Discussion Papers of the
Max Planck Institute for
Research on Collective Goods
2017/6

Probability Measures on Product Spaces with Uniform Metrics

Martin F. Hellwig

MAX PLANCK
SOCIETY

Discussion Papers of the
Max Planck Institute
for Research on Collective Goods 2017/6

Probability Measures on Product Spaces with Uniform Metrics

Martin F. Hellwig

May 2017

This version: May 2023

Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Str. 10, D-53113 Bonn
https://www.coll.mpg.de

Probability Measures on Product Spaces with Uniform Metrics*

Martin F. Hellwig
Max Planck Institute for Research on Collective Goods
Kurt Schumacher-Str. 10
D-53113 Bonn, Germany
hellwig@coll.mpg.de

May 3, 2023

Abstract

The paper provides mathematical foundations for a homeomorphism theorem à la Mertens and Zamir (1985) when the space of belief hierarchies of an agent has the uniform topology rather than the product topology. The Borel σ-algebra for the uniform topology being unsuitable, the theorem relies on the product σ-algebra but defines the topology of weak convergence on the space of measures on this σ-algebra with reference to the uniform topology on the underlying space. For a countable product of complete separable metric spaces, the paper shows that this topology on the space of measures on the product σ-algebra is metrizable by the Prohorov metric. The projection mapping from such measures to sequences of measures on the first ℓ factors, $\ell = 1, 2, ...$, is a homeomorphism if the range of this mapping is also given a uniform metric.

Key Words: Product spaces with uniform metrics, weak convergence of non-Borel measures, σ-algebras generated by the open balls, quasi-separable measures, Prohorov metric.

MSC Classification 60B05

JEL Classification: C02, C72

*Without implicating them, I thank Eduardo Faingold and Alia Gizatulina for helpful discussions and comments. I also thank an anonymous referee for very helpful advice, which uncovered an error in a previous version.

1 Introduction

This paper provides mathematical foundations for studying the question whether the homeomorphism theorem of Mertens and Zamir (1985) remains valid when the space of belief hierarchies of an agent is endowed with the topology generated by a uniform metric rather than the product topology. In games of incomplete information, a *belief hierarchy* of an agent is a list of beliefs of different orders: the agent's first-order belief is a subjective probability distribution over states of nature, the agent's second-order belief is a subjective probability distribution over states of nature and the other agents' first-order beliefs, the agent's third-order belief is a subjective probability distribution over states of nature and other agents' first- and second-order beliefs, and so on. A belief hierarchy is *consistent* if the implications of a higher-order belief for events that are also covered by lower-order beliefs coincide with the lower-order beliefs for these events.

A belief hierarchy is an element of the product of the spaces of beliefs of different orders. Mertens and Zamir (1985) imposed the topology of weak convergence on the space of beliefs of order k, for each k, and the associated product topology on the space of belief hierarchies.[1] With this specification of topologies, they showed there is a homeomorphism that maps consistent belief hierarchies of an agent into probability measures over states of nature and other agents' belief hierarchies.

Dekel et al. (2006) and Chen et al. (2010, 2017) argued that the product topology on the space of belief hierarchies is too coarse to capture all the continuity properties of strategic behaviour that one may be interested in. As an alternative, Chen et al. (2010, 2017) proposed the topology that is induced by a uniform metric on belief hierarchies. They showed that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if two hierarchies are δ-close under the uniform metric, an action that interim correlated ε-rationalizable for one hierarchy is also interim correlated ε'-rationalizable for any $\varepsilon' > \varepsilon$, where δ can be chosen uniformly over all games with a fixed payoff bound.[2] For brevity, I will refer to the topology induced by the uniform metric as the *uniform topology.*

[1] For an extension of this analysis, see Brandenburger and Dekel (1993).

[2] In fact they showed that the topology generated by their uniform metric is equivalent to the uniform strategic topology of Dekel et al. (2006), which is defined precisely in terms of upper and lower continuity properties of interim correlated ε-rationalizable behaviours. Rubinstein's (1989) email game shows that the desired lower continuity properties are not generally obtained if the space of belief hierarchies has the product topology. In the email game, beliefs of arbitrarily high orders can have a significant impact on strategic behaviour, which is incompatible with lower continuity with respect to the product topology.

What becomes of the Mertens-Zamir homeomorphism theorem when this topology is used? Because the change of topology concerns both the domain and the range of the homeomorphism in Mertens and Zamir (1985), the answer to this questions cannot simply be derived from the observation that the uniform topology is finer than the product topology.

In Hellwig (2016/2023), I prove a homeomorphism theorem of the Mertens-Zamir type when belief hierarchies have the uniform topology. The theorem involves the product σ-algebra on the space of belief hierarchies and, for measures on this σ-algebra, the topology of weak convergence that is defined by the uniform topology on belief hierarchies. In this topology, a sequence of measures converges to some limit if and only if, for every real-valued measurable function that is bounded and continuous with respect to the uniform topology on the underlying space, the sequence of integrals of the function with respect to the different measures converges to the integral of the function with respect to the limit measure.

Reliance on the product σ-algebra rather than the Borel σ-algebra for the chosen topology is unusual. Usually, one chooses a topology that is suitable for the analysis one is pursuing and then one works with the Borel σ-algebra for that topology. Under this procedure, any function that is continuous for the chosen topology is also measurable. However, this is no more than a matter of convenience. There are no deeper reasons for this subordination of the choice of a σ-algebra to the topology.

In the context of belief hierarchies, however, even if one prefers the uniform topology, the choice of the product σ-algebra is mandated by the substantive consideration that the Borel σ-algebra on the space of states of nature and other agents' belief hierarchies is much larger than the product σ-algebra and that the hierarchies of beliefs of different orders do not contain the information that is needed to assign probabilities to events in the larger σ-algebra that do not also belong to the smaller σ-algebra. In some cases, it may even be impossible to assign such probabilities to all events in the larger σ-algebra at all.[3]

The product σ-algebra is defined without reference to any topology. It happens to coincide with the Borel σ-algebra for the product topology but there is also a link to the uniform topology. To to see this, consider the

[3] Consider the infinite product $\{0,1\}^\infty$. The Borel σ-algebra for the topology induced by a uniform metric is just the set of all subsets of $\{0,1\}^\infty$. Consider a measure on the product σ-algebra that assigns the probability one half to each of the outcomes 0 and 1 for the k-th factor, for any k, with independence of the different factors (fair cointossing). This measure on the product σ-algebra cannot be extended to a measure on the σ-algebra of all subsets of $\{0,1\}^\infty$, i.e., the Borel σ-algebra for the uniform topology.

σ-algebra that is generated by the open *balls* in the uniform topology.[4] In contrast to the Borel σ-algebra for the uniform topology, this σ-algebra is actually smaller than the product σ-algebra. However, the difference is negligible because the measures on the product σ-algebra are just the completions of measures on the σ-algebra that is generated by the open balls in the uniform topology. For any measure on the product σ-algebra and any set in this σ-algebra, there is a set in the σ-algebra generated by the open *balls* in the uniform topology that is a subset of the first set and has the same measure.

For the proof of the homeomorphism theorem, a key issue is whether the topology on the space of measures is metrizable. For Borel measures, it is well known that the topology of weak convergence is metrizable if and only if it is metrizable by the so-called Prohorov metric. Moreover, such measures are metrizable by the Prohorov metric if they are separable in the sense that they assign probability one to a separable set.[5] The latter condition is trivially satisfied if the underlying space itself is separable. It is also satisfied if the underlying space is not separable but the cardinal of any discrete subset of the space is not atomlessly measurable.[6]

In the present context, these results cannot be used because the product σ-algebra is not a Borel σ-algebra for the uniform topology. For the space of probability measures on the product σ-algebra, the main result of this paper shows that, nevertheless, under a slightly stronger version of the set theoretic axiom of Billingsley (1968), the topology of weak convergence that is induced by the uniform topology on the underlying space is metrizable by a suitably adapted Prohorov metric.

The argument is similar to the argument for Borel measures. However, I replace the notion of separability by a notion of *quasi-separability.* I say that a measure on the product σ-algebra is *quasi-separable* in the uniform topology if and only if any family of open balls in the uniform topology to which the measure assigns probability one has a countable subfamily to which the measure also assigns probability one. Under the assumption that

[4]The σ-algebra that is generated by the open balls in the uniform topology was introduced by Dudley (1966, 1967)

[5]See Theorem 5, p. 238, in Billingsley (1968).

[6]See Theorem 2, p. 235, in Billingsley (1968). A cardinal said to be atomlessly measurable if there exists a set with cardinal no greater than the given cardinal and an atomless probability measure that is defined on all subsets of the given set. Billingsley uses the term "measurable". The more recent literature reserves the pair "measurable - nonmeasurable" for the case of binary measures taking the values zero and one, and uses the pair "atomlessly measurable - not atomlessly measurable" for the case considered by Billingsley.

the continuum is not atomlessly measurable, I show that every measure on the product σ-algebra is quasi-separable with respect to the uniform topology. Moreover, if the measures on this σ-algebra are quasi-separable, the topology of weak convergence on the space of these measures is metrizable by the Prohorov metric.

The analysis will proceed without reference to the game theoretic motivation. For an arbitrary countable product of complete separable metric spaces, Section 2 below introduces the σ-algebra that is generated by the open balls in the uniform topology and formulates the results on the quasi-separability of measures on this σ-algebra and the metrizability of the topology of weak convergence on the space of such measures. Quasi-separability is proved in Section 3, metrizability by the Prohorov metric in the appendix. Section 5 contains a homeomorphism theorem without reference to the game theory, however.

