m(1)m(2)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\begin{aligned}
& \text{Tr} \left[\mathbb{F}_{m(1)m(2)} \rho_{M_\ell}^{\otimes 2} \mathbb{F}_{m'(1)m'(2)} \right] \\
&= \delta_{[\lambda(1)-m(1)]_D, \bar{\lambda}(2)} \delta_{[\lambda(2)-m(2)]_D, \bar{\lambda}(1)} \delta_{[\lambda'(1)-m'(1)]_D, \bar{\lambda}'(2)} \times \\
&\quad \times \delta_{[\lambda'(2)-m'(2)]_D, \bar{\lambda}'(1)} \delta_{\lambda(1)\lambda'(1)} \delta_{\bar{\lambda}(1), \bar{\lambda}'(1)} \delta_{\lambda(2)\lambda'(2)} \delta_{\bar{\lambda}(2), \bar{\lambda}'(2)} \\
&= \delta_{[\lambda(1)-m(1)]_D, \bar{\lambda}(2)} \delta_{[\lambda(2)-m(2)]_D, \bar{\lambda}(1)} \delta_{[\lambda(1)-m'(1)]_D, \bar{\lambda}(2)} \times \\
&\quad \times \delta_{[\lambda(2)-m'(2)]_D, \bar{\lambda}(1)} \\
&= \delta_{[\lambda(1)-m(1)]_D, [\lambda(1)-m'(1)]_D} \delta_{[\lambda(2)-m(2)]_D, [\lambda(2)-m'(2)]_D} \\
&= D^2 \delta_{m(1)m'(1)} \delta_{m(2)m'(2)} \tag{3.23}
\end{aligned}$$

3. $\mathbb{1}_{m(1)m(2)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\begin{aligned}
& \text{Tr} \left[\mathbb{1}_{m(1)m(2)} \rho_{M_\ell}^{\otimes 2} \mathbb{F}_{m'(1)m'(2)} \right] \\
&= \delta_{[\lambda(1)-m(1)]_D, \bar{\lambda}(1)} \delta_{[\lambda(2)-m(2)]_D, \bar{\lambda}(2)} \delta_{[\lambda'(1)-m'(1)]_D, \bar{\lambda}'(2)} \times \\
&\quad \times \delta_{[\lambda'(2)-m'(2)]_D, \bar{\lambda}'(1)} \delta_{\lambda(1)\lambda'(1)} \delta_{\bar{\lambda}(1), \bar{\lambda}'(1)} \delta_{\lambda(2)\lambda'(2)} \delta_{\bar{\lambda}(2), \bar{\lambda}'(2)} \\
&= \delta_{[\lambda(1)-m(1)]_D, \bar{\lambda}(1)} \delta_{[\lambda(2)-m(2)]_D, \bar{\lambda}(2)} \delta_{[\lambda(1)-m'(1)]_D, \bar{\lambda}(2)} \\
&\quad \delta_{[\lambda(2)-m'(2)]_D, \bar{\lambda}(1)} \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
&= \delta_{[\lambda(1)-m(1)]_D, [\lambda(2)-m'(2)]_D} \delta_{[\lambda(2)-m(2)]_D, [\lambda(1)-m'(1)]_D} \\
&= \delta_{[\lambda(1)-m(1)+m'(2)]_D, [\lambda(1)-m'(1)+m(2)]_D} \tag{3.25}
\end{aligned}$$

$$= D \delta_{[m(1)+m(2)]_D, [m'(1)+m'(2)]_D} \tag{3.26}$$

On the boundary of region A , since at one end of the boundary link is $\mathbb{F}^{(A)} = \mathbb{F}_{00}$, so there are only two classes of situations

1. $\mathbb{F}^{(A)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\begin{aligned}
\text{Tr} \left[\mathbb{F}^{(A)} \rho_{M_\ell}^{\otimes 2} \mathbb{F}_{m'(1)m'(2)} \right] &= \text{Tr} \left[\mathbb{F}_{00} \rho_{M_\ell}^{\otimes 2} \mathbb{F}_{m'(1)m'(2)} \right] \\
&= D^2 \delta_{0m'(1)} \delta_{0m'(2)} \tag{3.27}
\end{aligned}$$

2. $\mathbb{F}^{(A)}$ and $\mathbb{1}_{m'(1)m'(2)}$

$$\begin{aligned}\mathrm{Tr} \left[\mathbb{F}^{(A)} \rho_{M_\ell}^{\otimes 2} \mathbb{1}_{m'(1)m'(2)} \right] &= \mathrm{Tr} \left[\mathbb{F}_{00} \rho_{M_\ell}^{\otimes 2} \mathbb{1}_{m'(1)m'(2)} \right] \\ &= D \delta_{0, [m'(1)+m'(2)]_D}\end{aligned}\quad (3.28)$$

On the boundary of region B , at one end of the boundary link is $\mathbb{1}^{(B)} = \mathbb{1}_{00}$, there exist two classed of situations,

1. $\mathbb{1}^{(B)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\begin{aligned}\mathrm{Tr} \left[\mathbb{1}^{(B)} \rho_{M_\ell}^{\otimes 2} \mathbb{F}_{m'(1)m'(2)} \right] &= \mathrm{Tr} \left[\mathbb{1}_{00} \rho_{M_\ell}^{\otimes 2} \mathbb{F}_{m'(1)m'(2)} \right] \\ &= D \delta_{0, [m'(1)+m'(2)]_D}\end{aligned}\quad (3.29)$$

2. $\mathbb{1}^{(B)}$ and $\mathbb{1}_{m'(1)m'(2)}$

$$\begin{aligned}\mathrm{Tr} \left[\mathbb{1}^{(B)} \rho_{M_\ell}^{\otimes 2} \mathbb{1}_{m'(1)m'(2)} \right] &= \mathrm{Tr} \left[\mathbb{1}_{00} \rho_{M_\ell}^{\otimes 2} \mathbb{1}_{m'(1)m'(2)} \right] \\ &= D^2 \delta_{0m'(1)} \delta_{0m'(2)}\end{aligned}\quad (3.30)$$

There are several main observations from the above calculations:

1. The calculations on averaging $Z_A^{(2)}$ and $Z_0^{(2)}$ over T is like to set a series of networks where each node is assigned with a matrix $\mathbb{1}_{m(1)m(2)}$ or $\mathbb{F}_{m(1)m(2)}$ and the boundary is assigned with $\mathbb{F}^{(A)}$ and $\mathbb{1}^{(B)}$ for $Z_A^{(2)}$, and $\mathbb{1}^{(A)}$ and $\mathbb{1}^{(B)}$ for $Z_0^{(2)}$.
2. For all $\mathbb{1}_{m(1)m(2)}$ and $\mathbb{F}_{m(1)m(2)}$, the ones with $[m(1)+m(2)]_D \neq 0$ will never contribute to $Z_A^{(2)}$ and $Z_0^{(2)}$. In fact, if there is one node has $[m(1) + m(2)]_D \neq 0$, it will make its neighboring node satisfying $[m(1) + m(2)]_D \neq 0$, and these nodes will make their neighboring nodes satisfying $[m(1) + m(2)]_D \neq 0$. Because all nodes connect to the boundary through a certain number of links, the consequence is that the boundary should be $[m(1) + m(2)]_D \neq 0$, but we assume that the boundary is assigned by $\mathbb{1}_{00}$ or \mathbb{F}_{00} , i.e. $[m(1) + m(2)]_D = 0$. Non of the matrix at each node will satisfy $[m(1) + m(2)]_D \neq 0$. All matrices should satisfy $[m(1) + m(2)]_D = 0$. So in the following discussion we only consider the matrices of $\mathbb{F}_{m(1)m(2)}$ and $\mathbb{1}_{m(1)m(2)}$ with $[m(1) + m(2)]_D = 0$
3. If a node is $\mathbb{F}_{m(1)m(2)}$, then its neighboring nodes can only be $\mathbb{F}_{m(1)m(2)}$ or $\mathbb{1}_{m'(1)m'(2)}$ with $[m'(1) + m'(2)]_D = 0$; vice versa. So the network is divided into several regions, where in each region, all nodes are associated with the same matrix. If a region is with $\mathbb{F}_{m(1)m(2)}$, its neighboring region can only be with $\mathbb{1}_{m'(1)m'(2)}$, vice versa. An example is shown in Figure (3). Each regions a labeled with \mathbb{F} or $\mathbb{1}$. The boundary of these regions are called the domain walls. The domain walls are the links, that one end of which is assigned with \mathbb{F} and the other ends is with $\mathbb{1}$.

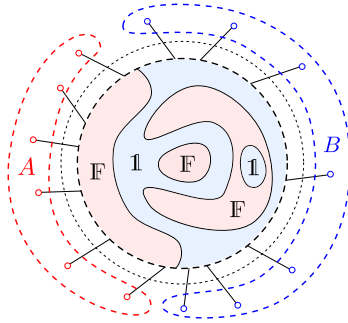


Figure 4. Network with boundary condition: $A = \mathbb{F}^{(A)}$ and $B = \mathbb{1}^{(B)}$ is divided into regions. The nodes in red regions are assigned with \mathbb{F} and the ones in blue are with $\mathbb{1}$.

4. As shown in this figure, the domain walls form different *patterns* \mathcal{P} for the network. For a given pattern, changing a region's label from $\mathbb{F}_{m(1)m(2)}$ to $\mathbb{F}_{m'(1)m'(2)}$ will not change the value of its corresponding term in $Z_A^{(2)}$ or $Z_0^{(2)}$. It introduces the degeneracy of the region. The degeneracy of the region that does not connected to the boundary is D , which is the number of the possible choice of the pair $(m(1), m(2))$. So

$$\mathbb{E}_T(Z_A^{(2)}) = \frac{1}{\mathbb{D}_2^N} \sum_{\mathcal{P}_A} d_{\mathcal{P}_A} Z_{\mathcal{P}_A}^{(2)}, \quad (3.31)$$

$$\mathbb{E}_T(Z_0^{(2)}) = \frac{1}{\mathbb{D}_2^N} \sum_{\mathcal{P}_0} d_{\mathcal{P}_0} Z_{\mathcal{P}_0}^{(2)} \quad (3.32)$$

where $d_{\mathcal{P}}$ is the degeneracy of the pattern, which is the product of the degeneracy of each region in these pattern. $Z_{\mathcal{P}}^{(2)}$ is given as

$$Z_{\mathcal{P}}^{(2)} = D^{2(L-L_{\mathcal{P}})} D^{L_{\mathcal{P}}} = D^{2L-L_{\mathcal{P}}} \quad (3.33)$$

where L is the total link number in a given network \mathcal{N} , including links across $\partial\mathcal{N}$; $L_{\mathcal{P}}$ is the links across the domain walls in \mathcal{P} .

5. The main contribution of $\mathbb{E}_T(Z_A^{(2)})$ is the pattern with the least number of links through the domain walls. We call this domain wall with least link number the *minimal surface*. One can show that this claim is true even the degeneracy $d_{\mathcal{P}}$ is taken into account. In fact, all patterns can be generated from the one only with minimal surface by deforming the minimal surface and adding new regions. Starting from the pattern with the minimal surface, no matter how to deform the minimal surface, it would not be minimal. So the only choice is to add more regions. However when you add a region, it will contribute to $d_{\mathcal{P}}$ with D but to the number of domain wall links at least $v > 1$, the valence of a single node, thus in total one has to product $D^{1-v} < 1$ to the original $Z_{\mathcal{P}}^{(2)}$, which makes the new one smaller than the original one. So the main contribution of $\mathbb{E}_T(Z_A^{(2)})$ should be from the pattern with only the minimal surface. And in this pattern there is only two regions which is labeled by

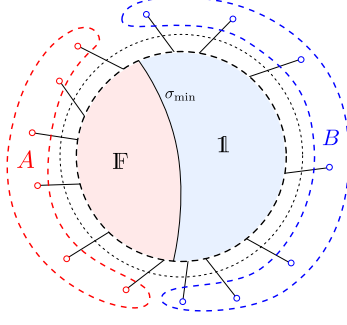


Figure 5. The pattern with only the minimal surface σ_{\min} . Boundary condition: $A = \mathbb{F}^{(A)}$ and $B = \mathbb{1}^{(B)}$.

$\mathbb{F}^{(A)}$ and $\mathbb{1}^{(B)}$, respectively.

$$\mathbb{E}_T(Z_A^{(2)}) = \frac{1}{\mathbb{D}_2^N} D^{2L-L_{\min}} (1 + O(D^{-1})) \quad (3.34)$$

6. The main contribution of $\mathbb{E}_T(Z_0^{(2)})$ is the pattern without any domain walls. This is because its boundary condition is $\mathbb{1}^{(A)} = \mathbb{1}^{(B)} = \mathbb{1}_{00}$. There exist the pattern without any domain wall and all nodes are assigned with $\mathbb{1}_{00}$. Then

$$\mathbb{E}_T(Z_0^{(2)}) = \frac{1}{\mathbb{D}_2^N} D^{2L} (1 + O(D^{-1})) \quad (3.35)$$

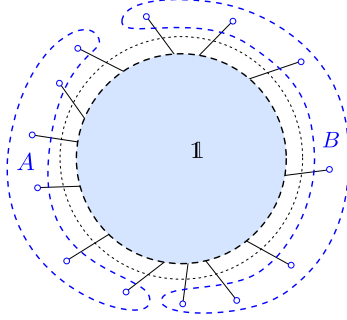


Figure 6. The pattern without the minimal surface. Boundary condition: $A = \mathbb{1}^{(A)}$ and $B = \mathbb{1}^{(B)}$.

The leading contribution of S_2 is given as

$$\mathbb{E}_T(e^{-S_2}) \sim \frac{\mathbb{E}_T(Z_A^{(2)})}{\mathbb{E}_T(Z_0^{(2)})} = D^{-L_{\min}} (1 + O(D^{-1})) \quad (3.36)$$

when $D \gg 1$,

$$S_2 = L_{\min} \ln D + O(D^{-1}) \quad (3.37)$$

This is the Ryu-Takayanagi formula for S_2 .

