No smooth beginning for spacetime

Job Feldbrugge,[1,*] Jean-Luc Lehners,[2,†] and Neil Turok[1,‡]

[1] *Perimeter Institute, 31 Caroline St N, Ontario, Canada*

[2] *Max–Planck–Institute for Gravitational Physics (Albert–Einstein–Institute), 14476 Potsdam, Germany*

We identify a fundamental obstruction to any theory of the beginning of the universe as a semi-classical quantum process, describable using complex, smooth solutions of the classical Einstein equations. The no boundary and tunneling proposals are examples of such theories. We argue that the Lorentzian path integral for quantum cosmology is meaningful and, with the use of Picard-Lefschetz theory, provides a consistent definition of the semi-classical expansion. Framed in this way, the no boundary and tunneling proposals become identified, and the resulting framework is unique. Unfortunately, the Picard-Lefschetz approach shows that the primordial fluctuations are out of control: larger fluctuations receive a higher quantum mechanical weighting. We prove a general theorem to this effect, in a wide class of theories. A semi-classical description of the beginning of the universe as a regular tunneling event thus appears to be untenable.

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Quantum gravity is supposed to provide a quantum description of spacetime. If so, it should address basic questions like: how did classical spacetime emerge? or even, how did the universe begin? Vilenkin's tunneling from nothing proposal [1] and Hartle and Hawking's no boundary proposal [2–4] represent two important attempts in this direction. Both are most naturally formulated in terms of sums over spacetime geometries, taken to have a certain form. However, the lack of a consistent definition for the Feynman path integral for quantum gravity has significantly hampered attempts to extract clear predictions from either proposal.

Hartle and Hawking take the Euclidean path integral for quantum gravity to be fundamental and attempt to describe the beginning of the universe using a complex, smooth, saddle point solution of the Einstein equations. In their picture, the real, Lorentzian universe is obtained as an analytic continuation from a compact Euclidean geometry. The big bang singularity is avoided because the Euclidean region closes off smoothly in the past (left panel, Fig. 1). Vilenkin in contrast adopted an entirely Lorentzian picture, in which the universe begins on a three-geometry of size zero, *i.e.*, a point. He assumed, without further justification, that there would be no freedom to input boundary data there.

In a previous paper [5] we showed that, in simple cosmological models, the Lorentzian path integral for quantum gravity is a meaningful, convergent quantity whereas the Euclidean version is not. Instead of performing a Wick rotation of a timelike coordinate, the convergence of the path integral is improved by analytically continuing in the fields. Picard-Lefschetz theory allows one to unambiguously identify the contributing saddle points and integration contours, thereby defining a consistent semiclassical expansion for the full quantum propagator.

We believe Picard-Lefschetz theory provides the most conservative and minimal approach to semiclassical quantum gravity, possessing considerable advantages over other approaches. This being the case, *any* proposal for the initial conditions of the universe, such as the tunneling or no boundary proposals, should be possible to formulate within the Picard-Lefschetz approach. Like Vilenkin, our starting point is the Lorentzian theory. However, like Hartle and Hawking, we demand that any contributing saddle point must be a smooth (albeit complex) solution of the Einstein equations. This condition is equally necessary, we believe, in Vilenkin's picture in order to avoid additional boundary data entering on the initial three-geometry. Remarkably, then, in the Picard-Lefschetz approach the tunneling and no boundary proposals become equivalent. In [5], we showed that implementing regularity *à la* Hartle and Hawking, results in a negative real part of the semiclassical exponent, disagreeing with them but in agreement with Vilenkin.

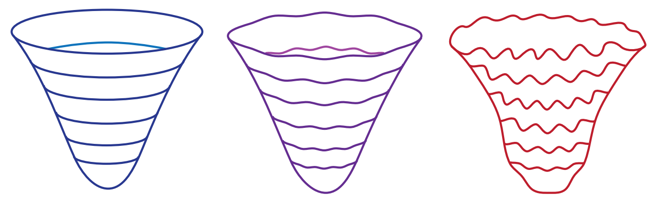

FIG. 1. *Left:* the smooth, regular picture of the no boundary background. *Middle:* the no boundary picture with hoped-for small fluctuations, in agreement with observations. *Right:* the fluctuations implied by the more rigorous, Lorentzian-Picard-Lefschetz approach developed here. Our analysis shows that, to leading semiclassical order, large fluctuations are preferred, leading to a breakdown of the theory.