2 The Main Results

Let $X_1, X_2, ...$ be non-singleton complete separable metric spaces with metrics $\rho_1, \rho_2, ...$. Suppose that the product

$$X = \prod_{k=1}^{\infty} X_k \tag{2.1}$$

has the topology induced by the uniform metric ρ^u where, for any x and $\hat{x}$ in X,

$$\rho^u(x, \hat{x}) = \sup_k \rho_k(\pi_k(x), \pi_k(\hat{x})) \tag{2.2}$$

and, for any k, π_k is the projection from X to X_k. I use the notation X^u to indicate that X has the topology induced by ρ^u.

Occasionally, I will also refer to the product topology on X. In those cases, I will use the notation X^p. As a product of complete separable metric spaces, X^p itself is a complete separable metric space.[7] A metric ρ^p for the product topology is given by the formula

$$\rho^p(x, \hat{x}) = \sum_k \alpha^k \rho_k(\pi_k(x), \pi_k(\hat{x})), \tag{2.3}$$

[7] See Problem 3, p. 42, Proposition 2.4.4, p. 50, and Theorem 2.5.7, p. 62, in Dudley (2002).

where α is some number strictly between zero and one and, for any k, π_k is again the projection from X to X_k.

The topology on X that is induced by the uniform metric ρ^u is strictly finer than the product topology. Indeed the space X^u is non-separable.[8] The non-separability of X^u causes difficulties for working with the Borel σ-algebra $\mathcal{B}(X^u)$ and the space $\mathcal{M}(X^u)$ of probability measures on $(X^u, \mathcal{B}(X^u))$.

As an alternative, I consider the product σ-algebra $\mathcal{B}_\pi(X)$ and the space $\mathcal{M}_\pi(X)$ of probability measures on $(X, \mathcal{B}_\pi(X))$. Both $\mathcal{B}_\pi(X)$ and $\mathcal{M}_\pi(X)$ are defined without reference to a topology on X. As is well known, however, $\mathcal{B}_\pi(X) = \mathcal{B}(X^p)$, the product σ-algebra is equal to the Borel σ-algebra that is induced by the product topology on X.

As mentioned in the introduction, I will also refer to the σ-algebra $\mathcal{B}_0(X^u)$ that is generated by the ρ^u-open balls and with the space $\mathcal{M}_0(X^u)$ of probability measures on $(X^u, \mathcal{B}_0(X^u))$.[9] This σ-algebra is obviously coarser then the Borel σ-algebra $\mathcal{B}(X^u)$.[10] The following lemma shows that it is also coarser than the product σ-algebra $\mathcal{B}_\pi(X)$.

Lemma 2.1 $\mathcal{B}_0(X^u) \subset \mathcal{B}_\pi(X)$.

Proof. Given that the product σ-algebra $\mathcal{B}_\pi(X)$ coincides with the Borel σ-algebra $\mathcal{B}(X^p)$ for the product topology, it suffices to show that any open ball in X^u can be represented as a countable intersection of countable unions of open balls in the product topology. To see this, consider any $x = \{x_k\}_{k=1}^\infty \in X$ and any $r > 0$. The ρ^u-open r-ball around x is given as

$$B^u(x,r) = \bigcup_{n=1}^{\infty} \prod_{k=1}^{\infty} B_k(x_k, \max(r - \frac{1}{n}, 0)), \tag{2.4}$$

where, for each k and $r' > 0$, $B_k(x_k, r')$ is the ρ_k-open r'-ball around x^k. Equation (2.4) can be rewritten as

$$B^u(x,r) = \bigcap_{\ell=1}^{\infty} \bigcup_{n=1}^{\infty} \left[\prod_{k=1}^{\ell} B_k(x_k, \max(r - \frac{1}{n}, 0)) \times \prod_{k=\ell+1}^{\infty} X_k \right], \tag{2.5}$$

[8] For $k = 1, 2, ...$, let x_k^* and $\hat{x}_k$ be two distinct elements of X_k. The set of sequences $\{x_k\}_{k=1}^\infty$ such that, for any k, $x_k \in \{x_k^*, \hat{x}_k\}$ is uncountable and is discrete in the topology induced by ρ^u.

[9] This σ-algebra was proposed by Dudley (1966, 1967) in order to avoid the difficulties in working with $\mathcal{B}(X^u)$.

[10] $\mathcal{B}_0(X^u)$ contains the ρ^u-open balls and is closed under countable unions and intersection, $\mathcal{B}(X^u)$ of ρ^u-open sets, i.e. uncountable unions of ρ^u-open balls and is closed under countable unions and intersections.

which is a countable intersection of countable unions of cylinder sets of the form

$$C = B_1 \times ... \times B_\ell \times X_{\ell+1} \times X_{\ell+2} \times ...., \tag{2.6}$$

where, for $k = 1, ...\ell$, B_k is a ρ_k-open subset of X_k. Since $\mathcal{B}(X^p)$ contains the cylinder sets and is closed under countable unions and intersections, (2.5) implies that $B^u(x,r) \in \mathcal{B}(X^p)$ and hence that $B^u(x,r) \in \mathcal{B}_\pi(X)$. ■

To take account of the fact that X has the uniform topology, I say that a sequence $\{\mu^r\}$ of measures in $\mathcal{M}_\pi(X^u)$ converges weakly to a measure $\mu \in \mathcal{M}_\pi(X^u)$ if and only if

$$\int_X f(x)d\mu^r(x) \to \int_X f(x)d\mu(x) \tag{2.7}$$

for all f in the space $\mathcal{C}_\pi(X^u)$ of bounded, ρ^u-continuous, and $\mathcal{B}_\pi(X)$-measurable real-valued functions on X. For measures in $\mathcal{M}_0(X^u)$, the same definition applies, except that the functions f must belong to the space $\mathcal{C}_0(X^u)$ of bounded, ρ^u-continuous, and $\mathcal{B}_0(X)$-measurable real-valued functions on X.

An important question concerns the metrizabilizy of the topology weak convergence. For Borel measures, Billingsley (1968) has shown that, if the topology of weak convergence is metrizable at all, it is metrizable by the Prohorov metric. Given the uniform topology on X, the associated (ρ^u-based) Prohorov distance between any two measures μ and $\hat{\mu}$ in $\mathcal{M}_\pi(X^u)$ is defined as the greatest lower bound on the set of $\varepsilon > 0$ such that

$$\mu(B) \leq \hat{\mu}(B^\varepsilon) + \varepsilon \text{ and } \hat{\mu}(B) \leq \mu(B^\varepsilon) + \varepsilon \tag{2.8}$$

for all sets $B \in \mathcal{B}_\pi(X)$. As shown by the following lemma, the set B^ε in (2.8) is always well defined.

Lemma 2.2 *For any $B \in \mathcal{B}_\pi(X)$ and any $\varepsilon > 0$, the set*

$$B^\varepsilon := \bigcup_{x \in B} \{x' \in X | \rho^u(x', x) < \varepsilon\} \tag{2.9}$$

is also an element of $\mathcal{B}_\pi(X)$.

Thus, the ρ^u-based Prohorov distance on $\mathcal{M}_\pi(X^u)$ is always well defined. The following result shows that it provides a suitable metric for the topology of weak convergence on $\mathcal{M}_\pi(X^u)$.

Theorem 2.3 *If the cardinal* **c** *of the continuum is not atomlessly measurable, the topology of weak convergence on* $\mathcal{M}_\pi(X^u)$ *is metrizable by the* ρ^u*-based Prohorov metric on* $\mathcal{M}_\pi(X^u)$.

For Borel measures, metrizability of the topology of weak convergence by the Prohorov metric is established in Appendix III of Billingsley (1968). The analysis there involves two steps. First, Theorem 2, p. 235, gives a condition under which any Borel measure is separable in the sense that any family of open sets to which the measure assigns probability one has a countable subfamily to which the measure also assigns probability one. Second, Theorem 5, p. 238, shows that, on a space of separable Borel measures, the topology of weak convergence is equivalent to the topology induced by the Prohorov metric.

Because $\mathcal{M}_\pi(X^u)$ is *not* a space of Borel measures, I cannot use Billingsley's arguments as such. However, if I replace the notion of separability of a measure by a notion of *quasi-separability*, I can use similar arguments. A measure in $\mathcal{M}_\pi(X^u)$ is quasi-separable if and only if any family of ρ^u-open balls to which the measure assigns probability one has a countable subfamily to which the measure also assigns probability one. The following result shows that quasi-separability is in fact enough to ensure that the topology of weak convergence is metrizable.

Proposition 2.4 *If the measures in* $\mathcal{M}_\pi(X^u)$ *are quasi-separable, the topology of weak convergence on* $\mathcal{M}_\pi(X^u)$ *is metrizable by the* ρ^u*-based Prohorov metric.*

The proofs of Lemma 2.2 and Proposition 2.4 are given in the appendix. The argument for Proposition 2.4 is step by step the same as the argument for Theorem 5, p. 238, in Billingsley (1968).

Given Proposition 2.4, Theorem 2.3 is an immediate consequence of the following proposition, which is proved in Section 3 below.

