

The BMS4 algebra at spatial infinity.

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ABSTRACT. We show how a global BMS4 algebra appears as the asymptotic symmetry algebra at spatial infinity. Using linearised theory, we then show that this global BMS4 algebra is the one introduced by Strominger as a symmetry of the S-matrix.

1 Introduction

In the last few years, following the work of Strominger and collaborators [1, 2, 3], a new understanding of infrared divergences in scattering processes has appeared. It was shown that soft theorems are related to Ward identities derived from conserved charges associated to asymptotic symmetries at null infinity (see for instance [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). One of the main examples where this relation appeared is between the BMS4 algebra of asymptotically flat space-times at null infinity and scattering processes of gravitons. It was shown that Weinberg's soft gravitons theorem can be seen as the Ward identity derived from super-translation charges.

More recently, Hawking, Perry and Strominger reconsidered the information loss problem in the evaporation of Black-holes [14, 15]. The existence of the infinite set of conserved charges associated to BMS4 means that part of the information about the system is retained in the form of global/soft gravitons. This will imply correlations between Hawking radiation produced by the black-hole and the status of the system before the collapse.

A key point of the results described above is the existence of a global BMS4 algebra and its associated conserved charges during an evolution process. This existence derives from a set of junction conditions at spatial infinity between various fields defined at past and future null infinity. While these hypotheses are well motivated and are equivalent to the soft graviton theorem [3, 6], their existence is surprising considering the non-differentiability of spatial infinity. These facts mean that both the symmetries and their associated charges are only defined at null infinity. They are properties of the initial and final states but we, in general, don't have a good understanding of them at finite times.

Recently, a description of the asymptotic symmetry algebra of electromagnetism has been done at spatial infinity by Campiglia and Eyheralde [16]. Their description provides a bridge between both asymptotic regimes and proves the junction condition in the context of electromagnetism.

In the case of gravity, it has been shown in [17] that the junction condition for the mass aspect between future and past null infinity is a consequence of the structure of spatial infinity. This result hints at the fact that there should also exist a description of spatial infinity for which the global BMS4 symmetry introduced by Strominger appears naturally as an asymptotic symmetry algebra.

In this paper, we consider the set of asymptotic conditions at spatial infinity introduced by Compère and Dehouck in [18]. These asymptotic conditions are a generalisation of the results obtained in [19, 20, 21] for the holographic renormalization of asymptotically flat space-times. Our main result is that a sub-algebra of the associated asymptotic symmetries is a global non extended BMS4. Using the linearised theory around flat space, we

then show that, when restricting the analysis to asymptotically flat space-times at null infinity, this global non extended BMS4 algebra defined at spatial infinity is identical to the one obtained by Strominger in his original analysis [2]. We also show that the associated conserved charges defined at spatial infinity reproduce the BMS4 charges at null infinity.

This paper is organized as follows. In section 2, we introduce the asymptotic conditions of Compère and Dehouck. In section 3, we study the asymptotic symmetry algebra and show that it contains BMS4. In the last section, we make the link with null infinity using the linearised theory and a description of space-like infinity introduced by Friedrich [22].

2 Spatial infinity

The set of metrics they consider have the form [18]:

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{2\sigma}{\eta} + \frac{\sigma^2}{\eta^2} + o(\eta^{-2})\right) d\eta^2 + o(\eta^{-1})d\eta dx^b + \left(\eta^2 h_{ab}^{(0)} + \eta h_{ab}^{(1)} + \log \eta i_{ab} + h_{ab}^{(2)} + o(\eta^0)\right) dx^a dx^b, \quad (2.1)$$

where the asymptotic gravitational fields, σ , k_{ab} , i_{ab} and $h_{ab}^{(n)}$ depend on x^a only. The boundary metric $h_{ab}^{(0)}$ is the metric on the unit hyperboloid with its associated covariant derivative \mathcal{D}_a . We will write it as:

$$x^a = (s, x^A), \quad h_{ab}^{(0)} dx^a dx^b = \frac{-1}{(1-s^2)^2} ds^2 + \frac{1}{1-s^2} \gamma_{AB} dx^A dx^B, \quad (2.2)$$

with x^A and γ_{AB} respectively coordinates and the unit metric on the 2-sphere. Following their work, we will also introduce the combination

$$k_{ab} = h_{ab}^{(1)} + 2\sigma h_{ab}^{(0)}, \quad (2.3)$$

and impose the extra conditions

$$k = k_a^a = 0, \quad \mathcal{D}^a k_{ab} = 0, \quad (2.4)$$

where indices are lowered and raised with $h_{ab}^{(0)}$ and its inverse. The leading order equations of motion are then

$$(\mathcal{D}_c \mathcal{D}^c + 3)\sigma = 0, \quad (\mathcal{D}_c \mathcal{D}^c - 3)k_{ab} = 0. \quad (2.5)$$

The explicit solutions to these equations are given in appendix A.

It has been shown by Compère and Dehouck that this set of conditions is sufficient to write a well defined action invariant under Poincaré. In particular, they don't need to

impose parity conditions on the asymptotic fields to obtain well defined Lorentz charges. One difference with the usual considerations is that these charges will be quadratic in the fields.

The asymptotic symmetry algebra contains Lorentz algebra, super-translations and logarithmic translations. The Poincaré algebra is a sub-algebra of the full asymptotic symmetry algebra with the translations being a sub-algebra of the super-translations. This structure is very similar to the one we have at null infinity however, the super-translations algebra present at spatial infinity is bigger than the BMS4 one and it is a priori not clear how the two are related.

With this set of asymptotic conditions, super-translations are parametrised by one function ω on the hyperboloid satisfying the condition:

$$(\mathcal{D}_c \mathcal{D}^c + 3)\omega = 0, \quad (2.6)$$

while the Lorentz sub-algebra is given by the killing vector fields of the boundary metric on the hyperboloid:

$$L_{\mathcal{Y}} h_{ab}^{(0)} = 0, \quad \mathcal{Y} = \mathcal{Y}^a (x^b) \partial_a. \quad (2.7)$$

Their corresponding asymptotic generators are given by

$$\xi^\eta = -\omega + \frac{1}{\eta} \omega^{(1)} + o(\eta^{-1}), \quad (2.8)$$

$$\xi^a = \mathcal{Y}^a - \frac{1}{\eta} \mathcal{D}^a \omega + \frac{1}{2\eta^2} (k^{ab} \partial_b \omega - 4\sigma \mathcal{D}^a \omega + \mathcal{D}^a \omega^{(1)}) + o(\eta^{-2}). \quad (2.9)$$

The sub-leading terms are necessary to preserve the form of the metric (2.1). The function $\omega^{(1)}$ is an arbitrary function on the hyperboloid and generates a proper gauge transformation. The associated conserved charges we will be interested in are super-translation charges. They are given by

$$\mathcal{Q}_\omega = \frac{1}{4\pi G} \oint_{S^2} d^2\Omega (\sigma \partial_s \omega - \omega \partial_s \sigma). \quad (2.10)$$

with the corresponding boundary conserved current

$$j_\omega^a = \frac{1}{4\pi G} \sqrt{-h^{(0)}} h^{(0)ab} (\omega \partial_b \sigma - \sigma \partial_b \omega). \quad (2.11)$$

The last elements of the asymptotic symmetry algebra are logarithmic translations. They are parametrized by a function $H(x^a)$ on the hyperboloid satisfying

$$\mathcal{D}_a \mathcal{D}_b H + h_{ab}^{(0)} H = 0. \quad (2.12)$$

Their bulk generators are given by

$$\xi^\eta = \log \eta H + \frac{1}{\eta} \log \eta \mathcal{D}^a H \partial_a \sigma + \frac{1}{\eta} \omega^{(1)} + o(\eta^{-1}), \quad (2.13)$$

$$\begin{aligned} \xi^a = & \frac{1}{\eta} (\log \eta + 1) \mathcal{D}^a H + \frac{1}{2\eta^2} (\log \eta + \frac{1}{2}) \left(\mathcal{D}^a (\mathcal{D}^b H \partial_b \sigma) + 4\sigma \mathcal{D}^a H - k^{ab} \partial_b H \right) \\ & + \frac{1}{2\eta^2} \mathcal{D}^a \omega^{(1)} + o(\eta^{-2}). \end{aligned} \quad (2.14)$$

As before, $\omega^{(1)}$ is arbitrary and generates a proper gauge transformation. On the hyperboloid, only four linearly independent functions satisfy equation (2.12):

$$H(x^a) = \frac{1}{\sqrt{1-s^2}} \left(H_0 s + H_1 Y_{1,-1}^0(x^A) + H_2 Y_{1,0}^0(x^A) + H_3 Y_{1,1}^0(x^A) \right), \quad (2.15)$$

where Y_{lm}^0 are usual spherical harmonics. When restricted to these four functions, logarithmic translations are symmetries of the asymptotic form of the metric given in (2.1). On the other hand, transformation of the form (2.13), (2.14) with an arbitrary function $H(x^a)$ are called logarithmic super-translations. They don't preserve the asymptotic form of the metric but we can use them to put $\sigma = 0$ at the price of the appearance of an extra logarithmic term in g_{ab} . These more general transformations will be useful in section 4 when we will make the link with null infinity.