In this Letter, we extend the discussion to linear fluctuations around such a background, and to slow-roll inflationary models. Contrary to the small perturbations hoped for in both proposals, we find large fluctuations are preferred. The middle and right panels in Fig. 1 are increasingly accurate descriptions of the no boundary or tunneling proposal. Perturbation theory and the normalizability of the perturbations break down, making a smooth, semi-classical beginning of spacetime impossible.

* jfeldbrugge@perimeterinstitute.ca

† jlehners@aei.mpg.de

‡ nturok@perimeterinstitute.ca

There is a basic conundrum at the heart of quantum cosmology, whose resolution underlies our main claim. The problem is that the scale factor of the universe (the so-called conformal factor) has a *negative* kinetic term, unlike all other bosonic fields. This simple, but fundamental fact prevents one from Wick rotating time so that the phase factor $e^{iS/\hbar}$ appearing in Lorentzian path integrals becomes a real suppression factor $e^{-S_E/\hbar}$ for all bosonic fields. Hence, our approach is to perform no Wick rotation at all, but instead use Picard-Lefschetz theory to make sense of the Lorentzian path integral as it stands. In doing so, we uncover an important subtlety. For the simplest case of quantum de Sitter spacetime, the relevant cosmological background is a round Euclidean four-sphere, as Hartle and Hawking claimed, but obtained from de Sitter via the conjugate continuation [5]. This inverts the semiclassical weighting factor, with the physically appealing consequence that large, rather than small, universes are suppressed.

However, as it turns out things are not so pleasant for the perturbations. Within the no boundary picture, the quantum amplitude for linearized perturbations is fixed by the complex, classical solution which equals a given perturbation amplitude on the final three-geometry and is regular on the past background four-geometry. Since the perturbation action is quadratic, the functional determinant is independent of the final perturbation amplitude. The Picard-Lefschetz construction ensures the convergence of the path integrals and determines the prefactor uniquely. However, as a result of the complex conjugate nature of the background solution already mentioned, arising from the negative sign of the scale factor's kinetic term, the final expression for the probability distribution of perturbations takes the form of an inverse Gaussian in the final perturbation. Hence, the distribution is unbounded and the perturbations are out of control. At the end of the paper we give a general topological argument indicating that this conclusion is unavoidable.

To set the stage, we briefly review the path integral computation of perturbations in the flat slicing of a *classical* de Sitter background. The line element is $a^2(\eta)(-d\eta^2 + d\vec{x}^2)$ with $a(\eta) = -1/(H\eta)$, (constant) Hubble parameter H and conformal time $-\infty < \eta < 0$. The Fourier modes of the perturbations decouple and can be treated independently. The quadratic action for a perturbation mode ϕ – for example, a gravitational wave – of wavenumber k takes the form $S^{(2)}_{0,1} = \frac{1}{2}\int_{\eta_0}^{\eta_1} \mathrm{d}\eta\, a^2(\eta)\left[(\phi_{,\eta})^2 - k^2\phi^2\right]$, with η_0 the initial and η_1 the final conformal time. We assume $|k\,\eta_0| \gg 1$ so that the perturbations start out in the local adiabatic vacuum at some early time η_0. For simplicity, we take $\eta_1 \to 0^-$, so the mode ends up frozen, with its physical wavelength far outside the Hubble radius. The amplitude for a final perturbation ϕ_1 is then given by

$$G^{(2)}_\phi[\phi_1] = \int \mathcal{D}\phi\, e^{iS^{(2)}_{0,1}[\phi]/\hbar - \frac{1}{2}ka_0^2\phi_0^2/\hbar}\,,$$

where the action $S^{(2)}_{0,1}$ incorporates the boundary conditions $\phi(\eta_{0,1}) = \phi_{0,1}$, and the functional measure includes an integral over ϕ_0. The second factor represents the initial (assumed) adiabatic ground state wavefunction.