Proposition 2.5 *If the cardinal* **c** *of the continuum is not atomlessly measurable, the measures in* $\mathcal{M}_\pi(X^u)$ *are quasi-separable.*

As a by-product of the analysis, one also obtains the following result on the relation between $\mathcal{M}_\pi(X^u)$ and the space $\mathcal{M}_0(X^u)$ of measures on the σ-algebra that is generated by the ρ^u-open balls.

Proposition 2.6 *If the cardinal* **c** *of the continuum is not atomlessly measurable, any measure in $\mathcal{M}_\pi(X)$ is the completion of a measure in $\mathcal{M}_0(X^u)$, i.e., for any measure $\bar{\mu} \in \mathcal{M}_\pi(X)$, there exists a measure $\mu \in \mathcal{M}_0(X^u)$ such that, for every $B \in \mathcal{B}_\pi(X)$, there exists $B_0 \in \mathcal{B}_0(X^u)$ such that $B_0 \subset B$ and $\bar{\mu}(B \backslash B_0) = 0$.*

In this analysis, the assumption that the cardinal **c** of the continuum is not atomlessly measurable replaces the assumption of Billingsley's (1968) that no discrete subset of the space has a cardinal that is atomlessly measurable. Whereas discrete sets depend on the topology, my somewhat stronger assumption does not depend on the topology of the underlying space.

By a theorem of Banach and Kuratowski (1929), the assumption that the cardinal **c** of the continuum is not atomlessly measurable is implied by the Continuum Hypothesis (CH). The older literature, such as Dudley (1967) or Billingsley (1968), invokes this fact to suggest that the assumption is unproblematic. Under the influence of Cohen (1966), CH has increasingly met with criticism in recent decades. It it therefore worth noting that CH is *not* necessary for the condition that **c** is not atomlessly measurable. Bartoszynski and Halbeisen (2003) point to the fact that, in Banach and Kuratoski (1929), the conclusion that **c** is not atomlessly measurable follows from the existence of a certain matrix, which Bartoszynski and Halbeisen call a *BK-matrix*. The existence of such a matrix is implied by CH but is also compatible with the negation of CH.[11]

3 Quasi-Separability of Measures in $\mathcal{M}_\pi(X^u)$

This section is devoted to the proof of Propositions 2.5 and 2.6.

Lemma 3.1 *For any k, the projection mapping π^k from X^u to $X_0 \times ... \times X_k$ is continuous and open.*

Proof. Continuity is trivial. To prove openness, I note that any ρ^u-open set $U \subset X^u$ can be written in the form

$$U = \bigcup_{\ell \in \mathcal{L}} V_\ell \tag{3.1}$$

[11] If the underlying space if the unit interval, a BK-matrix is a doubly infinite array of sets A_k^i, $i, k \in \mathbb{N}$, such that (i) for each $i \in \mathbb{N}$, $\cup_{k \in \mathbb{N}} A_k^i = [0, 1]$, (ii) for each $i \in \mathbb{N}$, and all $k, k' \in \mathbb{N}$, $A_k^i \cap A_{k'}^i = \emptyset$ if $k \neq k'$, and (iii) for any sequence $\{k_i\}$ in $\mathbb{N}$, the set $\cap_{i \in \mathbb{N}} (\cup_{k \leq k_i} A_k^i)$ is at most countable. Bartoszynski and Halbeisen show that such a matrix exists if and only if there exists a K-Lusin set of the size of the continuum.

where each $V_\ell, \ell \in \mathcal{L}$, is a ρ^u-open ball around an element $x^\ell = (x_0^\ell, x_1^\ell, ...)$ of U. Any one of these balls takes the form

$$V_\ell = \bigcup_{n=1}^{\infty} \prod_{j=0}^{\infty} B_j(x_j^\ell, r^\ell - \frac{1}{n}), \tag{3.2}$$

with projections

$$\pi^k(V_\ell) = \bigcup_{n=1}^{\infty} \prod_{j=0}^{k} B_j(x_j^\ell, r^\ell - \frac{1}{n}) = \prod_{j=0}^{k} B_j(x_j^\ell, r^\ell), \tag{3.3}$$

which are open in $X_0 \times ... \times X_k$, $k = 0, 1, 2, ...$ Since

$$\pi^k(U) = \bigcup_{\ell \in \mathcal{L}} \pi^k(V_\ell), \tag{3.4}$$

it follows that $\pi^k(U)$ is open if U is ρ^u-open. ■

Corollary 3.2 *For any k and any ρ^u-open set U, the set*

$$\Pi^k(U) = \pi^k(U) \times X_{k+1} \times X_{k+2} \times ... \tag{3.5}$$

is an element of $\mathcal{B}_\pi(X) = \mathcal{B}(X^p)$.

For any ρ^u-open set $U \subset X^u$ and all k, one obviously has $U \subset \Pi^{k+1}(U) \subset \Pi^k(U)$. Therefore, the set

$$V(U) = \bigcap_{k=0}^{\infty} \Pi^k(U) \tag{3.6}$$

is well-defined and satisfies

$$U \subset V(U). \tag{3.7}$$

Lemma 3.3 *Let $U, \hat{U}$ be two ρ^u-open subsets of X and suppose that, for some $\varepsilon > 0$,*

$$\rho^u(x, \hat{x}) \geq \varepsilon \tag{3.8}$$

for all $x \in U$ and $\hat{x}$ in $\hat{U}$. Then $V(U) \cap V(\hat{U}) = \emptyset$.

Proof. The premise of the lemma implies that, for every $\delta > 0$ and every $x \in U$ and $\hat{x}$ in $\hat{U}$, there exists k such that

$$\rho_k(\pi_k(x), \pi_k(\hat{x})) \geq \varepsilon - \delta. \tag{3.9}$$

If the lemma were false, then for some $x^* \in X$, one would have $x^* \in \Pi^k(U)$ for all k and $x^* \in \Pi^k(\hat{U})$ for all k. Since $\rho_k(\pi_k(x^*), \pi_k(x^*)) = 0$ for all k, this is incompatible with (3.9). ■

As a countable intersection of elements of $\mathcal{B}_\pi(X)$, $V(U)$ is also an element of $\mathcal{B}_\pi(X)$. Therefore, for any ρ^u-open set $U \subset X^u$ and any measure $\mu \in \mathcal{M}_\pi(X)$, the quantity $\mu(V(U))$ is well defined. By definition,

$$\mu(V(U)) \geq \mu^*(U), \tag{3.10}$$

where μ^*, the outer measure induced by the measure μ, is defined by the formula

$$\mu^*(U) := \inf \sum_{i=1}^{\infty} \mu(B_i), \tag{3.11}$$

where the infimum is taken over all sequences $\{B_i\}_{i=1}^{\infty}$ such that $B_i \in \mathcal{B}_\pi(X)$ for all i and $U \subset \bigcup_{i=1}^{\infty} B_i$.[12] The following result plays a similar role as Theorem 1 in Marczewski and Sikorski (1948); the proof idea is due to Banach (1930).

Lemma 3.4 *Assume that the cardinal* **c** *of the continuum is not atomlessly measurable, and let $\{U_i\}_{i \in I}$ be a collection of ρ^u-open subsets of X such that, for some $\varepsilon > 0$, for all i and j in I,*

$$\rho^u(x^i, x^j) \geq \varepsilon \tag{3.12}$$

for all $x^i \in U_i$ and all $x^j \in U_j$. Then for any $\mu \in \mathcal{M}_\pi(X)$, there exists a finite or countable set $I^* \subset I$ such that

$$\mu^*(U_i) > 0 \ \ \textit{for} \ \ i \in I^* \tag{3.13}$$

and

$$\mu^*\left(\bigcup_{i \in I \setminus I^*} U_i\right) = 0. \tag{3.14}$$

[12] For the definition of outer measure, see Dudley (2002), p. 89.

Proof. Before embarking on main argument, I recall that, when endowed with the product topology, the space X is a complete separable metric space. By the isomorphism theorem therefore the cardinal of X is no greater than c.[13] I claim that therefore also the cardinal of the index set I cannot be greater than c. To prove this claim, I note that any mapping $\varphi : I \to X$ such that $\varphi(i) \in U_i$ for all i is an injection because, by (3.12), $\rho^u(\varphi(i), \varphi(\hat{\imath})) \geq \varepsilon > 0$ for all i and $\hat{\imath}$ in I satisfying $i \neq \hat{\imath}$. The cardinal of I is therefore equal to the cardinal of the range of φ, which cannot exceed the cardinal of X.

Turning to the claim of the lemma, by (3.10), it suffices to prove that there exists a finite or countable set $I^{**} \subset I$ such that

$$\mu(V(U_i)) > 0 \quad \text{for} \quad i \in I^{**} \tag{3.15}$$

and

$$\mu\left(V\left(\bigcup_{i \in I \setminus I^{**}} U_i\right)\right) = 0. \tag{3.16}$$

The set I^{**} is defined as the union over n of the sets

$$I_n := \{i \in I | \mu(V(U_i)) > \frac{1}{n}\}.$$

I claim that each set I_n has at most n elements and, therefore, that I^{**}, as a countable union of these sets, is at most countable. For suppose that some set I_n has $n' > n$ elements, say $i_1, ..., i_{n'}$. Then, by Lemma 3.3 and the countable additivity of the measure μ,

$$\mu\left(\bigcup_{i=i_1}^{i_{n'}} V(U_i)\right) = \sum_{i=i_1}^{i_{n'}} \mu(V(U_i)) > n' \cdot \frac{1}{n} > 1,$$

which is impossible.