3 BMS4 algebra

We have seen in the previous section that, if we forget logarithmic translations, the asymptotic symmetry algebra is parametrised by a function ω and a vector \mathcal{Y}^a on the hyperboloid satisfying

$$(\mathcal{D}_c \mathcal{D}^c + 3)\omega = 0, \quad L_{\mathcal{Y}} h_{ab}^{(0)} = 0. \quad (3.1)$$

The general solution for ω is given in the appendix. It takes the form

$$\omega = \frac{1}{\sqrt{1-s^2}} (\widehat{\omega}^{even} + \widehat{\omega}^{odd}), \quad (3.2)$$

$$\widehat{\omega}^{even} = \sum_{l,m} \widehat{\omega}_{lm}^V V_l(s) Y_{lm}^0(x^A), \quad \widehat{\omega}^{odd} = \sum_{l,m} \widehat{\omega}_{lm}^W W_l(s) Y_{lm}^0(x^A), \quad (3.3)$$

where $\widehat{\omega}_{lm}^V$ and $\widehat{\omega}_{lm}^W$ are two sets of constants. The functions $\widehat{\omega}^{even}(s, x^A)$ and $\widehat{\omega}^{odd}(s, x^A)$ are respectively even and odd under a combination of time reversal, $s \rightarrow -s$, and the antipodal map, $x^A \rightarrow -x^A$ or $(\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi)$. This general solution is fully characterised by two functions on the 2-sphere:

$$T^{even}(x^A) = \lim_{s \rightarrow 1} \partial_s^2 \widehat{\omega}^{even}(s, x^A) = \sum_{l,m} \widehat{\omega}_{lm}^V Y_{lm}^0(x^A), \quad (3.4)$$

$$T^{odd}(x^A) = \lim_{s \rightarrow 1} \widehat{\omega}^{odd}(s, x^A) = \sum_{l,m} \widehat{\omega}_{lm}^W Y_{lm}^0(x^A). \quad (3.5)$$

The vectors \mathcal{Y}^a represent Lorentz algebra. The rotations are parametrized by killing vectors of the 2-sphere \mathcal{Y}_R^A with $\partial_s \mathcal{Y}_R^A = 0$ and the corresponding vectors on the hyperboloid are given by

$$\mathcal{Y}^s = 0, \quad \mathcal{Y}^A = \mathcal{Y}_R^A. \quad (3.6)$$

The boosts are parametrized by functions on the sphere $\psi(x^A)$ such that $\Delta\psi + 2\psi = 0$ with their associated vectors on the hyperboloid being:

$$\mathcal{Y}^s = -\frac{1}{2}(1 - s^2)\psi, \quad \mathcal{Y}^A = -\frac{1}{2}s\gamma^{AB}\partial_B\psi. \quad (3.7)$$

Rotations and boosts can also be encoded in global conformal killing vectors of the 2-sphere:

$$Y^A = \mathcal{Y}_R^A - \frac{1}{2}\gamma^{AB}\partial_B\psi, \quad D_A Y^A = \psi, \quad (3.8)$$

where D_A is the covariant derivative on the sphere. This relation forms an isomorphism between the algebra of killing vector fields of the hyperboloid and the algebra of global conformal killing vector fields of the 2-sphere.

The super-translations combined with the Lorentz algebra form a sub-algebra of the asymptotic symmetry algebra for which the bracket is given by:

$$[(\mathcal{Y}_1, \omega_1), (\mathcal{Y}_2, \omega_2)] = \left([\mathcal{Y}_1, \mathcal{Y}_2], \mathcal{Y}_1^a \partial_a \omega_2 - \mathcal{Y}_2^a \partial_a \omega_1 \right), \quad (3.9)$$

or, if we use the rescaled parameter $\widehat{\omega} = \sqrt{1 - s^2}\omega$,

$$[(\mathcal{Y}_1, \widehat{\omega}_1), (\mathcal{Y}_2, \widehat{\omega}_2)] = \left([\mathcal{Y}_1, \mathcal{Y}_2], \mathcal{Y}_1^a \partial_a \widehat{\omega}_2 - \frac{s}{2}\psi_1 \widehat{\omega}_2 - (1 \leftrightarrow 2) \right). \quad (3.10)$$

As the vector $\mathcal{Y}^a \partial_a$ is even under the combination of a time reversal and an antipodal map, its action on $\widehat{\omega}$ will not mix the even and odd parts. Parametrizing the super-translations with the two functions on the sphere $T^{even}(x^A)$ and $T^{odd}(x^A)$ and using the vectors Y^A to parametrize Lorentz algebra, we can write the bracket as

$$[(Y_1, T_1^{even}, T_1^{odd}), (Y_2, T_2^{even}, T_2^{odd})] = \left([Y_1, Y_2], T_{[1,2]}^{even}, T_{[1,2]}^{odd} \right), \quad (3.11)$$

with

$$T_{[1,2]}^{even} = Y_1^A \partial_A T_2^{even} + \frac{3}{2}\psi_1 T_2^{even} - Y_2^A \partial_A T_1^{even} - \frac{3}{2}\psi_2 T_1^{even}, \quad (3.12)$$

$$T_{[1,2]}^{odd} = Y_1^A \partial_A T_2^{odd} - \frac{1}{2}\psi_1 T_2^{odd} - Y_2^A \partial_A T_1^{odd} + \frac{1}{2}\psi_2 T_1^{odd}. \quad (3.13)$$

Let's consider the sub-algebra obtained by imposing $T^{even} = 0$. This algebra is a semi-direct product of an abelian algebra parametrized by an arbitrary function on the sphere $T^{odd}(x^A)$ with the Lorentz algebra parametrized by Y^A . In (3.13), we recognize the action of Lorentz algebra on the BMS4 super-translations [23]. This proves that this sub-algebra is isomorphic to the BMS4 algebra defined at null infinity. We will see in

the next section that, when evaluated at future or past null infinity, the asymptotic killing vectors associated to T^{odd} reduce to usual BMS4 super-translations.

We saw in the previous section that the conserved charges associated to super-translations are given by:

$$\mathcal{Q}_\omega = \frac{1}{4\pi G} \oint_{S^2} d^2\Omega (\sigma \partial_s \omega - \omega \partial_s \sigma). \quad (3.14)$$

Introducing the explicit solution we obtained for ω and σ , we showed in appendix A.1 that these charges take the form:

$$\mathcal{Q}_\omega = \frac{1}{8\pi G} \oint_{S^2} d^2\Omega (T^{odd} m^{even} - T^{even} m^{odd}), \quad (3.15)$$

where we used the decomposition of σ in terms of two functions on the sphere $m^{odd}(x^A)$ and $m^{even}(x^A)$ given in (A.32) and (A.33).

4 Null infinity

In this section, we will make the connection with null infinity. We will show that the BMS4 algebra we obtained at spatial infinity is the global BMS4 algebra introduced by Strominger in [2]. We will also recover a linearised version of the results of Herberthson and Ludvigsen relating the values of the mass aspect at future and past null infinity [17].

4.1 Structure close to i_0

The description we will use is inspired by the work of Friedrich in [22] where he introduced a description of spatial infinity based on conformal geodesics. The aim was to formulate an initial value problem for the conformal Einstein equations at spatial-infinity. As we will see, it is well adapted to the description of the scattering problem.