The functional integral is Gaussian so the saddle point approximation is exact. Stationarizing with respect to ϕ_0 and using the Hamilton-Jacobi equation $\partial S^{(2)}_{0,1}/\partial\phi_0 = -\pi_\phi(\eta_0) = -a^2\phi_{,\eta}(\eta_0)$, we find the saddle point solution to be "negative frequency" at early times. Solving the perturbation equation $\phi_{,\eta\eta} - (2/\eta)\phi_{,\eta} + k^2\phi = 0$, with the given boundary conditions, the classical solution is $\phi \approx \phi_1 e^{ik\eta}(1 - ik\eta)$. Evaluating the semiclassical exponent and carefully taking the limit $\eta_1 \to 0^-$, we find

$$G^{(2)}_\phi[\phi_1] \propto e^{-\frac{k^3}{2H^2}\phi_1^2 + i\frac{k^2}{2H^2\eta_1}\phi_1^2}\,. \tag{1}$$

The probability density is determined by the modulus squared of the amplitude. The divergent phase (which physically represents the final momentum of the mode) disappears and we recover the familiar result of a scale-invariant power spectrum for ϕ_1.

The same result can be obtained by analytic continuation from the Euclidean theory. First, we Weyl transform the line element to flat space, and ϕ to $\chi = a\phi$. After an integration by parts, the Lorentzian action becomes $S^{(2)}_{0,1} = \frac{1}{2}\int_{\eta_0}^{\eta_1}\mathrm{d}\eta\left[(\chi_{,\eta})^2 - (k^2 - 2/\eta^2)\chi^2\right]$. Now we pass to Euclidean time $X \equiv i\eta$ and $S_E \equiv -iS$, obtaining $S_E = \frac{1}{2}\int_{X_0}^{X_1}\mathrm{d}X\left[(\chi')^2 + (k^2 + 2/X^2)\chi^2\right]$, with $' \equiv d/dX$, *i.e.*, a positive Euclidean action. We compute $G_\chi[\chi[X_1]]$ from the Euclidean path integral over χ. Again, we seek a classical saddle point solution. Finiteness of S_E imposes regularity at $X \to -\infty$, automatically selecting the ground state wavefunction. The desired classical solution is $\chi(X) = \chi_1 f(X)/f(X_1)$, with $f(X) = e^{kX}(1/X - k)$. The on-shell action is $S_E(X_1) = \frac{1}{2}\chi\chi'(X_1) = \frac{1}{2}\chi_1^2 f'(X_1)/f(X_1)$. We continue back to Lorentzian time by setting $X_1 = i\eta_1$. Taking the limit $\eta_1 \to 0^-$ again yields (1), with an additional phase generated from the change of variables from ϕ to χ.

Let us now turn to a consistent semiclassical treatment of *both* the background and the perturbations in the no boundary proposal, in order to understand why this fails to yield the standard results just explained. We assume a homogeneous and isotropic, closed background cosmology: $ds^2 = -N_p(t_p)^2\mathrm{d}t_p^2 + a(t_p)^2\mathrm{d}\Omega_3^2$, with lapse function N_p, scale factor $a(t_p)$ and unit 3-sphere metric $\mathrm{d}\Omega_3^2$. The time t_p is the physical time if N_p is set to unity. We assume the only matter is a positive cosmological constant Λ. The Einstein-Hilbert action for the background is

$$S^{(0)}_{0,1} = 2\pi^2\int_0^1\left[-3a\frac{a_{,t_p}^2}{N_p} + N_p(3a - a^3\Lambda)\right]\mathrm{d}t_p\,,$$

where we have set $8\pi G = 1$. The quantum propagator to evolve from $a(0) = a_0$ to $a(1) = a_1$ is [6, 7]

$$G^{(0)}[a_1; a_0] = \int_{0^+}^{\infty}\mathrm{d}N\int\mathcal{D}a\, e^{iS^{(0)}_{0,1}[a,N]/\hbar}\,.$$

Re-defining the lapse and the time coordinate via $N_p\,\mathrm{d}t_p \equiv (N\,\mathrm{d}t)/a$ renders the action quadratic in $q \equiv a^2$,

$$S^{(0)} = 2\pi^2 \int_0^1 \left[-\frac{3}{4N}\dot{q}^2 + N(3-\Lambda q)\right]\mathrm{d}t\,. \qquad (2)$$