Interestingly, $\mathbb{E}_T(Z_A^{(2)})$ and $\mathbb{E}_T(Z_0^{(2)})$ can be written as the Ising model. Assign $S_{m(1)m(2)} = 1$ to node with $\mathbb{1}_{m(1)m(2)}$, and assigned $S_{m(1)m(2)} = -1$ to node with $\mathbb{F}_{m(1)m(2)}$. Assign $H = 1$ to the boundary with $\mathbb{1}$ and $H = -1$ to the boundary with \mathbb{F} . Then

$$\begin{aligned} \mathbb{E}_T(Z^{(2)}) &= \frac{1}{\mathbb{D}_2^N} \prod_{\langle nn' \rangle} (\delta_{SS'} \delta_{m(1)m'(1)} \delta_{m(2)m'(2)} + \delta_{S,-S'}) \\ &\times e^{-\frac{\ln D}{2} (SS'+3)} \prod_{\langle n\partial \rangle} (\delta_{HS} \delta_{0m(1)} \delta_{0m(2)} + \delta_{H,-S}) \times \\ &\times e^{-\frac{\ln D}{2} (HS+3)} \end{aligned} \tag{3.38}$$

4 Ryu-Takayanagi formula from Group Field Theory and spin networks

In this section we will rely on the structure of the Hilbert space of quantum gravity states in the context of the group field theory formalism, in turn very closely related to the kinematical Hilbert space of loop quantum gravity (see [64] for more details on the relation). The Rényi entropy of a state associated to a given open graph will be computed in both group and spin-network representations. We will obtain a formula very similar to the Ryu-Takayanagi one.

4.1 Rényi Entropy for GFT states in the group representation as generalised tensor networks

In this subsection, the Rényi entropy is calculated by using the techniques of the group field theory. The starting point is the state $|\Psi_\Gamma\rangle$ corresponding to a network with each node associated to a generalised tensor analogous to a group field. The calculation proceeds along the very same steps performed in the previous section, in the conventional tensor network formalism, as natural given the dictionary we have established between the two languages. This calculation is not in the full GFT setup, i.e. the state $|\Psi_\Gamma\rangle$ is not treated, in the calculation of the averaging over random (generalised) tensors, as an n -point function of a given GFT. This more complete calculation is postponed to a future analysis. Still, we apply several techniques from GFT and generalized the calculations in [23] based on our dictionary:

1. Tensors are generalized to group fields, from a finite dimensional object to a square integrable L^2 function, mapping from group manifolds to the complex numbers \mathbb{C} .
2. A gauge symmetry of the group field associated to each vertex as a vertex wave function is introduced in order to fit our setup more to the context of the quantum gravity theory.
3. The average over the N -replica of the wave functions (generalised tensors) associated to each network vertex is reinterpreted as a N -point correlations of a (simple) GFT model, which turns the Rényi entropy into an amplitude in GFT.

The last point can be seen as an approximation of a more complete calculation in which the (average over the) whole tensor network is understood as a GFT N-point function, and computed as such. This more complete calculation based on the full GFT setup is being explored [77]. We believe that the leading term of the entropy, at least for the entanglement entropy, would not be changed.

We divide the boundary $\partial\Gamma$ of the graph $\Gamma(V_\Gamma, L_\Gamma, L_{\partial\Gamma})$, with V_Γ nodes, L_Γ internal links and $L_{\partial\Gamma}$ boundary links, into two parts, called A and B . The Rényi entropy between A and B can be calculated as follows.

The density matrix ρ is defined as:

$$\rho = |\Psi_\Gamma\rangle \langle \Psi_\Gamma| = \text{Tr}_\ell \left[\bigotimes_\ell |M_\ell\rangle \langle M_\ell| \bigotimes_n |\psi_n\rangle \langle \psi_n| \right] \equiv \text{Tr}_\ell \left[\bigotimes_\ell \rho_\ell \bigotimes_n \rho_n \right] \quad . \quad (4.1)$$

The quantity we are interested in is the N th order Rényi entropy,

$$\begin{aligned} e^{(1-N)S_N} &= \frac{Z_N}{Z_0^N} = \frac{\mathbb{E}(\text{Tr}\rho_A^N)}{\mathbb{E}(\text{Tr}\rho)^N} = \frac{\mathbb{E}\text{Tr}[\rho^{\otimes N} \mathbb{P}(\pi_A^0; N, d)]}{\mathbb{E}(\text{Tr}\rho)^N} \\ &= \frac{\text{Tr} \left[\bigotimes_\ell \rho_\ell^N \bigotimes_n \mathbb{E}(\rho_n^N) \mathbb{P}(\pi_A^0; N, d) \right]}{\text{Tr} \left[\bigotimes_\ell \rho_\ell^N \bigotimes_n \mathbb{E}(\rho_n^N) \right]} \quad . \end{aligned} \quad (4.2)$$

The key step is to compute the quantity

$$\mathbb{E}(\rho_n^N) = \mathbb{E}[|\psi_n\rangle \langle \psi_n|^N] = \mathbb{E} \left[\left(\int \prod_a^N d\mathbf{g}_a d\bar{\mathbf{g}}_a \psi_n(\mathbf{g}_a) \overline{\psi_n(\bar{\mathbf{g}}_a)} |\mathbf{g}_a\rangle \langle \bar{\mathbf{g}}_a| \right) \right] \quad , \quad (4.3)$$

or, more precisely, the one inside the above integral

$$\mathbb{E} \left[\prod_a^N \psi_n(\mathbf{g}_a) \overline{\psi_n(\bar{\mathbf{g}}_a)} \right] \quad . \quad (4.4)$$

where $d\mathbf{g} \equiv \prod_i dg_i$, $\psi(\mathbf{g}) \equiv \psi(g_1, \dots, g_4)$ and $\bar{\mathbf{g}}$ is independent from \mathbf{g} , which denotes the arguments of $\bar{\psi}$. Now we define the averaging $\mathbb{E}[\dots]$ via the path integral of some group field theory model

$$\mathbb{E} [f[\psi, \bar{\psi}]] \equiv \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}] f[\psi, \bar{\psi}] e^{-iS[\psi, \bar{\psi}]} \quad (4.5)$$

where $S[\psi, \bar{\psi}]$ is the action of the given model of interest,

$$S[\psi, \bar{\psi}] = \int d\mathbf{g} d\bar{\mathbf{g}} \overline{\psi(\bar{\mathbf{g}})} \mathcal{K}(\mathbf{g}, \bar{\mathbf{g}}) \psi(\mathbf{g}) + \lambda S_{\text{int}}[\psi, \bar{\psi}] + cc. \quad (4.6)$$

where the first term on the right hand side is the kinetic term. In the following calculation we will consider the case in which

$$\mathcal{K}(\mathbf{g}, \bar{\mathbf{g}}) = \delta(\mathbf{g}^\dagger \bar{\mathbf{g}}) \quad , \quad (4.7)$$

thus a free part of the action of the simple form

$$S_0[\psi, \bar{\psi}] = \int d\mathbf{g} \bar{\psi}(\mathbf{g})\psi(\mathbf{g})^{10} \quad . \quad (4.8)$$

We further assume that the coupling constant λ is much smaller than 1, so the path integral $\mathbb{E}[f[\psi, \bar{\psi}]]$ can be perturbatively expanded in powers of λ

$$\begin{aligned} \mathbb{E}[f[\psi, \bar{\psi}]] &= \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}] f[\psi, \bar{\psi}] e^{-iS_0[\psi, \bar{\psi}]} (1 + \lambda S_{\text{int}}[\psi, \bar{\psi}] + \mathcal{O}(\lambda^2)) \\ &\equiv \mathbb{E}_0[f[\psi, \bar{\psi}]] + \mathcal{O}(\lambda) \quad . \end{aligned} \quad (4.9)$$

This is the regime of validity of the so-called spin foam expansion, seen from within the GFT formalism [1–9]. In the following calculation, we will only focus on the leading term $\mathbb{E}_0[f[\psi, \bar{\psi}]]^{11}$.

Because of the gauge symmetry $\psi(h\mathbf{g}) = \psi(\mathbf{g})$, the gauge equivalent paths in the above path integral have to be removed (via gauge fixing). In order to do so, we first introduce the following notation: if $\mathbf{g} = (g_1, g_2, g_3, g_4)$, then

$$[\mathbf{g}] \equiv g_1^{-1}\mathbf{g} = (\mathbf{1}, g_1^{-1}g_2, g_1^{-1}g_3, g_1^{-1}g_4) \quad . \quad (4.10)$$

Then, we insert the delta functional

$$\delta[\psi(\mathbf{g}) - \psi([\mathbf{g}])] \quad (4.11)$$

into the path integral, so that the average becomes

$$\mathbb{E}_0[f[\psi, \bar{\psi}]] \equiv \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}] f[\psi, \bar{\psi}] \delta[\psi(\mathbf{g}) - \psi([\mathbf{g}])] e^{-i \int d\mathbf{g} \bar{\psi}(\mathbf{g})\psi(\mathbf{g})} \quad . \quad (4.12)$$

Since this equation is simply the expectation value of $f[\psi, \bar{\psi}]$ in the free group field theory, we can immediately give the expectation value of (4.4) via Wick theorem:

$$\begin{aligned} \mathbb{E}_0 \left[\prod_a^N \psi(\mathbf{g}_a) \bar{\psi}(\underline{\mathbf{g}}_a) \right] &= \mathcal{C} \sum_{\pi \in \mathcal{S}_N} \prod_a^N \delta([\mathbf{g}_a][\underline{\mathbf{g}}_{\pi(a)}]^\dagger) \\ &= \mathcal{C} \sum_{\pi \in \mathcal{S}_N} \int \prod_a^N dh_a \prod_a^N \delta(h_a \mathbf{g}_a \underline{\mathbf{g}}_{\pi(a)}^\dagger) \quad , \end{aligned} \quad (4.13)$$

where $\underline{\mathbf{g}}$ is independent from \mathbf{g} , which denotes the arguments of $\bar{\psi}$; and

$$\delta([\mathbf{g}][\underline{\mathbf{g}}]^\dagger) \equiv \prod_{s=2}^4 \delta(g_1^\dagger g_s g_s^\dagger g_1) \quad \delta(h\mathbf{g}\underline{\mathbf{g}}) \equiv \prod_{s=1}^4 \delta(h g_s g_s^\dagger) \quad . \quad (4.14)$$

In the second equality, we re-introduce the gauge symmetry by inserting integrals of $h_a \in \text{SU}(2)$, $N = 1, 2, \dots, N$ into the delta functions such that g_{sa} on each leg of the node are on an

¹⁰Notice that several GFT models of quantum gravity [6–9] can be put in this form.

¹¹This, in turn, means that, from the point of view of the quantum gravity model, the quantum dynamics is not imposed and we confine ourselves to a purely kinematical evaluation of the observable of interest.

equal footing, unlike $g_1 = \mathbf{1}$ in the gauge fixing procedure. So in the following calculation, the network is without gauge fixing, i.e. all integrals of \mathbf{g} have to be performed.

Denote now $\prod_a^N \delta \left(h_a \underline{\mathbf{g}}_a \underline{\mathbf{g}}_{\pi(a)}^\dagger \right)$ as

$$\mathbb{P}_{\mathbf{h}}(\pi) \equiv \prod_a^N \delta \left(h_a \underline{\mathbf{g}}_a \underline{\mathbf{g}}_{\pi(a)}^\dagger \right) = \prod_{s=1}^4 \prod_a^N \delta \left(h_a g_{sa} \underline{g}_{s\pi(a)}^\dagger \right) \equiv \prod_s^4 \mathbb{P}_{\mathbf{h}}^s(\pi) \quad , \quad (4.15)$$

where \mathbf{h} denotes the set of h_a , $a = 1, \dots, N$. When $h_a = \mathbf{1}$ for all a from 1 to N ,

$$\mathbb{P}_{\mathbf{1}}(\pi) = \prod_a^N \delta \left(\underline{\mathbf{g}}_a \underline{\mathbf{g}}_{\pi(a)}^\dagger \right) = \mathbb{P}(\pi; N, D^4) = \prod_s^4 \mathbb{P}^s(\pi; N, D^4) \quad (4.16)$$

where $\mathbb{P}(\pi; N, D^4)$ and $\mathbb{P}^s(\pi; N, D^4)$ are the representations of $\pi \in \mathcal{S}_N$ on $\mathbb{H}^{\otimes 4}$ and \mathbb{H} , respectively.

Then, Z_N and Z_0^N become

$$\begin{aligned} Z_N &\approx \mathcal{C}^{\text{V}\Gamma} \sum_{\pi_n \in \mathcal{S}_N} \int \prod_n d\mathbf{h}_n \text{Tr} \left[\bigotimes_{\ell} \rho_{\ell}^N \bigotimes_n \mathbb{P}_{\mathbf{h}_n}(\pi_n) \mathbb{P}(\pi_A^0; N, d) \right] \\ &\equiv \mathcal{C}^{\text{V}\Gamma} \sum_{\pi_n \in \mathcal{S}_N} \int \prod_n d\mathbf{h}_n \mathcal{N}_A(\mathbf{h}_n, \pi_n) \end{aligned} \quad (4.17)$$

$$\begin{aligned} Z_0^N &= \mathcal{C}^{\text{V}\Gamma} \sum_{\pi_n \in \mathcal{S}_N} \int \prod_n d\mathbf{h}_n \text{Tr} \left[\bigotimes_{\ell} \rho_{\ell}^N \bigotimes_n \mathbb{P}_{\mathbf{h}_n}(\pi_n) \right] \\ &\equiv \mathcal{C}^{\text{V}\Gamma} \sum_{\pi_n \in \mathcal{S}_N} \int \prod_n d\mathbf{h}_n \mathcal{N}_0(\mathbf{h}_n, \pi_n) \quad , \end{aligned} \quad (4.18)$$

which means that Z_N and Z_0^N correspond to summations of the networks $\mathcal{N}_A(\mathbf{h}_n, \pi_n)$ and $\mathcal{N}_0(\mathbf{h}_n, \pi_n)$ where at each node n we have a contribution $\mathbb{P}_{\mathbf{h}_n}(\pi_n)$ and at each link ℓ we have a contribution ρ_{ℓ}^N . The only difference between these two networks is the boundary condition: where Z_N is defined with $\mathbb{P}(\pi_A^0; N, d)$ on A of $\partial\Gamma$ and $\mathbb{P}(\mathbf{1}; N, d)$ on \bar{A} of $\partial\Gamma$, and Z_0^N is defined with $\mathbb{P}(\mathbf{1}; N, d)$ for all boundary region $\partial\Gamma$.