For generic space-times, the asymptotic structure he obtained close to i_0 is as follows (see [22, 24, 25] for more details). If some smoothness conditions on the metric around spatial infinity are satisfied, there exists a patch of coordinates (ρ, s, x^A) in a neighbourhood of i_0 such that the curves obtained by keeping ρ and x^A constant are conformal geodesics. There exists smooth functions $\tilde{\Omega}(\rho, x^A)$ and $\omega(\rho, x^A)$ such that the rescaled metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ is continuous in the limit $s \rightarrow \pm\kappa(\rho)$ where

$$\Omega = \tilde{\Omega} \left(1 - \left(\frac{s}{\omega} \right)^2 \right), \quad \lim_{\rho \rightarrow 0} \rho^{-1} \tilde{\Omega} = 1, \quad \lim_{\rho \rightarrow 0} \omega = 1. \quad (4.1)$$

The two hypersurfaces $s = \pm\omega(\rho, x^A) = \pm 1 + o(\rho^0)$ are then future null infinity \mathcal{I}^+ and past null infinity \mathcal{I}^- while spatial infinity is located at $\rho = 0$. The rescaled metric

$\tilde{\eta}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$ diverges at spatial infinity but it is continuous at both null infinities \mathcal{I}^\pm . Slices of constant time s are spatial hyper-surfaces and in the limit s going to $\pm\omega$, these hypersurfaces asymptote to the corresponding \mathcal{I} (in a neighbourhood of i_0 as these coordinates don't cover the full manifold). Considering the evolution from a finite time $-\omega < s_0 < \omega$ to a finite time $-\omega < s_1 < \omega$ and taking the limit for infinite times $s_0 \rightarrow -\omega$ and $s_1 \rightarrow \omega$, we see that the in-state and out-state hypersurfaces will naturally contain \mathcal{I}^- and \mathcal{I}^+ respectively. A related feature of these coordinates is that the coordinates at null infinity are coming from spatial coordinates in the bulk namely (ρ, x^A) while, in the usual description in terms of the Bondi metric, one of the coordinates is the asymptote of a time coordinate. A few years later in [24, 25], Friedrich and Kannar made the explicit connection with quantities defined at null infinity. They for instance computed the Newman-Penrose constants at null infinity from the coefficients in the expansion around spatial infinity.

This description relies on a first order formalism of the conformal description of Einstein's equations. Unfortunately, the link between this formalism and the hyperbolic slicing in the metric formalism of i_0 is not simple. In the rest of this section, we will work in the linearised theory around flat space and we will describe the background metric using the coordinates obtained by Friedrich analysis.

In order to describe the structure of Minkowski close to i_0 , we start with the usual hyperbolic coordinates and introduce the following rescaled radial coordinates ρ :

$$\eta = \frac{1}{\rho\sqrt{1-s^2}}. \quad (4.2)$$

The flat metric then takes the form

$$\eta_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\rho^2(1-s^2)^2} \left(\frac{1-s^2}{\rho^2} d\rho^2 - 2\frac{s}{\rho} d\rho ds - ds^2 + \gamma_{AB} dx^A dx^B \right). \quad (4.3)$$

The curves obtained by keeping ρ and x^A constant are conformal geodesics [26, 22]. The conformal factor (4.1) is given in this case by

$$\Omega = \rho(1-s^2), \quad (4.4)$$

such that the hypersurfaces \mathcal{I}^\pm are located at $s = \pm 1$. The metric q_{ij} and the vector field n^i induced on these hypersurfaces by $\tilde{\eta}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$ and $\tilde{n}^\mu = \tilde{\eta}^{\mu\nu} \partial_\nu \Omega$ are given by

$$q_{ij} dx^i dx^j = \gamma_{AB} dx^A dx^B, \quad n^i \partial_i = 2\rho^2 \partial_\rho, \quad (4.5)$$

where $x^i = (\rho, x^A)$ are the induced coordinates. The usual retarded time coordinate on \mathcal{I}^+ is given by $u = -\frac{1}{2\rho}$ with $n^i \partial_i = \partial_u$.

Let's now consider the metrics introduced in section 2 and define $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ that we will treat as a linear perturbation. Under the change of coordinates (4.2), it takes the form

$$h_{\mu\nu} = \frac{1}{\rho^2(1-s^2)^2} \left\{ (2\rho\hat{\sigma} + o(\rho)) \left(\frac{1-s^2}{\rho^2} d\rho^2 - \frac{2s}{\rho} dsd\rho \right) + o(\rho) d\rho dx^a \right. \\ \left. + \rho\sqrt{1-s^2} h_{ab}^{(1)} dx^a dx^b + \rho \frac{2s^2}{(1-s^2)} \hat{\sigma} ds^2 + o(\rho) dx^a dx^b \right\}. \quad (4.6)$$

Using the explicit solutions obtained for σ and k_{ab} in appendix A, one can easily check that $\tilde{h}_{\mu\nu} = \Omega^2 h_{\mu\nu}$ diverges in the limit $s \rightarrow \pm 1$. This is related to the asymptotic gauge choice made in section 2. To avoid this problem, we will use a different set of coordinates:

$$\eta = \eta' - \sigma(x') \log \eta' + o(\eta'^0), \quad (4.7)$$

$$x^a = x'^a - \eta'^{-1} (\log \eta' + 1) (\mathcal{D}^a \sigma)(x') + o(\eta'^{-1}), \quad (4.8)$$

such that the metric (2.1) takes the form:

$$g'_{\mu\nu} dx'^{\mu} dx'^{\nu} = (1 + o(\eta'^{-2})) d\eta'^2 + o(\eta'^{-1}) d\eta' dx'^a + g'_{ab} dx'^a dx'^b, \quad (4.9)$$

$$g'_{ab} = \eta'^2 h_{ab}^{(0)} + \eta' (\log \eta' + 1) \left(-2\mathcal{D}_a \mathcal{D}_b \sigma - 2\sigma h_{ab}^{(0)} \right) + \eta' k_{ab} + o(\eta'), \quad (4.10)$$

where all asymptotic fields are evaluated at x'^a . The leading part of the transformation (4.7)-(4.8) is a logarithmic super-translation while the subleading terms have to be adapted to reach the asymptotic gauge condition chosen here. Super-translations at spatial infinity then take the following form:

$$\xi'^{\eta'} = -\omega + o(\eta'^{-1}), \quad (4.11)$$

$$\xi'^a = -\frac{1}{\eta'} \mathcal{D}^a \omega - \frac{1}{\eta'^2} \left(\log \eta' + \frac{3}{2} \right) (\mathcal{D}^a \mathcal{D}^b \sigma + \sigma h^{(0)ab}) \partial_b \omega + \frac{1}{2\eta'^2} k^{ab} \partial_b \omega + o(\eta'^{-2}). \quad (4.12)$$

In the rest of this section, we will work with these new coordinates while removing the primes. One remark is in order: while doing this change of coordinates, we have used elements of the asymptotic symmetry group to remove a few degrees of freedom. One can check easily that the final metric (4.10) is independent of $\hat{\sigma}_{00}^W$ and $\hat{\sigma}_{1m}^W$. These modes are the ones on which the logarithmic translations act. As the transformation generated by (4.7) and (4.8) is a generalization of a logarithmic translation, we can see its action as putting the four modes $\hat{\sigma}_{l<2,m}^W$ to zero while transferring all the other modes of σ to the components g_{ab} of the metric. These four modes being absent, the conjugated super-translations generated by $\hat{\omega}_{l<2,m}^V$ will have charges equal to zero. In this case, they are proper gauge transformations.

