Round-Trip of an Algorithm*

Rita Meyer-Spasche,
Max Planck Institute for Plasma Physics, Boltzmannstr. 2
85748 Garching, Germany; meyer-spasche@ipp-garching.mpg.de

Abstract

In this article we will discuss the development of an algorithm which is today best known in Numerical Analysis under the name *Inverse Iteration*, for the computation of one eigenvalue of a general complex matrix. It was invented, however, by engineers for the stability analysis of mechanical structures, i.e. for the computation of that eigenvalue of a two-point boundary value problem at which instability occurs. In later years, when the stability analysis of plasma configurations became important which are governed by quasilinear partial differential equations, the matrix method was adapted to these boundary value problems, probably without knowledge about its origin at ordinary boundary value problems. It thus performed a round-trip *boundary value problems - matrices - boundary value problems* within ca 80 years. During these approximately 80 years - from 1893 to 1970/1977 - computational methods were dramatically improved. This, of course, is also true for the treatment of boundary value problems accompanying the application of the algorithm.

Zusammenfassung: In diesem Artikel wird die Entwicklung eines Algorithmus diskutiert, der heute vor allem unter dem Namen *Inverse Iteration* in der Numerik bekannt ist, zur Berechnung eines Eigenwertes einer beliebigen komplexen Matrix. Er wurde jedoch von Ingenieuren zur Stabilitätsanalyse von mechanischen Strukturen erfunden, d.h. zur Berechnung desjenigen Eigenwertes eines zwei-Punkt-Randwertproblems, bei dem Instabilität auftritt. Später, als die Stabilitätsanalyse von Plasmakonfigurationen wichtig wurde, die durch quasilineare partielle Differentialgleichungen beschrieben werden, wurde die für Matrizen übliche Methode auf diese Randwertprobleme verallgemeinert: vermutlich ohne Wissen um die Ursprünge dieser Methode für Randwertprobleme. Der Algorithmus machte also eine Rundreise *Randwertprobleme - Matrizen - Randwertprobleme* in ca 80 Jahren. In diesen etwa 80 Jahren – von 1893 bis 1970/1977 – wurden numerische

*slightly revised version of pp. 195-204 in: Christa Binder (ed): Namenspatrone und Taufpaten - Wie mathematische Begriffe zu ihrem Namen kamen Proc. *XIII. Österreichisches Symposium zur Geschichte der Mathematik,* Miesenbach NÖ, 1.-7. Mai 2016, Institut für Analysis und Scientific Computing, TU Wien 2017

Methoden dramatisch verbessert. Dies gilt natürlich auch für die numerische Behandlung von Randwertproblemen, die die Anwendung des Algorithmus begleitet hat.

1 Introduction

In a recent article this author wrote:

> We will not discuss here historical differences of names in dependence on scientific discipline or country. [...] We will also not discuss here the round-trip of an algorithm which changed its name when used in different mathematical applications – this case will be treated elsewhere [MSp16b].

Considering the examples (Gauss elimination, Hopf bifurcation, Zorn's lemma) treated in that article, there seems to be general international and interdisciplinary agreement about their names today. Here we will look at the round-trip of the algorithm and see a very different picture. We will observe a number of changes of names: *Engesser method, Vianello method, Stodola method, Engesser-Vianello method*, *Stodola-Vianello method*; *Iterationsverfahren, Vektoriteration, von-Mises-Iteration, Power method*; *Gebrochene Iteration, Inverse Iteration*; *Continuation method, Fortsetzungsverfahren*. Many of these names are still in use today.

In passing, we will also observe that the use of the terms Eigenwert – charakteristische Zahl and Eigenlösung – Eigenvektor got changed during the same years.

During the approximately 80 years of this development - from 1893 to 1970/1977 - computational methods were dramatically improved [MSp16a]. This, of course, is also true for the numerical treatment of boundary value problems accompanying the application of the algorithm. For easier reading we discuss the various versions of the algorithm in a unified language, using the terms of today.