Since at each node $\mathbb{P}_{\mathbf{h}_n}(\pi_n)$ is decoupled among the incident legs, because of (4.15), the value of the networks $\mathcal{N}_A(\mathbf{h}_n, \pi_n)$ and $\mathcal{N}_0(\mathbf{h}_n, \pi_n)$ can be written as products factorised over links:

$$\mathcal{N}_A(\mathbf{h}_n, \pi_n) = \prod_{\ell \in \Gamma} \mathcal{L}_{\ell}(\pi_n, \pi_{n'}; \mathbf{h}_n, \mathbf{h}_{n'}) \prod_{\ell \in A} \mathcal{L}_{\ell}(\pi_n, \pi_A^0; \mathbf{h}_n) \prod_{\ell \in \bar{A}} \mathcal{L}_{\ell}(\pi_n, \mathbf{1}; \mathbf{h}_n) \quad (4.19)$$

$$\mathcal{N}_0(\mathbf{h}_n, \pi_n) = \prod_{\ell \in \Gamma} \mathcal{L}_{\ell}(\pi_n, \pi_{n'}; \mathbf{h}_n, \mathbf{h}_{n'}) \prod_{\ell \in \partial\Gamma} \mathcal{L}_{\ell}(\pi_n, \mathbf{1}; \mathbf{h}_n) \quad . \quad (4.20)$$

Because the \mathcal{L}_{ℓ} on the boundary are special cases of the \mathcal{L}_{ℓ} in the graph Γ , it is enough to calculate the \mathcal{L}_{ℓ} on the internal links. In general, $\mathcal{L}(\pi, \pi', \mathbf{h}, \mathbf{h}')$ can be written as a trace of a modified representation of a permutation group element $\varpi \equiv (\pi')^{-1}\pi$ as

$$\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}') = \text{Tr} \left[\mathbb{P}_{\mathbf{h}}(\pi) \rho_{\ell}^N \mathbb{P}_{\mathbf{h}'}(\pi) \right] = \text{Tr} \left[\mathbb{P}_{\mathbf{H}} \left((\pi')^{-1}\pi \right) \right] \equiv \text{Tr} \left[\mathbb{P}_{\mathbf{H}}(\varpi) \right] \quad , \quad (4.21)$$

where

$$\mathbf{H} = \left\{ H_a \mid H_a \equiv \left(h'_{\varpi(a)} \right)^\dagger h_a, \forall a = 1, \dots, N \right\} \quad (4.22)$$

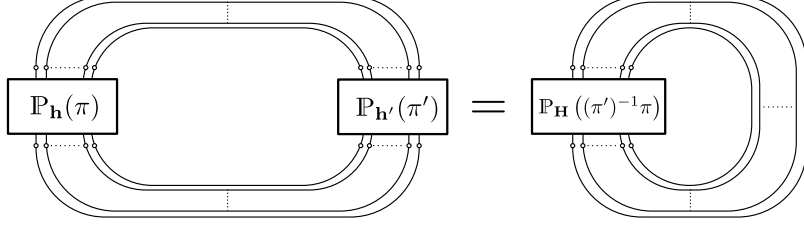


Figure 7. $\mathcal{L}(\pi, \pi', \mathbf{h}, \mathbf{h}')$

When $\pi = \pi'$, we have $\varpi = \mathbb{1}$ and $\mathbf{H} = (\mathbf{h}')^\dagger \mathbf{h}$, and then

$$\mathcal{L}(\pi, \pi; \mathbf{h}, \mathbf{h}') = \text{Tr} [\mathbb{P}_{\mathbf{h}}(\pi) \rho_\ell^N \mathbb{P}_{\mathbf{h}'}(\pi)] = \text{Tr} [\mathbb{P}_{(\mathbf{h}')^\dagger \mathbf{h}}(\mathbb{1})] \quad (4.23)$$

$$\begin{aligned} &= \prod_a^N \int dg_a dg'_a dg_{\pi(a)} dg'_{\pi(a)} \delta \left(h_a g_a g_{\pi(a)}^\dagger \right) \delta \left(g_{\pi(a)} g'_{\pi(a)}^\dagger \right) \times \\ &\quad \times \delta \left(h'_a g'_a g'_{\pi(a)}^\dagger \right) \delta \left(g'_a g_a^\dagger \right) \\ &= \prod_a^N \delta \left((h'_a)^\dagger h_a \right) = \prod_a^N \delta \left(H_a \right) \quad , \end{aligned} \quad (4.24)$$

The above equation can be depicted graphically as in Fig.8

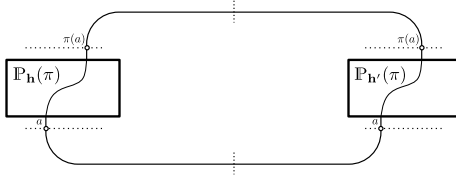


Figure 8. $\mathcal{L}(\pi, \pi, \mathbf{h}, \mathbf{h}')$

When $\pi \neq \pi'$, we have

$$\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}') = \text{Tr} [\mathbb{P}_{\mathbf{H}}(\varpi)] \quad . \quad (4.25)$$

In order to perform the computation, it is necessary to use some facts about the permutation group \mathcal{S}_N , which we recall briefly, before proceeding.

- Any element $\varpi \in \mathcal{S}_N$ can be expressed as the product of disjoint cycles \mathcal{C}_i

$$\varpi \equiv \prod_i^{\chi(\varpi)} \mathcal{C}_i \quad (4.26)$$

where $1 \leq \chi(\varpi) \leq N$ is the number of cycles in ϖ , which is 1 when ϖ is a 1-cycle and is N only when $\varpi = \mathbb{1}$. For instance, the permutation $\varpi = \{3241\}$ can be

expressed as a product of two cycles (134)(2), in which $\varpi(1) = 3, \varpi(3) = 4, \varpi(4) = 1$ and $\varpi(2) = 2$. (132) is a 3-cycle, because there are three elements in the cycle. We denote the number of elements in the cycle \mathcal{C}_i as r_i , which is also called the length of the cycle. We also have $\sum_i r_i = N$. Although the cycles \mathcal{C}_i commute with each other, we order the cycles such that

$$1 \leq \dots \leq r_i \leq r_{i+1} \leq \dots \leq N \quad . \quad (4.27)$$

We denote a_k^i , where k is from 1 to r_i , the elements of \mathcal{C}_i , and then we furthermore assume that

$$\varpi(a_k^i) = a_{[k]_{r_i}+1}^i \quad . \quad (4.28)$$

Thus, the cycle can be written as

$$\mathcal{C}_i = (a_1^i a_2^i \dots a_{r_i}^i) \quad . \quad (4.29)$$

- The trace of $\mathbb{P}_{\mathbf{H}}(\varpi)$ can be expressed as the product of the traces of the individual cycles \mathcal{C}_i

$$\text{Tr} [\mathbb{P}_{\mathbf{H}}(\varpi)] = \prod_i \text{Tr} [\mathbb{P}_{\mathbf{H}}(\mathcal{C}_i)] \quad . \quad (4.30)$$

Using the definition of $\mathbb{P}_{\mathbf{H}}$, one can immediately obtain the trace of the cycle \mathcal{C}_i as

$$\text{Tr} [\mathbb{P}_{\mathbf{H}}(\mathcal{C}_i)] = \int \prod_{k=1}^{r_i} dg_{a_k^i} \delta \left(H_{a_k^i} g_{a_k^i} g_{a_{[k]_{r_i}+1}^i}^\dagger \right) = \delta \left(\overleftarrow{\prod}_{k=1}^{r_i} H_{a_k^i} \right) \quad , \quad (4.31)$$

where

$$\overleftarrow{\prod}_{k=1}^{r_i} H_{a_k^i} \equiv H_{a_{r_i}^i} \dots H_{a_2^i} H_{a_1^i} \quad . \quad (4.32)$$

Then the trace of $\mathbb{P}_{\mathbf{H}}(\varpi)$ is

$$\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}') = \text{Tr} [\mathbb{P}_{\mathbf{H}}(\varpi)] = \prod_i^{\chi(\varpi)} \delta \left(\overleftarrow{\prod}_{k=1}^{r_i} H_{a_k^i} \right) \quad . \quad (4.33)$$

- On the boundary of \mathcal{N}_0 and B of \mathcal{N}_A , $\mathcal{L}(\pi, \mathbf{1}; \mathbf{h})$ is a very special case of $\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}')$ where $\pi' = \mathbf{1}$ and $\mathbf{h}' = \mathbf{1}$

$$\mathcal{L}(\pi, \mathbf{1}; \mathbf{h}) \equiv \mathcal{L}(\pi, \mathbf{1}; \mathbf{h}, \mathbf{1}) = \text{Tr} [\mathbb{P}_{\mathbf{h}}(\pi)] = \prod_i^{\chi(\pi)} \delta \left(\overleftarrow{\prod}_{k=1}^{r_i} h_{a_k^i} \right) \quad (4.34)$$

On the boundary A of \mathcal{N}_A , $\mathcal{L}(\pi, \pi_A^0; \mathbf{h})$ corresponds also to a special case of $\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}')$, where $\mathbf{h}' = \mathbf{1}$ and $\pi' = \pi_A^0 = \mathcal{C}_0$, which is the N -cycle that for any integer k from 1 to N , $\mathcal{C}_0(k) = [k]_N + 1$

$$\mathcal{L}(\pi, \pi_A^0; \mathbf{h}) \equiv \mathcal{L}(\pi, \mathcal{C}_0; \mathbf{h}, \mathbf{1}) = \text{Tr} [\mathbb{P}_{\mathbf{h}}(\mathcal{C}_0^{-1}\pi)] = \prod_i^{\chi(\mathcal{C}_0^{-1}\pi)} \delta \left(\overleftarrow{\prod}_{k=1}^{r_i} h_{a_k^i} \right) \quad . \quad (4.35)$$

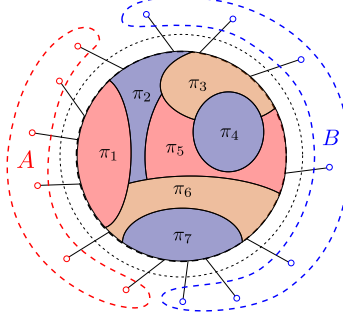


Figure 9. An example of pattern

Altogether, for a given network $\mathcal{N}(\mathbf{h}_n, \boldsymbol{\pi}_n)$, defining the new variables $\varpi \equiv (\pi')^{-1}\pi$ and \mathbf{H} given by (4.22) for each link, the corresponding link value is a product of $\chi(\varpi)$ delta function

$$\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}') \equiv \mathcal{L}(\varpi; \mathbf{H}) = \text{Tr} [\mathbb{P}_{\mathbf{H}}(\varpi)] = \prod_i \chi(\varpi) \delta \left(\prod_{k=1}^{\overleftarrow{r}_i} H_{a_k^i} \right) \quad (4.36)$$

In particular, when $\pi = \pi'$, the link value $\mathcal{L}(\pi, \pi; \mathbf{h}, \mathbf{h}')$ is given by a product of N delta functions as shown in (4.23) and we re-present it here

$$\mathcal{L}(\pi, \pi; \mathbf{h}, \mathbf{h}') = \prod_a^N \delta \left((h'_a)^\dagger h_a \right) = \prod_a^N \delta(H_a) \quad , \quad (4.37)$$

which is non-zero only when $\mathbf{h} = \mathbf{h}'$.

So in the end the network is divided into several regions, in each of which π_n and \mathbf{h}_n are the same. The links which connect different regions identify boundaries between each pair of different regions, called again domain walls. Corresponding to different domain walls and different assignments of permutation groups to each region, we have different patterns for the given network. We introduce pattern functions $\mathcal{P}_A(\boldsymbol{\pi}_n)$ and $\mathcal{P}_0(\boldsymbol{\pi}_n)$ such that

$$\mathcal{P}_A(\boldsymbol{\pi}_n) \equiv \int \prod_n d\mathbf{h}_n \mathcal{N}_A(\mathbf{h}_n, \boldsymbol{\pi}_n) \quad (4.38)$$

$$\mathcal{P}_0(\boldsymbol{\pi}_n) \equiv \int \prod_n d\mathbf{h}_n \mathcal{N}_0(\mathbf{h}_n, \boldsymbol{\pi}_n) \quad . \quad (4.39)$$

Given a set of $\{\pi_n\}$, $\mathcal{P}_A(\boldsymbol{\pi}_n)$ and $\mathcal{P}_0(\boldsymbol{\pi}_n)$ correspond to a certain network pattern with fixed boundary conditions, illustrated in the following figure.

More explicitly,

$$\mathcal{P}_A(\boldsymbol{\pi}_n) = \int \prod_n d\mathbf{h}_n \prod_{\ell \in \Gamma} \left[\prod_i \chi(\varpi_\ell) \delta \left(\prod_{k=1}^{\overleftarrow{r}_i} H_{\ell a_k^i} \right) \right] \prod_{\ell \in A} \left[\prod_i \chi(\mathcal{C}_0^{-1} \pi_{n\ell}) \delta \left(\prod_{k=1}^{\overleftarrow{r}_i} h_{\ell a_k^i} \right) \right] \prod_{\ell \in \bar{A}} \left[\prod_i \chi(\pi_{n\ell}) \delta \left(\prod_{k=1}^{\overleftarrow{r}_i} h_{\ell a_k^i} \right) \right] \quad (4.40)$$

$$\mathcal{P}_0(\boldsymbol{\pi}_n) = \int \prod_n d\mathbf{h}_n \prod_{\ell \in \Gamma} \left[\prod_i \chi(\varpi_\ell) \delta \left(\prod_{k=1}^{\overleftarrow{r}_i} H_{\ell a_k^i} \right) \right] \prod_{\ell \in \partial\Gamma} \left[\prod_i \chi(\pi_{n\ell}) \delta \left(\prod_{k=1}^{\overleftarrow{r}_i} h_{\ell a_k^i} \right) \right] \quad (4.41)$$

They are exactly the amplitudes of a topological BF field theory, with given boundary condition, discretized on a specific 2-complex among the N replica of networks, with each different pattern \mathcal{P} corresponding to a different 2-complex. Each edge of the 2-complex is associated with a holonomy h_{na} that is on node n and the a th replica. The two ends of the holonomy are the vertices of the 2-complex. The h_{na} inside a delta function form a loop holonomy, the corresponding edges of which form the face of the 2-complex. Then Z_N and Z_0^N are sum of BF amplitudes with different 2-complexes.

$$Z_N \equiv \mathcal{C}^{V_\Gamma} \sum_{\pi_n \in \mathcal{S}_N} \mathcal{P}_A(\pi_n), \quad Z_0^N \equiv \mathcal{C}^{V_\Gamma} \sum_{\pi_n \in \mathcal{S}_N} \mathcal{P}_0(\pi_n) \quad (4.42)$$

It is important to notice that this simple form of the various functions entering the calculation of the entropy, with the emergence of BF-like amplitudes, is *not* generic. It follows from the choice of GFT kinetic term, from the approximation used in the calculation of expectation values (neglecting GFT interactions) and from the special type of tensor network, in GFT language, that we have chosen (with simple delta functions associated to the links of the network). More involved, and interesting, cases could be considered.

What we are interested in is the leading term of Z_N and Z_0^N , while the dimension D of Hilbert space \mathbb{H} is much larger than 1. This leads us to seek the most divergent term of $\mathcal{P}_A(\pi_n)$ and $\mathcal{P}_0(\pi_n)$. In other words, we need to know the degree of divergence of $\mathcal{P}_A(\pi_n)$ and $\mathcal{P}_0(\pi_n)$. The divergence degree of BF amplitudes discretized on a lattice has been the subject of a number of works, both in the spin foam and GFT literature (see for example [78–80]), the most complete analysis being [81–83].