In these new coordinates, the perturbation $h_{\mu\nu}$ takes the form

$$h_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\rho^2(1-s^2)^2} \left\{ o(1) d\rho^2 + o(\rho) d\rho dx^a + \tilde{h}_{ab} dx^a dx^b \right\}, \quad (4.13)$$

where

$$\begin{aligned} \tilde{h}_{AB} = & -2\rho(1-s^2) \left(1 - \log(\rho\sqrt{1-s^2}) \right) (D_A D_B \hat{\sigma} + \gamma_{AB} \hat{\sigma} - s\gamma_{AB} \partial_s \hat{\sigma}) \\ & + \rho(1-s^2) \hat{k}_{AB} + o(\rho), \end{aligned} \quad (4.14)$$

$$\tilde{h}_{As} = -2\rho(1-s^2) \left(1 - \log(\rho\sqrt{1-s^2}) \right) \partial_A \partial_s \hat{\sigma} + \rho(1-s^2) \hat{k}_{sA} + o(\rho), \quad (4.15)$$

$$\tilde{h}_{ss} = -2\rho(1-s^2) \left(1 - \log(\rho\sqrt{1-s^2}) \right) \partial_s^2 \hat{\sigma} + \rho(1-s^2) \hat{k}_{ss} + o(\rho), \quad (4.16)$$

with $\hat{k}_{ab} = \sqrt{1-s^2} k_{AB}$ and $\hat{\sigma} = \sqrt{1-s^2} \sigma$. The leading part of the perturbation is now continuous as we approach null infinity and we have

$$\lim_{s \rightarrow \pm 1} \tilde{h}_{ab} = o(\rho). \quad (4.17)$$

In appendix B, we computed the behaviour of the linearized Weyl tensor of $\tilde{h}_{\mu\nu} = \Omega^2 h_{\mu\nu}$ in the limit $s \rightarrow \pm 1$. The component that will be relevant for the definition of the super-momentum charges is given by

$$\tilde{C}_{\rho s \rho s} = -\rho^{-1} (1-s^2) \partial_s^2 \hat{\sigma} + o(\rho^{-1}). \quad (4.18)$$

The leading term goes to zero in the limits $s \rightarrow \pm 1$, however, its rescaled version $K_{\rho s \rho s} = \Omega^{-1} \tilde{C}_{\rho s \rho s}$ will in general diverge logarithmically. This divergence breaks the structure of \mathcal{I}^\pm and we expect that, in these cases, the full metric $g_{\mu\nu}$ does not describe asymptotically flat space-times at null infinity. In the following, we will restrict our analysis to perturbations $h_{\mu\nu}$ for which the leading term of the rescaled linearized Weyl tensor $K_{\mu\nu\alpha\beta}$ has a well defined limit when $s \rightarrow \pm 1$. As shown in the appendix, this restriction imposes $\hat{\sigma}_{lm}^W = 0$ for all $l > 1$ (it will also impose $\mathcal{R}\alpha_{lm}^Q = 0$ for all l where $\mathcal{R}\alpha_{lm}^Q$ characterise part of the solution of \hat{k}_{ab}). Remark that, as we already put $\hat{\sigma}_{00}^W = 0 = \hat{\sigma}_{1m}$, we have that the odd part of the function $\hat{\sigma}$ is zero:

$$\hat{\sigma}^{odd}(s, x^A) = 0 \quad \Leftrightarrow \quad m^{odd}(x^A) = 0. \quad (4.19)$$

This extra restriction is the equivalent of the one made in [17], where the authors discard one branch of solutions because of logarithmic divergences in the Weyl tensor when approaching null infinity.

4.2 Global BMS4 algebra

At null infinity, super-translations are vector fields ξ_α^+ on \mathcal{I}^+ or ξ_β^- on \mathcal{I}^- such that:

$$(\xi_\alpha^+)^i = \alpha n^i|_{g^+}, \quad n^i \partial_i \alpha|_{g^+} = 0, \quad (\xi_\beta^-)^i = -\beta n^i|_{g^-}, \quad n^i \partial_i \beta|_{g^-} = 0. \quad (4.20)$$

In our case, they are given explicitly by

$$\xi_\alpha^+ = 2\alpha \rho^2 \partial_\rho, \quad \xi_\beta^- = -2\beta \rho^2 \partial_\rho. \quad (4.21)$$

Both functions α and β are arbitrary functions on the sphere and are the BMS4 super-translation parameters at future and past null infinity. The sign we used in the definition of ξ_β^- is due to our choice of coordinates: ∂_ρ always points away from spatial infinity.

Around Minkowski and up to a proper gauge transformation, the super-translations defined at spatial infinity in (4.11)-(4.12) are given by:

$$\xi^\rho = \rho^2 \left((1 + s^2) \widehat{\omega} + s(1 - s^2) \partial_s \widehat{\omega} \right), \quad (4.22)$$

$$\xi^s = \rho(1 - s^2) \left((1 - s^2) \partial_s \widehat{\omega} + s \widehat{\omega} \right), \quad (4.23)$$

$$\xi^A = -\rho(1 - s^2) \gamma^{AB} \partial_B \widehat{\omega}. \quad (4.24)$$

Taking the limit $s \rightarrow \pm 1$, we get

$$\lim_{s \rightarrow s^\pm} \xi^\rho = \rho^2 \lim_{s \rightarrow \pm 1} \widehat{\omega}, \quad \lim_{s \rightarrow s^\pm} \xi^s = 0, \quad \lim_{s \rightarrow s^\pm} \xi^A = 0, \quad (4.25)$$

while, using the form of $\widehat{\omega}$ we obtained in section 3, we have

$$\lim_{s \rightarrow 1} \widehat{\omega}(s, x^A) = T^{odd}(x^A) + \widehat{\omega}^{even}(1, x^A), \quad (4.26)$$

$$\lim_{s \rightarrow -1} \widehat{\omega}(s, x^A) = -T^{odd}(-x^A) + \widehat{\omega}^{even}(1, -x^A). \quad (4.27)$$

Only the first four spherical harmonics contribute to the even part:

$$\widehat{\omega}^{even}(1, x^A) = \widehat{\omega}_{00}^V - \frac{1}{3} \sum_{m=-1}^1 \widehat{\omega}_{1m}^V Y_{1m}^0(x^A). \quad (4.28)$$

Due to the change of coordinates we did at the beginning of the section, these four transformations have zero charges and are proper gauge transformations. We will then focus on the contribution from the odd part.

We saw in section 3 that the asymptotic symmetry algebra at spatial infinity is parametrized by two functions on the sphere, T^{odd} and T^{even} , combined with Lorentz transformations as the four logarithmic translations were removed by the change of coordinates we did in (4.7)-(4.8). When we restrict our analysis to asymptotically flat space-times at null infinity, we have to impose $m^{odd} = 0$. This implies that the super-translations parametrized by T^{even} become proper gauge transformations. The asymptotic symmetry algebra then reduces to the BMS4 algebra obtained in section 3, where the super-translations are parametrized by T^{odd} . Equations (4.25), (4.26) and (4.27) show that, on \mathcal{I}^+ and \mathcal{I}^- , these super-translations defined at spatial infinity correspond to super-translations ξ_α^+ and ξ_β^- defined respectively at future and past null infinity with

$$\alpha(x^A) = T^{odd}(x^A), \quad \beta(x^A) = T^{odd}(-x^A). \quad (4.29)$$

This proves that the BMS4 algebra of asymptotic symmetries existing at spatial infinity when using the asymptotic behaviour described in (2.1) is the global BMS4 algebra obtained by A. Strominger in [2]. In our case, the antipodal map between the super-translation parameter at \mathcal{I}^+ and \mathcal{I}^- is a consequence of the asymptotics prescribed at spatial infinity.

4.3 Super-translation charges

Other obvious quantities of interest are the associated charges: Bondi 4-momentum and super-translation charges. The component of the linearised Weyl tensor relevant for the definition of the super-momentum charges is given by

$$K_{\rho s \rho s} = \Omega^{-1} \tilde{C}_{\rho s \rho s} = -\rho^{-2} \partial_s^2 \hat{\sigma} + o(\rho^{-1}). \quad (4.30)$$

At null infinity, we get

$$\lim_{s \rightarrow \pm 1} K_{\rho s \rho s} = -\rho^{-2} \lim_{s \rightarrow \pm 1} \partial_s^2 \hat{\sigma} + o(\rho^{-2}) = -\rho^{-2} m^{even}(\pm x^A) + o(\rho^{-2}), \quad (4.31)$$

where $m^{even}(x^A)$ is a function on the sphere controlling the even part of $\hat{\sigma}$. For the Bondi super-momentum charges, we will use the expression given in [27]:

$$P_\alpha^+ = \frac{1}{32\pi G} \oint_{S^2} dx^A dx^B \sqrt{-\tilde{\eta}} \epsilon_{\mu\nu\gamma\delta} K^{\gamma\delta}{}_{AB} \alpha \tilde{n}^\mu l^\nu \Big|_{\mathcal{G}^+}, \quad \tilde{n}^\mu = \tilde{\eta}^{\mu\nu} \partial_\nu \Omega, \quad (4.32)$$

where we kept only the terms contributing at the linearised level. The vector l^μ is given on \mathcal{G}^+ by $l = \frac{1}{4} \partial_\rho + \frac{1}{2\rho} \partial_s$ and $\epsilon_{\alpha\beta\mu\nu}$ is antisymmetric with $\epsilon_{\rho s \zeta \bar{\zeta}} = 1$. Evaluating this expression in our case directly leads to:

$$P_\alpha^+ = \frac{1}{16\pi G} \oint_{S^2} \epsilon_{AB} dx^A dx^B \sqrt{\gamma} \rho \alpha \tilde{n}^\rho l^s K_{\rho s \rho s} \Big|_{\mathcal{G}^+} \quad (4.33)$$

$$= \frac{1}{8\pi G} \oint_{S^2} d^2\Omega \left(\alpha(x^A) m^{even}(x^A) + o(\rho^0) \right). \quad (4.34)$$