For any set $\hat{I} \subset I$, the union $\bigcup_{i \in \hat{I}} U_i$ is an open set in X^u. By the argument given above, it follows that, for any $\hat{I} \subset I$, the set $V\left(\bigcup_{i \in \hat{I}} U_i\right)$ belongs to the σ-algebra $\mathcal{B}(X^p)$ and therefore also to the σ-algebra $\mathcal{B}_0(X^u)$. One may therefore define a set function ν by setting

$$\nu(\hat{I}) = \mu\left(V\left(\bigcup_{i \in \hat{I}} U_i\right)\right) \quad \text{for} \quad \hat{I} \subset I.$$

[13] See Theorem 13.1.1, p. 487, in Dudley (2002).

Notice that this set function is defined on all subsets of I.

By the definition of I^{**},one obviously has

$$\nu(\{i\}) = \mu(V(U_i)) = 0 \text{ for } i \in I \backslash I^{**}, \tag{3.17}$$

i.e. the set function ν has no atoms in $I \backslash I^{**}$.

I also claim that ν is countably additive. Let $I_j, j = 1, 2, ...$ be any sequence of disjoint subsets of I. Then, by Lemma 3.3 and the countable additivity of μ,

$$\nu\left(\bigcup_{j=1}^{\infty} I_j\right) = \mu\left(V\left(\bigcup_{j=1}^{\infty}\bigcup_{i\in I_j} U_i\right)\right) = \sum_{j=1}^{\infty}\mu\left(V\left(\bigcup_{i\in I_j} U_i\right)\right) = \sum_{j=1}^{\infty}\nu(I_j),$$

which proves countable additivity of ν.

Because the cardinal of I is at most c, the cardinal of $I \backslash I^{**}$ is also at most **c**. By the assumption that the continuum is not atomlessly measurable, it follows that $\nu(I \backslash I^{**}) = 0$ and hence that

$$\mu\left(V\left(\bigcup_{i\in I\backslash I^{**}} U_i\right)\right) = \nu(I \backslash I^{**}) = 0,$$

which proves (3.16). The lemma follows immediately. ■

Proposition 3.5 *If the cardinal* **c** *of the continuum is not atomlessly measurable, then for any measure $\mu \in \mathcal{M}_\pi(X)$ and any family $\mathcal{G}$ of ρ^u-open balls covering X, there exists a countable subfamily $\mathcal{G}^* \subset \mathcal{G}$ such that $\mu(G^*) = 1$, where G^* is the union of the sets in $\mathcal{G}^*$.*

Proof. The proof follows along similar lines as the proof of Theorem 2, p. 235, in Billingsley (1968). Let μ and $\mathcal{G}$ be as specified in the proposition. Because X^u is a metric space, Theorem 4.21, p. 129, in Kelley (1955), implies that $\mathcal{G}$ has a σ-discrete refinement, i.e., there exists a family $\mathcal{H}$ of ρ^u-open sets (not necessarily balls) covering X such that, for every $H \in \mathcal{H}$, there exists $G(H) \in \mathcal{G}$ such that $H \subset G(H)$ and, moreover, $\mathcal{H}$ can be written as a countable union

$$\mathcal{H} = \bigcup_{t=1}^{\infty} \mathcal{H}_t \tag{3.18}$$

where, for any t, any two sets H_{ti}, H_{tj} in $\mathcal{H}_t$, there exists $\varepsilon_t > 0$ such that $\rho^u(x^i, x^j) \geq \varepsilon_t$ for all $x^i \in H_{ti}$ and all $x^j \in H_{tj}$.

For any t, let I_t be the set of indices i such that $H_{ti} \in \mathcal{H}_t$. By Lemma 3.4, the set I_t has a finite or countable subset I_t^* such that

$$\mu^*(H_{ti}) > 0 \quad \text{for} \quad i \in I_t^* \tag{3.19}$$

and

$$\mu^* \left(\bigcup_{i \in I_t \backslash I_t^*} H_{ti} \right) = 0. \tag{3.20}$$

Let $\mathcal{H}_t^*$ be the family of sets H_{ti}, $i \in I_t^*$, and let

$$\mathcal{H}^* = \bigcup_{t=1}^{\infty} \mathcal{H}_t^*. \tag{3.21}$$

Then $\mathcal{H}^*$ is a countable union of finite or countable sets and is itself countable.

Recalling that, for each $H \in \mathcal{H}$, there exists $G(H) \in \mathcal{G}$ such that $H \subset G(H)$, define $\mathcal{G}^* \subset \mathcal{G}$ so that $G \in \mathcal{G}^*$ if and only if $G = G(H)$ for some $H \in \mathcal{H}^*$. Since $\mathcal{H}^*$ is countable, so is $\mathcal{G}^*$. Moreover, because, by Lemma 2.1, the elements of $\mathcal{G}^*$ belong to $\mathcal{B}_\pi(X)$, so does the countable union

$$G^* = \bigcup_{H \in \mathcal{H}^*} G(H).$$

Hence $\mu(G^*)$ is well defined and so is $\mu(X \backslash G^*) = 1 - \mu(G^*)$.

I claim that $\mu(G^*) = 1$. For suppose that $\mu(G^*) < 1$. Then also $\mu(X \backslash G^*) > 0$. Hence also $\mu^*(X \backslash G^*) > 0$, where μ^* is the outer measure defined by μ , in accordance with (3.11) above. Since $H \subset G(H)$ for all H, we also have $X \backslash G^* \subset X \backslash H^*$, where H^* is the union of the sets H_{ti} over all t and all $i \in I_t^*$. Thus, $\mu^*(X \backslash G^*) > 0$ implies $\mu^*(X \backslash H^*) > 0$. Because the refinement $\mathcal{H}$ of the family $\mathcal{G}$ covers X , the set $X \backslash H^*$ is a subset of the union of the sets H_{ti} over all t and all $i \in I_t \backslash I_t^*$. By the subadditivity of outer measure, it follows that

$$\mu^*(X \backslash H^*) \le \mu^* \left(\bigcup_{t=1}^{\infty} \bigcup_{i \in I_t \backslash I_t^*} H_{ti} \right) \le \sum_{t=1}^{\infty} \mu^* \left(\bigcup_{i \in I_t \backslash I_t^*} H_{ti} \right). \tag{3.22}$$

By (3.20) the right-hand side of (3.22) is zero. Therefore, $\mu^*(X \backslash H^*) = 0$, which is incompatible with $\mu(G^*) < 1$. This completes the proof of the proposition. ■

Proposition 2.5 follows immediately. If the cardinal $\mathbf{c}$ of the continuum is not atomlessly measurable, the measures in $\mathcal{M}_\pi(X^u)$ are quasi-separable.

The arguments used to prove Proposition 3.5 can also be used to prove Proposition 2.6, concerning the relation of the space $\mathcal{M}_\pi(X^u)$ of measures on the product σ-algebra on X to the space $\mathcal{M}_0(X^u)$ of measures on the σ-algebra generated by the ρ^u-open balls. For any measure $\bar{\mu} \in \mathcal{M}_\pi(X^u)$, the restriction of $\bar{\mu}$ to the smaller σ-algebra $\mathcal{B}_0(X^u)$ belongs to the space $\mathcal{M}_0(X^u)$ of measures on $(X^u, \mathcal{B}_0(X^u))$. Proposition 2.6 is therefore equivalent to the following result.

Proposition 3.6 *If the cardinal* $\mathbf{c}$ *of the continuum is not atomlessly measurable, then, for every measure* $\mu \in \mathcal{M}_\pi(X)$ *and every set* $B \in \mathcal{B}_\pi(X)$, *there exists* $B_0 \in \mathcal{B}_0(X^u)$ *such that* $B_0 \subset B$ *and* $\mu(B \backslash B_0) = 0$.

Proof. The product σ-algebra $\mathcal{B}_\pi(X)$ is the smallest σ-algebra that is closed under countable unions and countable intersections and that contains the cylinder sets of the form

$$C(B^k) = B^k \times X_{k+1} \times X_{k+2} \times ...$$

for $k = 1, 2, ...$ and $B^k \in \mathcal{B}(X_1 \times ... \times B_k)$. Therefore it suffices to prove that the claim of the proposition is true for every measure $\mu \in \mathcal{M}_\pi(X)$ and every cylinder set $C(B^k)$, for $k = 1, 2, ...$ and $B^k \in \mathcal{B}(X_1 \times ... \times X_k)$. Fix a measure $\mu \in \mathcal{M}_\pi(X)$ and a cylinder set $C(B^k)$. Without loss of generality, one may assume that B^k is an open subset of the product $X_1 \times ... \times X_k$.

If $\mu(C(B^k)) = 0$, the claim of the proposition is trivially true because for any $B_0 \in \mathcal{B}_0(X^u)$ such that $B_0 \subset C(B^k)$, one has $\mu(C(B^k) \backslash B_0) \leq \mu(C(B^k)) = 0$. Suppose therefore that $\mu(C(B^k)) > 0$.