2 Engesser/Stodola/Vianello Method

It is well known, at least to engineers and architects, that mechanical structures bend, buckle or break if they are overloaded. This can even be caused by their own weight if their shape is inappropriate. Already in 1744 *Leonhard Euler (1707-1783)* gave a formula for the *critical load* of a rod as a function of the mechanical properties of the rod, i.e. for the maximum load that does not cause buckling. This is known as *Euler's Buckling Formula* [Gau08, p.22]. In German, the critical load for a rod is called *Euler'sche Knicklast*.

Around 1900 it became very important for engineers and architects to compute in an efficient way the critical loads of the structures they designed. Mathematically speaking, the simplest case consists in computing the smallest eigenvalue of the parameter-dependent ordinary boundary value problem

$$y'' = -\frac{P}{EJ_x}y \qquad \text{with 2 given boundary conditions.} \tag{1}$$

In 1893 the engineer *Friedrich Engesser* published the following method [Eng93]: guess the buckled shape $y_0(x)$ by engineer's intuition and solve a short sequence of linear boundary value problems

$$y''_{\nu+1} = -\frac{P}{EJ_x} y_\nu, \qquad \text{with 2 given boundary conditions,} \quad \nu = 0, 1, 2, \ldots \tag{2}$$

with known r.h.sides $y_\nu(x)$. If $y_0(x)$ is a good guess, only very few iterations will be needed to find the solution $y(x) = y_{\nu+1}(x)$. How much $\mu = y_\nu(x)/y_{\nu+1}(x)$ depends on x is a measure for the quality of the solution.

In 1898 the engineer *Luigi Vianello* published [Via98] essentially the same iteration scheme, with a different, a graphical, integration method for the linear boundary value problems (2). Probably in 1903 *Aurel Stodola* published a similar iteration scheme, combined with a graphical method, for solving the free vibration problem of a rotating shaft, i.e. for computing an eigenvalue (a critical rotational frequency) for a 4th order ordinary boundary value problem:

> A similar method was used by Vianello to solve buckling problems, and Delaporte describes a related method in "Revue de mécanique" 1903, vol III, p. 517 [Sto24, p. 381ff]. [1]

The first edition of Stodola's book appeared in 1903.

This iteration method was also applied to other engineering problems involving ordinary eigenvalue problems, and several authors investigated its convergence properties, sometimes calling it Vianello's method, sometimes Stodola's method [Poh21, Ko27]. Since all three authors were important personalities and wrote important text books on different problems in structural mechanics [Eng92, Via05, Sto24], all three were cited as inventors of the method.

Friedrich Engesser (1848-1931) was professor at Technische Hochschule Karlsruhe (Karlsruhe Institute of Technology). *Aurel Stodola (1859-1942)* was professor at ETH Zürich. *Luigi Vianello (1862-1907)* was born near Venice. His place of birth belonged to the Austrian-Hungarian Empire at the time of his birth. He studied engineering in Northern Italy and then took jobs with German companies. Most of the time he worked with *Siemens & Halske* on the construction of railways. He was concerned with the construction of the S-Bahn in Berlin (particularly with 'Gleisdreieck') and with the Schwebebahn in Wuppertal.

Today, the names *Engesser-Vianello method* and *Stodola-Vianello method* seem to dominate among structural engineers. In 1955 Kollbrunner and Meister commented on this: Engesser invented the method, Vianello made it popular [KoM55, p.31].

3 From 'Vianello's Method' to 'Inverse Iteration'

This part of the development lasted from 1929 until ca 1950.

[1] Citations in English of sentences from publications in German were translated by the present author.