We are interested in the values of the charges when one approaches spatial infinity. This corresponds to the limit $\rho \rightarrow 0$. We see that:

$$\lim_{\rho \rightarrow 0} P_\alpha^+ = \frac{1}{8\pi G} \oint_{S^2} d^2\Omega \alpha(x^A) m^{even}(x^A). \quad (4.35)$$

These charges are identical to the ones defined at spatial infinity if we take into account the link between BMS4 super-translation parameters at spatial and future null infinity: $\alpha(x^A) = T^{odd}(x^A)$. A similar computation on \mathcal{G}^- leads to

$$P_\beta^- = \frac{1}{32\pi G} \oint_{S^2} dx^A dx^B \sqrt{-\tilde{\eta}} \epsilon_{\mu\nu\gamma\eta} K^{\gamma\eta}{}_{AB} (-\beta \tilde{n}^\mu) l^\nu \Big|_{\mathcal{G}^-}, \quad (4.36)$$

$$\lim_{\rho \rightarrow 0} P_\beta^- = \frac{1}{8\pi G} \oint_{S^2} d^2\Omega \beta(x^A) m^{even}(-x^A), \quad (4.37)$$

where we used $l|_{\mathcal{G}^-} = -\frac{1}{4} \partial_\rho - \frac{1}{2\rho} \partial_s$. At spatial infinity, the BMS4 super-translations charges defined at future and past null infinities are equal up to an antipodal map. Explicitly, we have:

$$\beta(x^A) = \alpha(-x^A) \quad \Rightarrow \quad \lim_{\rho \rightarrow 0} P_\alpha^+ = \lim_{\rho \rightarrow 0} P_\beta^-. \quad (4.38)$$

This identity is equivalent to the antipodal boundary conditions imposed on the mass parameter by A. Strominger in [2]. We have shown here that, in the linearised theory, it is a consequence of the boundary conditions imposed at spatial infinity if we remove the space-times for which the differentiable structure at null infinity is not strong enough to define the Bondi super-momentum charges.

This result is a linearised version of a similar result already obtained in [17] by Herberthson and Ludvigsen. In their derivation, they used a generalization of the conformal description of i_0 introduced by Ashtekar and Hansen in [28]. It would be interesting to see how their boundary structure is related to the boundary condition used in section 2 to describe spatial infinity.

5 Conclusions

In this work, we have shown how a global BMS4 algebra appears as part of the asymptotic symmetry algebra at spatial infinity. We then used linearised theory around Minkowski to show that it corresponds to the diagonal algebra considered by Strominger at null infinity. While obtained in the lagrangian formalism, this is the gravitational equivalent of the results obtained in [16] for electromagnetism.

The BMS4 charges constructed here are defined on Cauchy slices. It means that a Hamiltonian description of these charges should also be possible. This would put this infinite set of conserved charges on the same footing as the ADM mass.

In section 4, we had to rely on linearised theory as the coordinates used to describe spatial infinity are not adapted to null infinity. In order to have the full non-linear picture, it would be of particular interest to rewrite the asymptotic conditions used in section 2 in the formalism introduced by Friedrich [26, 22].

In [29, 23], it was argued that the relevant asymptotic symmetry algebra at null infinity should not only contain Lorentz algebra but the full conformal algebra on the 2-sphere. In that case, it has been shown that the relevant structure is an algebroid and that the associated algebroid of charges closes up to a central extension [30, 31, 32]. It would be interesting to see if one can reproduce this structure at spatial infinity.

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A Solution to some differential equations

In this appendix, we will solve the various partial differential equations relevant for our asymptotic analysis. The equation of motion for σ as well as the equation satisfied by super-translation parameter are

$$(\mathcal{D}_a \mathcal{D}^a + 3)\sigma = 0, \quad (\mathcal{D}_a \mathcal{D}^a + 3)\omega = 0, \quad (\text{A.1})$$

while the asymptotic field k_{ab} satisfies

$$k^a_a = 0, \quad \mathcal{D}^a k_{ab} = 0, \quad (\mathcal{D}_a \mathcal{D}^a - 3)k_{bc} = 0. \quad (\text{A.2})$$

As in section 3 and 4, we will use the rescaled quantities:

$$\hat{\sigma} = \sqrt{1-s^2}\sigma, \quad \hat{\omega} = \sqrt{1-s^2}\omega, \quad \hat{k}_{ab} = \sqrt{1-s^2}k_{ab}. \quad (\text{A.3})$$

In order to solve these equations, we will use complex coordinates on the sphere $\zeta = \cot \frac{\theta}{2} e^{i\phi}$ for which the metric takes the form

$$\gamma_{AB} dx^A dx^B = 2P^{-2} d\zeta d\bar{\zeta}, \quad P = \frac{1 + \zeta\bar{\zeta}}{\sqrt{2}}. \quad (\text{A.4})$$

Tensors on the sphere can be encoded in spin weighted functions η of spin s_η and the covariant derivative is then given by the operators

$$\bar{\partial}\eta = P^{1-s_\eta} \partial_{\bar{\zeta}}(P^{s_\eta}\eta), \quad \bar{\partial}\eta = P^{1+s_\eta} \partial_{\zeta}(P^{-s_\eta}\eta), \quad (\text{A.5})$$

where $\bar{\partial}, \partial$ respectively raises and lowers the spin weight by one unit (see [33, 34] for more details). They satisfy

$$[\bar{\partial}, \partial]\eta = s_\eta \eta, \quad (\text{A.6})$$

and the Laplace operator on the sphere can be written as

$$\Delta = \bar{\partial}\partial + \partial\bar{\partial}. \quad (\text{A.7})$$

The asymptotic fields $\hat{\sigma}$ and $\hat{\omega}$ are spin weighted functions of spin zero while the tensor \hat{k}_{ab} can be encoded in the following spin weighted functions:

$$\hat{k}_{ss} = \kappa, \quad \hat{k}_{s\zeta} = P^{-1}\alpha, \quad \hat{k}_{s\bar{\zeta}} = P^{-1}\bar{\alpha}, \quad (\text{A.8})$$

$$\hat{k}_{\zeta\zeta} = P^{-2}\beta, \quad \hat{k}_{\bar{\zeta}\bar{\zeta}} = P^{-2}\bar{\beta}, \quad \hat{k}_{\zeta\bar{\zeta}} = \frac{P^{-2}}{2}(1-s^2)\kappa, \quad (\text{A.9})$$

where we used the first equation of (A.2). The spin weights are given by

$$s_\kappa = 0, \quad s_\alpha = -1, \quad s_{\bar{\alpha}} = 1, \quad s_\beta = -2, \quad s_{\bar{\beta}} = 2. \quad (\text{A.10})$$

The functions κ and σ are real while $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ satisfy $\alpha^* = \bar{\alpha}$ and $\beta^* = \bar{\beta}$ where the star denotes complex conjugation.

