New BCJ representations for one-loop amplitudes in gauge theories and gravity

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We explain a procedure to manifest the Bern–Carrasco–Johansson duality between color and kinematics in n-point one-loop amplitudes of a variety of supersymmetric gauge theories. Explicit amplitude representations are constructed through a systematic reorganization of the integrands in the Cachazo–He–Yuan formalism. Our construction is independent on the amount of supersymmetry or the number of spacetime dimensions. The cancellations from supersymmetry multiplets in the loop as well as the resulting power counting of loop momenta is manifested along the lines of the corresponding superstring computations. The setup is used to derive the one-loop version of the Kawai–Lewellen–Tye formula for the loop integrands of gravitational amplitudes.

June 2017

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Contents

1. Introduction

Recent progress on the study of scattering amplitudes has uncovered novel properties and symmetries of individual theories, as well as surprising connections between them. A notable example is the Bern–Carrasco–Johansson (BCJ) duality between color and kinematics in gauge theories, and double-copy relations to corresponding gravity theories [1,2].

The BCJ duality states that gauge-theory amplitudes can be expressed such that their kinematic dependence closely mirrors their color dependence, in which case the kinematic contributions from trivalent diagrams are known as *BCJ numerators*. Their most remarkable property is that gravity amplitudes can be obtained from gauge-theory ones by simply substituting color factors for another copy of BCJ numerators. This procedure to construct gravity amplitudes was known as the double copy, and has led to great advances in the study of the ultraviolet behavior of supergravity amplitudes [2,3,4].

The BCJ duality has been proved at tree level [5], where the double copy is equivalent to the field-theory limit of the famous Kawai-Lewellen-Tye (KLT) relations between open- and closed-string amplitudes [6]. At loop level, despite of strong evidence [2,3,7,8,9,10,11], the duality remains a conjecture and the principle behind it is poorly understood. More recently, there has been progress in trying to double copy without explicit BCJ numerators [12], which may shed light on the longstanding problem of finding the five-loop four-point integrand of maximal supergravity.

At one-loop level, a generalized KLT formula has been proposed for all-multiplicity integrands in gauge and gravity theories [13], which does not rely on BCJ numerators or any particular representation of these integrands. An important goal of the current paper is to prove this proposal for the one-loop KLT formula. And it will be shown that the formula is in fact equivalent to a cubic-diagram expansion involving double copies of BCJ numerators in a new representation of Feynman integrals.

Moreover, we will describe an algorithmic procedure to obtain all-multiplicity BCJ numerators for one-loop amplitudes in supersymmetric gauge and gravity theories. This becomes possible thanks to the interplay of two closely-related approaches to tree and loop amplitudes: the approach based on scattering equations and that based on string amplitudes. The first approach has been originally proposed by Cachazo, Yuan and one of the present authors (CHY) as a new formulation for tree amplitudes in gauge theory and gravity [14,15]. It expresses tree amplitudes as localized integrals over the moduli space of punctured Riemann spheres, and the prescription turned out to extend flexibly to a variety of other theories[1], such as the bi-adjoint scalars [18], Einstein–Yang–Mills (EYM) [19], Born–Infeld, non-linear sigma models (NLSM) and special Galileons [20] as well as couplings thereof [21]. Elegant worldsheet models that underpin the CHY formulation have been proposed, based on ambitwistor strings [22,23,24] including a manifestly supersymmetric pure-spinor version [25,26].

Already at tree level, it has become clear that the CHY approach is very closely related to the string-theory approach to field-theory amplitudes. The reduced Pfaffian, which is the central object of the CHY integrand for gauge theory and gravity [18], can be recast in a form that coincides with open-superstring correlators [27]. This can be seen at the level of operator product expansions of vertex operators, where the pure-spinor CHY setup of [25] is equivalent to superstring result [27,28] as shown in [29]. More recently, the CHY formulation for the NLSM [20,21] has found a natural counterpart in form of low-energy limits of the disk integrals in open-string amplitudes [30,31], including couplings to biadjoint scalars [32]. Both approaches have provided important insights to the BCJ duality and double copy at tree level. The first explicit local expressions for BCJ numerators of gauge theories were derived in [33] from the pure-spinor formulation of superstring theory [34]. As shown in [18], in the CHY formulation, BCJ duality and double copy, as well as the KLT formulae for tree amplitudes become completely natural, which has also led to a variety of new theories related by double copy [20].

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[1] More formulae have been found for gauge-theory and gravity amplitudes with insertions of higher-dimensional operators [16] as well as QCD and Higgs amplitudes [17] etc..

The ambitwistor theory was first generalized to higher genus in [23] (see also [26] for a pure-spinor version). This has led the extension of scattering equations and the CHY formulation to loop level using nodal Riemann spheres [35], which yield loop amplitudes in a new representation of their Feynman integrals with propagators linear in loop momenta. The equivalence with usual Feynman-integral representations can be seen via partial-fraction manipulations and shifts in the loop momenta [36,35], which can also be naturally understood as forward limits of tree amplitudes [37]. In this way, CHY-like formulae have been written down for one-loop gauge and gravity theories [38], for biadjoint scalars [39] and more recently for two-loop amplitudes of super-Yang–Mills (SYM) and supergravity [40] as well as scalar theories [41]. See [42] and [43] for closely-related constructions for loop-level scattering equations and CHY formulae.

In this paper, we exploit that the close interplay of the two approaches continues at loop level and apply a variety of results from the recent string-theory literature to supersymmetric one-loop gauge-theory and gravity amplitudes. Significant progress on loop amplitudes of the pure-spinor superstring has been driven by the framework of multiparticle superfields [44,45] which gave rise to explicit BCJ numerators at loop level [9,10,11]. Previously, these building blocks have been used to determine one-loop amplitudes for a BRST-invariant subsector of ten-dimensional open superstring [46] which yields the complete all-multiplicity results for four-dimensional MHV helicities as well [11]. Moreover, multiparticle superfields have been used to determine complete one-loop six-point [10] results and partial two-loop five-point [47] and three-loop four-point [48] results for open and closed strings. Likewise, a component version of multiparticle superfields has been used to streamline the kinematic factors in one-loop open- and closed-string amplitudes with reduced supersymmetry [49,50].

However, in one-loop six- and four-point amplitudes with maximal and reduced supersymmetry, respectively, the above approach faced difficulties in constructing BCJ numerators [9,50]. It will be shown how the new representation of Feynman integrals emerging from the CHY formulation of loop amplitudes surpasses these obstacles and reconciles the BCJ duality with the hexagon anomaly of ten-dimensional SYM.

The main results of the current paper are threefold and may be summarized as follows.

(A) Based on the CHY-inspired representations of supersymmetric gauge-theory and gravity amplitudes, we present a general proof of one-loop BCJ and KLT relations proposed in [13].

(B) An all-multiplicity procedure to determine BCJ numerators for one-loop amplitudes is derived from the RNS version of ambitwistor-string and superstring correlators on a nodal Riemann sphere. Our method works for external bosons in presence of any amount of supersymmetry as well as for both parity-even and parity-odd sectors. The powercounting of loop momenta ℓ is manifested in a manner that is well-known from superstrings: Correlators with maximal and reduced supersymmetry are identified as degree-$(n-4)$ and degree-$(n-2)$ polynomials in ℓ and the Green function on the nodal sphere, respectively.

(C) At multiplicities $n \leq 6$, these BCJ numerators are supersymmetrized such as to address any combination of external bosons and fermions. These expressions are obtained from the field-theory limit of the pure-spinor superstring.

1.1. Outline

The paper is organized as follows. In section 2, we review the color-kinematics duality and KLT relations in the tree-level CHY setup, as well as the one-loop CHY prescription and the resulting representations of Feynman integrals. Section 3 is devoted to our main result (A): The notion of "partial integrands" for gauge-theory amplitudes is introduced, and their BCJ relations as well as their combinations to yield one-loop KLT relations are derived from the scattering equations.

The proof of one-loop KLT relations relies on new representations of correlators on a nodal Riemann sphere which are obtained within the RNS formalism in section 4: For external bosons, all-multiplicity techniques are introduced to simplify correlators with any amount of supersymmetry and to derive the BCJ numerators of (B) along with their powercounting in ℓ. Some of the steps are known from the superstring literature [51,52,53,54] but nevertheless spelt out in a CHY context for the sake of a self-contained presentation.

In section 5, we proceed to (C) and derive supersymmetric generalizations of the CHY correlator from the pure-spinor superstring. Particular emphasis will be placed on the resolution of earlier difficulties in finding six-point BCJ numerators in ten-dimensional SYM. An analogous discussion of correlators and BCJ numerators with reduced supersymmetry (along with a suitable infrared regularization scheme) is given in section 6.

2. Review

In this section, we first review, within the tree-level CHY setup, the color-kinematics duality and double copy, as well as the BCJ and KLT amplitude relations. The presentation is kept very explicit to later on connect with the analogous structures at one loop. Furthermore, a brief reminder of the one-loop CHY prescription as well as the new form of Feynman integrals therein will be given. Throughout this work, our conventions for Mandelstam invariants $s_{12\ldots p}$ and multiparticle momenta $k_{12\ldots p}$ are as follows:

$$k_{12\ldots p} \equiv k_1 + k_2 + \ldots + k_p \ , \quad s_{12\ldots p} \equiv \sum_{i<j}^{p} k_i \cdot k_j \ , \quad s_{12\ldots p,\pm\ell} \equiv \sum_{i<j}^{p} k_i \cdot k_j \pm \ell \cdot k_{12\ldots p} \tag{2.1}$$

2.1. CHY at tree level and doubly-partial amplitudes

Tree-level scattering amplitudes in the CHY formulation are represented by integrals over the moduli space of punctured Riemann spheres [14,15,18] parametrized by $\sigma_i \in \mathbb{C}$

$$\mathcal{M}^{\text{tree}}_{L\otimes R} = \int \mathrm{d}\mu^{\text{tree}}_n \, \mathcal{I}^{\text{tree}}_L \, \mathcal{I}^{\text{tree}}_R \ , \qquad \mathrm{d}\mu^{\text{tree}}_n \equiv \frac{\mathrm{d}\sigma_1 \, \mathrm{d}\sigma_2 \ldots \mathrm{d}\sigma_n}{\mathrm{vol}\, SL(2,\mathbb{C})} \prod_{i=1}^{n}{}' \delta\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i \cdot k_j}{\sigma_{ij}} \Big) \ . \tag{2.2}$$

This formula applies to theories $L \otimes R$ that exhibit a double-copy structure such that the integrand factorizes into two pieces $\mathcal{I}^{\text{tree}}_L$ and $\mathcal{I}^{\text{tree}}_R$ which depend on the scattering data (momenta or polarizations) as well as the punctures $\sigma_{ij} \equiv \sigma_i - \sigma_j$. The delta functions in the measure $\mathrm{d}\mu^{\text{tree}}_n$ impose the scattering equations

$$E_i \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0 \ , \tag{2.3}$$

and thereby localize the integrals to their $(n-3)!$ solutions. Both of the theory-dependent "half-integrands" $\mathcal{I}^{\text{tree}}_L$ and $\mathcal{I}^{\text{tree}}_R$ are designed to transform with weight two under Möbius transformations $\sigma_i \rightarrow \frac{a\sigma_i + b}{c\sigma_i + d}$, with a, b, c, d forming an $SL(2,\mathbb{C})$ matrix. As indicated by $(\mathrm{vol}\, SL(2,\mathbb{C}))^{-1}$ and $\prod{}'$, this symmetry is taken into account by fixing any three punctures to $(0, 1, \infty)$ and by dropping three redundant scattering equations, see [14,15] for details.

The generic theory $L \otimes R$ in the CHY prescription (2.2) can be adapted to biadjoint scalars with gauge group $U(N) \times U(\tilde{N})$ by choosing $\mathcal{I}^{\text{tree}}_L$ and $\mathcal{I}^{\text{tree}}_R$ as [18]

$$\mathcal{I}^{\text{tree}}_{U(N)} = \sum_{\rho \in S_{n-1}} \mathrm{Tr}(t^{a_1} t^{a_{\rho(2)}} t^{a_{\rho(3)}} \ldots t^{a_{\rho(n)}}) \mathrm{PT}(1, \rho(2, 3, \ldots, n)) \ , \tag{2.4}$$

where t^{a_j} denotes the $U(N)$ generator associated with the $j^{\rm th}$ leg. Given that the dependence on the punctures is captured by the Parke–Taylor factors

$$\mathrm{PT}(1,2,3,\ldots,n{-}1,n) \equiv \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{n-1,n}\sigma_{n1}} \ , \tag{2.5}$$

the most general integral appearing in the tree-level S-matrix of the $U(N)\times U(\tilde N)$ theory is the doubly-partial amplitude

$$m^{\rm tree}[\rho(1,2,\ldots,n)\,|\,\tau(1,2,\ldots,n)] \equiv \int \mathrm{d}\mu_n^{\rm tree} \ \mathrm{PT}(\rho(1,2,\ldots,n))\mathrm{PT}(\tau(1,2,\ldots,n)) \ . \tag{2.6}$$

It accompanies the product of traces $\mathrm{Tr}(t^{a_{\rho(1)}}t^{a_{\rho(2)}}\ldots t^{a_{\rho(n)}})\mathrm{Tr}(\tilde t^{b_{\tau(1)}}\tilde t^{b_{\tau(2)}}\ldots \tilde t^{b_{\tau(n)}})$ (with possibly distinct permutations $\rho,\tau\in S_n$) in the expression (2.2) for $\mathcal{M}^{\rm tree}_{U(N)\times U(\tilde N)}$. The doubly-partial amplitude $m^{\rm tree}[\rho(\ldots)\,|\,\tau(\ldots)]$ assembles the propagators $s^{-1}_{i_1i_2\ldots i_p}$ of all the cubic diagrams compatible with the cyclic orderings ρ and τ and can be computed through the algorithm in [18] or a Berends–Giele recursion [55] (see also [56]).