The path integral over q can now be performed exactly[1]. The classical solution satisfying $q(0)=q_0$, $q(1)=q_1$ is

$$q(t) = \frac{\Lambda}{3}N^2t^2 + \left[q_1 - \frac{\Lambda}{3}N^2\right]t + q_0.$$

The no boundary condition is implemented by specifying $q_0 = 0$ and requiring the complex classical solution to be a regular, locally Euclidean metric there, *i.e.*, that near $t=0$ one can choose coordinates in which $ds^2 \approx \mathrm{d}\sigma^2 + \sigma^2\mathrm{d}\Omega_3^2$. The propagator reduces to:

$$G^{(0)}[q_1;q_0] = \sqrt{\frac{3\pi i}{2\hbar}}\int_{0^+}^{\infty}\frac{\mathrm{d}N}{N^{1/2}}e^{2\pi^2 iS^{(0)}[q_1;q_0;N]/\hbar};$$
$$S^{(0)}[q_1;q_0;N] = N^3\,\frac{\Lambda^2}{36} + N(3-\Lambda\,q_+) - \frac{3q_-^2}{N}\,,$$

with $q_\pm \equiv (q_1 \pm q_0)/2$. This oscillatory integral is then evaluated by deforming the integration contour into the complex N-plane, using Picard-Lefschetz theory [9, 10] to identify the relevant saddle points, and evaluating the integral in a semiclassical expansion.

The on-shell background action $S^{(0)}[q_1;0;N]$ has four saddle points, each located in a different quadrant of the complex N-plane. The relevant saddle is located at

$$N_s = \frac{3}{\Lambda}\left(i + \sqrt{\frac{\Lambda}{3}q_1 - 1}\right),$$

yielding for the no boundary propagator

$$G[q_1;0] \propto e^{-\frac{12\pi^2}{\Lambda} - i4\pi^2\sqrt{\frac{\Lambda}{3}}\left(q_1 - \frac{3}{\Lambda}\right)^{3/2}}\,.$$

As discussed in [5], Picard-Lefschetz theory implies semiclassical suppression $|G[q_1;0]| < 1$, in agreement with Vilenkin but not with Hartle and Hawking.

We have performed the analogous calculation with a slow-roll inflaton field φ whose potential is well-approximated by $V(\varphi) \approx \Lambda - \frac{1}{2}m^2\varphi^2$ near $\varphi = 0$. We find that, as one would naively expect, for small φ_1,

$$G[q_1,\varphi_1;0,0] \propto e^{-\frac{12\pi^2}{V(\varphi_1)}} \times \textit{phase}$$

so there is a higher weighting for a larger initial potential energy $V(\varphi)$ [11] . Given that the radius of the universe is approximately $\sqrt{3/V(\varphi)}$ when space and time become classical, this supports the intuition that it is easier to nucleate a small rather than a large universe.

[1] Modulo issues regarding operator ordering and the path integral measure, and the restriction $q \geq 0$, further discussed in [5, 8].

The same results can be obtained in physical time t_p using the correspondence

$$\sinh(Ht_p) = H^2\,N\,t - i\,, \qquad (3)$$

where we define $H = \sqrt{\frac{\Lambda}{3}}$ and $a(t_p) = \frac{1}{H}\cosh(Ht_p)$. The no boundary point $t=0$ corresponds to $Ht_p = -\frac{\pi}{2}i$.

Let us now extend our analysis to include perturbations – for example, gravitational waves – treated at leading (linearized) order. The full propagator is

$$G[q_1,\phi_1;q_0,\phi_0] = \int_{0^+}^{\infty}\mathrm{d}N\int\mathcal{D}q\int\mathcal{D}\phi\,e^{iS[q,\phi,N]/\hbar}\,,$$

where the action $S = S^{(0)} + S^{(2)}$ and the boundary conditions are implicit. In the q coordinates the perturbation action for gravitational waves reads

$$S^{(2)} = \frac{1}{2}\int N_s\mathrm{d}t\,\mathrm{d}^3x\left[q^2\left(\frac{\dot{\phi}}{N_s}\right)^2 - l(l+2)\phi^2\right],$$

with l the principal quantum number on the 3-sphere. For tensor perturbations, $l \geq 2$. (For more general situations, one may also have scalar or vector perturbations, with $l \geq 0$ and $l \geq 1$ respectively: see, *e.g.*, Ref. [12]).