The two equations in (A.1) can then be written as

$$-(1-s^2)\partial_s^2\hat{\sigma} - 2s\partial_s\hat{\sigma} + 2\hat{\sigma} + (\partial\bar{\partial} + \bar{\partial}\partial)\hat{\sigma} = 0, \quad (\text{A.11})$$

with the same equation for $\hat{\omega}$. The various equations for k_{ab} take the form

$$(1-s^2)\partial_s\kappa = \partial\alpha + \bar{\partial}\bar{\alpha}, \quad (\text{A.12})$$

$$(1-s^2)\partial_s\alpha + s\alpha = \frac{1}{2}(1-s^2)\bar{\partial}\kappa + \bar{\partial}\beta, \quad (\text{A.13})$$

associated with

$$-(1-s^2)\partial_s^2\kappa + 2s\partial_s\kappa + (\partial\bar{\partial} + \bar{\partial}\partial)\kappa = 0, \quad (\text{A.14})$$

$$-(1-s^2)\partial_s^2\alpha + 2s\partial_s\alpha - \alpha + (\partial\bar{\partial} + \bar{\partial}\partial)\alpha = 2s\bar{\partial}\kappa, \quad (\text{A.15})$$

$$-(1-s^2)\partial_s^2\beta + 2s\partial_s\beta - 2\beta + (\partial\bar{\partial} + \bar{\partial}\partial)\beta = 4s\bar{\partial}\alpha. \quad (\text{A.16})$$

To this set, we have to add the equivalent of equations (A.13), (A.15) and (A.16) for the barred quantities.

In order to solve these equations, we will expand our spin weighted functions in spin weighted spherical harmonics $Y_{lm}^s(x^A)$ where $Y_{lm}^0(x^A)$ are the usual spherical harmonics. Spin weighted spherical harmonics are only defined for $l > |s|$ and $l > |m|$. They form a complete set for each value of the spin s . The main properties we will be using are:

$$\partial Y_{lm}^s = -\sqrt{\frac{(l-s)(l+s+1)}{2}} Y_{lm}^{s+1}, \quad \bar{\partial} Y_{lm}^s = \sqrt{\frac{(l+s)(l-s+1)}{2}} Y_{lm}^{s-1}, \quad (\text{A.17})$$

$$(\partial\bar{\partial} + \bar{\partial}\partial) Y_{lm}^s = -[(l+1)l - s^2] Y_{lm}^s, \quad (Y_{lm}^s)^* = (-1)^{m+s} Y_{l,-m}^{-s}. \quad (\text{A.18})$$

A.1 Solution for σ and ω

We will focus on the solution to the equation of motion for σ as the equation satisfied by the super-translation parameter ω is identical.

Introducing the spherical harmonic decomposition $\hat{\sigma} = \sum_{l,m} \hat{\sigma}_{lm}(s) Y_{lm}^0$, equation (A.11) becomes

$$-(1-s^2)\partial_s^2\hat{\sigma}_{lm} - 2s\partial_s\hat{\sigma}_{lm} + 2\hat{\sigma}_{lm} - l(l+1)\hat{\sigma}_{lm} = 0, \quad \forall l, m. \quad (\text{A.19})$$

This equation is related to Legendre equation:

$$-(1-s^2)\partial_s^2\psi_l + 2s\partial_s\psi_l - l(l+1)\psi_l = 0 \quad (\text{A.20})$$

for which the general solution is given in terms of Legendre polynomials $P_l(s)$ and Legendre functions of the second kind $Q_l(s)$:

$$\psi_l(s) = \psi_l^P P_l(s) + \psi_l^Q Q_l(s), \quad (\text{A.21})$$

$$Q_l(s) = P_l(s) \frac{1}{2} \log \left(\frac{1+s}{1-s} \right) + \tilde{Q}_l(s), \quad (\text{A.22})$$

where \tilde{Q}_l are polynomials (see [35]). One can easily show that if ψ_l satisfies (A.20) then $(1-s^2)\partial_s^2\psi_l$ will satisfy equation (A.19). If we define

$$V_l(s) = (l-1)l(l+1)(l+2)(1-s^2)^2\partial_s^2 P_l, \quad W_l(s) = \frac{1}{2}(1-s^2)^2\partial_s^2 Q_l, \quad \forall l > 1, \quad (\text{A.23})$$

the general solution to (A.19) is then given by

$$\hat{\sigma}_{lm}(s) = \hat{\sigma}_{lm}^V V_l(s) + \hat{\sigma}_{lm}^W W_l(s), \quad \forall l > 1. \quad (\text{A.24})$$

The normalisations of V_l and W_l were chosen for future convenience. For $l < 2$, this procedure gives us only one of the two independent solutions namely:

$$W_0(s) = s, \quad W_1(s) = 1. \quad (\text{A.25})$$

The other one is easily constructed:

$$V_0 = \frac{1}{2}(s^2 + 1), \quad V_1 = \frac{1}{6}(s^3 - 3s). \quad (\text{A.26})$$

The general solution to the equation of motion for $\hat{\sigma}$ then takes the form

$$\hat{\sigma}(s, x^A) = \sum_{l,m} (\hat{\sigma}_{lm}^V V_l(s) + \hat{\sigma}_{lm}^W W_l(s)) Y_{lm}^0(x^A). \quad (\text{A.27})$$

The functions V_l and W_l inherit the parity properties of P_l and Q_l :

$$P_l(-s) = (-1)^l P_l(s) \quad \Rightarrow \quad V_l(-s) = (-1)^l V_l(s), \quad (\text{A.28})$$

$$Q_l(-s) = -(-1)^l Q_l(s) \quad \Rightarrow \quad W_l(-s) = -(-1)^l W_l(s). \quad (\text{A.29})$$

This means that under the combined action of a time reversal $s \rightarrow -s$ and an antipodal map $x^A \rightarrow -x^A$, the general solution (A.27) separates into an odd and an even part:

$$\hat{\sigma} = \hat{\sigma}^{even} + \hat{\sigma}^{odd}, \quad \hat{\sigma}^{even}(-s, -x^A) = \hat{\sigma}^{even}, \quad \hat{\sigma}^{odd}(-s, -x^A) = -\hat{\sigma}^{odd}, \quad (\text{A.30})$$

$$\hat{\sigma}^{even} = \sum_{l,m} \hat{\sigma}_{lm}^V V_l(s) Y_{lm}^0(x^A), \quad \hat{\sigma}^{odd} = \sum_{l,m} \hat{\sigma}_{lm}^W W_l(s) Y_{lm}^0(x^A). \quad (\text{A.31})$$

Each these parts can be parametrized by a function on the sphere:

$$m^{even}(x^A) \equiv \lim_{s \rightarrow 1} \sum_{l,m} \hat{\sigma}_{lm}^V \partial_s^2 V_l(s) Y_{lm}^0(x^A) = \sum_{l,m} \hat{\sigma}_{lm}^V Y_{lm}^0(x^A), \quad (\text{A.32})$$

$$m^{odd}(x^A) \equiv \lim_{s \rightarrow 1} \sum_{l,m} \hat{\sigma}_{lm}^W W_l(s) Y_{lm}^0(x^A) = \sum_{l,m} \hat{\sigma}_{lm}^W Y_{lm}^0(x^A), \quad (\text{A.33})$$

where we used the following identities

$$\lim_{s \rightarrow 1} \partial_s^2 V_l = 1, \quad \lim_{s \rightarrow 1} W_l = 1. \quad (\text{A.34})$$

They can be easily shown using $P_l(1) = 1$, the explicit form of Q_l given in (A.22) and Legendre equation (A.20). Doing an asymptotic expansion around $s = 1$ of both parts of the solution, we get

$$\widehat{\sigma}^{odd}(s, x^A) = m^{odd}(x^A) + O((1-s)), \quad (\text{A.35})$$

$$\widehat{\sigma}^{even}(s, x^A) = \widehat{\sigma}_{00}^V - \frac{1}{3} \sum_{m=-1}^{m=1} \widehat{\sigma}_{1m}^V Y_{1m}^0(x^A) + O((1-s)). \quad (\text{A.36})$$

A similar expansion can be done around $s = -1$.