2.2. Tree-level BCJ numerators from CHY

The CHY formula (2.2) describes (possibly supersymmetric) Yang–Mills theory and gravity if one or both of the half-integrands $\mathcal{I}_L^{\rm tree}$ and $\mathcal{I}_R^{\rm tree}$ are identified with a gauge invariant function $\mathcal{I}^{\rm tree}_{\rm SYM}\equiv\mathcal{K}_n^{\rm tree}$ of the polarizations in the gauge multiplet. For external bosons, the realization of $\mathcal{K}_n^{\rm tree}$ as the (reduced) Pfaffian of an antisymmetric $2n\times 2n$ matrix was presented in [15]. Despite the lack of a Pfaffian-like representation, the supersymmetric completion is known from the pure-spinor version of the CHY setup [25].

As pointed out in the ambitwistor setting in [22], and detailed in [29] in a pure-spinor context, $\mathcal{K}_n^{\rm tree}$ is identical to the field-theory limit $\alpha'\to 0$ of the n-point correlation function of open-string vertex operators (which sets the Koba–Nielsen factor to the identity). This equivalence of CHY integrands and superstring correlators holds on the support of scattering equations, or integration-by-parts relations of the string worldsheet. Hence, one can import the manifestly supersymmetric results on the superstring tree-level correlators obtained in [27,33,28], and we will later use the analogous correspondence at one loop.

The superstring version of $\mathcal{K}_n^{\rm tree}$ was shown in [33] to be organized in terms of $(n{-}2)!$ Parke–Taylor factors (2.5),

$$\mathcal{K}_n^{\rm tree} = \sum_{\rho\in S_{n-2}} \mathrm{PT}(1,\rho(2,3,\ldots,n{-}1),n)\, N^{\rm tree}_{1|\rho(2,3,\ldots,n-1)|n} \ , \tag{2.7}$$

and the same form can be attained in the CHY setting [18] by applying scattering equations to the Pfaffian representation of its bosonic components [15]. The kinematic numerators $N^{\rm tree}_{1|\rho(2,3,\ldots,n-1)|n}$ refer to the cubic diagrams of half-ladder topology with fixed endpoints 1 and n, see Fig. 1. Their explicit realization in pure-spinor superspace [33] is based on superfields of ten-dimensional SYM [57], and the components involving gluon polarization vectors e^m and gaugino wave functions χ^α can be conveniently extracted using the streamlined θ-expansions of [58], also see section 5 for more details.

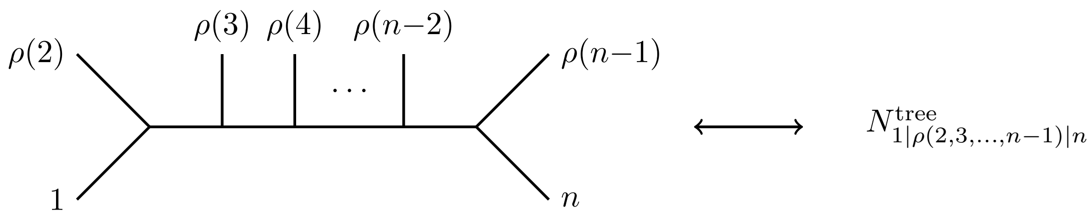

Fig. 1 Half-ladder diagrams with legs 1 and n attached to opposite endpoints and BCJ master numerators $N^{\rm tree}_{1|\rho(2,3,\ldots,n-1)|n}$ determine any other cubic diagram via kinematic Jacobi relations.

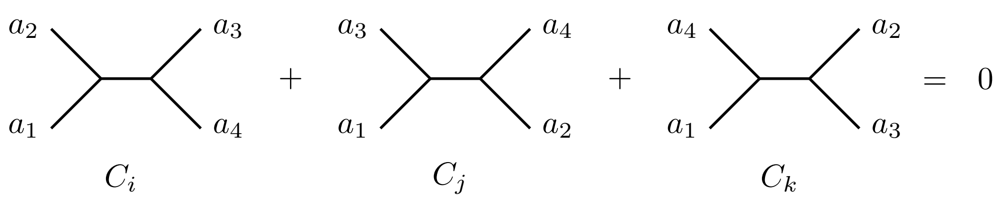

Fig. 2 The Jacobi identity implies the vanishing of the color factors associated to a triplet of cubic graphs, $C_i{+}C_j{+}C_k = 0$. In the above diagrams, the legs a_1, a_2, a_3 and a_4 may represent arbitrary subdiagrams. The BCJ duality states that their corresponding kinematic numerators $N_i(\ell)$ can be chosen such that $N_i(\ell) + N_j(\ell) + N_k(\ell) = 0$.

As emphasized in [33,18], the representation (2.7) implies that numerators for all the other cubic diagrams besides the $(n{-}2)!$ master graphs in Fig. 1 are determined by the BCJ duality between color and kinematics [1]: In the same way as any triplet of graphs as shown in Fig. 2 are related by a group-theoretic Jacobi identity $f^{ba_1[a_2}f^{a_3a_4]b} = 0$ among their color factors, one can arrange the kinematic dressings of these graphs such that they satisfy the same Jacobi identities. When computing color-ordered SYM amplitudes $A^{\rm tree}(\ldots)$ from the CHY prescription,

$$\begin{aligned} A^{\rm tree}(\tau(1,2,\ldots,n)) &= \mathcal{M}^{\rm tree}_{{\rm SYM}\otimes U(N)}\, \big|_{{\rm Tr}(t^{a_{\tau(1)}}t^{a_{\tau(2)}}\ldots t^{a_{\tau(n)}})} \\ &= \int {\rm d}\mu^{\rm tree}_n \; {\rm PT}(\tau(1,2,\ldots,n))\, \mathcal{K}^{\rm tree}_n \\ &= \sum_{\rho\in S_{n-2}} m^{\rm tree}[\tau(1,2,\ldots,n)\,|\,1,\rho(2,\ldots,n{-}1),n] N^{\rm tree}_{1|\rho(2,3,\ldots,n-1)|n} \ , \end{aligned} \tag{2.8}$$

the expansion of $\mathcal{K}_n^{\rm tree}$ in (2.7) and the form of the doubly-partial amplitudes guarantee that each cubic-diagram numerator is a linear combination of $N^{\rm tree}_{1|\rho(2,\ldots,n-1)|n}$ with coefficients $\in \{0,1,-1\}$. Their $(n{-}2)!$-counting agrees with the number of master numerators under kinematic Jacobi identities, and it follows from the arguments in [33,18] that the linear combinations of $N^{\rm tree}_{1|\rho(2,\ldots,n-1)|n}$ in (2.8) satisfy kinematic Jacobi identities. In summary, the expansion of the tree-level correlator (2.7) in terms of $(n{-}2)!$ Parke–Taylor factors PT$(\ldots)$ allows to read off a set of BCJ master numerators.

2.3. BCJ and KLT relations from CHY

At tree level, a manifestly gauge invariant double-copy expression for gravity amplitudes is given by the KLT formula

$$\mathcal{M}^{\rm tree}_{\rm SYM\otimes SYM} = \sum_{\rho,\tau\in S_{n-3}} \tilde{A}^{\rm tree}(1,\rho(2,\ldots,n{-}2),n,n{-}1)\, S[\rho|\tau]_1\, A^{\rm tree}(1,\tau(2,\ldots,n{-}2),n{-}1,n) \tag{2.9}$$

derived from tree-level scattering of open and closed strings [6]. The $(n{-}3)!\times(n{-}3)!$ matrix $S[\rho|\tau]_1 \equiv S[\rho(2,\ldots,n{-}2)|\tau(2,\ldots,n{-}2)]_1$ with entries of order $\sim s^{n-3}$ has been firstly pinpointed to all multiplicity in [59] and was later on studied in the momentum-kernel formalism [60]. A recursive formula for its entries is given by [30]

$$S[A,j|B,j,C]_i = k_j\cdot(k_i+k_B)\, S[A|B,C]_i \ , \qquad S[\emptyset|\emptyset]_i = 0 \ , \tag{2.10}$$

see (2.1) for the multiparticle momenta k_B associated with $B=b_1b_2\ldots b_p$. Permutation invariance of (2.9) follows from BCJ relations among partial amplitudes [1]

$$\sum_{j=2}^{n-1}(k_1\cdot k_{23\ldots j})A^{\rm tree}(2,3,\ldots,j,1,j{+}1,\ldots n) = 0 \tag{2.11}$$

which have been elegantly derived from monodromy properties of the open-string worldsheet [61]. In the CHY setup, BCJ relations emerge from the scattering equations (2.3) which relate Parke–Taylor factors in complete analogy to (2.11) [62,14]

$$\sum_{j=2}^{n-1}(k_1\cdot k_{23\ldots j}){\rm PT}(2,3,\ldots,j,1,j{+}1,\ldots n) = 0 \ {\rm mod}\ E_i \ , \tag{2.12}$$

and they also hold for both entries of the doubly-partial amplitudes (2.6). Note that the string-theory correlator (2.7) can be simplified to a BCJ basis of $(n{-}3)!$ worldsheet integrals using integration by parts on the string worldsheet [27]. This result was later on identified to reproduce the structure of the KLT formula (2.9) [63]

$$\mathcal{K}_n^{\rm tree} = \sum_{\rho,\tau \in S_{n-3}} {\rm PT}(1,\rho(2,\ldots,n{-}2),n,n{-}1)\, S[\rho|\tau]_1 \, A^{\rm tree}(1,\tau(2,\ldots,n{-}2),n{-}1,n) \ . \tag{2.13}$$

Insertion into (2.8) identifies doubly-partial amplitudes (2.6) in a suitable basis as the inverse of the momentum kernel (2.10) [63,18],

$$m^{\rm tree}[1,\rho(2,\ldots,n{-}2),n,n{-}1\,|\,1,\tau(2,\ldots,n{-}2),n{-}1,n] = S^{-1}[\rho|\tau]_1 \ . \tag{2.14}$$

Then, the KLT formula (2.9) follows from insertion of (2.13) in the CHY prescription (2.2),

$$\mathcal{M}^{\rm tree}_{\rm SYM\otimes SYM} = \int {\rm d}\mu_n^{\rm tree} \ \mathcal{K}_n^{\rm tree} \, \tilde{\mathcal{K}}_n^{\rm tree} \ , \tag{2.15}$$

where it is convenient to exchange the roles of n and $n{-}1$ in the formula (2.13) for $\tilde{\mathcal{K}}_n^{\rm tree}$.

Notice that (2.15) also makes the BCJ double-copy relations manifest, which are equivalent to KLT relations at tree level: By plugging (2.7) into (2.15), it follows that the (super-)gravity amplitude is given by sum of all cubic diagrams with numerators given by the double copy $N^{\rm tree}\,\tilde{N}^{\rm tree}$. This is the major advantage of having a representation of gauge-theory amplitude with numerators satisfying the BCJ color-kinematics duality [1].

2.4. CHY at one loop

In the ambitwistor-string version of the CHY formalism, g-loop amplitudes in various theories are written as integrals over the moduli space of punctured genus-g surfaces [23]. At one loop, the surface of interest is a torus with modular parameter τ in the upper half plane such that its complex coordinate z is identified with $z{+}1$ and $z{+}\tau$. Apart from the torus punctures $z_{i=1,2,\ldots,n}$, also the inequivalent choices of τ in the fundamental domain of the modular group with $-\frac{1}{2} \leq {\rm Re}\, \tau \leq \frac{1}{2}$ and $|\tau| > 1$ are integrated over.

However, one of the scattering equations at genus one can be exploited [35] to localize the τ integral at the cusp $\tau \rightarrow i\infty$ where the torus degenerates to a nodal sphere. Then, after a change of variables $\sigma = e^{2\pi i z}$, one-loop amplitudes of (possibly supersymmetric) gravity and gauge theories in D spacetime dimensions simplify to [35]

$$\mathcal{M}_{L\otimes R} = \int \frac{{\rm d}^D \ell}{\ell^2} \int \prod_{i=2}^n {\rm d}\sigma_j \, \delta\Big(\frac{(\ell\cdot k_i)}{\sigma_i} + \sum_{\substack{j=1\\ j\neq i}}^n \frac{k_i\cdot k_j}{\sigma_{ij}} \Big) \, \widehat{\mathcal{I}}_L(\ell) \, \widehat{\mathcal{I}}_R(\ell) \ . \tag{2.16}$$

Note that translation invariance in the z-variable allows to insert another integration $\mathrm{d}\sigma_1$ along with a delta function e.g. $\delta(\sigma_1 - 1)$, and the corresponding scattering equation

$$\frac{(\ell \cdot k_i)}{\sigma_i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{k_i \cdot k_j}{\sigma_{ij}} = 0 \tag{2.17}$$

for $i = 1$ does not need to be enforced separately because it follows by adding the remaining equations for $i = 2, 3 \ldots, n$.