3.1 Vianello's Method – Iterationsverfahren – Power Method

In part 2 of their paper *Praktische Verfahren zur Gleichungsauflösung* [MPG29-2], *Richard von Mises (1883-1953)* and *Hilda Pollaczek-Geiringer (1893-1973)* discussed in 1929 the application of Vianello's method to symmetric real $n \times n$ matrices A. They proved convergence in the case of these matrices and remarked that their proof is analogous to the proof by Koch and more or less also to the proof by Pohlhausen [Ko27, Poh21], though Pohlhausen treated a generalised matrix eigenvalue problem $Bx = \lambda Ax$, with special non-singular real $n \times n$ matrices A and B.

v.Mises/Pollaczek-Geiringer considered the vector iterations

$$z^{\nu+1} = \mu^\nu A z^\nu, \qquad A = A^t, \quad z^o \text{ and all } \mu^\nu \text{ given}, \quad \nu = 0, 1, 2, \ldots \tag{3}$$

which converge to the eigenvalue λ of smallest modulus and the associated eigenvector *(Eigenlösung - eigensolution)* of

$$x = \lambda A x, \qquad \text{i.e.} \quad \lambda^{-1} x = Ax, \tag{4}$$

if that eigenvalue is positive and simple (i.e. has multiplicity one) and if there is no negative eigenvalue with the same modulus (their Theorem 11). Then they considered more complicated cases: higher multiplicity of the eigenvalue with smallest modulus; positive as well as negative eigenvalues of smallest modulus; and also, how to compute higher eigenpairs (λ, x) if those of smaller modulus are known.

Also, they give an example how to compute the smallest eigenvalue of a self-adjoint two-point boundary value problem[2]

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \lambda k(x) y(x) = 0, \qquad y(a) = y(b) = 0, \tag{5}$$

combining discretization by the Ritz method with their *Iterationsverfahren*, i.e. with the iterations (3) on the discrete system.

In the following years, this 'Iterationsverfahren' for solving the matrix eigenvalue problem was considered and further developed by several authors, always giving the v.Mises/Pollaczek-Geiringer paper as reference.

In his important, much read book of 1950 on matrices, *Zurmühl (1904-1966)* called the solutions (λ, x) of equation (4) eigenvalues and eigenvectors *(Eigenvektoren)*, and the solutions (μ, x), $\mu = 1/\lambda$, of

$$Ax = \mu x \tag{6}$$

characteristic numbers *(charakteristische Zahlen)* and eigenvectors [Zu50, p.285ff], probably following the already existing convention about eigenvalues of differential equations and characteristic values of the equivalent integral equations. Today, we call the μs the eigenvalues of matrix A, and the λs the eigenvalues of A^{-1}. We will not investigate here when and by whom these changes of names were introduced.

Zurmühl considered the iteration

$$z^{\nu+1} = Az^\nu, \qquad z^o \text{ given}, \qquad \nu = 0, 1, 2, 3, \ldots \tag{7}$$

[2] In this case all eigenvalues are positive, and the smallest eigenvalue is always simple.

or rather

$$z_*^\nu = \frac{1}{z_j^\nu} z^\nu, \qquad z^o \text{ given}, \qquad j = 1, \ldots, n \tag{8}$$

$$z^{\nu+1} = A z_*^\nu, \qquad \nu = 0, 1, 2, 3, \ldots, \tag{9}$$

where the z_j^ν are the components of z^ν, for solving

$$\mu x = Ax, \qquad \text{i.e.} \qquad x = \mu^{-1} Ax. \tag{10}$$

This is a considerable simplification of the Iterationsverfahren by v.Mises/Pollaczek-Geiringer, though the division by the components z_j^ν in eq. (8) might lead to division by zero. It thus got soon replaced by a division by some norm $||z^\nu||$.