We will have the same kind of expressions for the super-translation parameter: $\widehat{\omega} = \widehat{\omega}^{even} + \widehat{\omega}^{odd}$ with

$$\widehat{\omega}^{even} = \sum_{l,m} \widehat{\omega}_{lm}^V V_l(s) Y_{lm}^0(x^A), \quad \widehat{\omega}^{odd} = \sum_{l,m} \widehat{\omega}_{lm}^W W_l(s) Y_{lm}^0(x^A), \quad (\text{A.37})$$

$$T^{even}(x^A) \equiv \sum_{l,m} \widehat{\omega}_{lm}^V Y_{lm}^0(x^A), \quad T^{odd}(x^A) \equiv \sum_{l,m} \widehat{\omega}_{lm}^W Y_{lm}^0(x^A). \quad (\text{A.38})$$

Super-translation charges given in (2.10) can be rewritten as

$$\mathcal{Q}_\omega = \frac{1}{4\pi G} \oint_{S^2} d^2\Omega \frac{1}{1-s^2} (\widehat{\sigma} \partial_s \widehat{\omega} - \widehat{\omega} \partial_s \widehat{\sigma}). \quad (\text{A.39})$$

Inserting the general solutions we obtained, we get

$$\mathcal{Q}_\omega = \frac{1}{4\pi G} \sum_{lm} \left(\overline{\widehat{\omega}_{lm}^W} \widehat{\sigma}_{lm}^V - \overline{\widehat{\omega}_{lm}^V} \widehat{\sigma}_{lm}^W \right) C_l, \quad (\text{A.40})$$

$$C_l = \frac{1}{1-s^2} (V_l \partial_s W_l - W_l \partial_s V_l). \quad (\text{A.41})$$

The quantity C_l is conserved $\partial_s C_l = 0$ and its value is easily computed asymptotically: $C_l = \lim_{s \rightarrow 1} C_l = \frac{1}{2}$. Plugging this into the value of the charges and using the functions introduced in (A.32), (A.33) and (A.38), we get

$$\mathcal{Q}_\omega = \frac{1}{8\pi G} \oint_{S^2} d^2\Omega T^{odd}(x^A) m^{even}(x^A) - \left(T^{even}(x^A) m^{odd}(x^A) \right). \quad (\text{A.42})$$

A.2 Solution for k_{ab}

Let's now have a look at the equations of k_{ab} : equations (A.12) to (A.16). We will introduce the corresponding spherical harmonic expansions:

$$\kappa = \frac{1}{1-s^2} \widehat{k} = \sum_{l,m} \kappa_{lm} Y_{lm}^0, \quad \alpha = \sum_{l>0,m} \alpha_{lm} Y_{lm}^{-1}, \quad \bar{\alpha} = \sum_{l>0,m} \bar{\alpha}_{lm} Y_{lm}^1, \quad (\text{A.43})$$

$$\beta = \sum_{l>1,m} \beta_{lm} Y_{lm}^{-2}, \quad \bar{\beta} = \sum_{l>1,m} \bar{\beta}_{l,m} Y_{lm}^2, \quad (\text{A.44})$$

where the reality conditions imply

$$(\kappa_{lm})^* = (-1)^m \kappa_{l,-m}, \quad (\alpha_{lm})^* = -(-1)^m \bar{\alpha}_{l,-m}, \quad (\beta_{lm})^* = (-1)^m \bar{\beta}_{l,-m}. \quad (\text{A.45})$$

Inserting this into equations (A.14) and (A.15), we get

$$-(1-s^2)\partial_s^2 \kappa_{lm} + 2s\partial_s \kappa_{lm} - l(l+1)\kappa_{lm} = 0, \quad \forall l, m, \quad (\text{A.46})$$

$$-(1-s^2)\partial_s^2 \alpha_{lm} + 2s\partial_s \alpha_{lm} - l(l+1)\alpha_{lm} = s\sqrt{2l(l+1)}\kappa_{lm}, \quad \forall l > 0, m, \quad (\text{A.47})$$

$$-(1-s^2)\partial_s^2 \bar{\alpha}_{lm} + 2s\partial_s \bar{\alpha}_{lm} - l(l+1)\bar{\alpha}_{lm} = -s\sqrt{2l(l+1)}\kappa_{lm}, \quad \forall l > 0, m, \quad (\text{A.48})$$

while equation (A.12) gives

$$(1-s^2)\partial_s \kappa_{00} = 0, \quad (1-s^2)\partial_s \kappa_{lm} = \sqrt{\frac{(l+1)l}{2}}(\bar{\alpha}_{lm} - \alpha_{lm}), \quad \forall l > 0, m. \quad (\text{A.49})$$

Both κ_{lm} and $\alpha_{lm} + \bar{\alpha}_{lm}$ satisfy Legendre equation for which the general solution is given in terms of Legendre polynomials P_l and Legendre functions of the second kind Q_l :

$$Q_l(s) = P_l(s) \frac{1}{2} \log \left(\frac{1+s}{1-s} \right) + \tilde{Q}_l(s), \quad (\text{A.50})$$

where \tilde{Q}_l are polynomials. The general solution to equations (A.46), (A.47) and (A.49) is then given by:

$$\kappa_{00}(s) = \kappa_{00}^P, \quad \kappa_{lm}(s) = \kappa_{lm}^P P_l(s) + \kappa_{lm}^Q Q_l(s) \quad \forall l > 0, m, \quad (\text{A.51})$$

$$\mathcal{R}\alpha_{lm}(s) = \frac{1}{2}(\alpha_{lm}(s) + \bar{\alpha}_{lm}(s)) = \mathcal{R}\alpha_{lm}^P P_l(s) + \mathcal{R}\alpha_{lm}^Q Q_l(s) \quad \forall l > 0, m, \quad (\text{A.52})$$

$$\alpha_{lm}(s) = \frac{-1}{\sqrt{2l(l+1)}}(1-s^2)\partial_s \kappa_{lm} + \mathcal{R}\alpha_{lm}, \quad \forall l > 0, m. \quad (\text{A.53})$$

Developing equation (A.13), we then get

$$(1-s^2)\partial_s \alpha_{lm} + s\alpha_{lm} - \frac{1}{2}\sqrt{\frac{l(l+1)}{2}}(1-s^2)\kappa_{lm} = -\sqrt{\frac{(l+2)(l-1)}{2}}\beta_{lm}, \quad \forall l > 1, m, \quad (\text{A.54})$$

$$(1-s^2)\partial_s \alpha_{1m} + s\alpha_{1m} - \frac{1}{2}(1-s^2)\kappa_{1m} = 0. \quad (\text{A.55})$$

which, when associated to their barred equivalent, lead to

$$\kappa_{1m}^Q = 0, \quad \mathcal{R}\alpha_{1m}^P = 0, \quad \mathcal{R}\alpha_{1m}^Q = 0, \quad (\text{A.56})$$

$$\beta_{lm}(s) = \sqrt{\frac{2}{(l-1)(l+2)}} \left[\frac{1}{2} \frac{(1-s^2)^2}{\sqrt{2l(l+1)}} \partial_s^2 \kappa_{lm} - ((1-s^2)\partial_s + s) \mathcal{R}\alpha_{lm} \right], \quad (\text{A.57})$$

where the last line is valid for $l > 1$. One can check that equation (A.16) is then automatically satisfied. Equations (A.51)-(A.53) with equations (A.56) and (A.57) give the complete solution to the system of equations (A.12) to (A.16).

B Weyl tensor of the unphysical metric

This appendix contains various useful results about geometric quantities associated to the unphysical metric $\tilde{g}_{\mu\nu} = \tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}$ written in equations (4.13) to (4.16):

$$h_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\rho^2(1-s^2)^2} \left\{ o(1)d\rho^2 + o(\rho)d\rho dx^a + \hat{h}_{ab} dx^a dx^b \right\}, \quad (\text{B.1})$$

where

$$\begin{aligned} \tilde{h}_{AB} = & -2\rho(1-s^2) \left(1 - \log(\rho\sqrt{1-s^2}) \right) (D_A D_B \hat{\sigma} + \gamma_{AB} \hat{\sigma} - s\gamma_{AB} \partial_s \hat{\sigma}) \\ & + \rho(1-s^2) \hat{k}_{AB} + o(\rho), \end{aligned} \quad (\text{B.2})$$

$$\tilde{h}_{As} = -2\rho(1-s^2) \left(1 - \log(\rho\sqrt{1-s^2}) \right) \partial_A \partial_s \hat{\sigma} + \rho(1-s^2) \hat{k}_{sA} + o(\rho), \quad (\text{B.3})$$

$$\tilde{h}_{ss} = -2\rho(1-s^2) \left(1 - \log(\rho\sqrt{1-s^2}) \right) \partial_s^2 \hat{\sigma} + \rho(1-s^2) \hat{k}_{ss} + o(\rho). \quad (\text{B.4})$$