For gauge theories, one of the integrands $\widehat{\mathcal{I}}_L(\ell) \to \widehat{\mathcal{I}}_{U(N)}(\ell)$ is a sum of color traces[2]

$$\widehat{\mathcal{I}}_{U(N)}(\ell) = \sum_{\rho \in S_{n-1}} \mathrm{Tr}(t^{a_1} t^{a_{\rho(2)}} t^{a_{\rho(3)}} \ldots t^{a_{\rho(n)}}) \, \widehat{\mathrm{PT}}^{(1)}(1, \rho(2,3,\ldots,n)) \ , \tag{2.18}$$

accompanied by one-loop analogues $\widehat{\mathrm{PT}}^{(1)}(\ldots)$ of the Parke–Taylor factors (2.5),

$$\widehat{\mathrm{PT}}^{(1)}(1,2,\ldots,n) \equiv \frac{1}{\sigma_1 \sigma_{12} \sigma_{23} \ldots \sigma_{n-1,n}} + \mathrm{cyc}(1,2,\ldots,n) \ . \tag{2.19}$$

The polarization-dependent integrand $\widehat{\mathcal{I}}_{\mathrm{SYM}}(\ell)$ is the $\tau \to i\infty$ degeneration of the genus-one correlation function involving n gauge-multiplet vertex operators $V(\sigma)$ to be discussed in later sections 4 and 5,

$$\widehat{\mathcal{I}}_{\mathrm{SYM}}(\ell) \equiv \frac{(-1)^n \mathcal{K}_n(\ell)}{\sigma_1 \sigma_2 \ldots \sigma_n} \ , \qquad \mathcal{K}_n(\ell) \equiv \lim_{\tau \to i\infty} \langle V_1(\sigma_1) V_2(\sigma_2) \ldots V_n(\sigma_n) \rangle_\tau \ . \tag{2.20}$$

The inverse σ_i can be traced back to the change of variables $\sigma = e^{2\pi i z}$ with $\mathrm{d}z = \frac{1}{2\pi i} \frac{\mathrm{d}\sigma}{\sigma}$, and the prescription for evaluating the correlation function $\langle \ldots \rangle_\tau$ is left generic at this point to later on import results from both the RNS and pure-spinor superstring. In terms of the two integrands (2.18) and (2.20), one-loop amplitudes (2.16) in gauge theory and gravity are obtained as $\mathcal{M}_{U(N)\otimes\mathrm{SYM}}$ and $\mathcal{M}_{\mathrm{SYM}\otimes\mathrm{SYM}}$, respectively.

2.5. New representations of one-loop integrals

It turns out that Feynman integrals arise in a non-standard representation when integrating over the σ_j in (2.16): Instead of conventional propagators $(\ell + K)^2$ quadratic in ℓ (with

[2] We suppress double traces in (2.18), and their accompanying color-stripped amplitudes can be recovered from linear combinations of single-trace subamplitudes [64].

some linear combination K of external momenta), the σ_j-integrals yield the results of repeated partial fraction [35]. The massless n-gon, for instance, appears in the form of

$$\int \frac{2^{n-1}\ \mathrm{d}^D\ell}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2\dots(\ell{+}k_{12\dots n-1})^2} = \sum_{i=0}^{n-1}\int \frac{2^{n-1}\ \mathrm{d}^D\ell}{(\ell{+}k_{12\dots i})^2}\prod_{j\neq i}\frac{1}{(\ell{+}k_{12\dots j})^2-(\ell{+}k_{12\dots i})^2}$$
$$= \sum_{i=0}^{n-1}\int \frac{\mathrm{d}^D\ell}{\ell^2}\prod_{j=0}^{i-1}\frac{1}{s_{j+1,j+2,\dots,i,-\ell}}\prod_{j=i+1}^{n-1}\frac{1}{s_{i+1,i+2,\dots,j,\ell}}\ , \tag{2.21}$$

where the loop momentum ℓ in the i^{th} term has been shifted by $k_{12\dots i}$ in passing to the last line to ensure that the only quadratic propagator is a pure ℓ^2 in each term. Each term in the sum over i singles out one way of cutting open the n-gon, and the result can be thought of as n tree diagrams involving off-shell momenta $\pm\ell$ [37], see Fig. 3. Each of these cubic diagrams will have a priori different kinematic numerators, leaving a total of $n!$ inequivalent n-gon numerators.

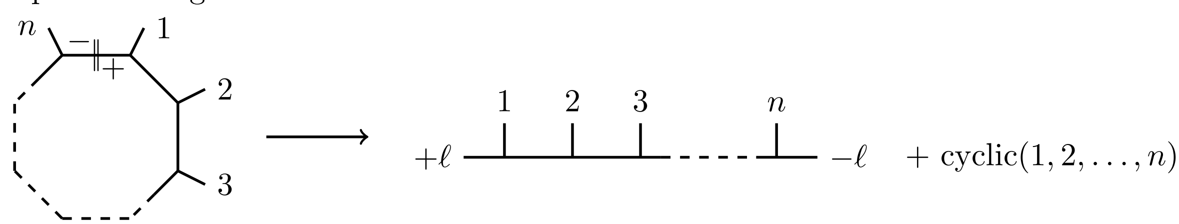

Fig. 3 Interpretation of the partial-fraction representation of loop integrals as (n+2)-point tree-level diagrams.

The manipulations in (2.21) straightforwardly generalize to integrals with tree-level subdiagrams, e.g. a box integral with massive momenta k_A, k_B, k_C and k_D allows for the following four-term representation:

$$\int \frac{8\ \mathrm{d}^D\ell}{\ell^2(\ell{+}k_A)^2(\ell{+}k_{AB})^2(\ell{+}k_{ABC})^2} = \int \frac{\mathrm{d}^D\ell}{\ell^2}\Big(\frac{1}{s_{A,\ell}s_{AB,\ell}s_{D,-\ell}} + \mathrm{cyc}(A,B,C,D)\Big)\ . \tag{2.22}$$

In this way, the one-loop integrand for color-ordered single-trace amplitudes can be split into n terms, similar to that of (2.21) for the n-gon. Each of the n terms can be interpreted as the forward limit of (n+2)-point trees with off-shell momenta, e.g. the momenta of the two legs between n and 1 being identified as ℓ and $-\ell$. The off-shell momenta can be viewed as on-shell, higher-dimensional ones, and the one-loop CHY formula (2.16) was obtained as the forward limit of such higher-dimensional tree amplitudes [37]. Although it is non-trivial to perform loop integrations, the new representation of loop integrals has to give the same result as the canonical Feynman integrals.

These integrals not only naturally appear in the CHY formalism, but also play an important role in the Q-cut representation of loop amplitudes [65]. The new representation provides a well-defined notion of "loop integrands" for generic, non-planar theories, which can be exploited to reveal structures of loop amplitudes. In particular, as conjectured recently [13], in the new representation it is natural to generalize KLT and BCJ relations, (2.9) and (2.11), to one loop. In section 3.6, we prove these new relations, as well as the color-kinematics duality and double copy at the one-loop level in this new representation.

3. BCJ and KLT at one-loop

3.1. One-loop correlators in generic $SL(2,\mathbb{C})$ frames

The expressions in the above review of the one-loop CHY setup are adapted to a particular $SL(2,\mathbb{C})$ frame where two additional punctures $\sigma_+ = 0$ and $\sigma_- \to \infty$ are identified on the nodal sphere and associated with momenta $k_\pm = \pm\ell$. This $SL(2,\mathbb{C})$-fixing is reflected in the hat notation for the integrands $\widehat{\mathcal{I}}_{U(N)}(\ell)$ and $\widehat{\mathcal{I}}_{\rm SYM}(\ell)$ in (2.18) and (2.20) as well as the one-loop Parke–Taylor factors $\widehat{\rm PT}^{(1)}(1,2,\ldots,n)$ in (2.19). In this subsection, we shall give the analogous expressions for "unhatted" quantities $\mathcal{I}_{U(N)}(\ell)$, $\mathcal{I}_{\rm SYM}(\ell)$ and ${\rm PT}^{(1)}(1,2,\ldots,n)$ in a generic frame: Requiring $SL(2,\mathbb{C})$-weight two in each puncture $\sigma_{j=1,2,\ldots,n}$ and σ_+,σ_- yields unique $SL(2,\mathbb{C})$-covariant uplifts, and we will introduce a method to express both $\mathcal{I}_{U(N)}(\ell)$ and $\mathcal{I}_{\rm SYM}(\ell)$ in terms of $(n{+}2)$-point tree-level Parke–Taylor factors (2.5).

For instance, σ_j-independent contributions from the correlators $\mathcal{K}_n(\ell)$ to the gauge-theory integrands (2.20) can expressed via $SL(2,\mathbb{C})$-fixed tree-level Parke–Taylor factors with $\sigma_+ = 0$ and $\sigma_- \to \infty$ [38],

$$
\begin{aligned}
\int \prod_{j=1}^n \frac{{\rm d}\sigma_j}{\sigma_j} &= \int \prod_{j=1}^n \frac{{\rm d}\sigma_j}{\sigma_{j,+}}\Big|_{\sigma_+=0} = (-1)^n \sum_{\rho\in S_n} \int \frac{{\rm d}\sigma_1\,{\rm d}\sigma_2\ldots{\rm d}\sigma_n}{\sigma_{+,\rho(1)}\sigma_{\rho(1),\rho(2)}\cdots\sigma_{\rho(n-1),\rho(n)}}\Big|_{\sigma_+=0} \\
&= (-1)^n \lim_{\sigma_-\to\infty} \sum_{\rho\in S_n} \int \frac{{\rm d}\sigma_1\,{\rm d}\sigma_2\ldots{\rm d}\sigma_n\ (-\sigma_-^2)}{\sigma_{+,\rho(1)}\sigma_{\rho(1),\rho(2)}\cdots\sigma_{\rho(n-1),\rho(n)}\sigma_{\rho(n),-}\sigma_{-,+}}\Big|_{\sigma_+=0} \\
&= (-1)^n \sum_{\rho\in S_n} \int \frac{{\rm d}\sigma_-\,{\rm d}\sigma_+\prod_{j=1}^n{\rm d}\sigma_j}{{\rm vol}\,SL(2,\mathbb{C})}{\rm PT}(+,\rho(1,2,\ldots,n),-)\ , \qquad (3.1)
\end{aligned}
$$

or in short

$$
\prod_{j=1}^n \frac{1}{\sigma_j} = (-1)^n \lim_{\sigma_-\to\infty}(-\sigma_-^2) \lim_{\sigma_+\to 0} \sum_{\rho\in S_n} {\rm PT}(+,\rho(1,2,\ldots,n),-)\ . \qquad (3.2)
$$

Likewise, one-loop Parke–Taylor factors $\widehat{\rm PT}^{(1)}(\ldots)$ in (2.19) were defined in [35] from their $SL(2,\mathbb{C})$-covariant uplifts ${\rm PT}^{(1)}(\ldots)$,

$$\begin{aligned}
{\rm PT}^{(1)}(1,2,\ldots,n) &\equiv {\rm PT}(+,1,2,\ldots,n,-) \ + \ {\rm cyc}(1,2,\ldots,n) &(3.3)\\
\widehat{\rm PT}^{(1)}(1,2,\ldots,n) &= \lim_{\sigma_-\to\infty}(-\sigma_-^2)\lim_{\sigma_+\to 0}{\rm PT}^{(1)}(1,2,\ldots,n) \ , &(3.4)
\end{aligned}$$

which implies the following form for the $U(N)$ integrand in a generic $SL(2,\mathbb{C})$-frame,

$$\mathcal{I}_{U(N)}(\ell) = \sum_{\rho\in S_{n-1}} {\rm Tr}(t^{a_1}t^{a_{\rho(2)}}t^{a_{\rho(3)}}\ldots t^{a_{\rho(n)}})\,{\rm PT}^{(1)}(1,\rho(2,3,\ldots,n)) \ . \tag{3.5}$$

As will be detailed in the next subsection, also a generic correlator $\mathcal{K}_n(\ell)$ with non-trivial σ_j-dependence admits a unique $SL(2,\mathbb{C})$-covariant uplift $\mathcal{I}_{\rm SYM}(\ell)$ for the SYM integrand (2.20). Regardless of the details of $\mathcal{I}_{U(N)}(\ell)$ and $\mathcal{I}_{\rm SYM}(\ell)$, the one-loop CHY prescription (2.16) in a generic $SL(2,\mathbb{C})$-frame can be boiled down to the tree-level measure (2.2),

$$\mathcal{M}_{L\otimes R} = \int \frac{{\rm d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\int {\rm d}\mu^{\rm tree}_{n+2}\ \mathcal{I}_L(\ell)\,\mathcal{I}_R(\ell) \ , \tag{3.6}$$

in lines with the degeneration of the torus to a nodal Riemann sphere as $\tau\to i\infty$. Note in particular that the one-loop scattering equations (2.17) descend from their (n+2)-point tree-level instances (2.3) in the limit $\sigma_-\to\infty$ and $\sigma_+=0$ with $k_\pm\to\pm\ell$,

$$\frac{(k_+\cdot k_i)}{\sigma_{i,+}} + \frac{(k_-\cdot k_i)}{\sigma_{i,-}} + \sum_{\substack{j=1\\ j\neq i}}^n \frac{k_i\cdot k_j}{\sigma_{ij}}\Big|_{k_\pm\to\pm\ell} \to 0 \ . \tag{3.7}$$

In the same way as only $n-3$ scattering equations are independent in the n-point tree-level prescription (2.2), the $n-1$ scattering equations in (2.16) are sufficient for the situation in one-loop amplitudes (3.6) with n+2 punctures.