Let us first focus on a sub-region R of the network such that $\pi_n = \pi$ for all nodes n inside of R . Suppose that there are L_i links inside R and L_e links connecting with other regions. Since we only consider 4-valent nodes, the number of nodes inside R is

$$V \equiv \frac{1}{4} (2L_i + L_e) = \frac{L_i}{2} + \frac{L_e}{4} \quad (4.43)$$

A minimum spanning tree (MST) T , which contains $\#_T = V - 1$ links, can be found in R .

$$T \equiv \{\ell | \ell \in \text{MST}\} \quad (4.44)$$

According to (4.37), since $\pi_n = \pi$, there are N delta functions on each link. The integrals over \mathbf{h}_n would eliminate the $(V - 1)N$ deltas associated to the MST and leave only one set of N integrals over $\mathbf{h} = \{h_a\}$ and $(L_i/2 - L_e/4 + 1)N$ $\delta(\mathbf{1})$'s. Here we keep indicating the divergent factor as the delta function evaluation originating it, but of course it should be understood more properly as a function of the cut-off used to regularize it. The pattern

function of region R is then

$$\begin{aligned}
\mathcal{P}_R(\pi) &\equiv \int \prod_{n \in R} d\mathbf{h}_n \prod_{\ell}^{L_i} \prod_a^N \delta(H_{\ell a}) \prod_{\ell}^{L_e} \left[\prod_i^{\chi(\varpi_{\ell})} \delta \left(\prod_{k=1}^{\overleftarrow{r_i}} H_{\ell a_k^i} \right) \right] \\
&= \int \prod_{n \in R} d\mathbf{h}_n \prod_{\ell \in \text{MST}} \prod_a^N \delta(H_{\ell a}) \prod_{\ell \notin \text{MST}} \prod_a^N \delta(H_{\ell a}) \prod_{\ell}^{L_e} \left[\prod_i^{\chi(\varpi_{\ell})} \delta \left(\prod_{k=1}^{\overleftarrow{r_i}} H_{\ell a_k^i} \right) \right] \\
&= [\delta(\mathbb{1})]^{\left(\frac{L_i}{2} - \frac{L_e}{4} + 1\right)N} \int d\mathbf{h} \prod_{\ell}^{L_e} \left[\prod_i^{\chi(\varpi_{\ell})} \delta \left(\prod_{k=1}^{\overleftarrow{r_i}} H_{\ell a_k^i} \right) \right] . \tag{4.45}
\end{aligned}$$

In the calculation, we have used

$$\int \prod_{n \in R} d\mathbf{h}_n \prod_{\ell \in \text{MST}} \prod_a^N \delta(H_{\ell a}) = \int d\mathbf{h} \tag{4.46}$$

and ($\mathbf{h}_n = \mathbf{h}$)

$$\prod_{\ell \notin \text{MST}} \prod_a^N \delta(H_{\ell a}) = [\delta(\mathbb{1})]^{\left(\frac{L_i}{2} - \frac{L_e}{4} + 1\right)N} . \tag{4.47}$$

The above calculation shows that we can coarse-grain the region R into one single L_e -valent node which is colored by π and \mathbf{h} .

$$\int \prod_{n \in R} d\mathbf{h}_n \int_{R(\pi, \mathbf{h}_n)} = \int \prod_{n \in R} d\mathbf{h}_n \int_{R(\pi, \mathbf{h}_n)}^{\text{MST}} = \int d\mathbf{h} \int_{\pi, \mathbf{h}} [\delta(\mathbb{1})]^{\left(\frac{L_i}{2} - \frac{L_e}{4} + 1\right)N} \tag{4.48}$$

So the degree of divergence in region R is: the number of internal links $\#_i = L_i$ subtracted the number of links in the MST $\#_T = V - 1$, and then times the number of replica N ,

$$\#_R = (\#_i - \#_T)N = (L_i - V + 1)N = \left(\frac{L_i}{2} - \frac{L_e}{4} + 1\right)N . \tag{4.49}$$

Since the boundary condition of \mathcal{N}_0 is $\pi = \mathbb{1}$ and $\mathbf{h} = \mathbb{1}$, the boundary of \mathcal{N}_0 can be coarse-grained into a single node with $\pi = \mathbb{1}$ and $\mathbf{h} = \mathbb{1}$. The same consideration holds for \mathcal{N}_A : its boundary can be coarse-grained into two nodes, one of which corresponds to A with $\pi = \mathcal{C}_0$, $\mathbf{h} = \mathbb{1}$ and the other to B with $\pi = \mathbb{1}$ and $\mathbf{h} = \mathbb{1}$. The corresponding closed graphs are denoted as Γ_0 and Γ_{AB} . A certain pattern $\mathcal{P}(\boldsymbol{\pi}_n)$ divides Γ_0 and Γ_{AB} into M regions that can be coarse-grained into M nodes, each of which is colored with permutation group π_m and N integrals over \mathbf{h}_m . Denote the graph with pattern $\mathcal{P}(\boldsymbol{\pi}_n)$ as $\Gamma_0(\boldsymbol{\pi}_m)$ and $\Gamma_{AB}(\boldsymbol{\pi}_m)$, and denote the corresponding coarse-grained graphs as $\Gamma_0^c(\boldsymbol{\pi}_m)$ and $\Gamma_{AB}^c(\boldsymbol{\pi}_m)$. One can show that, for Γ_0 , the pattern in which all nodes have assigned the same permutation group $\pi = \mathbb{1}$ has the highest degree of divergence $\#_0$.

$$\#_0 = (\#\ell \in \Gamma_0 - \#_{T_{\Gamma_0}})N \tag{4.50}$$

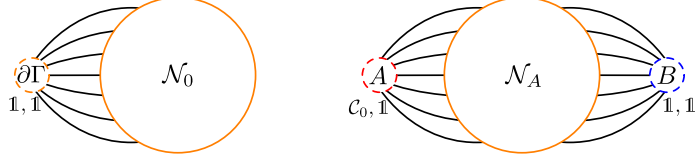


Figure 10. The networks in which the boundaries are coarse-grained into nodes.

where $\#_{\ell \in \Gamma_0}$ is the number of links in graph Γ_0 . Let us consider a coarse-grained graph $\Gamma_0^c(\boldsymbol{\pi}_m)$. Denote the number of links in region m , between regions m and m' , and between region m and boundary $\partial\Gamma$ are L_m , $L_{mm'}$ and L_{m0} , respectively. The proof goes as follows:

1. The permutation group on links between coarse-grained nodes m and m' is $\varpi_{mm'} \equiv \pi_m^{-1} \pi_{m'}$. As given by (4.36), the number of the delta functions on one of the links is the number of the disjoint cycles in ϖ , which is $\chi(\varpi_{mm'}) < N$. Since all links between m and m' are identical, having the same link value, which is given by (4.36), when one integrate over \mathbf{h}_m and $\mathbf{h}_{m'}$, only $\chi(\varpi_{mm'})$ deltas will be eliminated and left with $\delta(\mathbf{1})$ to the order of $\chi(\varpi_{mm'})(L_{mm'} - 1)$ and $2N - \chi(\varpi_{mm'})$ \mathbf{h} integrals. In fact

$$\begin{aligned}
\int d\mathbf{h} d\mathbf{h}' (\text{Tr} [\mathbf{P}_{\mathbf{H}}(\varpi)])^L &= \int d\mathbf{h} d\mathbf{h}' \prod_i^{\chi(\varpi)} \left[\delta \left(\prod_{k=1}^{\overleftarrow{r}_i} H_{a_k^i} \right) \right]^L \\
&= \int d\mathbf{h} \prod_i^{\chi(\varpi)} \left[\delta \left(\prod_{k=1}^{\overleftarrow{r}_i} H_{a_k^i} \prod_{k=1}^{\overrightarrow{r}_i} H_{a_k^i}^\dagger \right) \right]^{L-1} \\
&= [\delta(\mathbf{1})]^{\chi(\varpi)(L-1)} \int d\mathbf{h} \tag{4.51}
\end{aligned}$$

2. MST can be chosen for $\Gamma_0(\boldsymbol{\pi}_m)$, $\Gamma_0^c(\boldsymbol{\pi}_m)$ and M regions. It is obvious that, given a MST T_m for each of the M regions and a MST $T_{\Gamma_0^c(\boldsymbol{\pi}_m)}$ for $\Gamma_0^c(\boldsymbol{\pi}_m)$, rooting from the coarse-grained boundary node $\partial\Gamma$, a MST $T_{\Gamma_0(\boldsymbol{\pi}_m)}$ of $\Gamma_0(\boldsymbol{\pi}_m)$ can be constructed.

$$T_{\Gamma_0(\boldsymbol{\pi}_m)} = \bigcup_m^M T_m \cup T_{\Gamma_0^c(\boldsymbol{\pi}_m)} \tag{4.52}$$

The number of branches of the trees is

$$\#_{T_{\Gamma_0(\boldsymbol{\pi}_m)}} = \sum_m^M \#_{T_m} + \#_{T_{\Gamma_0^c(\boldsymbol{\pi}_m)}} \tag{4.53}$$

3. The degree of divergence of region m is given by (4.49)

$$\#_m = (L_m - \#_{T_m})N \tag{4.54}$$

Similarly, for the divergence degree $\#_{\Gamma_0^c(\mathbf{1})}$ of the pattern where all coarse-grained nodes have the same permutation $\pi_m = \mathbf{1}$ is

$$\#_{\Gamma_0^c(\mathbf{1})} = \left(\sum_{0 \leq m < m' \leq M} L_{mm'} - \#_{T_{\Gamma_0^c(\boldsymbol{\pi}_m)}} \right) N \tag{4.55}$$

The degree of divergence of $\Gamma_0^c(\boldsymbol{\pi}_m)$ is smaller than $\#\Gamma_0^c(\mathbf{1})$

$$\#\Gamma_0^c(\boldsymbol{\pi}_m) < \#\Gamma_0^c(\mathbf{1}) \quad (4.56)$$

This is because, after evaluating the delta functions on the MST $\Gamma_0^c(\boldsymbol{\pi}_m)$ in accordance with (4.51), there are still $MN - \sum_{(mm') \in T_{\Gamma_0^c(\boldsymbol{\pi}_m)}} \chi(\varpi_{mm'})$ integrals over \mathbf{h} . Performing these integrals makes the degree of divergence of $\Gamma_0^c(\boldsymbol{\pi}_m)$ not bigger than the following quantity

$$\#\Gamma_0^c(\boldsymbol{\pi}_m) \leq \sum_{0 \leq m < m' \leq M} L_{mm'} \chi(\varpi_{mm'}) - \sum_{(mm') \in T_{\Gamma_0^c(\boldsymbol{\pi}_m)}} \chi(\varpi_{mm'}) \quad (4.57)$$

$$= \sum_{(mm') \notin T_{\Gamma_0^c(\boldsymbol{\pi}_m)}} L_{mm'} \chi(\varpi_{mm'}) \quad (4.58)$$

$$+ \sum_{(mm') \in T_{\Gamma_0^c(\boldsymbol{\pi}_m)}} (L_{mm'} - 1) \chi(\varpi_{mm'}) \quad (4.59)$$

which is definitely smaller than $\#\Gamma_0(\mathbf{1})$ because $\chi(\varpi_{mm'}) < N$.

4. So the divergence degree of $\Gamma_0(\boldsymbol{\pi}_m)$ is smaller than the divergence degree $\#_0$ for the pattern where all nodes have the same permutation.

$$\#\Gamma_0(\boldsymbol{\pi}_m) = \#\Gamma_0^c(\boldsymbol{\pi}_m) + \sum_m \#_m < \#\Gamma_0^c(\mathbf{1}) + \sum_m \#_m = (\#\ell \in \Gamma_0 - \#_{T_{\Gamma_0}})N = \#_0 \quad (4.60)$$

The leading term of Z_0^N is $\mathcal{P}_0(\mathbf{1})$, whose divergence degree is $\#_0$.

$$Z_0^N = \mathcal{C}^{V_\Gamma} [\delta(\mathbf{1})]^{\#_0} [1 + \mathcal{O}(\delta^{-1}(\mathbf{1})) + \mathcal{O}(\lambda)] \quad (4.61)$$

For Z_N , since the boundary is separated into two parts, the most divergent pattern $\mathcal{P}_A(\boldsymbol{\pi}_n)$ is the one such that its corresponding coarse-grained graph has only two coarse-grained nodes A and B , which are connected by the minimum number of links $\min(\#\ell \in \partial_{AB})$, whose divergence degree is

$$\begin{aligned} \#_{AB} &= \#_A + \#_B + \min(\#\ell \in \partial_{AB}) \\ &= (\#\ell \in \Gamma_{AB} - \min(\#\ell \in \partial_{AB}) - \#_{T_A} - \#_{T_B})N + \min(\#\ell \in \partial_{AB}) \\ &= (\#\ell \in \Gamma_{AB} - \#_{T_A} - \#_{T_B})N + (1 - N) \min(\#\ell \in \partial_{AB}) \\ &= \#_0 + (1 - N) \min(\#\ell \in \partial_{AB}) \end{aligned} \quad (4.62)$$

where the second equality is in terms of (4.49) and the fourth equality is because $\#\ell \in \Gamma_{AB} = \#\ell \in \Gamma_0$ and $\#_{T_A} + \#_{T_B} = \#_{T_{\Gamma_0}}$ ¹².