Define a new product space $\hat{X} = \prod_{\ell=1}^{\infty} \hat{X}_\ell$ by setting $\hat{X}_1 = B^k$ and, for $\ell > 1$, $\hat{X}_\ell = X_{k+\ell-1}$. One easily verifies that the product σ-algebra $\mathcal{B}_\pi(\hat{X})$ on $\hat{X}$ consists of sets of the form $\hat{B} = B \cap C(B^k)$, $B \in \mathcal{B}_\pi(X)$. Any such set also belongs to $\mathcal{B}_\pi(X)$. One can therefore set

$$\hat{\mu}(\hat{B}) = \frac{\mu(\hat{B})}{\mu(C(B^k))} \quad \text{for } \hat{B} \in \mathcal{B}_\pi(\hat{X}),$$

which yields a measure $\hat{\mu}$ on $(\hat{X}, \mathcal{B}_\pi(\hat{X}))$. Upon applying Proposition 3.5 to the space $(\hat{X}, \mathcal{B}_\pi(\hat{X}))$, one finds that there exists $\hat{B}_0 \in \mathcal{B}_0(\hat{X}^u)$ such that $\hat{\mu}(\hat{B}_0) = 1$ and therefore $\mu(C(B^k)) = \mu(\hat{B}_0)$, or $\mu(C(B^k) \backslash \hat{B}_0) \leq \mu(C(B^k)) - \mu(\hat{B}_0) = 0$. ■

Whereas Proposition 3.6 starts from a measure on the product σ-algebra on X and uses Proposition 3.6 to show that any set to which this measure gives positive weight has a subset in $\mathcal{B}_0(X^u)$ to which the measure gives the same weight, I conjecture that one might also start from a measure on $(X^u, \mathcal{B}_0(X^u))$, take the completion of that measure and show that this completion is a measure on the product σ-algebra. For any measure on $(X^u, \mathcal{B}_0(X^u))$ that can be extended to $(X^u, \mathcal{B}_\pi(X))$, this conjecture would follow from the arguments given here. However, if there exist measures on $(X^u, \mathcal{B}_0(X^u))$ that cannot be extended to $(X^u, \mathcal{B}_\pi(X))$, the argument cannot be used because the proof of Lemma 3.4 presume that the measure with which one is concerned be defined on the product σ-algebra.

4 A Homeomorphism Theorem

As mentioned in the introduction, I came across the spaces $\mathcal{M}_\pi(X^u)$ and $\mathcal{M}_0(X^u)$ when I was trying to prove a homeomorphism theorem for the universal type space with the uniform topology. For reasons discussed in the introduction, there are problems in working with the Borel σ-algebra induced by the uniform topology. In Hellwig (2016/2023), I therefore work with the space $\mathcal{M}_\pi(X^u)$ of measures on the product σ-algebra with the topology of weak convergence that is induced by the uniform topology on X. With this modification, I show that the homeomorphism and embedding theorems of Mertens and Zamir (1985) remain intact when the product topology on the spaces of belief hierarchies is replaced by the uniform topology. The argument makes essential use of the equivalence of the topology of weak convergence and the topology induced by the Prohorov metric on $\mathcal{M}_\pi(X^u)$.

Without going into the game-theoretic analysis, I briefly sketch the mathematical argument. For $\ell = 1, 2, ...$, let π^ℓ be the projection from X to the finite product

$$X^\ell = \prod_{k=1}^{\ell} X_k. \tag{4.1}$$

For any $\mu \in \mathcal{M}_\pi(X^u)$ and any ℓ, let

$$\Pi^\ell(\mu) = \mu \circ (\pi^\ell)^{-1} \in \mathcal{M}\left(X^\ell\right), \tag{4.2}$$

and let

$$\Pi^\infty(\mu) = (\Pi^1(\mu), \Pi^2(\mu), ...). \tag{4.3}$$

Proposition 4.1 *Suppose that each of the spaces* $\mathcal{M}\left(X^\ell\right)$, $\ell = 1, 2, ...,$ *is endowed with the Prohorov metric, denoted as* d_ℓ, *and that the product space* $\prod_{\ell=1}^{\infty} \mathcal{M}\left(X^\ell\right)$ *is endowed with the uniform metric* d^u *such that, for any two sequences* $\{\mu^\ell\}_{\ell=1}^{\infty}, \{\hat{\mu}^\ell\}_{\ell=1}^{\infty}$,

$$d^u(\{\mu^\ell\}_{\ell=1}^{\infty}, \{\hat{\mu}^\ell\}_{\ell=1}^{\infty}) = \sup_\ell d_\ell(\mu^\ell, \hat{\mu}^\ell). \tag{4.4}$$

If $\mathcal{M}_0(X^u)$ *has the topology of weak convergence, then, under the assumption that the cardinal* $\mathbf{c}$ *of the continuum is not atomlessly measurable, the mapping* Π^∞ *is a homeomorphism between* $\mathcal{M}_0(X^u)$ *and the subspace* $H^u \subset \prod_{\ell=1}^{\infty} \mathcal{M}\left(X^\ell\right)$ *that consists of those sequences* $\{\mu^\ell\}_{\ell=1}^{\infty}$ *that are mutually consistent in that* $\mu^{\ell+1} \circ (\pi^\ell)^{-1} = \mu^\ell$ *for all* ℓ.

Proof. By Kolmogorov's extension theorem, there exists a mapping from the space H of sequences $\{\mu^\ell\}_{\ell=1}^{\infty}$ that are mutually consistent to the space $\mathcal{M}_\pi(X)$ of measures on $(X, \mathcal{B}_\pi(X))$. Moreover, this mapping is injective and onto.

The mapping Π^∞ in the proposition is just the inverse of the Kolmogorov mapping. Because the Kolmogorov mapping is injective and onto, it suffices to show that Π^∞ and $(\Pi^\infty)^{-1}$ are both continuous.

Continuity of Π^∞ is straightforward: For any ℓ and any set $W^\ell \in \mathcal{B}(X^\ell)$, the cylinder set

$$\hat{W}^\ell = W^\ell \times X_{\ell+1} \times X_{\ell+2} \times ...$$

belongs to $\mathcal{B}_\pi(X)$, and, for any $\varepsilon > 0$, the cylinder set

$$\hat{W}^{\ell\varepsilon} = (W^\ell)^\varepsilon \times X_{\ell+1} \times X_{\ell+2} \times ...$$

that is defined by the ε-neighbourhood $(W^\ell)^\varepsilon$ of W^ℓ in X^ℓ is an ε-neighbourhood of $\hat{W}^\ell$ in X^u. If the Prohorov distance $p^u(\mu, \hat{\mu})$ between two measures $\mu, \hat{\mu}$ in $\mathcal{M}_\pi(X^u)$ is less than ε, we have

$$\mu(\hat{W}^\ell) < \hat{\mu}(\hat{W}^{\ell\varepsilon}) + \varepsilon$$

and

$$\hat{\mu}(\hat{W}^\ell) < \mu(\hat{W}^{\ell\varepsilon}) + \varepsilon.$$

By the definition of the marginal distributions $\mu^\ell = \Pi^\ell(\mu)$ and $\hat{\mu}^\ell = \Pi^\ell(\hat{\mu})$, it follows that

$$\mu^\ell(W^\ell) < \hat{\mu}^\ell(W^{\ell\varepsilon}) + \varepsilon$$

and

$$\hat{\mu}^\ell(W^\ell) < \mu^\ell(W^{\ell\varepsilon}) + \varepsilon.$$

Because the choice of $W^\ell \in \mathcal{B}(X^\ell)$ was arbitrary, it follows that the Prohorov distance $d_\ell(\mu^\ell, \hat{\mu}^\ell)$ between $\mu^\ell = \Pi^\ell(\mu)$ and $\hat{\mu}^\ell = \Pi^\ell(\hat{\mu})$ in $\Pi^\ell(\mathcal{M}_\pi(X^u))$ is no greater than ε. Since ε may be taken to be arbitrarily close to the Prohorov distance $p^u(\mu, \hat{\mu})$ between μ and $\hat{\mu}$ in $\mathcal{M}_\pi(X^u)$, it follows that the Prohorov distance between $\mu^\ell = \Pi^\ell(\mu)$ and $\hat{\mu}^\ell = \Pi^\ell(\hat{\mu})$ in $\Pi^\ell(\mathcal{M}_0(X^u))$ is no greater than $p^u(\mu, \hat{\mu})$. Since this is true for all ℓ, it follows that

$$d^u(\Pi^\infty(\mu), \Pi^\infty(\hat{\mu})) = \sup_\ell d_\ell(\Pi^\ell(\mu), \Pi^\ell(\hat{\mu})) \leq p^u(\mu, \hat{\mu}).$$

Continuity of the map Π^∞ from $\mathcal{M}_\pi(X^u)$ to H^u follows immediately.

Next, consider the map $\beta := (\Pi^\infty)^{-1}$ from H^u to $\mathcal{M}_\pi(X^u)$ that is given by Kolmogorov's extension theorem. Proceeding indirectly, suppose that β is not continuous. Then there exist sequences $h^r = \{\mu^{\ell r}\}_{\ell=1}^\infty, r = 1, ..., \infty$, and $h = \{\mu^\ell\}_{\ell=1}^\infty$ in H^u such that h^r converges to $h \in H^u$ but $\beta(h^r)$ does not converge to $\beta(h)$ in $\mathcal{M}_0(X^u)$. Convergence of h^r to h implies that

$$\lim_{r\to\infty} \sup_\ell d_\ell(\mu^{\ell r}, \hat{\mu}^\ell) = 0. \tag{4.5}$$

Non-convergence of $\beta(h^r)$ to $\beta(h)$ implies that, for some $\varepsilon > 0$ and some subsequence $\{h^{r'}\}$ of $\{h^r\}$,

$$p^u(\beta(h^r), \beta(h)) \geq \varepsilon$$

for all r'. Thus, for every r', there exists a set $B_{r'} \in \mathcal{B}_0(X^u)$ such that

$$\beta(B_{r'}|h^r) > \beta(B_{r'}^\varepsilon|h) + \varepsilon \tag{4.6}$$

or

$$\beta(B_{r'}|h) > \beta(B_{r'}^\varepsilon|h^r) + \varepsilon, \tag{4.7}$$

where $B_{r'}^\varepsilon \in \mathcal{B}_\pi(X)$ is the ε-neighbourhood of $B_{r'}$ in X^u.