In theories with reduced supersymmetry, the kinematic regime with $k_\pm\to\pm\ell$ gives rise to singularities upon integration over σ_j, and we will later comment on their regularization.

3.2. The σ-dependence of gauge-theory correlators

This subsection is devoted to the structure of σ_j-dependent correlators $\mathcal{K}_n(\ell)$ which carry the state-dependence in the SYM integrand (2.20). The expressions for $\mathcal{K}_n(\ell)$ can be imported from the superstring correlator in the field-theory limit, see sections 4 and 5 for explicit examples in the RNS and pure-spinor formalism. As is well-known from superstring theory, singularities of genus-one correlators at generic values of τ arise from the holomorphic torus Green function $\partial_z \log \theta_1(z,\tau)$, where θ_1 denotes the odd Jacobi theta function

$$\theta_1(z,\tau) \equiv 2iq^{1/8}\sin(\pi z)\prod_{j=1}^{\infty}(1-q^j)(1-e^{2\pi i z}q^j)(1-e^{-2\pi i z}q^j) = -\theta_1(-z,\tau) \tag{3.8}$$

with a simple pole at the origin and $q \equiv e^{2\pi i\tau}$. We recall the change of variables $\sigma = e^{2\pi i z}$ between the punctures σ in (3.6) and the torus coordinates with identifications of z with $z+1$ and $z+\tau$. By the localization of CHY correlators at the cusp $\tau \to i\infty$, we will only be interested in the limit [35]

$$\frac{1}{2\pi i}\lim_{\tau\to i\infty}\partial_z \log\theta_1(z_i - z_j,\tau) = G_{ij} \equiv \frac{\sigma_i+\sigma_j}{2\sigma_{ij}} \ . \tag{3.9}$$

In terms of the Green function G_{ij}, the one-loop scattering equations (2.17) (and also the $\tau \to i\infty$ degeneration of integration-by-parts relations in string theory) can be written as

$$(\ell\cdot k_i) + \sum_{\substack{j=1\\ j\neq i}}^{n}(k_i\cdot k_j)G_{ij} = 0 \ . \tag{3.10}$$

Note that the partial-fraction identity $(\sigma_{ij}\sigma_{ik})^{-1} + \mathrm{cyc}(i,j,k) = 0$ among nested products of the tree-level Green function σ_{ij}^{-1} does not carry over to G_{ij},

$$G_{ij}G_{ik} + \mathrm{cyc}(i,j,k) = \frac{[\sigma_{jk}(\sigma_i+\sigma_j)(\sigma_i+\sigma_k) + \mathrm{cyc}(i,j,k)]}{4\sigma_{ij}\sigma_{ik}\sigma_{jk}} = \frac{1}{4} \ . \tag{3.11}$$

This result follows from the field-theory limit of the corresponding genus-one Fay identities studied in [66,54].

As will be proven in section 4, any one-loop gauge-theory correlator $\mathcal{K}_n(\ell)$ can be written as a polynomial in G_{ij} and ℓ, regardless of the multiplicity and the amount of supersymmetries. The degree of this polynomial will be shown to vary with the amount of supersymmetry, the highest power of Green functions being G_{ij}^{n-4} in presence of maximal supersymmetry, G_{ij}^{n-2} in gauge theories with 8 or 4 supercharges and G_{ij}^{n} in non-supersymmetric cases. Of course, the G_{ij} do not appear with homogeneous degree since integration by parts (3.10) interchanges combinations of G_{ij} with loop momenta, and the Fay identity (3.11) mixes powers of G_{ij}^k, G_{ij}^{k-2}, $G_{ij}^{k-4}, \ldots$ along with a given ℓ-dependence, see the examples in section 5.

With less than n powers of G_{ij}, i.e. in presence of at least 4 supercharges, one can furthermore use the scattering equations in their form (3.10) to eliminate closed subcycles of Green functions such as $G_{12}^2 = -G_{12}G_{21}$ and $G_{12}G_{23}G_{31}$. In other words, when drawing an edge between vertices i and j for each factor of G_{ij}, the pattern of G_{ij} in supersymmetric $\mathcal{K}_n(\ell)$ can be represented as a Cayley graph. This is always possible at any multiplicity, see appendix A below (for similar algorithms at tree-level, see [67]).

3.3. Gauge-theory correlators in terms of Parke–Taylor factors

The central result of this section concerns the interplay of such G_{ij} with the Parke–Taylor structure (3.2) seen in the case of σ_j-independent $\mathcal{K}_n(\ell)$, where it is convenient to define

$$\mathcal{Z}_{i_1 i_2 i_3 \ldots i_{q-1} i_q} \equiv \frac{1}{\sigma_{i_1 i_2}\sigma_{i_2 i_3}\ldots\sigma_{i_{q-1}i_q}} \, . \tag{3.12}$$

In the presence of G_{ij} factors with no subcycles, it will be proven in the appendix B that the sum in the right-hand side of (recall that $\sigma_+ = 0$)

$$\prod_{j=1}^{n} \frac{1}{\sigma_j} = (-1)^n \sum_{\rho \in S_n} \mathcal{Z}_{+\rho(1,2,3,\ldots,n)} \, , \tag{3.13}$$

is modified by ρ-dependent signs,

$$\mathrm{sgn}_{ij}^{\rho} \equiv \begin{cases} +1 & : \ i \text{ is on the right of } j \text{ in } \rho(1,2,\ldots,n) \\ -1 & : \ i \text{ is on the left of } j \text{ in } \rho(1,2,\ldots,n) \end{cases} \, . \tag{3.14}$$

More explicitly, with m factors of G_{ij} without subcycles,

$$G_{i_1 j_1} G_{i_2 j_2} \ldots G_{i_m j_m} \prod_{j=1}^{n} \frac{1}{\sigma_j} = \frac{(-1)^n}{2^m} \sum_{\rho \in S_n} \mathrm{sgn}_{i_1 j_1}^{\rho} \mathrm{sgn}_{i_2 j_2}^{\rho} \ldots \mathrm{sgn}_{i_m j_m}^{\rho} \mathcal{Z}_{+\rho(1,2,\ldots,n)} \, . \tag{3.15}$$

Given that

$$\mathcal{Z}_{+\rho(1,2,\ldots,n)} = \lim_{\sigma_- \to \infty} (-\sigma_-^2) P(+,\rho(1,2,\ldots,n),-) \, , \tag{3.16}$$

the net effect of G_{ij} in converting the correlator $\mathcal{K}_n(\ell)$ to a Parke–Taylor expansion of the gauge-theory integrand (2.20) is captured by the prescription $G_{ij} \to \frac{1}{2}\mathrm{sgn}_{ij}^{\rho}$,

$$\mathcal{I}_{\mathrm{SYM}}(\ell) = \sum_{\rho \in S_n} \mathrm{PT}(+,\rho(1,2,\ldots,n),-) \left(\mathcal{K}_n(\ell) \Big|_{G_{ij} \to \frac{1}{2}\mathrm{sgn}_{ij}^{\rho}} \right) \, . \tag{3.17}$$

This generalizes the expansion (2.7) of the tree-level correlator in terms of $(n-2)!$ Parke–Taylor factors $\mathrm{PT}(1,\rho(2,3,\ldots,n-1),n)$ to the one-loop order: With $n+2$ punctures on the nodal Riemann sphere – $\sigma_\pm$ and $\sigma_{j=1,2,\ldots,n}$ – the analogous family of Parke–Taylor factors has $n!$ elements $\mathrm{PT}(+,\rho(1,2,\ldots,n),-)$. By analogy with (2.7), it is tempting to introduce a notation

$$N_{+|\rho(1,2,\ldots,n)|-}(\ell) \equiv \mathcal{K}_n(\ell)\Big|_{G_{ij}\to\frac{1}{2}\mathrm{sgn}^\rho_{ij}} \tag{3.18}$$

for the kinematic coefficients of the Parke–Taylor factors, and it will be argued in the next subsection that the resulting expansion

$$\mathcal{I}_{\rm SYM}(\ell) = \sum_{\rho\in S_n} \mathrm{PT}(+,\rho(1,2,\ldots,n),-)\, N_{+|\rho(1,2,\ldots,n)|-}(\ell) \tag{3.19}$$

identifies the $N_{+|\rho(1,2,\ldots,n)|-}(\ell)$ in (3.18) as BCJ master numerators of n-gon graphs. The counting of Parke–Taylor factors in (3.19) matches the $n!$ inequivalent n-gon diagrams in the partial-fraction representation of loop integrals, realizing the permutations of $1,2,\ldots,n$ in Fig. 3. Note, however, that (3.17) to (3.19) are based on representations of $\mathcal{K}_n(\ell)$ without any closed subcycles of G_{ij} which are known to exist for theories with at least four supercharges. It is not clear in which way the results extend to non-supersymmetric cases.

3.4. Analytic evaluation of CHY integrals and BCJ master numerators

Already at tree level, a central advantage of expressing the kinematic integrand $\mathcal{I}^{\rm tree}_{\rm SYM}$ in terms of Parke–Taylor factors is the availability of doubly-partial amplitudes (2.6) to evaluate the CHY integrals. Similarly, the Parke–Taylor form of the one-loop kinematic integrand (3.19) and the $\mathrm{d}\mu^{\rm tree}_{n+2}$ measure in (3.6) allow one to derive the one-loop propagators from doubly-partial amplitudes at tree-level with $(n{+}2)$ legs

$$\begin{aligned}\int \mathrm{d}\mu^{\rm tree}_{n+2}\ &\mathrm{PT}(\alpha(1,2,\ldots,n,+,-))\mathrm{PT}(\beta(1,2,\ldots,n,+,-))\\ &= \lim_{k_\pm\to\pm\ell} m^{\rm tree}[\alpha(1,2,\ldots,n,+,-)\,|\,\beta(1,2,\ldots,n,+,-)]\ .\end{aligned} \tag{3.20}$$

Thanks to the Berends–Giele recursion for $m^{\rm tree}[\cdot\,|\,\cdot]$ [55], this makes analytic evaluations of gauge-theory and gravity amplitudes tractable for a large number of external legs,

$$\begin{aligned}A(1,2,\ldots,n) &= \int \frac{\mathrm{d}^D\ell}{\ell^2}\int \mathrm{d}\mu^{\rm tree}_{n+2}\ \mathrm{PT}^{(1)}(1,2,\ldots,n)\,\mathcal{I}_{\rm SYM}(\ell)\\ &= \int \frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\sum_{\rho\in S_n} N_{+|\rho(1,2,\ldots,n)|-}(\ell)\end{aligned} \tag{3.21}$$

$$
\times \sum_{i=0}^{n-1} m^{\rm tree}[+,i{+}1,\ldots,n,1,2,\ldots,i,-\,|\,+,\rho(1,2,\ldots,n),-]
$$

$$
\mathcal{M}_{\rm SYM\otimes SYM} = \int \frac{{\rm d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \sum_{\rho,\tau\in S_n} N_{+|\rho(1,2,\ldots,n)|-}(\ell)\, \tilde N_{+|\tau(1,2,\ldots,n)|-}(\ell) \times m^{\rm tree}[+,\rho(1,2,\ldots,n),-\,|\,+,\tau(1,2,\ldots,n),-]\ . \tag{3.22}
$$

It is important to perform the limit $k_\pm \to \pm\ell$ *after* summing the permutations ρ,τ because the conspiration of different $N_{+|\rho(\ldots)|-}(\ell)$ leads to cancellations among spurious divergent propagators. In absence of maximal supersymmetry, forward-limit divergences will arise in (3.21), and a regularization scheme for cases with at least four supercharges is given around (3.24) as well as section 6.

As an example for a smooth forward limit $k_\pm \to \pm\ell$, let us reproduce the scalar box integral (2.22) in the four-point one-loop amplitude from a sum of six-point doubly-partial amplitudes at tree level following from (3.21) [35,39,38]

$$
\begin{aligned}
&\int \frac{{\rm d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \sum_{\rho\in S_4} \Big(m^{\rm tree}[+,1,2,3,4,-\,|\,+,\rho(1,2,3,4),-] + {\rm cyc}(1,2,3,4)\Big) \\
&= \int \frac{{\rm d}^D\ell}{\ell^2} \Big(\frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} + {\rm cyc}(1,2,3,4)\Big) = \int \frac{8\,{\rm d}^D\ell}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell+k_{123})^2}\ .
\end{aligned} \tag{3.23}
$$

This example illustrates that the kinematic limit must be performed *after* combining the permutations ρ: Several choices of ρ introduce divergent tadpole propagators such as s_{1234}^{-1} in $m^{\rm tree}[+,1,2,3,4,-\,|\,+,2,1,4,3,-] = (s_{12}s_{34}s_{12,\ell})^{-1} + (s_{12}s_{34}s_{1234})^{-1}$ which drop out after summing over ρ.