¹²Since the boundary is coarse-grained into two nodes in Γ_{AB} , there are one more node in Γ_{AB} than in Γ_0 ,

$$V_{\Gamma_{AB}} = V_{\Gamma_0} + 1 \quad (4.63)$$

Thus the number of the branches of the MST in A and B is equal to the number of the MST branches in Γ_0

$$\#_A + \#_B = (V_A - 1) - (V_B - 1) = V_{\Gamma_{AB}} - 2 = V_{\Gamma_0} - 1 = \#_{T_{\Gamma_0}} \quad (4.64)$$

Let us consider a graph $\Gamma_{AB}(\boldsymbol{\pi}_m)$ and its corresponding coarse-grained graph $\Gamma_{AB}^c(\boldsymbol{\pi}_m)$. The divergence degree of $\Gamma_{AB}(\boldsymbol{\pi}_m)$ is given as

$$\#\Gamma_{AB}(\boldsymbol{\pi}_m) = \#\Gamma_{AB}^c(\boldsymbol{\pi}_m) + \sum_{m=\{1,\dots,M,A,B\}} \#_m \quad (4.65)$$

where $\#_m$ is given by (4.49)

$$\#_m = (L_m - \#T_m)N \quad (4.66)$$

Adapting the same argument as for Z_0^N , because of the integral over \mathbf{h}_n , $\#\Gamma_{AB}^c(\boldsymbol{\pi}_m)$ should not be bigger than the following quantity

$$\begin{aligned} \#\Gamma_{AB}^c(\boldsymbol{\pi}_m) &\leq \sum_{(mm') \notin T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^A, T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^B} L_{mm'} \chi(\varpi_{mm'}) \\ &+ \sum_{(mm') \in T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^A \text{ or } T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^B} (L_{mm'} - 1) \chi(\varpi_{mm'}) \end{aligned} \quad (4.67)$$

where we assume $m < m'$ in order to avoid double counting, and $T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^A$ and $T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^B$ are the MST rooting from coarse-grained nodes A and B , respectively. The right hand side of the above formula corresponds to the divergence degree of pattern $\mathcal{P}_A(\boldsymbol{\pi}_m)$ on a graph $\overline{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}$ with all $\mathbf{h}_n = \mathbf{1}$, which differs from $\Gamma_{AB}^c(\boldsymbol{\pi}_m)$ by $T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^A$ and $T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^B$, i.e.

$$\overline{\Gamma_{AB}^c(\boldsymbol{\pi}_m)} \equiv \Gamma_{AB}^c(\boldsymbol{\pi}_m) \setminus \{T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^A, T_{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}^B\} \quad (4.68)$$

As presented in section 2, the major difference between [23, 27] and our paper is that we are considering the gauge transformation \mathbf{h}_n on each node n . When all \mathbf{h}_n are set to be the identity, our Z_N and Z_0^N simplify to the ones in [23, 27] up to overall normalization. In this case, as shown in [23, 27], the patterns which gives only one domain wall for Γ_{AB} have higher divergence degree than the divergent degree of multi-domain walls, which in our language means that the patterns whose corresponding coarse-grained graph contains only two coarse-grained nodes are more divergent than the patterns $\mathcal{P}_A(\boldsymbol{\pi}_m)$, which give more than two coarse-grained nodes. So the divergence degree of the pattern $\mathcal{P}_A(\boldsymbol{\pi}_m)$ on the graph $\overline{\Gamma_{AB}^c(\boldsymbol{\pi}_m)}$ is not bigger than the pattern $\mathcal{P}_A(\mathbf{1}, \mathcal{C}_0)$. So we have

$$\begin{aligned} \#\Gamma_{AB}(\boldsymbol{\pi}_m) &= \#\Gamma_{AB}^c(\boldsymbol{\pi}_m) + \sum_{m=\{1,\dots,M,A,B\}} \#_m \\ &\leq \#\overline{\Gamma_{AB}^c(\boldsymbol{\pi}_m)} + \sum_{m=\{1,\dots,M,A,B\}} \#\bar{m} \\ &\leq \#_A + \#_B + \#\ell \in \partial_{AB} = \#_0 + (1 - N) \#\ell \in \partial_{AB} \\ &\leq \#_{AB} = \#_0 + (1 - N) \min(\#\ell \in \partial_{AB}) \end{aligned} \quad (4.69)$$

It follows that the amplitude Z_N is

$$Z_N = \mathcal{C}^{V_\Gamma} [\delta(\mathbf{1})]^{\#_0 + (1-N) \min(\#\ell \in \partial_{AB})} [1 + \mathcal{O}(\delta^{-1}(\mathbf{1})) + \mathcal{O}(\lambda)] \quad . \quad (4.70)$$

Finally, the N th order Rényi entropy S_N is then:

$$e^{(1-N)S_N} = \frac{Z_N}{Z_0^N} = [\delta(\mathbf{1})]^{(1-N)\min(\#\ell \in \partial_{AB})} [1 + \mathcal{O}(\delta^{-1}(\mathbf{1})) + \mathcal{O}(\lambda)] \quad . \quad (4.71)$$

When N goes to 1, S_N becomes the entanglement entropy S_{EE} . The leading term of the entanglement entropy S_{EE} is therefore

$$S_{EE} = \min(\#\ell \in \partial_{AB}) \ln \delta(\mathbf{1}) \quad , \quad (4.72)$$

which can be understood as the Ryu-Takayanagi formula in a GFT context, with the same interpretation for the area of the minimal surface that we have mentioned in the previous section, concerning the tensor network techniques.

Before moving on to a different derivation of the same result, we want to clarify the interpretation of this calculation.

The definition of the expectation value (4.5) in the GFT language shows that the exponential of S_N can be interpreted as a GFT $2N$ -point function, at least within the limits of the approximation made, focusing on the average over group field functions at each node, without recasting the whole generalized tensor network as a GFT correlation function. As shown in previous sections, the GFT amplitudes can in turn be written, by standard perturbative expansion, as a sum of Feynman amplitudes associated to Feynman diagrams, each of which corresponds to a different discretized “space-time” with fixed boundary, with the Feynman amplitude defining (for quantum gravity models) a lattice path integral for gravity discretised on the corresponding cellular complex. This allows a tentative (and partial) interpretation of the entropy formula we have derived, in geometric spatiotemporal terms. It implies, in fact, that, in the calculation of the entropy, not only the information of a time-slice of a space-time is considered, as encoded in a given network, but also its full quantum dynamics. This, at least, is true when the complete GFT partition function (for quantum gravity models) is employed in the computation of the entropy. The leading term, the free GFT amplitude, captures only a sector of that full quantum dynamics. With the specific (trivial) choice of kinetic term we have used, the quantum dynamics can at best correspond to (summing over) static space-times. When N goes to 1, in particular, the amplitude becomes the trivial propagation of GFT states, with any given network propagating to itself. This corresponds exactly to the context (static space-time) in which the Ryu-Takayanagi formula is usually derived. In other words, our calculation provides a realization of the Ryu-Takayanagi formula, at least in one extremely simple case, within the full dynamics of a non-perturbative approach to quantum gravity, the group field theory formalism, which can also be seen as a different definition of loop quantum gravity. Our result also shows that the same formalism allows to compute non-perturbative quantum gravity corrections to the Ryu-Takayanagi formula, by including the contributions from the GFT interaction term into the amplitude (as well as considering different choices for the GFT kinetic term).

4.2 Rényi Entropy from Spin-Networks

We now perform a similar calculation of the Ryu-Takanayagi entropy using a different truncation of a generic GFT state, reformulated as a tensor network. We use a given linear combination of spin networks, corresponding to a specific assignment of spins to the links of the network, and thus to the tensors associated to its nodes.

As presented in Section 2, the spin representation of a GFT network is spin-network, in which each node is colored by a tensor $\phi_{\mathbf{m}}^{\mathbf{j}i}$

$$\phi_{\mathbf{m}}^{\mathbf{j}i} = \sum_{\mathbf{p}} \bar{i}_{\mathbf{p}} \psi_{\mathbf{p}\mathbf{m}}^{\mathbf{j}}, \quad |\phi^{\mathbf{j}i}\rangle = \sum_{\mathbf{m}} \phi_{\mathbf{m}}^{\mathbf{j}i} |\mathbf{j}, \mathbf{m}\rangle \in \bigotimes_{\ell} \mathbb{H}_{j_{\ell}} \quad (4.73)$$

and each link is colored by matrix $M_{mm'}^j$

$$|M^j\rangle = \sum_{mm'} M_{mm'}^j |j, m\rangle \otimes |j, m'\rangle \in \mathbb{H}_j^{\otimes 2}, \quad (4.74)$$

where \mathbb{H}_j is the spin- j irreducible representation of $SU(2)$.

A spin-network has a clear geometric interpretation. The graph Γ is the dual of a 3d cellular complex. When all nodes are 4-valent, the graph is dual to a 3d simplicial complex. Each node is dual to a tetrahedron and each link is dual to a triangle. The area of the triangle is given by the spin- j irreducible representation associated with the dual link of the triangle. More precisely, the area \mathcal{A}_{ℓ} is

$$\mathcal{A}_{\ell} = 8\pi\gamma\sqrt{j_{\ell}(j_{\ell} + 1)}\ell_p^2, \quad (4.75)$$

where Γ is the Barbero-Immirzi parameter and ℓ_p is the Planck length (while this results follows both from a canonical quantization of General Relativity in the continuum, and from the geometric quantization of simplicial geometries, the identification of the length scale with the Planck length is, of course, natural from the first perspective only).

A detailed analysis (see e.g. [84–86]) shows that the semi-classical regime of loop quantum gravity states, in which the Regge-Einstein gravity can be recovered, at least at the kinematical level, in the sense of approximating smooth geometries with simplicial ones, is at a scale intermediate between the Planck scale ℓ_p and the average background curvature scale L_{Λ} , which means that if we are working on this regime, area \mathcal{A}_{ℓ} of the triangle should be

$$\ell_p^2 \ll \mathcal{A}_{\ell} \ll L_{\Lambda}^2 \quad (4.76)$$

Together with the relation $\mathcal{A}_{\ell}/L_{\Lambda}^2 \sim \gamma^{-1}j^{1/2} \ll 1$ uncovered in [84], the above regime is equivalent to

$$\frac{1}{j} \ll \gamma \ll \frac{1}{j^{1/2}} \quad (4.77)$$

In a semi-classical regime, then, one has $\mathcal{A}_{\ell} \approx \gamma j_{\ell} \ell_p^2$.

In [51], a special choice of $M_{mm'}^j$

$$M_{mm'}^j = \langle j, m | n^{\dagger} e^{-\pi\gamma L_z - \frac{\exp(1-2\pi\gamma L_z)}{4\pi\gamma}} n' | j, m' \rangle \quad (4.78)$$

has been considered, with the property that the leading order of the entanglement entropy between the two \mathbb{H}_j on a link is proportional to the same area $\mathcal{A}_\ell \approx \gamma j_\ell \ell_p^2$ in the semi-classical regime. In (4.78), n and n' are $SU(2)$ elements; L_z is the $SU(2)$ generator in z -axis. We use the same choice for $M_{mm'}^j$ in our calculation to obtain the Ryu-Takayanagi formula.

Considering the same graph Γ as in the previous subsection, the spin-network state $|\Psi_\Gamma^{\mathbf{j}i}\rangle$ and its corresponding density matrix ρ are given as

$$|\Psi_\Gamma^{\mathbf{j}i}\rangle \equiv \bigotimes_{\ell} \langle M^{j_\ell} | \bigotimes_n |\phi_n^{\mathbf{j}n i_n}\rangle, \quad \rho \equiv |\Psi_\Gamma^{\mathbf{j}i}\rangle \langle \Psi_\Gamma^{\mathbf{j}i}| \quad . \quad (4.79)$$

Just as in the previous calculation, we divide boundary $\partial\Gamma$ into two parts A and B . The N th Rényi entropy is

$$e^{(1-N)S_N} = \frac{Z_N}{Z_0^N} = \frac{\mathbb{E}\text{Tr}[\rho^{\otimes N} \mathbb{P}(\pi_A^0; N, d)]}{\mathbb{E}(\text{Tr}\rho)^N} = \frac{\text{Tr}[\bigotimes_{\ell} \rho_\ell^N \bigotimes_n \mathbb{E}(\rho_n^N) \mathbb{P}(\pi_A^0; N, d)]}{\text{Tr}[\bigotimes_{\ell} \rho_\ell^N \bigotimes_n \mathbb{E}(\rho_n^N)]} \quad . \quad (4.80)$$

The first key step is to calculate $\mathbb{E}(\rho_n^N)$. Because the gauge symmetry is already encoded in the intertwiner \bar{i} for $\phi_{\mathbf{m}}^{\mathbf{j}i}$, $\phi_{\mathbf{m}}^{\mathbf{j}i}$ is not a gauge symmetric tensor, which is in the invariant space of $\mathbb{H}^{\otimes 4}$ as introduced in Section 2, but rather an ordinary tensor in $\otimes_{\ell} \mathbb{H}_{j_\ell}$. So the average over ρ_n^N can be performed in the same way as the one shown in [51]:

$$\mathbb{E}(\rho_n^N) \equiv \int d\phi^{\mathbf{j}i} f(\phi^{\mathbf{j}i}) \left(|\phi^{\mathbf{j}i}\rangle \langle \phi^{\mathbf{j}i}| \right)^{\otimes N} \equiv \int_{U(D)} dU f(\phi^{\mathbf{j}i}) \left(U |\phi_0^{\mathbf{j}i}\rangle \langle \phi_0^{\mathbf{j}i}| U^\dagger \right)^{\otimes N} \quad , \quad (4.81)$$

where $f(\phi^{\mathbf{j}i})$ is a distribution of $\phi^{\mathbf{j}i}$ and U is the group element in the unitary group $U(D)$, in which $D = \prod_{\ell \in n} d_{j_\ell}$. $f(\phi^{\mathbf{j}i})$ is invariant under the transformation of $U(D)$ and in our following calculation we focus on either the uniform or the Gaussian distribution, which keep the main calculation unchanged up to an overall normalization that will be canceled in the final result.

Because of Schur's lemma, $\mathbb{E}(\rho_n^N)$ is the invariant tensor in $(\otimes_{\ell} \mathbb{H}_{j_\ell})^{\otimes N}$, which can be written as a sum of permutations

$$\mathbb{E}(\rho_n^N) = \mathcal{C} \sum_{\pi_n \in \mathcal{S}_N} \mathbb{P}(\pi_n; N, D) = \mathcal{C} \sum_{\pi_n \in \mathcal{S}_N} \prod_{\ell \in n} \mathbb{P}(\pi_n; N, d_{j_\ell}) \quad (4.82)$$

where \mathcal{C} is an normalization factor which depends on the distribution. Then Z_N and Z_0^N can be written as a sum of different patterns $\mathcal{P}(\boldsymbol{\pi}_n)$

$$Z_N = \mathcal{C}^{\text{Vr}} \sum_{\boldsymbol{\pi}_n \in \mathcal{S}_N} \mathcal{P}_A(\boldsymbol{\pi}_n), \quad Z_0^N = \mathcal{C}^{\text{Vr}} \sum_{\boldsymbol{\pi}_n \in \mathcal{S}_N} \mathcal{P}_0(\boldsymbol{\pi}_n) \quad (4.83)$$

where $\#$ is the number of nodes in Γ . $\mathcal{P}_A(\boldsymbol{\pi}_n)$ and $\mathcal{P}_0(\boldsymbol{\pi}_n)$ can be written as products of link values $\mathcal{L}(\pi_n, \pi'_n)$

$$\mathcal{P}_A(\boldsymbol{\pi}_n) = \prod_{\ell \in \Gamma} \mathcal{L}_\ell(\pi_n, \pi_{n'}) \prod_{\ell \in A} \mathcal{L}_\ell(\pi_n, \pi_A^0) \prod_{\ell \in B} \mathcal{L}_\ell(\pi_n, \mathbf{1}) \quad (4.84)$$

$$\mathcal{P}_0(\boldsymbol{\pi}_n) = \prod_{\ell \in \Gamma} \mathcal{L}_\ell(\pi_n, \pi_{n'}) \prod_{\ell \in \partial\Gamma} \mathcal{L}_\ell(\pi_n, \mathbb{1}) \quad (4.85)$$

where $\mathcal{L}_\ell(\pi, \pi')$ is defined as

$$\mathcal{L}_\ell(\pi, \pi') \equiv \text{Tr}[\mathbb{P}(\pi; N, d_{j_\ell}) \rho_\ell^N \mathbb{P}(\pi'; N, d_{j_\ell})] \quad (4.86)$$