For any ℓ, let $B_{r'}^\ell = \pi^\ell(B_{r'})$ be the projection of $B_{r'}$ to $X^\ell = \pi^\ell(X^u)$ and let $(B_{r'}^\ell)^\varepsilon$ be an ε-neighbourhood of $B_{r'}^\ell$ in X^ℓ. Let

$$\hat{B}_{r'}^\ell := B_{r'}^\ell \times X_{\ell+1} \times X_{\ell+2} \times ...$$

and

$$(\hat{B}_{r'}^\ell)^\varepsilon := (B_{r'}^\ell)^\varepsilon \times X_{\ell+1} \times X_{\ell+2} \times ...$$

be the cylinder sets in X^u that are defined by $B^\ell_{r'}$ and $(B^\ell_{r'})^\varepsilon$. One easily verifies that the sequences $\{\hat{B}^\ell_{r'}\}$ and $\{(\hat{B}^\ell_{r'})^\varepsilon\}$ are nonincreasing and that

$$B_{r'} = \bigcap_{\ell=1}^{\infty} \hat{B}^\ell_{r'} \text{ and } B^\varepsilon_{r'} = \bigcap_{\ell=1}^{\infty} (\hat{B}^\ell_{r'})^\varepsilon \tag{4.8}$$

for all r'. By elementary measure theory,[14] for any r' and any $\delta > 0$, there exists an integer $L^{r'}(\delta)$ such that, for $\ell > L^{r'}(\delta)$,

$$\beta(B^\varepsilon_{r'}|h) \geq \beta((\hat{B}^\ell_{r'})^\varepsilon|h) - \delta \tag{4.9}$$

and

$$\beta(B^\varepsilon_{r'}|h^{r'}) \geq \beta((\hat{B}^\ell_{r'})^\varepsilon|h^{r'}) - \delta. \tag{4.10}$$

Moreover, by (4.8),

$$\beta(B_{r'}|h) \leq \beta(\hat{B}^\ell_{r'}|h) \tag{4.11}$$

and

$$\beta(B_{r'}|h^{r'}) \leq \beta(\hat{B}^\ell_{r'}|h^{r'}). \tag{4.12}$$

Set $\delta = \frac{\varepsilon}{2}$ and combine (4.9) - (4.12) with (4.6) and (4.7). Thereby one finds that, for all r', all $\delta > 0$, and all $\ell > L^{r'}(\delta)$, either

$$\beta(\hat{B}^\ell_{r'}|h^{r'}) > \beta((\hat{B}^\ell_{r'})^\varepsilon|h) + \frac{\varepsilon}{2} \tag{4.13}$$

or

$$\beta(\hat{B}^\ell_{r'}|h) > \beta((\hat{B}^\ell_{r'})^\varepsilon|h^{r'}) + \frac{\varepsilon}{2}. \tag{4.14}$$

Since $(\hat{B}^\ell_{r'})^{\frac{\varepsilon}{2}} \subset (\hat{B}^\ell_{r'})^\varepsilon$, it follows that, for all r', all $\delta > 0$, and all $\ell > L^{r'}(\delta)$, either

$$\beta(\hat{B}^\ell_{r'}|h^{r'}) > \beta((\hat{B}^\ell_{r'})^{\frac{\varepsilon}{2}}|h) + \frac{\varepsilon}{2} \tag{4.15}$$

or

$$\beta(\hat{B}^\ell_{r'}|h) > \beta((\hat{B}^\ell_{r'})^{\frac{\varepsilon}{2}}|h^{r'}) + \frac{\varepsilon}{2}. \tag{4.16}$$

By the definition of β as the inverse of $\Pi^\infty = (\Pi^1, \Pi^2, ...)$, and the cylinder nature of the sets $\hat{B}_{r'}$ and $(\hat{B}^\ell_{r'})^{\frac{\varepsilon}{2}}$, we also have

$$\beta(\hat{B}^\ell_{r'}|h^{r'}) = \Pi^\ell(B^\ell_{r'}|\beta(h^{r'})) = \mu^{\ell r'}(B^\ell_{r'}), \tag{4.17}$$

$$\beta(\hat{B}^\ell_{r'}|h) = \Pi^\ell(B^\ell_{r'}|\beta(h)) = \mu^\ell(B^\ell_{r'}), \tag{4.18}$$

[14] Theorem 3.1.1, p. 86, in Dudley (2002).

$$\beta((\hat{B}^{\ell}_{r'})^{\frac{\varepsilon}{2}}|h^{r'}) = \Pi^{\ell}((B^{\ell}_{r'})^{\frac{\varepsilon}{2}}|\beta(h^{r'})) = \mu^{\ell r'}((B^{\ell}_{r'})^{\frac{\varepsilon}{2}}), \tag{4.19}$$

$$\beta((\hat{B}^{\ell}_{r'})^{\frac{\varepsilon}{2}}|h) = \Pi^{\ell}((B^{\ell}_{r'})^{\frac{\varepsilon}{2}}|\beta(h)) = \mu^{\ell}((B^{\ell}_{r'})^{\frac{\varepsilon}{2}}). \tag{4.20}$$

Thus, (4.15) and (4.16) can be rewritten as

$$\mu^{\ell r'}(B^{\ell}_{r'}) > \mu^{\ell}((B^{\ell}_{r'})^{\frac{\varepsilon}{2}}) + \frac{\varepsilon}{2} \tag{4.21}$$

and

$$\mu^{\ell}(B^{\ell}_{r'}) > \mu^{\ell r'}((B^{\ell}_{r'})^{\frac{\varepsilon}{2}}) + \frac{\varepsilon}{2}, \tag{4.22}$$

and one of (4.21), (4.22) must hold if r', $\delta = \frac{\varepsilon}{2}$, and $\ell > L^{r'}(\delta)$. But then, for such r', δ, and ℓ, $d_{\ell}(\mu^{\ell r'}, \hat{\mu}^{\ell}) \geq \frac{\varepsilon}{2}$, contrary to (4.5). The assumption that $\beta = (\Pi^{\infty})^{-1}$ is not continuous has thus led to a contradiction and must be false. ■

A Metrizability of the Topology of Weak Convergence on $\mathcal{M}_0(X^u)$

In this appendix, I provide a proof of Proposition 2.4. As mentioned in the text, the proof is step by step an analogue to the proof of Theorem 5, p. 238, in Billingsley (1968), the analogue of Proposition 2.4 for Borel measures.

As discussed in the introduction, I rely on the ρ^u-based Prohorov metric, which specifies the distance between any two measures μ and $\hat{\mu}$ in $\mathcal{M}_{\pi}(X)$ as the greatest lower bound on the set of $\varepsilon > 0$ such that

$$\mu(B) \leq \hat{\mu}(B^{\varepsilon}) + \varepsilon \text{ and } \hat{\mu}(B) \leq \mu(B^{\varepsilon}) + \varepsilon$$

for all sets $B \in \mathcal{B}_{\pi}(X)$, where

$$B^{\varepsilon} := \bigcup_{x \in B} \{x' \in X | \rho^u(x', x) < \varepsilon\}. \tag{A.1}$$

Lemma 2.2 in the text indicates that, for any $B \in \mathcal{B}_{\pi}(X)$ and any $\varepsilon > 0$, the set B^{ε} is also an element of $\mathcal{B}_{\pi}(X)$ so that, for any μ and $\hat{\mu}$ in $\mathcal{M}_{\pi}(X)$, not only $\mu(B)$ and $\hat{\mu}(B)$ but also $\mu(B^{\varepsilon})$ and $\hat{\mu}(B^{\varepsilon})$ are well defined.

Proof of Lemma 2.2. Consider the class $\mathcal{C}$ of sets for which the lemma is true. For $B^k \in \mathcal{B}(X_1 \times ... \times B_k)$, let

$$C(B^k) = B^k \times X_{k+1} \times X_{k+2} \times ...$$

be the cylinder set defined by B^k. Then

$$C(B^k)^\varepsilon = (B^k)^\varepsilon \times X_{k+1} \times X_{k+2} \times ...,$$

where

$$(B^k)^\varepsilon := \bigcup_{(x_1,...,x_k)\in B^k} \{(x'_1, ..., x'_k) \in X_1 \times ... \times X_k | \max_j \rho_j(x'_j, x_j) < \varepsilon\}.$$

Since, obviously, $(B^k)^\varepsilon \in \mathcal{B}(X_1 \times ... \times X_k)$, it follows that $C(B^k)^\varepsilon \in \mathcal{B}_\pi(X)$ and therefore that $C(B^k) \in \mathcal{C}$. The class $\mathcal{C}$ contains all cylinder sets in X.