A more delicate treatment is needed for half- and quarter-maximal supersymmetry, where one factor of G_{12} occurs in the three-point correlator, and (3.15) leads to

$$
\begin{aligned}
\lim_{k_\pm\to\pm\ell} \sum_{\rho\in S_3} {\rm sgn}^\rho_{12}\, m^{\rm tree}[+,1,2,3,-\,|\,+,\rho(1,2,3),-] &= \frac{2}{s_{12}s_{12,\ell}} + \frac{1}{s_{1,\ell}s_{12,\ell}} \\
\lim_{k_\pm\to\pm\ell} \sum_{\rho\in S_3} {\rm sgn}^\rho_{13}\, m^{\rm tree}[+,1,2,3,-\,|\,+,\rho(1,2,3),-] &= \frac{1}{s_{1,\ell}s_{12,\ell}} \\
\lim_{k_\pm\to\pm\ell} \sum_{\rho\in S_3} {\rm sgn}^\rho_{23}\, m^{\rm tree}[+,1,2,3,-\,|\,+,\rho(1,2,3),-] &= \frac{2}{s_{23}s_{1,\ell}} + \frac{1}{s_{1,\ell}s_{12,\ell}}
\end{aligned} \tag{3.24}
$$

via five-point doubly-partial amplitudes. In the kinematic phase space of three massless particles, we obtain divergences from the pole in $s_{12} = \frac{1}{2}[k_3^2 - k_2^2 - k_1^2] = 0$. However, a compensating numerator of s_{12} can be extracted from the kinematic factor along with G_{12} [49], see Fig. 4. Hence, in a suitable regularization scheme due to Minahan [68] which is detailed in section 6, one can extract finite bubble contributions [50] from the terms $\sim (s_{12}s_{12,\ell})^{-1}$ and $\sim (s_{23}s_{1,\ell})^{-1}$ in (3.24).

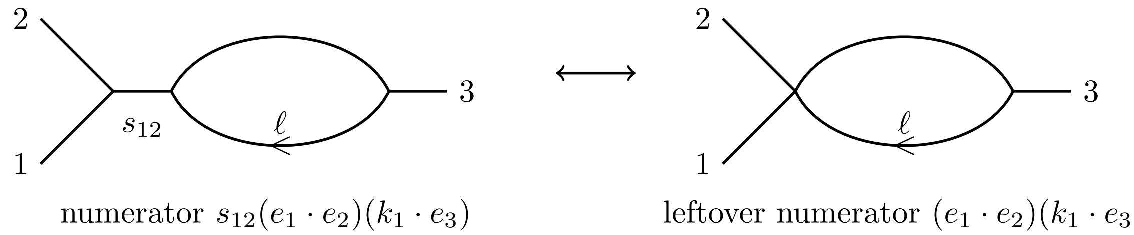

Fig. 4 The divergent propagator s_{ij}^{-1} in external bubbles is cancelled by a formally vanishing factor of s_{ij} in the kinematic numerator.

3.4.1. The BCJ duality in the new representation of Feynman integrals

Given that the expression (2.8) for n-point gauge-theory trees is known to yield cubic-diagram numerators which satisfy kinematic Jacobi identities [33,18], its forward limit in (3.21) must also realize the BCJ duality between color and kinematics [1,2]. In particular, by restricting the tree-level arguments of [5] to the forward limit, the cubic-diagram numerators in the representation (3.22) of the gravity amplitude are the double copies of the kinematic gauge-theory numerators from (3.21).

We emphasize that the present realization of the BCJ duality is adapted to the new representation (2.21) of Feynman integrals with all ℓ-dependent propagators but one linear in the loop momentum. In the original formulation of the loop-level BCJ duality [2] with propagators quadratic in ℓ, each cyclically inequivalent n-gon graph is counted as a single cubic diagram. As explained in section 2.5, the results of the CHY integrals in (3.20) organize one-loop amplitudes into n distinct cubic diagrams per cyclically inequivalent n-gon. They are interpreted as distinct tree-level diagrams with two extra legs at the n possible positions of ℓ, and their kinematic numerators are a priori unrelated.

Accordingly, the cubic-diagram expansion of one-loop gauge-theory and gravity amplitudes obtained from (3.21) and (3.22) takes the schematic form

$$\mathcal{M}_{\mathrm{SYM}\otimes U(N)} = \int \frac{\mathrm{d}^D \ell}{\ell^2} \sum_{i \in \Gamma_{n+2}} \frac{C_i\, N_i(\ell)}{\prod_{\mathrm{edges}\ \alpha_i} P^2_{\alpha_i}(\ell)} \tag{3.25}$$

$$\mathcal{M}_{\mathrm{SYM}\otimes\mathrm{SYM}} = \int \frac{\mathrm{d}^D \ell}{\ell^2} \sum_{i \in \Gamma_{n+2}} \frac{N_i(\ell)\, \tilde{N}_i(\ell)}{\prod_{\mathrm{edges}\ \alpha_i} P^2_{\alpha_i}(\ell)} \ , \tag{3.26}$$

where Γ_{n+2} denotes the set of $(n{+}2)$-point tree-level graphs i. The propagators $P^{-2}_{\alpha_i}(\ell)$ are linear in ℓ, and the color factors C_i are obtained by dressing each cubic vertex with f^{abc} while contracting the two extra legs $\pm\ell$ with a Kronecker delta. Note that all the n cubic diagrams in the partial-fraction decomposition of an n-gon yield identical color factors.

The numerators $N_i(\ell)$ are linear combinations of the $N_{+|\rho(1,2,\ldots,n)|-}(\ell)$ in (3.18) and (3.19) such as to solve the kinematic Jacobi relations depicted in Fig. 2. Of course, these $N_i(\ell)$ vanish for tadpole graphs in supersymmetric theories, and also for bubble- and triangle graphs in case of maximal supersymmetry. In summary, the expression for supersymmetric n-point correlators (3.19) in terms of Parke–Taylor factors identifies the kinematic coefficients $N_{+|\rho(1,2,\ldots,n)|-}(\ell)$ as BCJ master numerators of n-gon diagrams.

Since physical properties such as unitarity cuts and UV divergences are currently more evident in the standard representations of loop integrals in terms of propagators $(\ell{+}K)^{-2}$, it would be interesting to study the systematic recombination of the loop integrals in (3.21) and (3.22) to the standard form. Moreover, it would be desirable to preserve the color-kinematics duality in this recombination process. We have checked that the local five-point BCJ numerators of [9] for the conventional $(\ell + K)^{-2}$ propagators are reproduced in this recombination, and the situation at six points is discussed in section 5.5.

3.5. Partial integrands and one-loop BCJ-relations

The above construction of one-loop BCJ-representations was greatly alleviated by the tight analogy with tree level. In defining gauge invariant building blocks, however, this analogy is broken by the definition of color-ordered one-loop amplitudes $A(\ldots)$ of SYM through the sum (3.21) of *several* $(n{+}2)$-particle Parke–Taylor factors in $\mathrm{PT}^{(1)}(\ldots)$. In order to arrive at a manifestly gauge and diffeomorphism invariant formulation of the BCJ duality and double copy, it is convenient to study a more elementary quantity, the *partial integrand* [13]

$$a(\tau(1,2,\ldots,n,+,-)) \equiv \int \mathrm{d}\mu^{\mathrm{tree}}_{n+2}\ \mathrm{PT}(\tau(1,2,\ldots,n,+,-))\,\mathcal{I}_{\mathrm{SYM}}(\ell)\ . \tag{3.27}$$

Partial integrands isolate a single tree-level Parke–Taylor factor from their sum in $\mathrm{PT}^{(1)}(\ldots)$, still enjoy gauge invariance by the properties of $\mathcal{I}_{\mathrm{SYM}}(\ell)$ and allow to reconstruct color-stripped single-trace amplitudes (3.21) via

$$A(1,2,\ldots,n) = \int \frac{\mathrm{d}^D\ell}{\ell^2}\sum_{i=1}^{n} a(1,2,\ldots,i,-,+,i{+}1,\ldots,n)\,. \tag{3.28}$$

Choices of $\tau \in S_{n+2}$ with non-adjacent $+$ and $-$ appear in the CHY description of non-planar amplitudes

$$A(1,2,\ldots,j\,|\,j{+}1,\ldots,n) = \int \frac{\mathrm{d}^D \ell}{\ell^2} \sum_{\substack{\rho \in \mathrm{cyc}(1,2,\ldots,j)\\ \tau \in \mathrm{cyc}(j+1,\ldots,n)}} a(+,\rho(1,2,\ldots,j),-,\tau(j{+}1,\ldots,n)) \tag{3.29}$$

associated with double traces $\mathrm{Tr}(t^1 t^2 \ldots t^j)\mathrm{Tr}(t^{j+1}\ldots t^n)$. The partial integrands in (3.29) can be reduced to the cases in (3.28) with $+,-$ adjacent via Kleiss–Kuijf relations [69]

$$a(C,+,B,-) = (-1)^{|C|} a(+,(B \sqcup\!\sqcup C^t),-) \ , \tag{3.30}$$

where C^t and $|C|$ denote the transpose $c_p \ldots c_2 c_1$ and length p of the word $C \equiv c_1 c_2 \ldots c_p$, respectively. This reproduces the amplitude relations of [64] to express double-trace contributions at one loop in terms of single-trace amplitudes.

While the definition and Kleiss–Kuijf relations of the partial integrand are valid in absence of supersymmetry, we shall now explore the interplay with the Parke–Taylor organization of the supersymmetric gauge-theory integrands. Inserting (3.19) into (3.27) leads to the following cubic-diagram expansion analogous to (3.21),

$$a(\tau(1,\ldots,n,+,-)) = \lim_{k_\pm \to \pm \ell} \sum_{\rho \in S_n} N_{+|\rho(1,\ldots,n)|-}(\ell)\, m^{\mathrm{tree}}[\tau(1,\ldots,n,+,-)\,|\,+,\rho(1,\ldots,n),-] \, . \tag{3.31}$$

As an example with maximal supersymmetry, permutation invariance of the box numerator $N^{\mathrm{box}} \equiv s_{12}s_{23}A^{\mathrm{tree}}(1,2,3,4)$ [70] gives rise to the following diagrams in the four-point partial integrand with 16 supercharges [13]

$$\begin{aligned} a^{\max}(1,2,3,4,-,+) &= \frac{N^{\mathrm{box}}}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} \qquad (3.32)\\ a^{\max}(1,2,3,-,4,+) &= \frac{N^{\mathrm{box}}}{s_{1,\ell}s_{12,\ell}s_{4,\ell}} + \frac{N^{\mathrm{box}}}{s_{1,\ell}s_{12,\ell}s_{3,\ell}} + \frac{N^{\mathrm{box}}}{s_{1,\ell}s_{14,\ell}s_{3,\ell}} + \frac{N^{\mathrm{box}}}{s_{4,\ell}s_{14,\ell}s_{3,\ell}} \ . \end{aligned}$$

Moreover, three external gluons with polarization vectors e_i^m yield the following partial integrands with half-maximal supersymmetry [50,13]

$$\begin{aligned} a^{1/2}(1,2,3,-,+) &= \frac{\ell_m \big[e_1^m (k_2\cdot e_3)(k_3\cdot e_2) + (1\leftrightarrow 2,3)\big]}{s_{1,\ell}s_{12,\ell}} - \frac{(e_1\cdot e_2)(k_1\cdot e_3)}{s_{12,\ell}} - \frac{(e_2\cdot e_3)(k_2\cdot e_1)}{s_{1,\ell}}\\ a^{1/2}(1,2,-,3,+) &= 0 \ . \qquad (3.33) \end{aligned}$$

The bubble contributions $\sim (e_i \cdot e_j)$ are crucial for gauge invariance, and they stem from the cancellation of the (formally vanishing) invariant s_{12} between the doubly-partial amplitudes in (3.24) and the G_{12}-coefficient $s_{12}(e_1 \cdot e_2)(k_1 \cdot e_3)$ in $\mathcal{K}_3^{1/2}(\ell)$, see section 6.

In comparison to the color-ordered amplitude $A(1, 2, \ldots, n)$, the partial integrand $a(1, 2, \ldots, i, -, +, i{+}1, \ldots, n)$ only contains the subset of cubic diagrams with the loop momentum inserted between legs i and $i{+}1$. Hence, a single partial integrand cannot suffice to recombine to a Feynman integral with propagators of conventional $(\ell{+}K)^{-2}$-form.

However, as a major virtue of partial integrands, they inherit BCJ symmetry from their definition (3.27) via Parke–Taylor factor: In the same way as tree-level scattering equations yield the BCJ relations (2.12) for Parke–Taylor factors and thereby $A^{\rm tree}(\ldots)$, the one-loop scattering equations as a forward limit of their tree-level counterparts imply the one-loop BCJ relations:

$$\sum_{i=1}^{n-1} (\ell \cdot k_{12\ldots i}) \, \mathrm{PT}(1, 2, \ldots, i, +, i{+}1, \ldots, n, -) = 0 \tag{3.34}$$

Hence, the partial integrands (3.27) generalize the tree-level BCJ relations (2.11) to [13]

$$\sum_{i=1}^{n-1} (\ell \cdot k_{12\ldots i}) \, a(1, 2, \ldots, i, +, i{+}1, \ldots, n, -) = 0 \ , \tag{3.35}$$

as well as another topology mixing different orders of ℓ [13]

$$\sum_{i=2}^{n-1} (k_1 \cdot k_{23\ldots i}) \, a(2, 3, \ldots, i, 1, i{+}1, \ldots, n, -, +) = (\ell \cdot k_1) \, a(2, 3, \ldots, n, -, 1, +) \ . \tag{3.36}$$

One can immediately check that these BCJ relations are obeyed by the three- and four-point partial integrands (3.32) and (3.33). As will be detailed in section 5.5, the BCJ relations among partial integrands still hold in presence of anomalies: Since permutation invariance of $\mathcal{I}_{\rm SYM}(\ell)$ is broken by anomalies, all partial integrands must then be defined with respect to the same expression for $\mathcal{I}_{\rm SYM}(\ell)$ in (3.27).