3.2 The Basic Idea of a Proof of Convergence

Assume that the non-singular real $n \times n$ matrix A is symmetric, i.e. $A \in I\!R^{n\times n},\ A = A^t$. Then there exist n real numbers $\mu_i,\ i = 1, 2, \ldots, n$ and n vectors $x_1, x_2, \ldots, x_n \in I\!R^n$ which are pairwise linearly independent and perpendicular, i.e. $x_i \perp x_j$ for $i \neq j$, which satisfy

$$\mu_i x_i = Ax_i, \qquad i = 1, 2, \ldots, n.$$

Assume that $\mu_1 \gg \mu_2 \geq \mu_3 \ldots \geq \mu_n > 0$.
Now *let* $z^o \in I\!R^n,\ z^o \neq 0$ be given. Then there exist coefficients $c_i \in I\!R$ such that

$$z^o = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n,$$

and thus

$$\begin{aligned} z^1 &= Az^o = \mu_1 c_1 x_1 + \mu_2 c_2 x_2 + \ldots + \mu_n c_n x_n, \\ &= Az^o = \mu_1 \left(c_1 x_1 + \frac{\mu_2}{\mu_1} c_2 x_2 + \ldots + \frac{\mu_n}{\mu_1} c_n x_n \right), \\ z^\nu &= A^\nu z^o = (\mu_1)^\nu \left(c_1 x_1 + (\frac{\mu_2}{\mu_1})^\nu c_2 x_2 + \ldots + (\frac{\mu_n}{\mu_1})^\nu c_n x_n \right), \qquad \nu = 1, 2, \ldots. \end{aligned} \tag{11}$$

Since $(\mu_i/\mu_1)^\nu$ tend to zero for $i = 2, \ldots, n$ and $\nu \to \infty$, we see that non-zero $z_i^{\nu+1}/z_i^\nu$ converge to μ_1 for $i = 1, \ldots, n$, and that the z^ν are parallel to x_1 for large enough ν.

Now it is obvious that the convergence behavior is changed very much if μ_1 is not positive, simple and dominant. Assume, for instance, that the eigenvalue with largest modulus is negative: then we see from equations (11) that the sequence $\{z^\nu\}_{\nu=o}^\infty$ is alternating. More complicated are the cases $\mu_1 = \mu_2$ or $\mu_1 \approx \mu_2$ or $\mu_n = -\mu_1 < 0$ or .... Even much more complicated patterns are produced by the iterations if the matrix A is real but not symmetric. In that case eigenvalues and eigenvectors may be complex, and real iterations produce real results that are hardly understandable without looking at them in the framework of complex numbers. A very complete, very impressive analysis of all different cases that are possible for non-symmetric matrices was published by *Helmut Wielandt (1910-2001)* [Wie44a] in 1944 (sic!).

Today, a variety of other methods for computing eigenvalues is available. Thus using this elementary iteration scheme does make sense only if it is known in advance that $\mu_1 \gg \mu_i > 0,\ i = 2, \ldots, n$ and if only (μ_1, x_1) are needed. In such cases, this iteration method (called *Power Method, Power Iteration - Potenzmethode* today) is quite efficient.

3.3 Further Development of Names and Methods

As already mentioned, the name first used for the application of this iterative method to matrices was *Iterationsverfahren* [Wie44a, Zu50]. In later years, the names *von-Mises-Iteration*[3] and *Vektoriteration* became common in the German numerical-analysis community, see various textbooks in German, particularly editions 2-4 of Zurmühl's book. In the 5th edition in 1986, the co-author Sigurd Falk of the late Rudolf Zurmühl used the name *Potenziteration nach von Mises* [Zu50]. Each of the seven editions of that book was a revised and enlarged version of the previous one, so the historical development may be observed more closely by comparing the various editions. In English, the names *Power Method* (numerical analysis) and *Power Iterations* (Markov chains) became common, and that probably led to the names *Potenzmethode* and *Potenziteration*.

The changes of the names of the algorithm is only a side-effect and indicates that the computation of eigenpairs is an important problem in many applications. Thus it is not surprizing that the power method was a germ for a rich family of algorithms which are taylored for different patterns of spectra. Already in 1944 (sic!) Helmut Wielandt introduced an algorithm which he called *Gebrochene Iteration* [Wie44b], today mostly called *Inverse Iteration*, both in German and in English. This is essentially the power method applied to the inverse of the matrix. It turns out that this reformulation has big advantages, see subsection 3.4 and the review article by Kerner [Ke89].