Moreover, $\mathcal{C}$ is closed under countable unions: If $B_r, r = 1, 2, ...,$ is any countable family of sets in $\mathcal{C}$, a point x belongs to the ε-neighbourhood of $\cup_r B_r$ if and only if it belongs to B_r^ε for some r. The ε-neighbourhood of $\cup_r B_r$ is therefore equal to the union $\cup_r B_r^\varepsilon$. Since $B_r \in \mathcal{C}$ implies $B_r^\varepsilon \in \mathcal{B}_\pi(X)$ and $\mathcal{B}_\pi(X)$ is closed under countable unions, it follows that $\cup_r B_r^\varepsilon \in \mathcal{B}_\pi(X)$ and hence that $\cup_r B_r \in \mathcal{C}$.

Finally, $\mathcal{C}$ is also closed under countable intersectons: If $B_r, r = 1, 2, ...,$ is any countable family of sets in $\mathcal{C}$, a point x belongs to the ε-neighbourhood of $\cap_r B_r$ if and only if it belongs to B_r^ε for all r. The ε-neighbourhood of $\cap_r B_r$ is therefore equal to the intersection $\cap_r B_r^\varepsilon$. Since $B_r \in \mathcal{C}$ implies $B_r^\varepsilon \in \mathcal{B}_\pi(X)$ and $\mathcal{B}_\pi(X)$ is closed under countable intersections, it follows that $\cap_r B_r^\varepsilon \in \mathcal{B}_\pi(X)$ and hence that $\cap_r B_r \in \mathcal{C}$.

Since $\mathcal{B}_\pi(X)$ is the smallest σ-algebra that is closed under countable unions and countable intersections and that contains the cylinder sets in X, it follows that $\mathcal{B}_\pi(X) \subset \mathcal{C}$. Since, trivially, $\mathcal{B}_\pi(X) \supset \mathcal{C}$, it follows that $\mathcal{B}_\pi(X) = \mathcal{C}$. ∎

Turning to the proof of Proposition 2.4, I note that the proof of the analogous result in Billingsley (1968) comes in two distinct steps. The first step (Theorem 3, p. 236) specifies several families of sets of measures and shows that each family is a base for the topology of weak convergence. The second step (Theorem 5, p. 238) uses this finding to establish the equivalence of the topology of weak convergence with the topology generated by the Prohorov metric.

The argument here has the same structure. For the first step of the argument, I note that the family $\mathcal{F}_0$ of sets taking the form

$$\left\{\nu \in \mathcal{M}_\pi(X) | \left| \int_{X^u} f_i(x) d\nu(x) - \int_{X^u} f_i(x) d\mu(x) \right| < \varepsilon,\ i = 1, ..., k \right\} \quad \text{(A.2)}$$

for some $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and $f_1, ..., f_k$ in $\mathcal{C}_0(X^u)$ is a base for the topology of weak convergence on $\mathcal{M}_\pi(X^u)$. I also consider families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of sets taking the forms

$$\{\nu \in \mathcal{M}_\pi(X^u)|\ \nu(F_i) < \mu(F_i) + \varepsilon,\ i = 1, ..., k\} \tag{A.3}$$

for some $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and ρ^u-closed sets $F_1, ..., F_k$ belonging to $\mathcal{B}_\pi(X)$ in the case of $\mathcal{F}_1$,

$$\{\nu \in \mathcal{M}_\pi(X^u)|\ \nu(G_i) < \mu(G_i) + \varepsilon,\ i = 1, ..., k\} \tag{A.4}$$

for some $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and ρ^u-open sets $G_1, ..., G_k$ belonging to $\mathcal{B}_\pi(X)$ in the case of $\mathcal{F}_2$, and

$$\{\nu \in \mathcal{M}_\pi(X^u)|\ |\nu(A_i) - \mu(A_i)| < \varepsilon,\ i = 1, ..., k\} \tag{A.5}$$

for some $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and μ-continuity sets $A_1, ..., A_k$ belonging to $\mathcal{B}_\pi(X^u)$ in the case of $\mathcal{F}_3$. Each one of the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ is the base for a topology on $\mathcal{M}_\pi(X^u)$. The following result provides an analogue of Theorem 3, p. 236, in Billingsley (1968).

Proposition A.1 *Each of the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ is a base for the topology of weak convergence on $\mathcal{M}_\pi(X^u)$.*

Proof. The proof has two parts. The first part shows that the topologies induced by the families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are equivalent to each other. The second part shows that the topology induced by the family $\mathcal{F}_1$ is equivalent to the topology induced by the family $\mathcal{F}_0$. For the first part, I refer to the argument of Billingsley (1968, p. 237), which goes through without any change. Because of Lemma 2.2, the requirement that the sets F_i, G_i, A_i must all belong to $\mathcal{B}_\pi(X)$ plays no role in the argument.

For the second part, the requirement that the sets F_i in $\mathcal{F}_1$ must belong to $\mathcal{B}_\pi(X)$ does play a role. Therefore I give the adapted argument in detail.

I first show that any element of $\mathcal{F}_1$ contains an element of $\mathcal{F}_0$ as a subset. Any set $N \in \mathcal{F}_1$ takes the form (A.3) for some $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and ρ^u-closed sets $F_1, ..., F_k$ in $\mathcal{B}_\pi(X)$. By Lemma 2.2, for any $\delta > 0$, the sets $F_i^\delta = \{x \in X|\rho^u(x, F_i) < \delta\}$, $i = 1, ..., k$, also belong to $\mathcal{B}_\pi(X)$. Let $\delta > 0$ be such that, for $i = 1, ..., k$, the set F_i^δ is a μ-continuity set and, moreover,

$$\mu(F_i^\delta) < \mu(F_i) + \frac{\varepsilon}{2}. \tag{A.6}$$

Next, let $\varphi : \mathbb{R}_+ \to [0,1]$ be a continuous function such that, for any $t \in \mathbb{R}$,

$$\varphi(t) = 1 - t \text{ if } t \in [0,1) \text{ and } \varphi(t) = 1 \text{ if } t \geq 1.$$

For any i, define a function $f_i : X \to [0,1]$ by setting

$$f_i(x) = \varphi\left(\frac{1}{\delta} \cdot \rho^u(x, F_i)\right), \tag{A.7}$$

where again $\rho^u(x, F_i) = \min_{\hat{x} \in F_i} \rho^u(x, \hat{x})$. The function f_i is obviously bounded and ρ^u-continuous. Since $F_i \in \mathcal{B}_\pi(X)$, by Lemma 2.2, f_i is also measurable with respect to $\mathcal{B}_\pi(X)$.[15] We also have $f_i(x) = 0$ for $x \in X \backslash F_i^\delta$ and $f_i(x) = 1$ for $x \in F_i$. For any $\nu \in \mathcal{M}_\pi(X^u)$, therefore,

$$\int_{X^u} f_i(x) d\nu(x) < \int_{X^u} f_i(x) d\mu(x) + \frac{\varepsilon}{2}$$

implies

$$\nu(F_i) \leq \int_{X^u} f_i(x) d\nu(x) < \int_{X^u} f_i(x) d\mu(x) + \frac{\varepsilon}{2} \leq \mu(F_i^\delta) + \frac{\varepsilon}{2} < \mu(F_i) + \varepsilon$$

For any $\nu \in \mathcal{M}_\pi(X^u)$ satisfying

$$\left| \int_{X^u} f_i(x) d\nu(x) - \int_{X^u} f_i(x) d\mu(x) \right| < \frac{\varepsilon}{2}$$

for $i = 1, ..., k$, we therefore have

$$\nu(F_i) < \mu(F_i) + \varepsilon$$

for $i = 1, ..., k$. Thus the set $N \in \mathcal{F}_1$ of measures $\nu \in \mathcal{M}_\pi(X^u)$ that satisfy (A.3) for the given $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and $F_1, ..., F_k$ contains a set of measures $\nu \in \mathcal{M}_\pi(X^u)$ that satisfy (A.2) for the same $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and the specified functions $f_i, i = 1, ..., k$. The latter set is an element of $\mathcal{F}_0$. Thus every element of $\mathcal{F}_1$ contains an element of $\mathcal{F}_0$.

For the claim that every element of $\mathcal{F}_0$ also contains an element of $\mathcal{F}_1$, the argument in Billingsley (1968) applies with hardly any change. For example, let $N \in \mathcal{F}_0$ be the set of measures $\nu \in \mathcal{M}_\pi(X^u)$ such that, for given $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and $f \in \mathcal{C}_0(X^u)$,

$$\left| \int_{X^u} f(x) d\nu(x) - \int_{X^u} f(x) d\mu(x) \right| < \varepsilon. \tag{A.8}$$

[15] $f_i(x) > f_i(\hat{x})$ implies $\rho^u(x, F_i) < \rho^u(\hat{x}, F_i)$ so that, for some $\eta \in (\rho^u(x, F_i), \rho^u(\hat{x}, F_i))$, $x \in F_i^\eta$ and $\hat{x} \in X \backslash F_i^\eta$.