Suppose $\varpi \equiv (\pi')^{-1}\pi = \prod_i \mathcal{C}_i$, where \mathcal{C}_i is an r_i -cycle, and impose (4.78) into (4.86). $\mathcal{L}(\pi, \pi')$ becomes

$$\mathcal{L}(\pi, \pi') = \prod_{i=1}^{\chi(\varpi)} \chi_j \left(e^{-r_i 2\pi\gamma L_z - r_i \frac{\exp(1-2\pi\gamma L_z)}{2\pi\gamma}} \right) \quad (4.87)$$

In the semi-classical regime (4.77), the leading contribution of $\mathcal{L}(\pi, \pi')$ is obtained as

$$\begin{aligned} \mathcal{L}(\pi, \pi') &\approx \prod_{i=1}^{\chi(\varpi)} \frac{1}{r_i} e^{-1+(1-r_i)2\pi\gamma j - r_i \frac{\exp(1-2\pi\gamma j)}{2\pi\gamma}} \\ &= e^{-\chi(\varpi) + (\chi(\varpi) - N)2\pi\gamma j - N \frac{\exp(1-2\pi\gamma j)}{2\pi\gamma}} \prod_{i=1}^{\chi(\varpi)} \frac{1}{r_i} \end{aligned} \quad (4.88)$$

A detailed calculation from (4.87) to (4.88) can be found in the appendix. When $\varpi = \mathbb{1}$, i.e. $\pi = \pi'$ and $\chi(\varpi) = N$, $\mathcal{L}(\pi, \pi)$ is then

$$\mathcal{L}(\pi, \pi) \approx e^{-N - N \frac{\exp(1-2\pi\gamma j)}{2\pi\gamma}} \quad (4.89)$$

It is straightforward to check that $\mathcal{L}(\pi, \pi) \geq \mathcal{L}(\pi_1, \pi_2)$. In fact because the sum of r_i equals to N , $\mathcal{L}(\pi, \pi)$ can be rewritten as

$$\mathcal{L}(\pi, \pi) = \prod_{i=1}^{\chi(\varpi)} e^{-r_i - r_i \frac{\exp(1-2\pi\gamma j)}{2\pi\gamma}} \quad (4.90)$$

Then the ratio between $\mathcal{L}(\pi_1, \pi_2)$ and $\mathcal{L}(\pi, \pi)$ is

$$\frac{\mathcal{L}(\pi_1, \pi_2)}{\mathcal{L}(\pi, \pi)} = \prod_{i=1}^{\chi(\varpi)} \frac{e^{-1+(1-r_i)2\pi\gamma j - r_i \frac{\exp(1-2\pi\gamma j)}{2\pi\gamma}}}{r_i e^{-r_i - r_i \frac{\exp(1-2\pi\gamma j)}{2\pi\gamma}}} = \prod_{i=1}^{\chi(\varpi)} \frac{e^{(1-r_i)(2\pi\gamma j - 1)}}{r_i} \leq 1 \quad . \quad (4.91)$$

The last inequality holds because $r_i \geq 1$ and in the regime (4.77) $\gamma j \gg 1$. The equality holds if and only if $\pi_1 = \pi_2$.

If we assume that all j_ℓ are in the same order of magnitude, because of (4.91), one can observe immediately that the leading term of Z_0^N is $\mathcal{P}_0(\mathbb{1})$, i.e. the permutation group for all nodes is $\pi_n = \mathbb{1}$. Suppose there are L_i internal links and L_e external links in Γ , then

$$Z_0^N \approx \mathcal{C}^{V_\Gamma} \prod_{\ell} e^{-N - N \frac{\exp(1-2\pi\gamma j_\ell)}{2\pi\gamma}} \quad . \quad (4.92)$$

The N th order Rényi entropy becomes

$$e^{(1-N)S_N} = \frac{Z_N}{Z_0^N} \approx \sum_{\boldsymbol{\pi}_n} \prod_{\ell} \prod_i^{\chi(\varpi_\ell)} \frac{e^{(1-r_i)(2\pi\gamma j - 1)}}{r_i} \quad . \quad (4.93)$$

As shown in [27], in order for the single domain wall pattern to contribute the most to the Rényi entropy, when three domain walls intersect, they should satisfy

$$\prod_i^{\chi(\varpi_1)} \frac{e^{(1-r_i)(2\pi\gamma j-1)}}{r_i} \geq \prod_i^{\chi(\varpi_2)} \frac{e^{(1-r_i)(2\pi\gamma j-1)}}{r_i} \prod_i^{\chi(\varpi_3)} \frac{e^{(1-r_i)(2\pi\gamma j-1)}}{r_i} \quad , \quad (4.94)$$

where $\varpi_1\varpi_2\varpi_3 = 1$. The above inequality can be simplified to

$$e^{[C(\varpi_2)+C(\varpi_3)-C(\varpi_1)](2\pi\gamma j-1)} \frac{\prod_i^{\chi(\varpi_2)} r_i \prod_i^{\chi(\varpi_3)} r_i}{\prod_i^{\chi(\varpi_1)} r_i} \geq 1 \quad , \quad (4.95)$$

where $C(\varpi)$ is the Cayley weight of a permutation ϖ which satisfies the triangular inequality $C(\varpi_1\varpi_2) \leq C(\varpi_1) + C(\varpi_2)$. In general, when $C(\varpi_1) < C(\varpi_2) + C(\varpi_3)$, the above inequality is satisfied because when $\gamma j \gg 1$ the exponential part of the inequality dominant. When $C(\varpi_1) = C(\varpi_2) + C(\varpi_3)$, one can check that the inequality is satisfied at least for $N \leq 3$ ¹³. Since we are only interested in the entropy while taking the limit $N \rightarrow 1$, this inequality is well satisfied. The Rényi entropy S_N for small N is given as

$$e^{(1-N)S_N} \approx \prod_{\ell \in \partial_{AB}} \exp((1-N)(2\pi\gamma j_\ell - 1) - \ln N) \quad . \quad (4.97)$$

When N goes to zero, we have

$$S_{EE} \approx \sum_{\ell \in \partial_{AB}} \left[2\pi\gamma j_\ell - 1 - \lim_{N \rightarrow 1} \frac{\ln N}{1-N} \right] = \frac{\mathcal{A}_{\partial_{AB}}}{4\ell_p^2} \quad , \quad (4.98)$$

which is exactly the Ryu-Takayanagi formula. Comparing with the calculation in [27], we both reproduce the Ryu-Takayanagi formula from the spin-network state in the semi-classical regime 4.2 of loop quantum gravity and GFT states. This gives further support to the expectation that a classical gravitational theory can be recovered in this formalism. Differently from [27], however, our result directly relies on the fundamental degrees of freedom of the theory.

5 Randomness and Universality

The dictionary we have established between GFT states and (generalized) random tensor networks suggest the potential for useful cross-over of results across these two research

¹³Using the geometric inequality, the left hand side of the above inequality becomes

$$\begin{aligned} \frac{\prod_i^{\chi(\varpi_2)} r_i \prod_i^{\chi(\varpi_3)} r_i}{\prod_i^{\chi(\varpi_1)} r_i} &\geq \frac{(C(\varpi_2) + 1)(C(\varpi_1) - C(\varpi_2) + 1)}{\left(\frac{N}{N-C(\varpi_1)}\right)^{N-C(\varpi_1)}} \\ &\geq (C(\varpi_1) + 1) \left(\frac{N - C(\varpi_1)}{N}\right)^{N-C(\varpi_1)} \end{aligned} \quad (4.96)$$

This simplification is very rough since one has to keep $\frac{N}{N-C(\varpi_1)}$ to be integer. Even in this approximate situation, we could find that it is bigger than 1 when N is a bit smaller than 3.

areas. In particular, one can already envisage a direct application of results concerning the quantum dynamics of GFT models and the statistical properties of random tensor models to problems in statistical mechanics and condensed matter that can be formulated in terms of random tensor networks.

Indeed, our path integral analysis generalises the statistical derivation given in [23], where the random character of the tensors allowed to map the computation of typical Rényi entropies to the evaluation of partition functions of generalized Ising models with inverse temperature $\beta \propto \log D$, D being the dimension of each leg of each tensor in the network. Interestingly, in the original work, the form of the averaged entropies was derived only in the *large D limit*, where the fluctuations of the partition functions are effectively suppressed. In the large D (low temperature) limit, corresponding to the long-range ordered phase for the Ising models, the entropies of a boundary region can be directly related to the energy of a domain wall between different domains of the order parameter: the Ising action can be estimated by the lowest energy configuration and the minimal energy condition of the domain wall naturally leads to the RT formula.

One set of results that appears immediately useful in this context concerns *universality* properties of probability distributions over random tensors, in the limit of large D [87]. They represent a generalization to tensor distributions of the central limit theorem for ordinary probability distributions.

Indeed, a recently proved universality theorem for random tensor fields [87] states that a rank- d random tensor whose entries are N^d independent, identically distributed, complex random variables, and whose distribution is a trace invariant (of the type defining the interactions of *tensorial GFTs* as well), converges in distribution in the *large D limit* to the distributional limit of a Gaussian tensor model, namely a Gaussian tensor field theory. This is already quite remarkable. However, a second, stronger, universality result [87] states that under only the assumption that the joint probability distribution of tensor entries is invariant, assuming that the cumulants of this invariant distribution are uniformly bounded, the large D limit the tensor distribution again converges to the distributional limit of a Gaussian tensor model.

We expect these theorems to have direct applicability to random tensor networks, and even to the generalized class corresponding to the infinite dimensional group fields, where the large D limit refers to the regime in which any UV cut-off on group representations is removed.

The key point to be careful about is that such theorems generally apply to distributions of invariant tensor observables, constructed out of trace (bubble) invariants for bipartite d -colored graphs [10]. Therefore, it does not directly apply to simple products of tensors as we have dealt with in this paper. However, one may wonder how much of such universal behavior survives for generic graphs when distributions of generic tensor observable are considered, e.g. including polynomials made by contractions of tensors which leave some indices free, as for the case of a contracted tensor network state associated to an open graph.

Intuitively, if one randomizes tensors at the nodes independently of contractions, one can still rely on such results, to some extent, but the conclusions become much less solid,

because contractions do affect the scaling of the tensors. Much more solid would be to treat the whole tensor network as an observable in a random tensor or GFT model; then, for tensor networks associated to d -colored graphs (trace invariants), the universality theorems would apply, thereby indicating a new direction for further characterizations of the tensor network states. We postpone this type of evaluations to future work, alongside the complete reformulation of tensor network states and their statistical average within the 2nd quantized GFT framework.

6 Conclusions

Let us summarize our results in this paper. We have established a precise dictionary between GFT states and (generalized) random tensor networks. This dictionary also implies, under different restrictions on the GFT states, a correspondence between LQG spin network states and tensor networks, and a correspondence between random tensors models and tensor networks. Next, we have computed the Rényi entropy and derived the RT entropy formula, for random tensor networks with an additional gauge invariance property, as suggested by the correspondence with GFT. We have then computed the same quantity using directly GFT and spin network techniques, first using a simple approximation to a complete definition of a random tensor network evaluation seen as a GFT correlation function, but still using a truly generalized tensor network seen as a GFT state, and then considering directly a spin network state as a random tensor network. This elucidates further the correspondence and its potential. Finally, we have discussed how universality theorems for random tensors can be applied to tensor network states, as a first example of application of results from the theory of random tensors and GFT to tensor networks. We are convinced that these results can be just the beginning of many further developments, made possible by the fertile meeting between tensor networks and fundamental quantum gravity, along the lines we have established.

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A Average over the random tensor

Let us first consider the case without the gauge symmetry (2.44). Given a graph with only one node. The corresponding density matrix is

$$\rho_{\mu\bar{\mu}} \equiv T_{\mu}\bar{T}_{\bar{\mu}} \tag{A.1}$$

Consider n copies of the density matrix $\rho^{\otimes n}$. If T is uniformly distributed, then the average $\rho^{\otimes n}$ over T is

$$\mathbb{E}_T(\rho^{\otimes n}) \equiv \int_{U(D^v)} dU \prod_{s=1}^n (UT^0)_{\mu^{(s)}} \overline{(UT^0)}_{\bar{\mu}^{(s)}} \tag{A.2}$$

Because of the Schur's lemma, since \mathbb{H}_{D^v} is the irrep of $U(D^v)$, the result of the average is the identity matrix on the symmetric subspace of $\mathbb{H}_{D^v}^{\otimes n}$. When $\text{Tr}\rho = 1$, the result is

$$\begin{aligned}\mathbb{E}_T(\rho^{\otimes n}) &= \frac{1}{D^v[n]} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu^{(s)} \bar{\mu}^{(\pi(s))}} \\ &\equiv \frac{1}{D^v[n]} \sum_{\pi \in \mathcal{S}_n} \mathbb{P}(\pi; n, D^v)\end{aligned}\quad (\text{A.3})$$

where

$$D^v[n] \equiv D^v(D^v + 1) \cdots (D^v + n - 1) \quad (\text{A.4})$$

\mathcal{S}_n is the symmetric group on n objects and

$$\mathbb{P}(\pi; n, D^v) \equiv \prod_{s=1}^n \delta_{\mu^{(s)} \bar{\mu}^{(\pi(s))}} \quad (\text{A.5})$$

which is the representation matrix of $\pi \in \mathcal{S}_n$ on $\mathbb{H}_{D^v}^{\otimes n}$. Using the gauge (2.42) or (2.49), we can get the relation between representations on \mathbb{H}_{D^v} and $\mathbb{H}_D^{\otimes v}$

$$\delta_{\mu^{(s)} \bar{\mu}^{(\pi(s))}} \equiv \prod_{a=1}^v \delta_{\lambda_a^{(s)} \bar{\lambda}_a^{(\pi(s))}} \quad (\text{A.6})$$

Then we have

$$\mathbb{P}(\pi; n, D^v) = \prod_{a=1}^v \mathbb{P}(\pi_a; n, D) \quad (\text{A.7})$$

If T is an random Gaussian vector, then the average is

$$\mathbb{E}_T(\rho^{\otimes n}) \equiv \int \mathcal{D}T e^{-\beta|T|^2} \prod_{s=1}^n T_{\mu^{(s)}} \bar{T}_{\bar{\mu}^{(s)}} \quad (\text{A.8})$$