Algorithms adapted to matrices with eigenvalues and eigenvectors which are difficult to separate are for instance *Krylov subspace iterations*, *Lanczos iterations* and the *Arnoldi method*. With these methods, the difficult task of separating clustering eigenvalues and their eigenvectors is replaced by computing the subspaces spanned by those eigenvectors, and the sets of the related eigenvalues. For more information, see any modern textbook on matrix computations, for instance the one by Demmel, Dongarra et al [BDD00].

3.4 Gebrochene Iteration – Inverse Iteration

As we have seen in subsection 3.2, the iterations of the Power method converge in general (i.e. if all necessary assumptions are satisfied) to the eigenvalue of largest modulus, and the sequence of vectors is unbounded if $|\mu_1| > 1$. For one of the most important applications, however, for the solution of the stability problem, convergence to the eigenvalue of smallest modulus is needed. This led to the development of *Inverse Iteration*:

[3]This is an example both for the *Matthew-Effect*: 'The more famous person in a group gets all the credit' [Mer68] and of the *Mathilda-Effect*: 'Women are more often not mentioned for their achievements than men' [Ros93].

the Power method for A^{-1}, with normalization of the iterates. As already mentioned, it was introduced by Wielandt under the name *Gebrochene Iteration* [Wie44b]. In today's formulation for real matrices:
Let $A \in I\!R^{n\times n}$ and $z^o \in I\!R^n$ be given; iterate

$$\begin{aligned} z_*^\nu &:= \frac{z^\nu}{||z^\nu||}, \\ Az^{\nu+1} &= z_*^\nu, \qquad \mu^{\nu+1} = \frac{(z^{\nu+1})^t A z^{\nu+1}}{(z^{\nu+1})^t z^{\nu+1}}, \quad \nu = 0,1,2,\ldots \end{aligned} \tag{12}$$

for solving

$$Ax = \mu x, \qquad |\mu| < |\mu_i|,\ i = 2,\ldots n \tag{13}$$

Its big advantage over the power method: if a first guess $\tilde{\mu}$ for the desired eigenvalue μ is known, A may be replaced by the shifted matrix $B = A - \tilde{\mu}E$, where E is the identity matrix. Thus B is nearly singular, and the largest eigenvalue of B^{-1} is huge in comparison to all other eigenvalues. Thus convergence is very fast.

4 Inverse Iteration on Partial Differential Equations

Using a physicists intuition, Lackner [La70] generalized Inverse Iteration for application to nonlinear eigenvalue problems of type

$$Lu = \alpha f(u) \text{ in } D \subset I\!R^2, \qquad u|_{\partial D} = g, \tag{14}$$

where L are a linear differential operator and f a nonlinear operator $\mathcal{B} \to \mathcal{B}$, $\mathcal{B}$ some Banach function space. A simple model problem was

$$-\Delta u = \lambda e^u \text{ in } D \subset I\!R^2, \qquad u|_{\partial D} = 0. \tag{15}$$

The equations depend on several parameters which are not shown here. Treatment of these parameters was a problem: it was not clear which ones should be fixed and which ones should be left free to obtain a formulation that made sense both mathematically and physically and led to convergence of the iterations [La70, p.3184], [HL75, p.141]: such equations may have any number of solutions, depending on outer and inner parameters [MSp81, MSp99].