The path integral over the perturbations is again quadratic, so the saddle point approximation gives the ϕ_1 dependence exactly. The equation of motion for ϕ is $\ddot{\phi} + 2\frac{\dot{q}}{q}\dot{\phi} + \frac{N_s^2}{q^2}l(l+2)\phi = 0$, where we use the saddle point N_s of the background, neglecting backreaction. The solution satisfying $\phi_0 = 0$ is $\phi(t) = \phi_1 F(t)/F(1)$, with

$$F(t) = \left(1 + \frac{i}{H^2N_st - i}\right)^{\frac{l}{2}}\left(1 - \frac{i}{H^2N_st - i}\right)^{-\frac{l+2}{2}} \times \left(1 - \frac{i(l+1)}{H^2N_st - i}\right). \qquad (4)$$

Note $\phi(t) \propto t^{\frac{l}{2}}$ as $t \to 0$, implying ϕ is regular there.

The classical action for the perturbations is given by a surface term on the final boundary,

$$S^{(2)}[q_1,\phi_1] = \frac{1}{2}\int\mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{q^2}{N_s}\phi\dot{\phi}\right] = \frac{q_1^2}{2N_s}\phi_1^2\frac{\dot{F}(1)}{F(1)}$$
$$= \frac{\phi_1^2}{2}\left[-i\frac{l(l+2)}{H}\sqrt{q_1} + \frac{l(l+1)(l+2))}{H^2} + \mathcal{O}\left(\frac{1}{\sqrt{q_1}}\right)\right].$$

The full propagator for the perturbed background factorizes at this order $G[q_1,\phi_1;0,0] = G[q_1;0]G_\phi[\phi_1;0]$, with

$$G_\phi[\phi_1;0] \propto e^{\frac{l(l+1)(l+2)}{2\hbar H^2}\phi_1^2} \times \textit{phase}$$

corresponding to an *inverse* Gaussian distribution. A similar result is obtained for the curvature perturbations in slow-roll inflation [11].

In order to compare our results with the Bunch-Davies vacuum, we convert (4) to conformal time $\mathrm{d}\eta = \mathrm{d}t_p/a$.

The physical time and the conformal time are related by $\tan\left(\frac{\pi}{4}+\frac{\eta}{2}\right) = \tanh\left(\frac{Ht_p}{2}\right)$, where $-\infty < t_p < \infty$ corresponds to $-\pi < \eta < 0$. Thus, as $\eta \to 0$,

$$\sinh(Ht_p) = 2\frac{\tan(\frac{\pi}{4}+\frac{\eta}{2})}{1-\tan^2(\frac{\pi}{4}+\frac{\eta}{2})} \to -\frac{1}{\eta} + \frac{\eta}{3} + \frac{\eta^3}{45} + \ldots$$

which, using (3), leads to the late time approximation

$$\phi = \phi_1 \left[1 + \frac{1}{2}l(l+2)\eta^2 - \frac{i}{3}l(l+1)(l+2)\eta^3 + \ldots\right].$$

This is the late time expansion of the "positive frequency" mode function, confirming that the no boundary condition selects the "wrong" mode function as compared to the adiabatic ground state.

Having demonstrated our claim that the perturbations are out of control in the no boundary description of quantum de Sitter spacetime, we would like to establish how general the result is. To begin with, we shall consider a fluid more general than a cosmological constant, but which is still "adiabatic", namely, the background pressure P is a function of the energy density ρ so that there is a unique cosmological history parameterized by the scale factor a. Furthermore, we assume this classical evolution results in a smooth "bounce" of the scale factor such as occurs in the closed slicing of de Sitter spacetime.

From our discussion above, it is clear that the on-shell classical action is all that is needed to determine the semiclassical exponent in the quantum propagator both for the background and for the perturbations. In the no boundary solutions, $q = a^2$ runs from $q_0 = 0$ to q_1, a positive value. Thus q itself may be used as a time coordinate. The Friedmann constraint allows us to express the background line element as

$$ds^2 = -\frac{\mathrm{d}q^2}{4q\left(\frac{1}{3}\rho(q)q - 1\right)} + q\,\mathrm{d}\Omega_3^2, \tag{5}$$

where we allow the energy density $\rho(q)$ to vary with q.