If we ask $\mathbb{E}_T(\rho) = \mathbb{1}/D^v$ and $T = x\hat{T}$, where $|\hat{T}| = 1$, then $\beta = D^v$ and the average $\mathbb{E}_T(\rho^{\otimes n})$ becomes

$$\begin{aligned}& \int dx |x|^{2n} e^{-D^v|x|^2} \int dU \prod_{s=1}^n (U\hat{T}^0)_{\mu^{(s)}} \overline{(U\hat{T}^0)_{\bar{\mu}^{(s)}}} \\ &= (D^v)^{-n} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu^{(s)} \bar{\mu}^{(\pi(s))}}\end{aligned}\quad (\text{A.9})$$

Now let us consider the case with symmetry (2.44). The corresponding density matrix is

$$\rho_{\mu\mu'} \equiv T_\mu \bar{T}_{\bar{\mu}} = T_{\mu_i} \bar{T}_{\bar{\mu}_i} \equiv \tilde{\rho}_{\mu_i \bar{\mu}_i} \quad (\text{A.10})$$

The n copies of the density matrix is then written as

$$\rho^{\otimes n} = \prod_{s=1}^n \tilde{\rho}_{\mu_i^{(s)} \bar{\mu}_i^{(s)}} = \prod_{s=1}^n T_{\mu_i^{(s)}} \bar{T}_{\bar{\mu}_i^{(s)}} \quad (\text{A.11})$$

If T is uniformly distributed, then the average of $\rho^{\otimes n}$ over T is

$$\mathbb{E}_T(\rho^{\otimes n}) = \int_{U(D^{v-1})} dU \prod_{s=1}^n (UT^0)_{\mu_{i(s)}} \overline{(UT^0)_{\bar{\mu}_{\bar{i}(s)}}} \quad (\text{A.12})$$

As shown in the first case, the result of the integral is the identity matrix in the symmetric subspace of $\mathbb{H}_{D^{v-1}}^{\otimes n}$.

$$\begin{aligned} \mathbb{E}_T(\rho^{\otimes n}) &\propto \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(\pi(s))}} \\ &\equiv \sum_{\pi \in \mathcal{S}_n} \mathbb{P}_{\{i(s)\}\{\bar{i}(s)\}}(\pi; n, D^{v-1}) \end{aligned} \quad (\text{A.13})$$

The normalization will be calculated in the next subsection. $\mathbb{P}_{\{i(s)\}\{\bar{i}(s)\}}(\pi; n, D^{v-1})$ is the representation matrix of $\pi \in \mathcal{S}_n$ on $\mathbb{H}_{D^{v-1}}^{\otimes n}$ with a set of $\{i(s)\}\{\bar{i}(s)\}$. Similarly, when T_μ is a Gaussian vector, the result of the average is the same as the above equation up to a normalization.

By using the gauge (2.49), one can show the relation between representations μ_i and λ_a . Because of (2.49), $\delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(s')}} is not zero only when$

$$[\lambda_a(s) - i(s)]_D = [\bar{\lambda}_a(s') - \bar{i}(s')]_D \quad (\text{A.14})$$

because of the modular rules (2.46), the above equation can be rewritten as

$$[\lambda_a(s) - \bar{\lambda}_a(s')]_D = [i(s) - \bar{i}(s')]_D \quad (\text{A.15})$$

If $[i(s) - \bar{i}(s')]_D = \ell \in \mathbb{Z}/D$, then

$$\delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(s')}} = \prod_{a=1}^v \delta_{[\lambda_a(s) - \bar{\lambda}_a(s')]_D, \ell} = \prod_{a=1}^v \delta_{[\lambda_a(s) - \ell]_D, \bar{\lambda}_a(s')} \quad (\text{A.16})$$

One of the observation is that ℓ is a uniform shift for all legs of each node as long as the step $[i(s) - \bar{i}(s')]_D = \ell$ is fixed.

At the end of this subsection, let us define the trace on a tensor $\mathbb{T}_{\{\mu^{(s)}\}\{\bar{\mu}^{(s)}\}}$. The trace is given as

$$\text{Tr} \mathbb{T} = \mathbb{T}_{\{\mu^{(s)}\}\{\bar{\mu}^{(s)}\}} \prod_{s=1}^n \delta_{\mu^{(s)} \bar{\mu}^{(s)}} \quad (\text{A.17})$$

With the symmetry (2.44),

$$\text{Tr} \mathbb{T} = \sum_{\substack{\{i(s)\} \\ \{\bar{i}(s)\}}} \mathbb{T}_{\{\mu_{i(s)}\}\{\bar{\mu}_{\bar{i}(s)}\}} \prod_{s=1}^n \delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(s)}} \delta_{i(s) \bar{i}(s)} \quad (\text{A.18})$$

The definition will be used in the following text.

B Structure of the matrix of $\mathbb{E}_T(\rho^{\otimes n})$

In this subsection we will analysis the structure of the matrix in (A.13), i.e. the structure of

$$\mathbb{M} \equiv \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(\pi(s))}} = \sum_{\pi \in \mathcal{S}_n} \mathbb{P}_{\{i(s)\}\{\bar{i}(s)\}}(\pi; n, D^{v-1}) \quad (\text{B.1})$$

The sum over $\mathbb{P}(\pi; n, D^{v-1})$ is proportional to the projector operator $\mathbb{P}_{\text{sym}}^{n, D^{v-1}}$ which projects vectors in $\mathbb{H}_{D^{v-1}}^{\otimes n}$ into its symmetric subspace.

$$\mathbb{P}_{\text{sym}}^{n, D^{v-1}} = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \mathbb{P}(\pi; n, D^{v-1}) \quad (\text{B.2})$$

Given a set of $\{i(s)\} = \{i(1), i(2), \dots, i(n)\}$ and $\{\bar{i}(s)\} = \{\bar{i}(1), \bar{i}(2), \dots, \bar{i}(n)\}$ where $i(s)$ and $\bar{i}(s)$ are from 0 to $D - 1$, there is a projector $\mathbb{P}_{\text{sym}}^{n, D^{v-1}}$, which is a $D^{(v-1)n} \times D^{(v-1)n}$ matrix. Write (A.13) as a matrix, with $\{i(s)\}$ labeling its rows and $\{\bar{i}(s)\}$ labels its columns:

$$\mathbb{M} = n! \begin{pmatrix} \mathbb{P}_{\text{sym}}^{n, D^{v-1}} & \dots & \mathbb{P}_{\text{sym}}^{n, D^{v-1}} \\ \vdots & \ddots & \vdots \\ \mathbb{P}_{\text{sym}}^{n, D^{v-1}} & \dots & \mathbb{P}_{\text{sym}}^{n, D^{v-1}} \end{pmatrix} \quad (\text{B.3})$$

This matrix is a $D^n \times D^n$ block matrix.

The trace of \mathbb{M} is

$$\begin{aligned} \text{Tr} \mathbb{M} &= \sum_{\substack{\{i(s)\} \\ \{\bar{i}(s)\}}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(\pi(s))}} \delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(s)}} \delta_{i(s) \bar{i}(s)} \\ &= \sum_{\{i(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{i(s)} \bar{\mu}_{i(\pi(s))}} \delta_{\mu_{i(s)} \bar{\mu}_{i(s)}} \\ &= \sum_{\{i(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{i(s)} \mu_{i(\pi(s))}} \\ &= \sum_{\{i(s)\}} n! \text{Tr} \mathbb{P}_{\text{sym}}^{n, D^{v-1}} \\ &= D^n D^{v-1} [n] \end{aligned} \quad (\text{B.4})$$

where we use the trace of the projector $\mathbb{P}_{\text{sym}}^n$

$$\text{Tr} \mathbb{P}_{\text{sym}}^{n, D^{v-1}} = \binom{D^{v-1} + n - 1}{n} \quad (\text{B.5})$$

The matrix \mathbb{M} can be written as a sum of matrices

$$\mathbb{M} = \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{i(s)} \bar{\mu}_{\bar{i}(\pi(s))}} \quad (\text{B.6})$$

$$\begin{aligned} &= \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{\mu_{[i(s)-m(s)]_D} \bar{\mu}_{\bar{i}(\pi(s))}} \delta_{i(s) \bar{i}(\pi(s))} \\ &= \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \delta_{[\mu^{(s)}-m(s)]_{D^v-1} D^v, \bar{\mu}^{(\pi(s))}} \end{aligned} \quad (\text{B.7})$$

$$= \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^n \prod_{a=1}^v \delta_{[\lambda_a(s)-m(s)]_D, \bar{\lambda}_a(\pi(s))} \quad (\text{B.8})$$

Let us define a class of new matrices $\mathbb{P}(\pi)_{\{m(s)\}}$ as

$$\begin{aligned} \mathbb{P}(\pi)_{\{m(s)\}} &\equiv \prod_{s=1}^n \prod_{a=1}^v \delta_{[\lambda_a(s)-m(s)]_D, \bar{\lambda}_a(\pi(s))} \\ &= \prod_{s=1}^n \delta_{[\mu^{(s)}-m(s)]_{D^v-1} D^v, \bar{\mu}^{(\pi(s))}} \end{aligned} \quad (\text{B.9})$$

Then matrix \mathbb{M} becomes

$$\mathbb{M} = \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \mathbb{P}(\pi)_{\{m(s)\}} \quad (\text{B.10})$$

When $m(s) = 0$ for all s , $\mathbb{P}(\pi)_{\{0\}} = \mathbb{P}(\pi; n, D^v)$, the representation matrix of $\pi \in \mathcal{S}_n$ on $\mathbb{H}_{D^v}^{\otimes n}$.

C From (4.87) to (4.88)

In this appendix we perform the calculation from (4.87) to (4.88). $\mathcal{L}(\pi, \pi')$ is given by (4.87). Let us denote $2\pi\gamma$ as c for simplicity, then $\mathcal{L}(\pi, \pi')$ can be written as

$$\mathcal{L}(\pi, \pi') = \prod_{i=1}^{\chi(\varpi)} \chi_j \left(e^{-r_i c L_z - r_i \frac{\exp(1-cL_z)}{c}} \right) \equiv \prod_{i=1}^{\chi(\varpi)} I_{r_i} \quad (\text{C.1})$$

I_r can be written in terms of SU(2) coherent state as

$$\begin{aligned} I_r &= d_j \int dn \langle j, j | n^\dagger e^{-rcL_z - r \frac{\exp(1-cL_z)}{c}} n | j, j \rangle \\ &= d_j \sum_k^{\infty} \frac{(-)^k r^k e^k}{k! c^k} \int dn \langle j, j | n^\dagger e^{-(r+k)cL_z} n | j, j \rangle \end{aligned} \quad (\text{C.2})$$

$$\equiv d_j \sum_k^{\infty} \frac{(-)^k r^k e^k}{k! c^k} \int dn e^{S_r^{(k)}} \equiv d_j \int dn e^{S_r} \quad (\text{C.3})$$

where S_r is the total action and $S_r^{(k)} \equiv 2j \ln \langle \uparrow | n^\dagger e^{-c(r+k)L_z} n | \uparrow \rangle$ and $|\uparrow\rangle \equiv |\frac{1}{2}, \frac{1}{2}\rangle$. In the semi-classical regime of loop gravity, i.e. the large spin- j regime, the leading contribution of I_r is from the critical point of $S_r^{(k)}$, which is the solutions of the equations of motion

$$\delta_n S_r^{(k)} = 0 \quad \Rightarrow \quad n^\dagger e^{-c(r+k)L_z} n = e^{-\alpha L_z} \quad (\text{C.4})$$

One can obtain the solutions

$$n^\dagger L_z n = \pm L_z, \quad \alpha^\pm = \pm c(r+k) \quad (\text{C.5})$$

Bring the solutions back to I_r , we can get

$$I_r \sim d_j \sum_{\epsilon=\pm} \frac{e^{S_{r0}^\epsilon}}{\sqrt{\det H_r^\epsilon}} \equiv \sum_{\epsilon=\pm} I_r^\epsilon \quad (\text{C.6})$$

where S_{r0}^ϵ is the total action S_r on the critical point

$$S_{r0}^\epsilon \equiv -\epsilon r c j - r \frac{\exp(1 - \epsilon c j)}{\gamma} \quad (\text{C.7})$$

and H_r^ϵ is the Hessian matrix of S_r

$$H_r^\epsilon \equiv \frac{1}{2} \delta_n^2 S_r |_\epsilon \quad (\text{C.8})$$

After perform the second derivation on S_r , one can obtain

$$\det H^\epsilon = 4j^2 r^2 (-c + \exp(1 - \epsilon c j))^2 \quad (\text{C.9})$$

In the semi-classical and low energy limit

$$\det H^\epsilon \sim 4j^2 N^2 \exp 2(1 - \epsilon c j) \quad (\text{C.10})$$

Then I_r^ϵ becomes

$$I_r^\epsilon \sim \frac{\exp\left(-1 + \epsilon c j(1 - r) - r \frac{\exp(1 - \epsilon c j)}{c}\right)}{r} \quad (\text{C.11})$$

One can observe that $I_r^+ \gg I_r^-$ since when $\epsilon = -$, in the large spin regime I_r^- goes to zero. I_r thus becomes I_r^+ , which is one of the term in the product of (4.88).

$$I_r \approx I_r^+ = \frac{\exp\left(-1 + c j(1 - r) - r \frac{\exp(1 - c j)}{c}\right)}{r} \quad (\text{C.12})$$

References

- [1] A. Ashtekar and J. Lewandowski, *Background independent quantum gravity: A status report*, *Class. Quant. Grav.* **21** (2004) R53, [[gr-qc/0404018](#)].
- [2] C. Rovelli, *Quantum Gravity*. Cambridge University Press, London, 2004.
- [3] T. Thiemann, *Modern canonical quantum general relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, London, 2007.