Note that one-loop BCJ relations in the context of conventional $(\ell{+}K)^{-2}$ propagators have been discussed earlier in the field- [71] and string-theory literature [72]. As we will see in the following subsection, the partial integrands (3.27) along with the partial-fraction representation of loop integrals are tailored to enter a one-loop KLT formula. It would be interesting to reformulate the one-loop KLT formula in terms of $(\ell + K)^{-2}$ such as to incorporate the one-loop BCJ relations in [71,72], possibly along the lines of [73].

Given that the tree-level BCJ relations leave a basis of $(n-3)!$ independent permutations of $A^{\rm tree}(\ldots)$ [1], one may wonder about the analogous basis dimensions for partial integrands. The forward limit of the tree-level setup implies an upper bound of $(n-1)!$ independent partial integrands, but already the maximally supersymmetric four-point examples in (3.32) illustrate that this bound is usually not saturated: All the $a^{\max}(\tau(1,2,3,4,+,-))$ are proportional to $N^{\rm box} \equiv s_{12}s_{23}A^{\rm tree}(1,2,3,4)$, so they are all related by rational functions of k_j and ℓ. Similarly, we will find three linearly independent five-point partial integrands with maximal supersymmetry in section 5.4.

3.6. The correlator in a BCJ basis and one-loop KLT relations

We recall that integration by parts or scattering equations can be used to expand the tree-level correlator (2.7) in a BCJ basis of Parke–Taylor factors, leading to the KLT form (2.13). These steps will now be repeated at the one-loop order, assuming a minimum of four supercharges in one of the gauge theories.

Following the string calculations of [46], it is convenient to perform the integration-by-parts reduction of $\mathcal{I}_{\rm SYM}(\ell)$ at the level of the correlator $\mathcal{K}_n(\ell)$ whose σ-dependence is captured by the Green function G_{ij} in (3.9). After choosing a reference leg 1, the scattering equations (3.10) allow to eliminate all instances of G_{1j} with $j=2,3,\ldots,n$, i.e. the correlator $\mathcal{K}_n(\ell)$ is rendered independent on σ_1. This representation of $\mathcal{K}_n(\ell)$ without G_{1j} leaves no more freedom to apply further scattering equations without re-introducing σ_1, so all the kinematic factors must be gauge invariant. Moreover, all factors of ${\rm sgn}^\rho_{1j}=1$ disappear when converting to $\mathcal{I}_{\rm SYM}(\ell)$, see (3.17).

In absence of ${\rm sgn}^\rho_{1j}$, in turn, the coefficients of Parke–Taylor factors ${\rm PT}(+,\rho(1,\ldots,n),-)$ in $\mathcal{I}_{\rm SYM}(\ell)$ do not depend on the position of leg 1 within $\rho(1,2,\ldots,n)$. Hence, kinematic factors will be accompanied by

$$\begin{aligned}&{\rm PT}(+,1,2,3,\ldots,n,-)+{\rm PT}(+,2,1,3,\ldots,n,-)+{\rm PT}(+,2,3,1,\ldots,n,-)+\ldots \\ &\quad+{\rm PT}(+,2,3,\ldots,1,n,-)+{\rm PT}(+,2,3,\ldots,n,1,-)=-{\rm PT}(1,+,2,3,\ldots,n,-)\end{aligned} \tag{3.37}$$

and permutations in $2,3,\ldots,n$, using Kleiss–Kuijf relations in the second line. Hence, the elimination of G_{1j} in $\mathcal{K}_n(\ell)$ naturally leads to an $(n-1)!$-term expression for the correlator,

$$\mathcal{I}_{\rm SYM}(\ell)=-\sum_{\rho\in S_{n-1}}{\rm PT}(-,1,+,\rho(2,3,\ldots,n))\,\mathcal{C}_{+|\rho(2,3,\ldots,n)|-}(\ell)\ , \tag{3.38}$$

where $\mathcal{C}_{+|\rho(2,3,\ldots,n)|-}(\ell)$ can be viewed as a gauge invariant but non-local representation of an n-gon numerator. More precisely, $\mathcal{C}_{+|2,3,\ldots,n|-}(\ell)$ accompanies all the n diagrams where the external legs of the n-gon appear in the orders $2,3,\ldots,n$ with leg 1 inserted at an arbitrary position. The non-locality of $\mathcal{C}_{+|\rho(2,3,\ldots,n)|-}(\ell)$ stems from the elimination of G_{1j} via scattering equations, but this only generates poles in the *external* Mandelstam invariants $s_{1ij\ldots p}$, i.e. there are no ℓ-dependent propagators $s^{-1}_{ij\ldots p,\ell}$. Explicit four- to six-point expressions for $\mathcal{C}_{+|2,3,\ldots,n|-}(\ell)$ can be found in section 5.4, also see section 6.3 for examples with reduced supersymmetry.

In order to ensure that the correct partial integrands $a(+,\tau(2,3,\ldots,n),1,-)$ arise after performing CHY integrals over (3.38), the gauge invariant coefficients $\mathcal{C}_{+|\rho(2,3,\ldots,n)|-}(\ell)$ must by themselves be expressible in terms of partial integrands. The requirement is met by the expansion

$$\mathcal{C}_{+|\rho(2,3,\ldots,n)|-}(\ell) = S[\rho|\tau]_\ell \, a(+,\tau(2,3,\ldots,n),1,-) \tag{3.39}$$

which reproduces the pattern of the tree-level KLT formula upon insertion into (3.38):

$$\mathcal{I}_{\rm SYM}(\ell) = \sum_{\rho,\tau \in S_{n-1}} {\rm PT}(+,\rho(2,3,\ldots,n),-,1)\, S[\rho|\tau]_\ell \, a(+,\tau(2,3,\ldots,n),1,-) \ . \tag{3.40}$$

The $(n{-}1)!\times(n{-}1)!$-matrix $S[\rho|\tau]_\ell \equiv S[\rho(2,3,\ldots,n)|\tau(2,3,\ldots,n)]_\ell$ follows the functional form of the tree-level momentum kernel (2.10), where the loop momentum now enters as the pivot leg. We are using that, before performing the forward limit $k_\pm \to \pm\ell$ in (3.20), $S[\rho|\tau]_\ell$ is the inverse of the $(n{-}1)! \times (n{-}1)!$ matrix of doubly-partial amplitudes $m^{\rm tree}[+,\rho(2,3,\ldots,n),-,1\,|\,+,\tau(2,3,\ldots,n),1,-]$, see (2.14).

The KLT form (3.40) of the supersymmetric gauge-theory integrand can be used to derive the analogous KLT formula for loop integrands in supergravity. We are using the permutation symmetric and gauge invariant definition of a supergravity integrand $m_n(\ell)$ in the CHY framework,

$$m_n(\ell) \equiv \int {\rm d}\mu^{\rm tree}_{n+2}\, \mathcal{I}_{\rm SYM}(\ell)\, \tilde{\mathcal{I}}_{\rm SYM}(\ell) \ , \qquad \mathcal{M}_{\rm SYM\otimes SYM} = \int \frac{{\rm d}^D \ell}{\ell^2} m_n(\ell) \ , \tag{3.41}$$

where $\mathcal{I}_{\rm SYM}(\ell)$ and $\tilde{\mathcal{I}}_{\rm SYM}(\ell)$ may refer to different gauge theories. Similar to (3.28), the definition (3.41) amputates the overall quadratic propagator ℓ^{-2} in a partial-fraction representation of Feynman integrals [13]. Next, we insert the minimal $(n{-}1)!$ form (3.40) of the left-moving and supersymmetric gauge-theory integrand into (3.41),

$$m_n(\ell) = \sum_{\rho,\tau \in S_{n-1}} a(+,\rho(2,\ldots,n),1,-)\, S[\rho|\tau]_\ell \int {\rm d}\mu^{\rm tree}_{n+2}\, {\rm PT}(+,\tau(2,\ldots,n),-,1)\, \tilde{\mathcal{I}}_{\rm SYM}(\ell) \ . \tag{3.42}$$

Then, the Parke–Taylor factor on the right-hand side suggests to apply the definition (3.27) of the partial integrand for $\tilde{\mathcal{I}}_{\rm SYM}(\ell)$, whose validity does not rely on supersymmetry. In this way, one arrives at the one-loop KLT formula [13]

$$m_n(\ell) = \sum_{\rho,\tau \in S_{n-1}} a(+,\rho(2,3,\ldots,n),1,-)\, S[\rho|\tau]_\ell \, \tilde{a}(+,\tau(2,3,\ldots,n),-,1) \ , \qquad (3.43)$$

whose present derivation applies to any double copy of gauge theories with at least four supercharges on one side. For the case with zero supersymmetry, we expect that (3.43) still holds, but a careful proof including a suitable treatment of forward-limit divergences is relegated to the future.

4. One-loop RNS correlators for field-theory amplitudes

In this section, we will investigate one-loop correlators (2.20) for field-theory amplitudes in the RNS formulation of the underlying ambitwistor string or superstring. We will on the one hand point out universal structures that do not depend on the amount of supersymmetry and on the other hand describe the simplifications in supersymmetric theories. In particular, the simple dependence of supersymmetric correlators on the punctures which has been central to the discussion in sections 3.2 and 3.3 will be derived.

While external fermions will be addressed in section 5 by the supersymmetric correlators in pure-spinor superspace, we will focus the one-loop RNS correlators for external bosons in this section. Their multiparticle instances have been firstly discussed in [51] for maximally supersymmetric superstring theory (see also [52,53,74]), and four-point string amplitudes with reduced supersymmetry can be found in [75,76,49]. A major challenge in the RNS variables is to manifest the supersymmetry-induced simplifications when combining different spin structures, the boundary conditions for the worldsheet spinors $\psi^m(z)$ in the RNS formulation as $z \to z+1$ and $z \to z+\tau$.

Even before performing the sum over spin structures, we find that at the degeneration $\tau \to i\infty$ of the torus relevant for the field-theory limit, the correlators simplify significantly: They reduce to polynomials in the Green function G_{ij} on the nodal Riemann sphere defined in (3.9), with *local* functions of external polarizations as their coefficients. After performing the spin sum, the final form of the polynomials depends on the amount of supersymmetry, and we present complete correlators for gauge theories with maximal as well as half- (or quarter-)maximal supersymmetry in various dimensions.

Scattering equations and algebraic identities of G_{ij}'s can be used to reduce these monomials of G_{ij} to a basis which leads to the KLT relations described in section 3.6. In this way, we obtain a basis expansion of the correlator with *gauge invariant* but non-local coefficients for external bosons, which can be nicely packaged using Berends–Giele currents. Using the supersymmetrized version of these Berends–Giele currents (see section 5), we will later present explicit results for BCJ master numerators in pure-spinor superspace whose bosonic components can be matched with the one-loop correlators in this section.

4.1. Structure of RNS correlators on a torus

As a spurious difference between the correlators of the ambitwistor string and the superstring, the bosonic worldsheet fields x^m do not exhibit any two-point contractions in the former case [29,23]. At tree level this difference is known to wash out after removing double poles in σ_{ij} via integration by parts [29] and expanding the correlators in terms of Parke–Taylor factors. Since the same kind of integration by parts can be performed at arbitrary genus, there is no loss of generality in starting with the one-loop RNS correlator of the ambitwistor string for n external gluons [23], the same end results would have been obtained from the superstring.