The linearised Weyl tensor of $\tilde{h}_{\mu\nu}$ is given by

$$\tilde{C}_{\rho\alpha\rho b} = -\frac{1}{\rho} (1-s^2) \hat{\sigma}_{ab} + o(\rho^{-1}), \quad (\text{B.5})$$

$$\tilde{C}_{\rho abc} = (1-s^2) \left(\frac{3}{2} \sqrt{1-s^2} (\mathcal{D}_b k_{ac} - \mathcal{D}_c k_{ab}) + \frac{s}{1-s^2} (\delta_b^s \hat{\sigma}_{ac} - \delta_c^s \hat{\sigma}_{ab}) \right) + o(1). \quad (\text{B.6})$$

All the other components can be obtained using the properties of the Weyl tensor. The combination relevant for the description of null infinity is $K_{\mu\nu\alpha\beta} = \Omega^{-1} \tilde{C}_{\mu\nu\alpha\beta}$. If this tensor is not continuous at null infinity then the structure of \mathcal{G}^\pm is not differentiable enough to allow the definition of the BMS4 super-translation charges. Let's have a look at a few specific components:

$$K_{\rho s \rho s} = -\frac{1}{\rho^2} \partial_s^2 \hat{\sigma} + o(\rho^{-2}), \quad (\text{B.7})$$

$$K_{\rho s \zeta \bar{\zeta}} = \frac{1}{\rho} \frac{3}{2} P^{-2} (\bar{\partial} \bar{\alpha} - \partial \alpha) + o(\rho^{-1}) \quad (\text{B.8})$$

$$= -\frac{3}{\rho} P^{-2} \sum_{lm} \sqrt{\frac{l(l+1)}{2}} \left(\mathcal{R} \alpha_{lm}^P P_l(s) + \mathcal{R} \alpha_{lm}^Q Q_l(s) \right) Y_{lm}^0 + o(\rho^{-1}). \quad (\text{B.9})$$

In the limit $s \rightarrow \pm 1$, these components diverge logarithmically when $\hat{\sigma}_{lm}^W \neq 0$ and $\mathcal{R} \alpha_{lm}^Q \neq 0$.

References

- [1] A. Strominger, ‘‘Asymptotic Symmetries of Yang-Mills Theory,’’ *Journal of High Energy Physics* **1407** (2013) 151, 1308.0589.

- [2] A. Strominger, “On BMS Invariance of Gravitational Scattering,” *Journal of High Energy Physics* **1407** (2014) 152, 1312 . 2229.
- [3] T. He, V. Lysov, P. Mitra, and A. Strominger, “BMS supertranslations and Weinberg’s soft graviton theorem,” *Journal of High Energy Physics* **1505** (2015) 15, 1401 . 7026.
- [4] S. Hyun, S.-A. Park, and S.-H. Yi, “Quasi-local charges and asymptotic symmetry generators,” *Journal of High Energy Physics* **1406** (2014) 151, 1403 . 2196.
- [5] T. Adamo, E. Casali, and D. Skinner, “Perturbative gravity at null infinity,” *Classical and Quantum Gravity* **31** (2014), no. 22, 225008, 1405 . 5122.
- [6] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, “Semiclassical Virasoro Symmetry of the Quantum Gravity S-Matrix,” *Journal of High Energy Physics* **2014** (2014), no. 8, 58, 1406 . 3312.
- [7] M. Campiglia and A. Laddha, “New symmetries for the Gravitational S-matrix,” *Journal of High Energy Physics* **1504** (2015) 076, 1502 . 02318.
- [8] M. Campiglia and A. Laddha, “Asymptotic symmetries of QED and Weinberg’s soft photon theorem,” *Journal of High Energy Physics* **1507** (2015) 115, 1505 . 05346.
- [9] D. Kapec, M. Pate, and A. Strominger, “New Symmetries of QED,” 1506 . 02906.
- [10] S. G. Avery and B. U. W. Schwab, “Residual Local Supersymmetry and the Soft Gravitino,” *Phys* **116** (2015), no. 17, 171601, 1512 . 02657.
- [11] M. Campiglia and A. Laddha, “Asymptotic symmetries of gravity and soft theorems for massive particles,” *Journal of High Energy Physics* **1512** (2015) 094, 1509 . 01406.
- [12] S. G. Avery and B. U. W. Schwab, “BMS, String Theory, and Soft Theorems,” *Physical Review D* **93** (2015) 026003, 1506 . 05789.
- [13] A. Strominger, “Lectures on the Infrared Structure of Gravity and Gauge Theory,” 1703 . 05448.
- [14] S. W. Hawking, M. J. Perry, and A. Strominger, “Soft Hair on Black Holes,” *Physical Review Letters* **116** (2016), no. 23, 231301, 1601 . 00921.
- [15] S. W. Hawking, M. J. Perry, and A. Strominger, “Superrotation Charge and Supertranslation Hair on Black Holes,” 1611 . 09175.

- [16] M. Campiglia and R. Eyheralde, “Asymptotic $U(1)$ charges at spatial infinity,” 1703.07884.
- [17] M. Herberthson and M. Ludvigsen, “A Relationship Between Future and Past Null Infinity,” *General Relativity and Gravitation* **24** (1992), no. 11, 1185–1193.
- [18] G. Compère and F. Dehouck, “Relaxing the Parity Conditions of Asymptotically Flat Gravity,” *Classical and Quantum Gravity* **28** (2011), no. 24, 1–50, 1106.4045.
- [19] M. Park and R. B. Mann, “Holographic renormalization of asymptotically flat gravity,” *Journal of High Energy Physics* **2012** (2012), no. 12, 2927–2950, 0511096.
- [20] R. B. Mann, D. Marolf, and A. Virmani, “Covariant counter-terms and conserved charges in asymptotically flat spacetimes,” *Classical and Quantum Gravity* **23** (2006), no. 22, 6357–6378, 0607041.
- [21] R. B. Mann, D. Marolf, R. McNees, and A. Virmani, “On the stress tensor for asymptotically flat gravity,” *Classical and Quantum Gravity* **25** (2008), no. 22, 225019, 0804.2079.
- [22] H. Friedrich, “Gravitational fields near space-like and null infinity,” *Journal of Geometry and Physics* **24** (1998), no. 2, 83–163, 0209034v1.
- [23] G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence,” *Journal of High Energy Physics* **2010** (2010), no. 5, 62.
- [24] H. Friedrich and J. Kannar, “Calculating asymptotic quantities near space-like and null infinity from Cauchy data,” *Annalen der Physik (Leipzig)* **9** (2000), no. 3-5, 321–330, 9911103v1.
- [25] H. Friedrich and J. Kannar, “Bondi-type systems near spacelike infinity and the calculation of the Newman-Penrose constants,” *Journal of Mathematical Physics* **41** (2000), no. 4, 2195–2232, 9910077v1.
- [26] H. Friedrich, “Einstein equations and conformal structure: Existence of anti-de Sitter-type space-times,” *Journal of Geometry and Physics* **17** (1995), no. 2, 125–184.
- [27] A. Ashtekar and A. Magnon-Ashtekar, “Energy-momentum in general relativity,” *Physical Review Letters* **43** (1979), no. 3, 181–184.

- [28] A. Ashtekar and R. O. Hansen, “A unified treatment of null and spatial infinity in general relativity. I. Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity,” *Journal of Mathematical Physics* **19** (1978), no. 7, 1542.
- [29] G. Barnich and C. Troessaert, “Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited,” *Physical Review Letters* **105** (2010) 111103, 0909.2617.
- [30] G. Barnich and C. Troessaert, “BMS charge algebra,” *Journal of High Energy Physics* **2011** (2011), no. 12, 105, 1106.0213.
- [31] G. Barnich and C. Troessaert, “Comments on holographic current algebras and asymptotically flat four dimensional spacetimes at null infinity,” *Journal of High Energy Physics* **2013** (2013), no. 11, 3.
- [32] G. Barnich, “Centrally extended BMS4 Lie algebroid,” 1703.08704.
- [33] R. Penrose and W. Rindler, *Spinors and space-time, Volume 1: Two-spinor calculus and relativistic fields*. Cambridge University Press, Cambridge, 1984.
- [34] R. Penrose and W. Rindler, *Spinors and space-time, Volume 2: Spinor and twistor methods in space-time geometry*. Cambridge University Press, Cambridge, 1986.
- [35] G. Szegő, *Orthogonal Polynomials*. American Mathematical Society, Providence, Rhode Island, colloquium ed., 1991.