Without loss of generality, assume that f takes values in the unit interval. Choose k' so that $\frac{1}{k'} < \varepsilon$ and, for $i = 1, ..., k'$, let $F_i = \{x \in X^u | \frac{i}{k'} \leq f(x)\}$. Then, for any i, F_i is ρ^u-closed. Moreover, since f is $\mathcal{B}_\pi(X)$-measurable, F_i is in $\mathcal{B}_\pi(X)$. By standard arguments, therefore, $\nu(F_i) < \mu(F_i) + \varepsilon$ implies

$$\int_{X^u} f(x) d\nu(x) < \frac{1}{k'} + \frac{1}{k'} \sum_{i=1}^{k'} \nu(F_i) < \frac{1}{k'} + \frac{1}{k'} \sum_{i=1}^{k'} \mu(F_i) + \varepsilon < \int_{X^u} f(x) d\mu(x) + 2\varepsilon.$$

By a parallel argument for the function $x \to 1 - f(x)$, there also exist ρ^u-closed, $\mathcal{B}_\pi(X^u)$-measurable sets F_i, $i = k' + 1, ..., 2k'$, such that, for $i = k' + 1, ..., 2k'$, $\nu(F_i) < \mu(F_i) + \varepsilon$ implies

$$\int_{X^u} (1 - f(x)) d\nu(x) < \int_{X^u} (1 - f(x)) d\mu(x) + 2\varepsilon,$$

or, equivalently,

$$\int_{X^u} f(x) d\mu(x) < \int_{X^u} f(x) d\nu(x) + 2\varepsilon.$$

Upon combining these arguments, one finds that the set N of measures $\nu \in \mathcal{M}_\pi(X^u)$ that satisfy (A.8) for some given $\mu \in \mathcal{M}_\pi(X^u), \varepsilon > 0$, and $f \in \mathcal{C}_0(X^u)$, an element of $\mathcal{F}_0$, contains a subset consisting of measures that satisfy $\nu(F_i) < \mu(F_i) + \varepsilon$ for the given $\mu, \varepsilon, k = 2k'$, and ρ^u-closed, $\mathcal{B}_\pi(X)$-measurable sets $F_i, i = 1, ..., k$. Thus N has a subset that belongs to $\mathcal{F}_1$.

By taking intersections of sets like N, with different functions f, the argument can be generalized to all sets in $\mathcal{F}_0$. Thus every element of $\mathcal{F}_0$ contains an element of $\mathcal{F}_1$ as a subset. This completes the proof of the claim that the topologies induced by $\mathcal{F}_0$ and $\mathcal{F}_1$ are equivalent. ■

The following result provides an analogue of Billingsley's (1968) Theorem 5, p.238.

Proposition A.2 *For any measure $\mu \in \mathcal{M}_\pi(X^u)$ that is quasi-separable, the topology induced by the Prohorov metric and the topology of weak convergence on $\mathcal{M}_0(X^u)$ are equivalent at μ.*

Proof. Let $N(\mu)$ be any $\mathcal{F}_1$-neighbourhood of μ, characterized by $\varepsilon > 0$ and ρ^u-closed, $\mathcal{B}_\pi(X)$-measurable sets $F_i, i = 1, ..., k$, such that $\nu \in N(\mu)$ if and only if ν satisfies (A.3) for μ, ε, and $F_1, ..., F_k$. Let $\delta < \varepsilon$ be such that, for $i = 1, ..., k$, the set $F_i^\delta = \{x \in X^u | \rho^u(x, F_i) < \delta\}$ is a μ-continuity set and, moreover, $\mu(F_i^\delta) < \mu(F_i) + \frac{\varepsilon}{2}$. For any $\nu \in \mathcal{M}_0(X^u)$ such that $p(\nu, \mu) < \delta$, one has

$$\nu(F_i) < \mu(F_i^\delta) + \delta < \mu(F_i) + \varepsilon,$$

so the p-open ball with radius δ around μ is a subset of $N(\mu)$. Therefore the topology on $\mathcal{M}_\pi(X^u)$ that is induced by the Prohorov metric is at least as fine at μ as the topology of weak convergence.

Next, suppose that μ is quasi-separable. I will show that for any $\varepsilon > 0$, the p-open ball with radius ε around μ contains an $\mathcal{F}_3$-neighbourhood $N(\mu)$ of μ. Fix a cover of X by ρ^u-open, μ-continuity balls with diameters less than δ, where $\delta < \frac{\varepsilon}{3}$. Appealing to quasi-separability, pass to a countable subfamily $\{B_i\}_{i=1}^\infty$ such that $\mu(\cup_i B_i) = 1$. Construct disjoint μ-continuity stes $A_1, A_2, ...$ by setting $A_1 = B_1$ and, for $i > 1$, $A_i = B_i \setminus \cup_{j<i} A_j$. Choose k so that

$$\mu(\cup_{i=1}^k A_i) > 1 - \delta \tag{A.9}$$

and let $\mathcal{A}$ be the set of unions of the sets A_i over subsets of the indices $i = 1, ..., k$. Then each $A \in \mathcal{A}$ is a μ-continuity set, and, by Proposition A.1, there is a neighbourhood $N(\mu) \in \mathcal{F}_3$ of μ such that, for any $\nu \in N(\mu)$,

$$|\nu(A) - \mu(A)| < \delta \quad \text{for all} \quad A \in \mathcal{A}. \tag{A.10}$$

I claim that $N(\mu)$ is contained in the p-open ball with radius ε around μ. To prove this claim, consider any $B \in \mathcal{B}_\pi(X)$. Let I_B be the set of indices i such that $A_i \cap B \neq \emptyset$ and let $A_B := \cup_{i \in I_B} A_i$. Then $B \subset A_B \cup (X^u \setminus \cup_{i=1}^k A_i)$. Moreover, $A_B \in \mathcal{A}$ and, because the sets A_i all have diameters less than δ, $A_B \subset B^\delta$. Using (A.9) and (A.10), one obtains

$$\mu(B) \leq \mu(A_B) + 1 - \mu(\cup_{i=1}^k A_i) < \nu(A_B) + 2\delta \leq \nu(B^\delta) + 2\delta < \nu(B^\varepsilon) + \varepsilon.$$

Similiarly, taking account of the fact that (A.9) and (A.10) imply

$$\nu(\cup_{i=1}^k A_i) > 1 - 2\delta,$$

one also obtains

$$\nu(B) \leq \nu(A_B) + 1 - \nu(\cup_{i=1}^k A_i) < \mu(A_B) + 3\delta \leq \mu(B^\delta) + 3\delta < \mu(B^\varepsilon) + \varepsilon.$$

Thus, for any $\nu \in N(\mu)$, the Prohorov distance between ν and μ is less than ε. The specified set $N(\mu) \in \mathcal{F}_3$ is contained in the p-open ball with radius ε around μ. At μ, therefore the topology of weak convergence on $\mathcal{M}_\pi(X^u)$ is at least as fine as the topology that is induced by the Prohorov metric. Given that the topology induced by the Prohorov metric is also at least as fine at μ as the topology of weak convergence, it follows that the two topologies are equivalent at μ. ■

Proposition 2.4 in the text is a straightforward corollary to Proposition A.2.

References

[1] Banach, S. (1930), Über additive Maßfunktionen in abstrakten Mengen, *Fundamenta Mathematicae* 15, 97-101.

[2] Banach, S., and C. Kuratowski (1929), Sur une généralisation du problème de la mesure, *Fundamenta Mathematicae* 14, 127-131.

[3] Bartoszynski, T., and L. Halbeisen (2003), On a Theorem of Banach and Kuratowski and K-Lusin Sets, *Rocky Mountain Journal of Mathematics* 33 (4), 1223-1231.

[4] Billingsley, P. (1968), *Convergence of Probability Measures*, Wiley, New York.

[5] Brandenburger, A., and E. Dekel (1993), Hierarchies of Beliefs and Common Knowledge, *Journal of Economic Theory* 59, 189-198.

[6] Chen, Y., A. Di Tillio, E. Faingold, and S. Xiong (2010), Uniform Topologies on Types, *Theoretical Economics* 5, 445 -478.

[7] Chen, Y., A. Di Tillio, E. Faingold, and S. Xiong (2017), Characterizing the Strategic Impcat of Misspecified Beliefs, *Review of Economic Studies* 84 (4), 1424-1471.

[8] Cohen, P.J. (1966), *Set Theory and the Continuum Hypothesis*, W.A. Benjamin, Reading, MA.

[9] Dekel, E., D. Fudenberg, and S. Morris (2006), Topologies on Types, *Theoretical Economics* 1, 275 - 309.

[10] Dudley, R.M. (1966), Weak Convergence of Probability Measures on Nonseparable Metric Spaces and Empirical Measures on Euclidean Spaces, *Illinois Journal of Mathematics* 10, 109-126.

[11] Dudley, R.M. (1967), Measures on Non-Separable Metric Spaces, *Illinois Journal of Mathematics* 11, 449-453.

[12] Dudley, R.M. (2002), *Real Analysis and Probability*, Cambridge Studies in Advanced Mathematics 74, Cambridge University Press, Cambridge, UK.

[13] Hellwig, M.F. (2016/23), A Homeomorphism Theorem for the Universal Type Space with the Uniform Weak Topology, Preprint 17/2016, Max Planck Institute for Research on Collective Goods, Bonn, revised May 2023. http://www.coll.mpg.de/pdf_dat/2016_17online.pdf.

[14] Kelley, J.L. (1955), *General Topology*, Van Nostrand, New York.

[15] Marczewski, E., and R. Sikorski (1948), Measures in Non-Separable Metric Spaces, *Colloquium Mathematicum 1, 133-139.*

[16] Mertens, J.F., and S. Zamir (1985), Formulation of Bayesian Analysis for Games with Incomplete Information, *International Journal of Game Theory* 14, 1–29.

[17] Rubinstein, A. (1989), The Electronic Mail Game: Strategic Behavior under 'Almost Common Knowledge', *American Economic Review* 79, 385-391.