Cauchy's theorem enables us to deform the time (or q) contour upon which we evaluate the classical action as long as it does not cross any singularity. In particular, we can deform it to one in which q is real everywhere. The line element (5) is Lorentzian for $q > 3/\rho(q)$ but Euclidean for $0 \le q < 3/\rho(q)$, and is easily checked to be regular at $q = 0$. At $q = 3/\rho(q)$, where $q = q_B$, the real, Lorentzian solution "bounces," and q therefore ceases to be a single-valued time coordinate. Our complex saddle point solution (3) passes *below* this point in the complex q-plane: it is precisely this topological fact which results in the suppression of the semiclassical amplitude, required by Picard-Lefschetz theory [5]. Using the Friedmann constraint, the classical action (2) gives $iS^{(0)} = -6\pi^2 i\int \mathrm{d}q\sqrt{\rho\,q/3-1}$. Since we start in the Lorentzian region we take the branch cut to run leftwards from the point q_B, the classical "bounce." Continuing the q integral below the branch cut to $q = 0$,

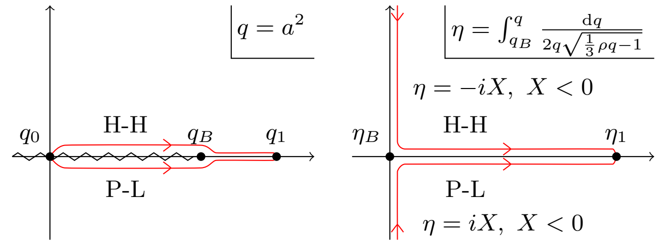

FIG. 2. *Left:* Analytic continuation contours (red) in the Hartle-Hawking (H-H) and Picard-Lefschetz (P-L) descriptions, above and below the branch cut in the complex q-plane. *Right:* Corresponding contours for the conformal time.

we obtain for the real part of the semiclassical exponent $-6\pi^2\int_0^{q_B}\sqrt{1-\rho\,q/3}$. For a cosmological constant $\rho(q) = \Lambda$, we obtain $-12\pi^2/\Lambda$. Continuing above the branch cut yields $+12\pi^2/\Lambda$, Hartle and Hawking's result, which is inconsistent with Picard-Lefschetz theory.

To analyze the perturbations, we pass to coordinates in which the metric is conformally static: for $q > q_B$, we set $d\eta = dq/(2q\sqrt{\rho\,q/3-1})$ to obtain the line element $q(\eta)(-d\eta^2 + \mathrm{d}\Omega_3^2)$. We take $\eta = 0$ to correspond to the "bounce," so η is positive in the Lorentzian region. Now, when q passes below the branch cut commencing at q_B, the square root in the definition of $d\eta$ means that η continues from the positive real η-axis onto the negative imaginary η-axis, $\eta = iX$ with $X < 0$ in the Euclidean region. Conversely, following X forward from the Euclidean region, it "turns right" into the Lorentzian region, whereas in the usual Wick rotation, assumed by Hartle and Hawking, it "turns left" (see Fig. 2). Taking the continuation implied by Picard-Lefschetz theory for the background, the Euclidean action for the perturbations has the "wrong" sign. We can still impose regularity of the modes in the Euclidean region, but the resulting semiclassical weighting factor will inherit the wrong sign.

As in our earlier discussion, it is convenient to go to a Weyl frame in which the kinetic terms are canonical. So we set $\phi = \chi/a$, obtaining for the Lorentzian action

$$iS^{(2)} = i\pi^2 \int d\eta \left[(\chi_{,\eta})^2 + \frac{a_{,\eta\eta}}{a}\chi^2 - l(l+2)\chi^2\right]. \tag{6}$$

The background equations imply that $a_{,\eta\eta}/a = \frac{1}{2}(\frac{1}{3} - w)\rho\,a^2 - 1$, where $w = P/\rho$ is the equation of state. Analytically continuing η back into the Euclidean region and then on to $q = 0$ (corresponding to $X = -\infty$), as explained above, we obtain the Euclidean action

$$-S_E^{(2)} = \pi^2 \int_{-\infty}^{0} dX \left[\chi'^2 + U(X)\chi^2\right], \tag{7}$$

where $\chi' \equiv d\chi/dX$ and $U(X) \equiv l(l+2) + 1 + \frac{1}{2}q\,(w_E - \frac{1}{3})\rho_E$. Here, w_E and ρ_E are the analytic continuations of their Lorentzian counterparts into the Euclidean region. Whatever the equation of state of the matter, $U(X)$ is positive at large l, since regularity demands that ρ_E remains finite, and correspondingly $w_E \to -1$, as $q \to 0$.