In the case of eq. (15) Hagenow-Lackner got satisfactory results with:
$M, \alpha^o \in I\!R,\ g, u_*^o \in \mathcal{B}$ given, $||u_*^o|| = M$, iterate

$$Lu^{\nu+1} = \alpha^\nu f(u_*^\nu), \qquad u^{\nu+1}|_{\partial D} = g, \tag{16}$$

$$u_*^{\nu+1} := u^{\nu+1}\frac{M}{||u^{\nu+1}||}, \qquad \alpha^{\nu+1} = \alpha^\nu \frac{M}{||u^{\nu+1}||}, \quad \nu = 0,1,2,\ldots \tag{17}$$

5 Continuation Method

All uncertainties about the treatment of parameters in the iterations were eliminated in a paper by *Herbert B. Keller (1925-2008)* [Ke77]:

Let $\mathcal{B}$ be a Banach space as before, rewrite

$$Lu = \lambda f(u) \text{ in } D \subset I\!R^2, \qquad u|_{\partial D} = g, \tag{18}$$

as $G(u, \lambda) = 0$. Introduce an additional condition on the solutions $N(u, \lambda)$. Make sure that system is solvable by introducing an additional parameter s:

$$G(u(s), \lambda(s)) = 0, \qquad N(u, \lambda, s) = 0. \tag{19}$$

$N(u, \lambda, s) = ||u|| - s$: This is the normalization of eqs. (17) with $s = M$.

$$N(u, \lambda, s) = ||du/ds||^2 + |d\lambda/ds|^2 \tag{20}$$

This condition defines arclength of the solution branch $(u(s), \lambda(s))$ if certain assumptions on eqs. (19) are satisfied. In the existence theorems for solutions of eqs. (19) [Ke77], the norm N is defined according to (20). In practical computations, a numerical approximation to arclength is much more convenient [Ke77, MK80]. The method was applied successfully to various parameter-dependent systems of partial differential equations, see for instance the very involved solution branches displayed in [MSp99], and the references therein.

6 Acknowledgement

The author is very grateful to all who improved this work by their criticism, questions and remarks, and to Christa Binder and Peter Schmitt for organizing the meeting and these proceedings.

References

[BDD00] Z. Bai, J. Demmel, J. Dongarra, A. Ruhe and H. van der Vorst (eds): *Templates for the solution of Algebraic Eigenvalue Problems: A Practical Guide*, SIAM, Philadelphia, 2000

[Eng93] Engesser, Friedrich (1893): Über die Berechnung auf Knickfestigkeit beanspruchter Stäbe aus Schweiß- und Guß-Eisen, Z. öst. Ing.- und Archit.-Ver. (1893), Heft 38, S. 506

[Eng92] Engesser, Friedrich (1892-93): Die Zusatzkräfte und Nebenspannungen eiserner Fachwerkbrücken. Teil 1 1892, Teil 2 1893, Springer Verlag Berlin, Heidelberg GmbH

[Gau08] Gautschi, Walter (2008): Leonhard Euler: His Life, the Man, and His Works, SIAM REVIEW vol. 50, No. 1, pp. 3-33

[HL75] Hagenow, Karl-Ulrich von, Lackner, Karl (1975): Computation of Axisymmetric MHD equilibria. *Proceedings of the Seventh Conference on Numerical Simulation of Plasmas*, Courant Institute, New York University, June 2-4, 1975

[Ke77] Keller, Herbert B. (1977): Numerical solution of bifurcation and nonlinear eigenvalue problems, pp. 359-384 in: Rabinowitz, P.H., ed: *Applications of Bifurcation Theory*, Wiley, New York,N.Y., 1977

[Ke89] Kerner, Wolfgang (1989): Large-scale complex eigenvalue problems, Journal of Computational Physics 85, pp. 1-85

[Ko27] Koch, J.J.: Bestimmung höherer kritischer Drehzahlen schnell laufender Wellen, pp. 213-218 in: Meissner, Ernst (ed.): Verhandlungen des 2. Internationalen Kongresses für Technische Mechanik / Comptes Rendus du 2^{ieme} Congrès Internationale de Mécanique Appliquée / Proceedings of the 2^{ND} International Congress for Applied Mechanics, Zürich 12.-17. September 1926, Orell Füssli Verlag Zürich und Leipzig, 1927