The parity-even part of the n-point RNS correlator $\mathbf{K}_n$ can be expanded in terms of $n!$ gauge invariant terms

$$\mathbf{K}_n(\ell|\tau) = \sum_{\rho\in S_n} \mathbf{R}_\rho(\ell|\tau) \ , \quad \text{with} \quad \rho = (i)\, \cdots\, (j)\, I\, \cdots\, J \ , \tag{4.1}$$

$$\mathbf{R}_{(i)\,\cdots\,(j)\,I\,\cdots\,J}(\ell|\tau) \,\equiv\, \mathbf{c}_i(\ell|\tau)\, \cdots\, \mathbf{c}_j(\ell|\tau)\, \mathrm{tr}(f_I)\, \cdots\, \mathrm{tr}(f_J)\, \mathbf{G}_{I,\ldots,J}(\tau) \ , \tag{4.2}$$

where the summand $\mathbf{R}_\rho$ is defined according to the unique decomposition of ρ into disjoint cycles[3]. Each length-one cycle or fixed point $(i), \cdots, (j)$ of ρ contributes a factor of

$$\mathbf{c}_i(\ell|\tau) \equiv 2\pi i\, (e_i\cdot \ell) + \sum_{\substack{j=1\\ j\neq i}}^{n} (e_i\cdot k_j)\, \partial \log\theta_1(z_{ij},\tau) \ , \tag{4.3}$$

[3] For example, the sum $\rho\in S_3$ relevant to $n=3$ is organized in terms of the cycles $\rho =$ (1)(2)(3), (1)(23), (2)(31), (3)(12), (123), (321), leading to the following expansion (4.1):

$$\begin{aligned}\mathbf{K}_3(\ell|\tau) &= \mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{G}_\emptyset + \mathrm{tr}(f_{(123)})\mathbf{G}_{(123)} + \mathrm{tr}(f_{(321)})\mathbf{G}_{(321)}\\ &\quad + \mathbf{c}_1\mathrm{tr}(f_{(23)})\mathbf{G}_{(23)} + \mathbf{c}_2\mathrm{tr}(f_{(31)})\mathbf{G}_{(31)} + \mathbf{c}_3\mathrm{tr}(f_{(12)})\mathbf{G}_{(12)}\ .\end{aligned}$$

which captures the contributions from the worldsheet bosons and is the only place that the loop momentum appears. Cycles of length two and larger, on the other hand, are denoted by $I,\ldots,J$, e.g. $I=(i_1 i_2\cdots i_a)$, $J=(j_1 j_2\cdots j_b)$ with $a,b\geq 2$. The associated kinematic functions of momenta and polarizations are traces (over Lorentz indices m,n) of linearized field strengths

$$\mathrm{tr}(f_I)\equiv -\frac{1}{2}\,\mathrm{tr}(f_{i_1}f_{i_2}\cdots f_{i_a})\,,\qquad \text{with}\quad f_i^{mn}=k_i^m e_i^n-e_i^m k_i^n\ . \tag{4.4}$$

Finally, the accompanying functions of the punctures boil down to two-point contractions

$$S_\nu(x,\tau)\equiv\frac{\theta_1'(0,\tau)\theta_\nu(x,\tau)}{\theta_1(x,\tau)\theta_\nu(0,\tau)} \tag{4.5}$$

of the worldsheet spinors with even spin structures $\nu=2,3,4$:

$$\mathbf{G}_N(x_1,x_2,\ldots,x_N|\tau)\equiv\sum_{\nu=2}^{4}(-1)^{\nu-1}\left[\frac{\theta_\nu(0,\tau)}{\theta_1'(0,\tau)}\right]^4 S_\nu(x_1,\tau)S_\nu(x_2,\tau)\ldots S_\nu(x_N,\tau)\ . \tag{4.6}$$

More precisely, cycles $I=(i_1 i_2\cdots i_a)$, $J=(j_1 j_2\cdots j_b)$ of length $a,b\geq 2$ yield

$$\mathbf{G}_{I,\ldots,J}(\tau)\equiv\mathbf{G}_{a+\ldots+b}(x_1,x_2,\ldots,x_a,\ldots,y_1,y_2,\ldots,y_b|\tau)\Big|^{x_k=z_{i_k}-z_{i_{k+1}}}_{y_k=z_{j_k}-z_{j_{k+1}}} \tag{4.7}$$

subject to cyclic identification $i_{a+1}=i_1$ and $j_{b+1}=j_1$. By (4.7), each cycle $I,\ldots,J$ in $\mathbf{G}_{I,\ldots,J}(\tau)$ yields a group of arguments x_j in (4.6) which add up to zero, $\sum_{k\in I}x_k=\sum_{k\in J}x_k=0$. In particular, we always have $x_1+x_2+\ldots+x_N$ in (4.6) and can derive all cases with multiple cycles from specializations of the single-cycle configuration.

The even Jacobi theta functions $\theta_{\nu=2,3,4}(z,\tau)$ entering (4.5) are regular at $z=0$ and defined in appendix C. They conspire in the sum over the even spin structures $\nu=2,3,4$ along with the partition functions $(\frac{\theta_\nu(0,\tau)}{\theta_1'(0,\tau)})^4$. Capturing the effect of spacetime supersymmetry, the expressions in (4.6) simplify drastically after performing the spin sums as in $0=\mathbf{G}_0(\emptyset|\tau)=\mathbf{G}_2(x_1,x_2|\tau)=\mathbf{G}_3(x_1,x_2,x_3|\tau)$ as well as [51]

$$\mathbf{G}_4(x_1,x_2,x_3,x_4|\tau)=1\ ,\qquad \mathbf{G}_5(x_1,x_2,\ldots,x_5|\tau)=\sum_{j=1}^{5}\partial\log\theta_1(x_j,\tau)\ . \tag{4.8}$$

The higher-multiplicity systematics has firstly been discussed in [51] (also see [52,74]) and was later organized via combinations of Eisenstein series and elliptic functions [54] explicitly known to all multiplicities. One can see from the results of the references that the spin sum in $\mathbf{G}_N(x_1,x_2,\ldots,x_N|\tau)$ only leaves $N-4$ simultaneous poles in the arguments x_k.

The contributions from the odd spin structure $\nu=1$ yield the parity odd part of $\mathbf{K}_n$ which vanishes in spacetime dimensions $D<9$ and will be discussed separately in section 4.6.

4.2. Structure of RNS correlators on a nodal Riemann sphere

The CHY integrands (2.20) for maximally supersymmetric gauge theories can be obtained from the degeneration limit

$$\mathcal{K}_n \equiv \frac{1}{(2\pi i)^{n-4}} \lim_{\tau \to i\infty} \mathbf{K}_n(\tau) \tag{4.9}$$

of (4.1) which preserves the expansion in terms of $n!$ separately gauge invariant terms

$$\mathcal{K}_n = \sum_{\rho \in S_n} \mathcal{R}_\rho \ , \quad \text{with} \quad \rho = (i) \cdots (j)\, I \cdots J \ , \tag{4.10}$$

$$\mathcal{R}_{(i) \cdots (j)\, I \cdots J} \equiv c_i(\ell) \cdots c_j(\ell) \operatorname{tr}(f_I) \cdots \operatorname{tr}(f_J)\, \mathcal{G}_{I,\ldots,J}(\sigma) \ . \tag{4.11}$$

The unique decomposition of ρ into disjoint cycles is explained below (4.1), so it remains to investigate the behavior of the τ-dependent constituents when the toroidal worldsheet degenerates to a nodal Riemann sphere. The loop-momentum dependent part (4.3) is easily seen to degenerate into

$$c_i(\ell) \equiv \frac{1}{2\pi i} \lim_{\tau \to i\infty} \mathbf{c}_i(\ell|\tau) = e_i \cdot \ell + \sum_{\substack{j=1 \\ j \neq i}}^{n} e_i \cdot k_j \ \frac{\sigma_i}{\sigma_{ij}} = e_i \cdot \ell + \sum_{\substack{j=1 \\ j \neq i}}^{n} e_i \cdot k_j \ G_{ij} \ , \tag{4.12}$$

which is manifestly gauge invariant on the support of the one-loop scattering equations (3.10). Note that we have used momentum conservation and transversality of e_i in passing to the representation in terms of G_{ij}.

The traces $\operatorname{tr}(f_I)$ over linearized field strengths, see (4.4), are accompanied by spin-summed correlators over worldsheet fermions detailed around (4.6) and (4.7),

$$\mathcal{G}_{I_1,I_2,\ldots,I_k} \equiv \frac{1}{(2\pi i)^{N-4}} \lim_{\tau \to i\infty} \mathbf{G}_N(x_1, x_2, \ldots, x_N|\tau)\Big|_{\sum_{i \in I_j} x_i = 0} \ . \tag{4.13}$$

The cycles $I_1, I_2, \ldots, I_k$ track the subsets of x_i that add up to zero, and the remarkable simplifications in the $\tau \to i\infty$ limit of the all-multiplicity spin sums (4.6) will be presented in the next subsection. Again, any instance of (4.13) with multiple cycles I_j can be obtained by specializing the single-cycle configuration

$$\mathcal{G}_N \equiv \mathcal{G}_{(12\ldots N)} = \frac{1}{(2\pi i)^{N-4}} \lim_{\tau \to i\infty} \mathbf{G}_N(x_1, x_2, \ldots, x_N|\tau)\Big|_{x_1+x_2+\ldots+x_N=0} \ . \tag{4.14}$$

It is useful to illustrate the expansion (4.10) of $\mathcal{K}_n$ with the $n = 3, 4$ examples,

$$
\begin{aligned}
\mathcal{K}_3 &= c_1c_2c_3\mathcal{G}_\emptyset + c_1\mathrm{tr}(f_{(23)})\mathcal{G}_{(23)} + c_2\mathrm{tr}(f_{(31)})\mathcal{G}_{(31)} + c_3\mathrm{tr}(f_{(12)})\mathcal{G}_{(12)} \\
&\quad + \mathrm{tr}(f_{(123)})\mathcal{G}_{(123)} + \mathrm{tr}(f_{(321)})\mathcal{G}_{(321)} \ , \qquad (4.15)\\
\mathcal{K}_4 &= c_1c_2c_3c_4\mathcal{G}_\emptyset + \big(c_1c_2\mathrm{tr}(f_{(34)})\mathcal{G}_{(34)} + 5 \text{ more}\big) + \big(c_1\mathrm{tr}(f_{(234)})\mathcal{G}_{(234)} + 7 \text{ more}\big) \\
&\quad + \big(\mathrm{tr}(f_{(1234)})\mathcal{G}_{(1234)} + 5 \text{ more}\big) + \big(\mathrm{tr}(f_{(12)})\mathrm{tr}(f_{(34)})\mathcal{G}_{(12)(34)} + 2 \text{ more}\big) \ ,
\end{aligned}
$$

and the analogous five-point expressions along with the resulting numerators are spelt out in appendix D.

Recall that the x's in $\mathbf{G}_N$ denote differences of z's on the torus; here in the $\tau \to i\infty$ limit, as a slight abuse of notation, we will continue to denote the arguments of $\mathcal{G}_N$ as $x_1, \ldots x_N$. Now they simply refer to N pairs of labels $x_1 = (i_1, j_1), \ldots, x_N = (i_N, j_N)$ determined by the cycle structure $I, \ldots, J$. For example, we have $x_1 = (1,2), x_2 = (2,1)$ for $\mathcal{G}_{(12)}$, $x_1 = (1,2), x_2 = (2,3), x_3 = (3,4), x_4 = (4,1)$ for $\mathcal{G}_{(1234)}$ and $x_1 = (1,2), x_2 = (2,1), x_3 = (3,4), x_4 = (4,3)$ for $\mathcal{G}_{(12)(34)}$.

4.3. Spin sums in maximally supersymmetric correlators

This section is devoted to the impact of the $\tau \to i\infty$ limit on the fermionic correlators $\mathbf{G}_N$ in the expansion of $\mathbf{K}_n(\tau)$: all the elliptic functions in the expressions (4.6) for $\mathbf{G}_N$ simplify drastically, and their degenerate versions $\mathcal{G}_N$ become *polynomials* in G_{ij}'s. More precisely, each cyclic structure $\{x_1, x_2, \ldots, x_N\}$ of length N gives rise to symmetric polynomials in $G_{x_1}, G_{x_2}, \ldots, G_{x_N}$ of degree $0 \leq k \leq N$, and it is convenient to introduce the notation

$$
\Sigma_k(x_1, x_2, \ldots, x_N) \equiv \sum_{1\leq\alpha_1<\alpha_2<\ldots<\alpha_k\leq N} G_{x_{\alpha_1}}\, G_{x_{\alpha_2}} \, \cdots \, G_{x_{\alpha_k}} \qquad (4.16)
$$

with $\Sigma_0(x_1, \ldots, x_N) \equiv 1$. For example, the symmetric polynomials at length $N = 2$ with $x_1 = (1,2), x_2 = (2,3)$ are $\Sigma_1(x_1, x_2) = G_{12} + G_{21} = 0$ and $\Sigma_2(x_1, x_2) = G_{12}G_{21} = -G_{12}^2$. For $N = 3$ with $x_1 = (1,2), x_2 = (2,3), x_3 = (3,1)$, we have $\Sigma_1(x_1, x_2, x_3) = G_{12} + G_{23} + G_{31}$ and $\Sigma_3(x_1, x_2, x_3) = G_{12}\, G_{23}\, G_{31}$, whereas $\Sigma_2(x_1, x_2, x_3)$ is a constant by the Fay identity (3.11):

$$
\Sigma_2(x_1, x_2, x_3) = G_{12}G_{23} + G_{23}G_{31} + G_{31}G_{12} = -\frac{1}{4} \ . \qquad (4.17)
$$

The maximally supersymmetric spin sums (4.13) turn out to yield extremely simple linear combinations of the polynomials $\Sigma_k \equiv \Sigma_k(x_1, x_2, \ldots, x_N)$ in (4.16). By taking the $\tau \to i\infty$ limit of the elliptic functions in $\mathbf{G}_N$ [54], we find

$$\begin{aligned}\mathcal{G}_N &= \Sigma_{N-4} + \frac{1}{2}\Sigma_{N-6} + \frac{3}{16}\Sigma_{N-8} + \frac{1}{16}\Sigma_{N-10} + \cdots \\ &= \sum_{m=0}^{\lfloor (N-4)/2 \rfloor} \frac{m+1}{4^m}\Sigma_{N-4-2m} \ ,\end{aligned} \tag{4.18}$$

where the all-multiplicity conjecture has been checked up to $N = 20$. In other words, the only contributing degrees in G_{ij} are $N{-}4$, $N{-}6$, $N{-}8, \ldots$, and all cases $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ below four points vanish, $\mathcal{G}_N = 0 \ \forall \ N < 4$. Let's spell out some simple non-vanishing examples, starting with $\mathcal{G}_4 = \Sigma_0 = 1$. The first non-trivial dependence on x_i arises at $N = 5$, where two types of inequivalent cycle structures occur,

$$\mathcal{G}_5 = \sum_{i=1}^{5} G_{x_i} \ , \quad \text{e.g.} \quad \mathcal{G}_{(12345)} = \sum_{i=1}^{5} G_{i,i+1} \ , \quad \mathcal{G}_{(12)(345)} = G_{34} + G_{45} + G_{53} \ , \tag{4.19}$$

in lines with the $\tau \to i\infty$ limit of (4.8). A complete expression for the resulting five-point correlator is assembled in appendix D.

