Conformal Anomaly and Off-Shell Extensions of Gravity

Krzysztof A. Meissner[1] and Hermann Nicolai[2]
[1] *Faculty of Physics, University of Warsaw*
ul. Pasteura 5, 02-093 Warsaw, Poland
[2] *Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut)*
Mühlenberg 1, D-14476 Potsdam, Germany

The gauge dependence of the conformal anomaly for spin-$\frac{3}{2}$ and spin-2 fields in non-conformal supergravities has been a long standing puzzle. In this Letter we argue that the 'correct' gauge choice is the one that follows from requiring all terms that would imply a violation of the Wess-Zumino consistency condition to be absent in the counterterm, because otherwise the usual link between the anomaly and the one-loop divergence becomes invalid. Remarkably, the 'good' choice of gauge is the one that confirms our previous result [1] that a complete cancellation of conformal anomalies in $D = 4$ can only be achieved for N-extended (Poincaré) supergravities with $N \geqslant 5$.

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1. Introduction. Conformal anomalies have been a subject of investigation for a long time [2–11]. While their relevance is obvious for theories that are manifestly classically conformal, they are equally relevant for *non*-conformal theories, in the same way that gauge anomalies are relevant even if gauge invariance is spontaneously broken. While the anomaly coefficients are unambiguous for spin $s \leqslant 1$ fields, it has long been known that in the context of (non-conformal) Poincaré or AdS supergravities the anomaly coefficients for spin-$\frac{3}{2}$ and spin-2 fields suffer from an apparent 'gauge-dependence', in the sense that they depend on how the theory is taken off the mass shell (defined in lowest order by $R_{\mu\nu} = 0$) [5, 6]. This is a puzzling feature, because the anomaly, being a physical quantity, should *not* depend on the choice of gauge, unlike the divergent counterterms that must be introduced to render the results finite. This is in marked contrast to gauge anomalies, where such a gauge dependence is not observed, a feature that is known to be absolutely crucial for the consistency of the Standard Model of particle physics. In this Letter we propose a simple prescription to resolve this puzzle and to arrive at a unique and physical answer also for conformal anomalies.

We refer to the papers cited above for reviews and basic results on conformal anomalies. We assume that there exists a (non-local) effective action functional $\Gamma_{\rm eff}$, depending on the metric $g_{\mu\nu}$ and various matter fields, such that the conformal anomaly $\mathcal{A}$ can be represented as the variational derivative

$$T^\mu{}_\mu(x) \equiv -\frac{1}{\sqrt{-g(x)}}\frac{\delta\Gamma_{\rm eff}}{\delta\sigma(x)} = \mathcal{A}(x) + \cdots \qquad (1)$$

where $\sigma(x) \equiv \sqrt{-g(x)}$ is the conformal factor. The dots on the r.h.s. stand for non-gravitational contributions due to matter interactions, or due to explicit or spontaneous breaking of conformal invariance (such as explicit mass terms $\propto m^2\varphi^2$) that are not relevant to our argument. We emphasize that (1) remains a perfectly valid definition of the trace of the quantized energy momentum tensor also for *non*-conformal theories. As a direct consequence of (1) the trace, and thus the anomaly, must satisfy the integrability (*alias* Wess-Zumino) consistency condition

$$\frac{\delta\big(\sqrt{-g}\mathcal{A}(x)\big)}{\delta\sigma(y)} = \frac{\delta\big(\sqrt{-g}\mathcal{A}(y)\big)}{\delta\sigma(x)} \qquad (2)$$

independently of whether we are dealing with a conformal or a non-conformal theory.

As is well known (see for instance [8]) the gravitational part of the conformal anomaly in four dimensional spacetime takes the form

$$\mathcal{A}(x) = \frac{1}{180\cdot 16\pi^2}\Big(a\,{\rm E}_4(x) + c\,C^2(x)\Big) \qquad (3)$$

where

$$C^2 \equiv C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$
$${\rm E}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \qquad (4)$$

and the coefficients a and c depend on the type of field coupling to gravity that induces the anomaly. ${\rm E}_4$ is the Gauss-Bonnet density, which is a total derivative. In writing (3) we omit a possible further term $\propto \Box R$ which can be removed by a local counterterm ($\propto R^2$). It is straightforward to check that the two terms displayed in (3) do satisfy the consistency condition (2), whereas the square of the scalar curvature R^2 by itself does not. Importantly, possible extra terms indicated by dots in (1), and more specifically any terms arising for non-conformal theories, must also satisfy (2).