- [4] A. Perez, *The Spin Foam Approach to Quantum Gravity*, *Living Rev.Rel.* **16** (2012) 3, [[arXiv:1205.2019](#)].
- [5] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2014.
- [6] D. Oriti, *Group Field Theory and Loop Quantum Gravity*, in *to appear in “Loop Quantum Gravity - 100 Years of General Relativity Series”* (A. Ashtekar and J. Pullin, eds.), 2014. [[arXiv:1408.7112](#)].
- [7] A. Baratin and D. Oriti, *Ten questions on Group Field Theory (and their tentative answers)*, *J. Phys. Conf. Ser.* **360** (2012) 012002, [[arXiv:1112.3270](#)].
- [8] D. Oriti, *The microscopic dynamics of quantum space as a group field theory*, in *Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity: Cape Town, South Africa*, pp. 257–320, 2011. [[arXiv:1110.5606](#)].
- [9] D. Oriti, *The Group field theory approach to quantum gravity: Some recent results*, *AIP Conf. Proc.* **1196** (2009) 209–218, [[arXiv:0912.2441](#)].
- [10] R. Gurau and J. P. Ryan, *Colored Tensor Models - a review*, *SIGMA* **8** (2012) 020, [[arXiv:1109.4812](#)].
- [11] R. Gurau, *Invitation to Random Tensors*, *SIGMA* **12** (2016) 094, [[arXiv:1609.06439](#)].
- [12] V. Rivasseau, *Random Tensors and Quantum Gravity*, *SIGMA* **12** (2016) 069, [[arXiv:1603.07278](#)].
- [13] R. Orus, *A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States*, *Annals Phys.* **349** (2014) 117–158, [[arXiv:1306.2164](#)].
- [14] J. C. Bridgeman and C. T. Chubb, *Hand-waving and Interpretive Dance: An Introductory Course on Tensor Networks*, [[arXiv:1603.03039](#)].
- [15] X. G. Wen, *Quantum field theory of many-body systems: From the origin of sound to an origin of light and electrons*. 2004.
- [16] X.-G. Wen, *Zoo of quantum-topological phases of matter*, [[arXiv:1610.03911](#)].
- [17] J. I. Cirac and F. Verstraete, *Renormalization and tensor product states in spin chains and lattices*, *J. Phys.* **A42** (2009) 504004.
- [18] F. Verstraete, V. Murg, and J. I. Cirac, *Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems*, *Advances in Physics* **57** (2008), no. 2 143–224.
- [19] R. Augusiak, F. Cucchietti, and M. Lewenstein, *Modern theories of many-particle systems in condensed matter physics*, *Lect. Not. Phys.* **843** (2012) 245–294.
- [20] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys.Rev.Lett.* **96** (2006) 181602, [[hep-th/0603001](#)].
- [21] B. Swingle, *Entanglement Renormalization and Holography*, *Phys. Rev.* **D86** (2012) 065007, [[arXiv:0905.1317](#)].
- [22] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, *Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence*, *JHEP* **06** (2015) 149, [[arXiv:1503.06237](#)].

- [23] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang, *Holographic duality from random tensor networks*, *JHEP* **11** (2016) 009, [[arXiv:1601.01694](#)].
- [24] G. Vidal, *Class of quantum many-body states that can be efficiently simulated*, *Physical review letters* **101** (2008), no. 11 110501.
- [25] S. Singh, R. N. C. Pfeifer, and G. Vidal, *Tensor network decompositions in the presence of a global symmetry*, *Phys. Rev.* **A82** (2010) 050301, [[arXiv:0907.2994](#)].
- [26] G. Evenbly and G. Vidal, *Tensor network states and geometry*, *Journal of Statistical Physics* **145** (2011), no. 4 891–918.
- [27] M. Han and L.-Y. Hung, *Loop Quantum Gravity, Exact Holographic Mapping, and Holographic Entanglement Entropy*, [[arXiv:1610.02134](#)].
- [28] B. Dittrich, S. Mizera, and S. Steinhaus, *Decorated tensor network renormalization for lattice gauge theories and spin foam models*, *New J. Phys.* **18** (2016), no. 5 053009, [[arXiv:1409.2407](#)].
- [29] C. Delcamp and B. Dittrich, *Towards a phase diagram for spin foams*, [[arXiv:1612.04506](#)].
- [30] B. Dittrich, F. C. Eckert, and M. Martin-Benito, *Coarse graining methods for spin net and spin foam models*, *New J. Phys.* **14** (2012) 035008, [[arXiv:1109.4927](#)].
- [31] B. Dittrich, E. Schnetter, C. J. Seth, and S. Steinhaus, *Coarse graining flow of spin foam intertwiners*, *Phys. Rev.* **D94** (2016), no. 12 124050, [[arXiv:1609.02429](#)].
- [32] S. Carrozza, D. Oriti, and V. Rivasseau, *Renormalization of a $SU(2)$ Tensorial Group Field Theory in Three Dimensions*, *Commun. Math. Phys.* **330** (2014) 581–637, [[arXiv:1303.6772](#)].
- [33] D. Benedetti, J. Ben Geloun, and D. Oriti, *Functional Renormalisation Group Approach for Tensorial Group Field Theory: a Rank-3 Model*, *JHEP* **03** (2015) 084, [[arXiv:1411.3180](#)].
- [34] S. Carrozza and V. Lahoche, *Asymptotic safety in three-dimensional $SU(2)$ Group Field Theory: evidences in the local potential approximation*, [[arXiv:1612.02452](#)].
- [35] S. Carrozza, *Flowing in Group Field Theory Space: a Review*, *SIGMA* **12** (2016) 070, [[arXiv:1603.01902](#)].
- [36] J. Ben Geloun, R. Martini, and D. Oriti, *Functional Renormalisation Group analysis of Tensorial Group Field Theories on \mathbb{R}^d* , *Phys. Rev.* **D94** (2016), no. 2 024017, [[arXiv:1601.08211](#)].
- [37] V. Lahoche and D. Oriti, *Renormalization of a tensorial field theory on the homogeneous space $SU(2)/U(1)$* , *J. Phys.* **A50** (2017), no. 2 025201, [[arXiv:1506.08393](#)].
- [38] B. Bahr, B. Dittrich, F. Hellmann, and W. Kaminski, *Holonomy Spin Foam Models: Definition and Coarse Graining*, *Phys. Rev.* **D87** (2013), no. 4 044048, [[arXiv:1208.3388](#)].
- [39] B. Bahr and S. Steinhaus, *Numerical evidence for a phase transition in 4d spin foam quantum gravity*, *Phys. Rev. Lett.* **117** (2016), no. 14 141302, [[arXiv:1605.07649](#)].
- [40] B. Bahr, *On background-independent renormalization of spin foam models*, [[arXiv:1407.7746](#)].
- [41] G. Chirco, F. Mele, D. Oriti, and P. Vitale, *Geometric entanglement and spin networks*, in to appear.

- [42] E. R. Livine and D. R. Terno, *Quantum black holes: Entropy and entanglement on the horizon*, *Nucl. Phys.* **B741** (2006) 131–161, [[gr-qc/0508085](#)].
- [43] E. R. Livine and D. R. Terno, *Reconstructing quantum geometry from quantum information: Area renormalisation, coarse-graining and entanglement on spin networks*, [gr-qc/0603008](#).
- [44] W. Donnelly, *Entanglement entropy in loop quantum gravity*, *Phys. Rev.* **D77** (2008) 104006, [[arXiv:0802.0880](#)].
- [45] J. Diaz-Polo and D. Pranzetti, *Isolated Horizons and Black Hole Entropy In Loop Quantum Gravity*, *SIGMA* **8** (2012) 048, [[arXiv:1112.0291](#)].
- [46] A. Perez, *Statistical and entanglement entropy for black holes in quantum geometry*, *Phys. Rev.* **D90** (2014), no. 8 084015, [[arXiv:1405.7287](#)]. [Addendum: *Phys. Rev.* **D90**, no. 8, 089907 (2014)].
- [47] E. Bianchi and R. C. Myers, *On the Architecture of Spacetime Geometry*, *Class. Quant. Grav.* **31** (2014), no. 21 214002, [[arXiv:1212.5183](#)].
- [48] G. Chirco, C. Rovelli, and P. Ruggiero, *Thermally correlated states in Loop Quantum Gravity*, *Class. Quant. Grav.* **32** (2015), no. 3 035011, [[arXiv:1408.0121](#)].
- [49] A. Ghosh and D. Pranzetti, *CFT/Gravity Correspondence on the Isolated Horizon*, *Nucl. Phys.* **B889** (2014) 1–24, [[arXiv:1405.7056](#)].
- [50] V. Bonzom and B. Dittrich, *3D holography: from discretum to continuum*, *JHEP* **03** (2016) 208, [[arXiv:1511.05441](#)].
- [51] A. Hamma, L.-Y. Hung, A. Marciano, and M. Zhang, *Area Law from Loop Quantum Gravity*, [arXiv:1506.01623](#).
- [52] E. Bianchi, L. Hackl, and N. Yokomizo, *Entanglement entropy of squeezed vacua on a lattice*, *Phys. Rev.* **D92** (2015), no. 8 085045, [[arXiv:1507.01567](#)].
- [53] M. Han, *Black Hole Entropy in Loop Quantum Gravity, Analytic Continuation, and Dual Holography*, [arXiv:1402.2084](#).
- [54] M. Han, *4d Quantum Geometry from 3d Supersymmetric Gauge Theory and Holomorphic Block*, *JHEP* **01** (2016) 065, [[arXiv:1509.00466](#)].
- [55] D. Oriti, D. Pranzetti, and L. Sindoni, *Horizon entropy from quantum gravity condensates*, *Phys. Rev. Lett.* **116** (2016), no. 21 211301, [[arXiv:1510.06991](#)].
- [56] A. Baratin and D. Oriti, *Group field theory with non-commutative metric variables*, *Phys. Rev. Lett.* **105** (2010) 221302, [[arXiv:1002.4723](#)].
- [57] A. Baratin and D. Oriti, *Quantum simplicial geometry in the group field theory formalism: reconsidering the Barrett-Crane model*, *New J. Phys.* **13** (2011) 125011, [[arXiv:1108.1178](#)].
- [58] A. Baratin and D. Oriti, *Group field theory and simplicial gravity path integrals: A model for Holst-Plebanski gravity*, *Phys. Rev.* **D85** (2012) 044003, [[arXiv:1111.5842](#)].
- [59] R. De Pietri, *Matrix model formulation of four-dimensional gravity*, *Nucl. Phys. Proc. Suppl.* **94** (2001) 697–700, [[hep-lat/0011033](#)]. [697(2000)].
- [60] R. De Pietri and C. Petronio, *Feynman diagrams of generalized matrix models and the associated manifolds in dimension 4*, *J. Math. Phys.* **41** (2000) 6671–6688, [[gr-qc/0004045](#)].
- [61] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli, *Barrett-Crane model from a*

- Boulatov-Ooguri field theory over a homogeneous space*, *Nucl. Phys.* **B574** (2000) 785–806, [[hep-th/9907154](#)].
- [62] D. Oriti, D. Pranzetti, J. P. Ryan, and L. Sindoni, *Generalized quantum gravity condensates for homogeneous geometries and cosmology*, *Class. Quant. Grav.* **32** (2015), no. 23 235016, [[arXiv:1501.00936](#)].
- [63] D. Oriti, *The universe as a quantum gravity condensate*, [arXiv:1612.09521](#).
- [64] D. Oriti, *Group field theory as the 2nd quantization of Loop Quantum Gravity*, *Class. Quant. Grav.* **33** (2016), no. 8 085005, [[arXiv:1310.7786](#)].
- [65] S. Singh, R. N. Pfeifer, and G. Vidal, *Tensor network states and algorithms in the presence of a global $u(1)$ symmetry*, *Physical Review B* **83** (2011), no. 11 115125.
- [66] D. N. Page, *Average entropy of a subsystem*, *Phys. Rev. Lett.* **71** (1993) 1291–1294, [[gr-qc/9305007](#)].
- [67] J. Deutsch, *Quantum statistical mechanics in a closed system*, *Physical Review A* **43** (1991), no. 4 2046.
- [68] M. Srednicki, *Chaos and quantum thermalization*, *Physical Review E* **50** (1994), no. 2 888.
- [69] D. Oriti, *Disappearance and emergence of space and time in quantum gravity*, *Stud. Hist. Phil. Sci.* **B46** (2014) 186–199, [[arXiv:1302.2849](#)].
- [70] C. Cao, S. M. Carroll, and S. Michalakis, *Space from Hilbert Space: Recovering Geometry from Bulk Entanglement*, [arXiv:1606.08444](#).
- [71] N. Seiberg, *Emergent spacetime*, in *The Quantum Structure of Space and Time: Proceedings of the 23rd Solvay Conference on Physics. Brussels, Belgium. 1 - 3 December 2005*, pp. 163–178, 2006. [hep-th/0601234](#).
- [72] T. A. Koslowski, *Dynamical Quantum Geometry (DQG Programme)*, [arXiv:0709.3465](#).
- [73] D. Oriti, *Group field theory as the microscopic description of the quantum spacetime fluid: A New perspective on the continuum in quantum gravity*, *PoS QG-PH* (2007) 030, [[arXiv:0710.3276](#)].
- [74] T. Konopka, F. Markopoulou, and L. Smolin, *Quantum Graphity*, 2006.
- [75] V. Rivasseau, *Quantum Gravity and Renormalization: The Tensor Track*, *AIP Conf. Proc.* **1444** (2011) 18–29, [[arXiv:1112.5104](#)].
- [76] H. Steinacker, *Emergent Geometry and Gravity from Matrix Models: an Introduction*, *Class. Quant. Grav.* **27** (2010) 133001, [[arXiv:1003.4134](#)].
- [77] G. Chirco, D. Oriti, and M. Zhang, *to appear*, .
- [78] L. Freidel, R. Gurau, and D. Oriti, *Group field theory renormalization - the 3d case: Power counting of divergences*, *Phys. Rev.* **D80** (2009) 044007, [[arXiv:0905.3772](#)].
- [79] J. Magnen, K. Noui, V. Rivasseau, and M. Smerlak, *Scaling behaviour of three-dimensional group field theory*, *Class. Quant. Grav.* **26** (2009) 185012, [[arXiv:0906.5477](#)].
- [80] J. W. Barrett and I. Naish-Guzman, *The Ponzano-Regge model and Reidemeister torsion*, in *Recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories. Proceedings, 11th Marcel Grossmann Meeting, MG11, Berlin, Germany, July 23-29, 2006. Pt. A-C*, pp. 2782–2784, 2006. [gr-qc/0612170](#).

- [81] V. Bonzom and M. Smerlak, *Bubble divergences from cellular cohomology*, *Lett. Math. Phys.* **93** (2010) 295–305, [[arXiv:1004.5196](#)].
- [82] V. Bonzom and M. Smerlak, *Bubble divergences from twisted cohomology*, *Commun. Math. Phys.* **312** (2012) 399–426, [[arXiv:1008.1476](#)].
- [83] V. Bonzom and M. Smerlak, *Bubble divergences: sorting out topology from cell structure*, *Annales Henri Poincare* **13** (2012) 185–208, [[arXiv:1103.3961](#)].
- [84] M. Han, *Semiclassical Analysis of Spinfoam Model with a Small Barbero-Immirzi Parameter*, *Phys. Rev.* **D88** (2013) 044051, [[arXiv:1304.5628](#)].
- [85] M. Han, *On Spinfoam Models in Large Spin Regime*, *Class.Quant.Grav.* **31** (2014) 015004, [[arXiv:1304.5627](#)].
- [86] M. Han, *Covariant Loop Quantum Gravity, Low Energy Perturbation Theory, and Einstein Gravity with High Curvature UV Corrections*, *Phys.Rev.* **D89** (2014), no. 12 124001, [[arXiv:1308.4063](#)].
- [87] R. Gurau, *Universality for Random Tensors*, *Ann. Inst. H. Poincare Probab. Statist.* **50** (2014), no. 4 1474–1525, [[arXiv:1111.0519](#)].