In fact $U(X)$ is positive for all tensor modes as long as $\rho_E > 0$ and $w_E > -17/3$.

As before, the quantum amplitude's dependence on the final perturbation χ_1 on the final three surface, where $q = q_1$, is given by the classical action. The no boundary prescription selects the mode $\chi = f(X)$ which is regular at $q = 0$, *i.e.*, which vanishes at $X = -\infty$ (in the large l limit, we simply have $f(X) \sim e^{\sqrt{l(l+2)}X}$). Using an integration by parts, from (7) we obtain the on-shell Euclidean action $-S_E^{(2)} = \pi^2\chi_1^2 f'(X_1)/f(X_1)$. The quantity $f'(X)/f(X)$ is positive at $X = -\infty$: as long as $U(X)$ is real and positive, the classical equation of motion for f implies $f'(X)/f(X)$ remains positive throughout the Euclidean region.

Continuing the conformal time into the Lorentzian region, we can show that the real part of the semiclassical exponent remains positive. Expressing the mode function in terms of its real and imaginary parts, $f(X) = R(X) + iI(X)$, we have shown that $\text{Re}[f'/f] = (RR' + II')/(R^2 + I^2) > 0$ at $X = 0$. When X turns in the negative imaginary direction, $X = -i\eta$, with η positive, the Cauchy-Riemann equations yield $R' + iI' = i(R_{,\eta} + iI_{,\eta})$. Therefore, at $X = \eta = 0$, we have $R_{,\eta} = I'$ and $I_{,\eta} = -R'$ and follows that the Wronskian $IR_{,\eta} - RI_{,\eta}$, which is independent of η, equals $(R^2+I^2)\text{Re}[f'/f]$ at $X = 0$, which is positive. Now, the real part of the semiclassical exponent, at a final Lorentzian time η_1 is similarly given, after an integration by parts, by $\pi^2\chi_1^2\text{Re}[i\, f_{,\eta}(\eta_1)/f(\eta_1)] = \pi^2\chi_1^2(IR_{,\eta} - RI_{,\eta})/(R^2+I^2)$ (in fact, I vanishes there by assumption). Since the Wronskian is positive, it follows that the semiclassical exponent for the perturbation χ_1 is positive, for all positive η.

In more general situations, the background pressure may not be expressible in terms of the density. In this case, it may not be possible to describe both the Euclidean and Lorentzian regions in terms of a real potential U. Nevertheless, even in this more general situation, where the "bounce" point q_B satisfying $q_B = 3/\rho_B$ is complex, we still need to pass below it in the complex q-plane to be consistent with Picard-Lefschetz theory. This topological result again implies that the conformal time η runs from $-i\infty$ in the region around $q = 0$ to positive, nearly real values in an approximately "Lorentzian" region. For modes of large l, the (in general complex) potential $U(X)$ is dominated by the l^2 term, and the no boundary solution is accurately described by the WKB Euclidean growing mode, so that $\text{Re}[f'/f] \sim \sqrt{l(l+2)} + O(l^{-1})$ at large l. The arguments above again demonstrate that the final semiclassical exponent has a positive real part. We conclude that the problem of unbounded perturbations, at small wavelengths, is unavoidable in the no boundary approach.

The no boundary and tunneling proposals had as their objective to provide theories of initial conditions for the universe, and in particular to explain the initial smoothness of the universe. As we have demonstrated here, when analyzing these proposals in a well-defined mathematical setting the picture that emerges is rather the opposite. Large perturbations are preferred, to such an extent that the propagator becomes non-normalizable and the entire framework fails. As detailed in a forthcoming publication [11], the situation cannot be improved by considering steeper inflationary potentials, even in cases where the lapse function becomes real at the saddle points (and even for negative ekpyrotic potentials the same problems arise). A smooth semiclassical beginning to the universe, where the big bang singularity is avoided, is thus not an option.

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