Existing calculations of the conformal anomaly rely on the relation between the conformal anomaly and the one-loop divergence. This was in particular explained in [3] in the framework of dimensional regularization, where, however, spin $s > 1$ fields were not considered. The divergence can be generally represented in the form

$$\Gamma_\infty = \frac{1}{16\pi^2}\int {\rm d}^4x\sqrt{-g}\,\mathcal{L}_\infty(x) \qquad (5)$$

where [6]

$$\mathcal{L}_\infty \;=\; b_0 L^4 + b_2 L^2 + b_4 \ln(L^2/\mu^2) \;+\; \dots \tag{6}$$

with the UV cutoff L, and where the dots stand for finite (non-local) contributions. The terms b_0 and b_2 are removed by appropriate counterterms, while the coefficient of the logarithmic divergence contains the physically relevant information. More precisely, using the notation from [6] which we follow throughout,

$$b_4 = \beta_1 \mathrm{E}_4 + \frac{\beta_2}{2}(C^2 - \mathrm{E}_4) + \frac{\beta_3}{3} R^2 + \cdots \tag{7}$$

where the ellipses denote further terms *e.g.* involving a cosmological constant, as well as possible matter field contributions. In the context of supergravity the coefficients β_i were computed in [6], building on earlier work in [5]. It was also pointed out there that the counterterm coefficients β_2 and β_3 depend on the gauge choice, while β_1 does not.

The link between $\mathcal{L}_\infty$ and the anomaly $\mathcal{A}$ is encapsulated in the relation

$$\mathcal{A}(x) \;\propto\; \lim_{L\to\infty} \left[L \frac{d}{dL}\big(\sqrt{-g}\mathcal{L}_\infty(x)\big)\right]_{\text{F.P.}} \tag{8}$$

where "F.P." denotes the finite part (after subtraction of quartic and quadratic infinities). The formula (8) captures the anomalous scale dependence of the theory in presence of a (scale breaking) cutoff L (and incidentally also explains the appearance of β-functions on the r.h.s. of (1) in the presence of matter interactions). However it is clear that the relation $b_4 = \mathcal{A}$ cannot hold if $\beta_3 \neq 0$ because β_3 multiplies the term R^2 which does not satisfy (2). Because β_3 is gauge dependent our proposal is therefore *to identify the 'good' gauge as the one where $\beta_3 = 0$, because only then do we obtain a consistent anomaly.*

Let us now examine existing results in view of the potential discrepancy for $\beta_3 \neq 0$. For low spins $s \leqslant 1$ the anomaly coefficients have been known for a long time [5, 6, 12–16], and there is no issue (and hence no gauge dependence) here, as the relevant actions are conformal also for non-conformal supergravities. For all of these one finds $\beta_3 = 0$. However, for spins $s \geqslant \frac{3}{2}$ there are two different possible actions. One is conformal (super-)gravity, with kinetic terms of higher order (cubic for spin-$\frac{3}{2}$ and fourth order for spin-2), whereas the non-conformal supergravity actions are of first and second order, respectively, just like the actions for spin $s \leqslant 1$. Higher derivative actions for higher spin fields have been investigated recently in [11], and for them (and thus for conformal supergravities) one gets again $\beta_3 = 0$, whence the relation (8) is preserved. For those theories it is found that a complete cancellation of conformal anomalies is possible only for the maximal $N = 4$ conformal supergravity coupled to an $N = 4$ Yang-Mills gauge theory with gauge group $SU(2) \times U(1)$ or $U(1)^4$ [7].

For non-conformal (Poincaré or AdS) (super)gravities the two explicit calculations with different off-shell formulations available in the literature give different results for spins $s \geqslant \frac{3}{2}$, including ones with non-vanishing β_3 [6]. In the Table we show the relevant coefficients together with the values for β_3 for the two different gauges, namely the Feynman gauge [6] and the harmonic gauge [11], respectively. As already mentioned, there is no ambiguity for spins $s \leqslant 1$, and $\beta_3 = 0$. By contrast, for $s \geqslant \frac{3}{2}$ the numbers do differ. In the first three columns of Table 1 we give the coefficients as extracted from [11], cf. appendix of [1]; for these β_3 vanishes for all spins. The coefficients in the remaining columns of the Table are based on Table 1 of [6], yielding $\beta_3 = -\frac{1}{3}$ for spin-$\frac{3}{2}$, and $\beta_0 = \frac{3}{4}$ for spin-2. We thus see that only the a and c coefficients from the first two columns are fully consistent with (2), whereas the ones from the fourth and fifth columns are not.

	[11]			[6]		
	c_s	a_s	β_3	c_s	a_s	β_3
0	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$	$-\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{9}{2}$	$-\frac{11}{4}$	0	$\frac{9}{2}$	$-\frac{11}{4}$	0
1	18	-31	0	18	-31	0
$\frac{3}{2}$	$-\frac{411}{2}$	$\frac{589}{4}$	0	$-\frac{231}{2}$	$\frac{229}{4}$	$-\frac{1}{3}$
2	783	-571	0	63	149	$\frac{3}{4}$

Table 1. Anomaly related coefficients for two gauges.