[KoM55] Kollbrunner, Curt F.; Meister, Martin (1955): *Knicken -Theorie und Berechnung von Knickstäben; Knickvorschriften*, Springer Verlag

[La70] Lackner, Karl (1970): Deformation of a magnetic dipole field by trapped particles, Journal of Geophysical Research *75*, 3180-3192

[Mer68] Merton, Robert K. (1968): The Matthew Effect in Science, *Science 149*

[MSp16a] Meyer-Spasche, Rita (2016a): Some remarks on the impact of computers on mathematics and physics, to appear in: *Vom Abakus zum Computer*, G. Wolfschmidt, ed., Proceedings of the Scriba Memorial Meeting – History of Mathematics, Hamburg, 12. - 13. Mai 2015, Nuncius Hamburgensis, Band 21, in preparation

[MSp16b] Meyer-Spasche, Rita (2016b): Hidden Authors, to appear in: *Scriba Memorial Meeting - Geschichte der Mathematik*, G. Wolfschmidt, ed., Proceedings of Tagung der Fachsektion *Geschichte der Mathematik* der *Deutschen Mathematiker-Vereinigung*, Sunderhof, 14.-17. Mai 2015 Nuncius Hamburgensis, Band 36, in preparation

[MSp99] Meyer-Spasche, Rita (1999): *Pattern Formation in Viscous Flows*, Birkhäuser Verlag, ISNM vol. 128, Basel, Boston, Berlin

[MSp81] Meyer-Spasche, Rita (1981): Numerical approximation of boundary value problems having several solutions, pp.97-121 in: *Metodi Matematici nella Dinamica dei Fluidi e dei Gas Ionizzati,* Seminario Fisico Matematico Trieste, Universita di Trieste

[MK80] Meyer-Spasche, Rita, Keller, Herbert B. (1980): 'Computations of the axisymmetric flow between rotating cylinders'. Journal of Computational Physics 35 (1980), 100-109

[MPG29-2] R. v. Mises, H. Pollaczek-Geiringer (1929b): Praktische Verfahren der Gleichungsauflösung, Teil 2, ZAMM *9*, 152-164

[Poh21] Pohlhausen, Ernst: Berechnung der Eigenschwingungen statisch-bestimmter Fachwerke, ZAMM - Zeitschrift für Angewandte Mathematik und Mechanik 1, (1921), 28-42

[Ros93] Rossiter, Margaret W. (1993): The Matthew-Matilda-Effect in Science, Sage Publ., London 23, pp. 325-341

[Sto24] Stodola, Aurel: *Dampf- und Gas-Turbinen*, 5. Auflage 1924,
Steam and Gas Turbines, P. Smith, New York, 1945

[Via05] Vianello, Luigi (1905): *Der Eisenbau*, Verlag Oldenbourg, München, mehrere Auflagen

[Via98] Vianello, Luigi: Graphische Untersuchung der Knickfestigkeit gerader Stäbe, Zeitschrift des Vereins Deutscher Ingenieure vol 42, no 52 (1898), 1436-1443

[Wie44a] Wielandt, Helmut (1944a): Das Iterationsverfahren bei nicht selbstadjungierten Eigenwertaufgaben. Math Z 50, p. 93-143

[Wie44b] Wielandt, Helmut (1944b): Bestimmung höherer Eigenwerte durch gebrochene Iteration, Ber. B44/J/37 der Aerodynamischen Versuchsanstalt Göttingen, 11 pages

[Zu50] Zurmühl, Rudolf: *Matrizen - Eine Darstellung für Ingenieure*, Springer Verlag, 1. Aufl. 1950, 2. Aufl. 1958, 3. Aufl. 1961, 4. Aufl. 1964;
Zurmühl, Rudolf † and Falk, Sigurd: *Matrizen und ihre technischen Anwendungen*, Teil 2: *Numerische Methoden*, 5., überarbeitete und erweiterte Auflage 1986, 6. Aufl. 1992, 7. Aufl. 1997