Remarkably, the cancellations exhibited in [1], to wit,

$$\begin{aligned} c_2 + 5c_{\frac{3}{2}} + 10c_1 + 11c_{\frac{1}{2}} + 10c_0 \;&=\; 0 \\ c_2 + 6c_{\frac{3}{2}} + 16c_1 + 26c_{\frac{1}{2}} + 30c_0 \;&=\; 0\,, \\ c_2 + 8c_{\frac{3}{2}} + 28c_1 + 56c_{\frac{1}{2}} + 70c_0 \;&=\; 0\,, \end{aligned} \tag{9}$$

work only with the numbers from [11], which are the ones yielding a consistent anomaly. Inspecting the differences in the coefficients for $s \geqslant \frac{3}{2}$, *viz.*

$$\Delta a_{3/2} = -\Delta c_{3/2} = -90\,, \quad \Delta a_2 = -\Delta c_2 = 720 \tag{10}$$

(which are in accord with the fact that only the sum $(a + c)$ is gauge invariant [4, 6]) we see that the cancellation persists for $N = 8$ supergravity (as already noted in [6]), but fails for lower N. From the present point of view, however, this is just an accident and the cancellations might not work for yet different ways of taking the theory off shell. Further clarification of the ambiguities could come from a Feynman diagram computation of a three-point correlator of energy-momentum tensors (as opposed to the determination of the conformal anomaly from the one-loop divergence, as in previous work), analogous to the textbook derivation of the axial anomaly. To the best of our knowledge such a calculation, which could exhibit the dependence of β_3 on a continuous gauge parameter, is currently not available.

The phenomenon that we describe is similar in spirit to the one discovered in [17]. There is a continuous family of α' actions of the string gravity (gravity + dilaton + antisymmetric tensor) that can be changed by the lower order equations of motion but only one action exhibits $O(d,d)$ symmetry [18] and this action was proposed in [17] as the 'correct' one.

We consider the cancellations in (9) as indicative of a hidden conformal structure of unknown type underlying these theories (and possibly M theory). In [1] we furthermore suggested a link between these cancellations and the cancellation of composite anomalies in extended supergravities [19], as well as the so far unexplained finiteness of $N \geqslant 5$ supergravities [20]; further evidence for such a link has been exhibited in very recent work [21, 22].

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[1] K.A. Meissner and H. Nicolai, arXiv:1607.07312[gr-qc].
[2] S. Deser, M.J. Duff and C. Isham, Nucl. Phys. **B111** (1976) 45.
[3] M.J. Duff, Nucl. Phys. **B125**(1977) 334
[4] S.M. Christensen and M.J. Duff, Phys. Lett. **76B** (1978) 571.
[5] S. M. Christensen and M. J. Duff, Nucl. Phys. **B154** (1979) 301.
[6] E. Fradkin and A. Tseytlin, Phys. Lett. **B117** (1982) 303.
[7] E. Fradkin and A. Tseytlin, Phys. Lett. **B134** (1984) 187.
[8] S. Deser and A. Schwimmer, Phys. Lett. **B309** (1993) 279
[9] A.O. Barvinsky, Y.V. Gusev, G.A. Vilkovisky and V.V. Zhytnikov, Nucl. Phys **B439** (1995) 561.
[10] J. Erdmenger and H. Osborn, Nucl. Phys. **B483** (1997) 431.
[11] A.A Tseytlin, Nucl. Phys. **B877** (2013) 598.
[12] L. Parker, in *Recent Developments in Gravitation*, Cargèse 1978.
[13] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rept. **66** (1980) 213.
[14] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, 1982.
[15] D. Vassilevich, Phys.Rept. **388** (2003) 279
[16] F. Bastianelli and P. van Nieuwenhuizen, *Path Integrals and Anomalies in Curved Space*, Cambridge University Press (2009).
[17] K.A. Meissner, Phys.Lett. **B392** (1997) 298.
[18] K.A. Meissner and G. Veneziano, Phys. Lett. **B267** (1991) 33.
[19] N. Marcus, Phys. Lett. **B157** (1985) 383.
[20] J.J.M. Carrasco, R. Kallosh, R. Roiban and A.A. Tseytlin, JHEP **1307** (2013) 029.
[21] R. Kallosh, Phys. Rev. **D95** (2017) 041701.
[22] D.Z. Freedman, R. Kallosh, D. Murli, A. Van Proeyen and Y. Yamada, arXiv:1703.03879 [hep-th].