The single-cycle expressions for $n = 6, 7$ are

$$\mathcal{G}_6 = \sum_{i<j}^{6} G_{x_i} G_{x_j} + \frac{1}{2} \ , \qquad \mathcal{G}_7 = \sum_{i<j<k}^{7} G_{x_i} G_{x_j} G_{x_k} + \frac{1}{2}\sum_{i=1}^{7} G_{x_i} \ , \tag{4.20}$$

which can be specialized to multiple cycles such as

$$\begin{aligned}\mathcal{G}_{(123)(456)} &= (G_{12} + G_{23} + G_{31})(G_{45} + G_{56} + G_{64}) \ , \\ \mathcal{G}_{(12)(34)(56)} &= \frac{1}{2} - G_{12}^2 - G_{34}^2 - G_{56}^2 \, , \\ \mathcal{G}_{(12)(3456)} &= \frac{1}{2} - G_{12}^2 + \big[G_{34}G_{45} + \text{cyc}(3,4,5,6)\big] + G_{34}G_{56} + G_{45}G_{63} \, , \\ \mathcal{G}_{(12)(34)(567)} &= \left(\frac{1}{2} - G_{12}^2 - G_{34}^2\right)(G_{56} + G_{67} + G_{75}) + G_{56}G_{67}G_{75} \, , \ \text{etc.} \ ,\end{aligned} \tag{4.21}$$

where the Fay identity (4.17) has been used in the first and last line.

As is well-known from the superstring literature, the spin sums (4.18) expose the structure of maximally supersymmetric field-theory correlators and manifest their power-counting of loop momenta: The n-point correlator (4.10) comprises spin-summed $\mathcal{G}_N$ of highest power G_{ij}^{N-4} along with $n-N$ factors of $c_i(\ell)$ defined in (4.12). Given that each $c_i(\ell)$ is linear in ℓ and G_{ij}, the correlator $\mathcal{K}_n$ is a polynomial of degree $n-4$ in ℓ and G_{ij}. More precisely, contributions with k powers of ℓ are accompanied by at most $n-4-k$ powers of G_{ij}.

Moreover, after evaluating the CHY integral, the bounds on G_{ij} imply the absence of triangles, bubbles and tadpoles in maximally supersymmetric amplitudes, see e.g. [70,53] for the analogous superstring discussion: In each term of the doubly-partial amplitudes (3.20), the number of *external* propagators (i.e. those independent on ℓ) is bounded by the powers of G_{ij}. With at most $n-4$ powers of G_{ij}, at least three of the propagators from the doubly-partial amplitudes depend on ℓ, corresponding to box diagrams and higher n-gons. Hence, the spin sum (4.18) is responsible for the famous no-triangle property [64].

4.4. CHY correlators with reduced supersymmetries

4.4.1. The anatomy of spin sums

In fact, some of the simplifications seen in the previous section even work for individual spin structures $\nu = 2, 3, 4$ before the spin sum, thus they also apply to cases with reduced supersymmetry. Once we denote the cycles of fermion Green functions (4.5) with spin structures ν by

$$\mathbf{S}_N^{(\nu)}(x_1, x_2, \ldots, x_N|\tau) \equiv S_\nu(x_1, \tau) S_\nu(x_2, \tau) \ldots S_\nu(x_N, \tau) \ , \tag{4.22}$$

it turns out that the $\tau \to i\infty$ limit of (4.6) picks out three independent contributions

$$\begin{aligned}
\mathcal{G}_N^{\rm f} &\equiv \frac{1}{(2\pi i)^N} \mathbf{S}_N^{(\nu=2)}(x_1, x_2, \ldots, x_N|\tau)\,\big|_{q^0} \\
\mathcal{G}_N^{\rm s} &\equiv \frac{1}{(2\pi i)^N} \mathbf{S}_N^{(\nu=3)}(x_1, x_2, \ldots, x_N|\tau)\,\big|_{q^0} \quad = +\frac{1}{(2\pi i)^N} \mathbf{S}_N^{(\nu=4)}(x_1, x_2, \ldots, x_N|\tau)\,\big|_{q^0} \\
\mathcal{G}_N^{\rm v} &\equiv \frac{1}{(2\pi i)^N} \mathbf{S}_N^{(\nu=3)}(x_1, x_2, \ldots, x_N|\tau)\,\big|_{q^{1/2}} = -\frac{1}{(2\pi i)^N} \mathbf{S}_N^{(\nu=4)}(x_1, x_2, \ldots, x_N|\tau)\,\big|_{q^{1/2}} \ ,
\end{aligned} \tag{4.23}$$

which remain inert for any amount of supersymmetry [35,38]. In an expansion w.r.t. $q \equiv e^{2\pi i \tau}$, the notation $|_{q^0}$ and $|_{q^{1/2}}$ in (4.23) refers to the coefficients of q^0 and $q^{1/2}$, respectively, and we have used the fact that these lowest orders of $\mathbf{S}_N^{(3)}$ and $\mathbf{S}_N^{(4)}$ are simply related to each other.

In terms of the symmetric polynomials Σ_k in (4.16), all of $\mathcal{G}_N^{\rm f}, \mathcal{G}_N^{\rm s}$ and $\mathcal{G}_N^{\rm v}$ can be identified as extremely simple linear combinations:

$$
\begin{aligned}
\mathcal{G}_N^{\rm f} &= \Sigma_N \ , \\
\mathcal{G}_N^{\rm s} &= \Sigma_N + \frac{1}{4}\Sigma_{N-2} + \frac{1}{16}\Sigma_{N-4} + \frac{1}{64}\Sigma_{N-6} + \ldots = \sum_{m=0}^{\lfloor N/2 \rfloor} \frac{\Sigma_{N-2m}}{4^m} \ , \\
\mathcal{G}_N^{\rm v} &= -2\Big(\Sigma_{N-2} + \frac{1}{2}\Sigma_{N-4} + \frac{3}{16}\Sigma_{N-6} + \ldots\Big) = -2 \sum_{m=0}^{\lfloor (N-2)/2 \rfloor} \frac{(m+1)\Sigma_{N-2-2m}}{4^{m+1}} \ ,
\end{aligned}
\tag{4.24}
$$

as can be straightforwardly checked through the leading q-orders of S_ν spelt out in (C.2). All contributions $\Sigma_{N-1}, \Sigma_{N-3}, \Sigma_{N-5}, \ldots$ whose parity is opposite to Σ_N drop out, so the sum extends down to Σ_1 for odd N and Σ_0 for even N.

The partition functions $(-1)^{\nu-1}\left[\frac{\theta_\nu(0,\tau)}{\theta_1'(0,\tau)}\right]^4$ which multiply (4.22) reflect maximal supersymmetry. Their leading orders in q are spelt out in (C.3) and combine the building blocks in (4.23) to

$$
\mathcal{G}_N = 16(\mathcal{G}_N^{\rm s} - \mathcal{G}_N^{\rm f}) + 2\mathcal{G}_N^{\rm v} \ . \tag{4.25}
$$

By inserting the expansions (4.24), one recovers the organization (4.18) of $\mathcal{G}_N$ in terms of symmetric polynomials Σ_k. The highest degree $k = N-4$ in $\mathcal{G}_N$ results from cancellation of both Σ_N and Σ_{N-2} due to the interplay between bosons and fermions as well as the GSO projection in the NS sector.

4.4.2. Super-Yang–Mills theories

Since supersymmetry breaking only affects the one-loop correlator through a modification of the partition function in (4.6), the structure of the correlator (4.10) and (4.11) is universal. In scenarios with reduced supersymmetry, we simply adjust the spin sums $\mathcal{G}_{I,\ldots,J} \rightarrow \mathcal{G}^*_{I,\ldots,J}$ to the particle content of the theory (indicated by the placeholder *):

$$
\mathcal{K}_n^* = \sum_{\rho \in S_n} \mathcal{R}_\rho^* \ , \quad \text{with} \quad \rho = (i)\cdots(j)\, I \cdots J \ , \tag{4.26}
$$

$$
\mathcal{R}^*_{(i)\cdots(j)\, I\cdots J} \equiv c_i(\ell)\cdots c_j(\ell)\,{\rm tr}(f_I)\cdots{\rm tr}(f_J)\,\mathcal{G}^*_{I,\ldots,J} \ . \tag{4.27}
$$

The four-point instances of the corresponding superstring amplitudes in compactifications with reduced supersymmetry have been discussed and simplified in [75,76,49]. In particular, a systematic method to express the all-multiplicity spin sums in terms of the Eisenstein series and elliptic functions of [74,54] has been given in [49].

The results of the previous subsection pave the way to extending the above analysis of spin sums to cases with less or no supersymmetries. As pointed out in [38], the linear combinations of $\mathcal{G}_N^{\rm f}, \mathcal{G}_N^{\rm s}$ and $\mathcal{G}_N^{\rm v}$ in (4.25) can be adjusted such as to describe any number of d-dimensional massless vectors, fermions and scalars running in the loop. Every scalar and fermionic degree of freedom, for instance, contributes with $2\mathcal{G}_N^{\rm s}$ and $-2\mathcal{G}_N^{\rm f}$ to the spin sum, respectively. A d-dimensional vector, one the other hand, yields the combination $2\mathcal{G}_N^{\rm v} + 2(d-2)\mathcal{G}_N^{\rm s}$ [38].

Accordingly, the spin sums of pure SYM theories are given by

$$\mathcal{G}_N^{\alpha-\rm SYM} = 2\mathcal{G}_N^{\rm v} + 16\alpha(\mathcal{G}_N^{\rm s} - \mathcal{G}_N^{\rm f}) \ , \qquad \alpha = 1, \frac{1}{2}, \frac{1}{4} \ , \tag{4.28}$$

where $\alpha = 1, \frac{1}{2}, \frac{1}{4}$ for maximal, half-maximal and quarter-maximal supersymmetry, respectively. These values of α control the number of fermions, and the rigid combinations of $\mathcal{G}_N^{\rm s} - \mathcal{G}_N^{\rm f}$ ensure the same number of bosonic degrees of freedom while keeping a single vector in the multiplet. Explicitly, the spin sums of half- and quarter-maximal SYM read

$$\begin{aligned} \mathcal{G}_N^{\frac{1}{2}-\rm SYM} &= 2\Sigma_{N-2} + \frac{3}{2}\Sigma_{N-4} + \frac{5}{8}\Sigma_{N-6} + \frac{7}{32}\Sigma_{N-8} + \ldots \\ &= 2\sum_{m=0}^{\lfloor (N-2)/2 \rfloor} \frac{2m+1}{4^m}\Sigma_{N-2-2m} \end{aligned} \tag{4.29}$$

$$\begin{aligned} \mathcal{G}_N^{\frac{1}{4}-\rm SYM} &= 3\Sigma_{N-2} + \frac{7}{4}\Sigma_{N-4} + \frac{11}{16}\Sigma_{N-6} + \frac{15}{64}\Sigma_{N-8} + \ldots \\ &= \sum_{m=0}^{\lfloor (N-2)/2 \rfloor} \frac{4m+3}{4^m}\Sigma_{N-2-2m} \end{aligned} \tag{4.30}$$

and can be reconciled with the string-theory results of [49] for certain choices of the compactification details. From the difference of (4.29) and (4.30), the contributions of a spin-$\frac{1}{2}$ multiplet with two scalar and fermionic degrees of freedom each is identified as

$$\mathcal{G}_N^{2+2} = 4\big(\mathcal{G}_N^{\rm s} - \mathcal{G}_N^{\rm f}\big) = \Sigma_{N-2} + \frac{1}{4}\Sigma_{N-4} + \frac{1}{16}\Sigma_{N-6} + \ldots = \sum_{m=0}^{\lfloor (N-2)/2 \rfloor} \frac{1}{4^m}\Sigma_{N-2-2m} \ . \tag{4.31}$$

Therefore, as long as a minimum of four supercharges is preserved, Σ_N always drops out from (4.28), and the degree k of the polynomials Σ_k does not exceed $N-2$ (with the additional cancellation of Σ_{N-2} in case of (4.25) with maximal supersymmetry).

4.4.3. Pure Yang–Mills theory

For pure Yang–Mills theory, i.e. in absence of supersymmetry, one is left with the bosonic truncation of (4.24), where the contribution from $\mathcal{G}^{\rm f}_N$ is set to zero. For a single gauge boson in d spacetime dimensions, the relevant spin sum is

$$\begin{aligned}\mathcal{G}^{d-\rm YM}_N &= 2\mathcal{G}^{\rm v}_N + 2(d-2)\mathcal{G}^{\rm s}_N \\ &= 2(d-2)\Sigma_N + \Big(\frac{d}{2}-5\Big)\Sigma_{N-2} + \Big(\frac{d}{8}-\frac{9}{4}\Big)\Sigma_{N-4} + \Big(\frac{d}{32}-\frac{13}{16}\Big)\Sigma_{N-6} + \ldots \\ &= 2(d-2)\Sigma_N + \sum_{m=0}^{\lfloor (N-2)/2\rfloor} \Big(\frac{d}{2}-4m-5\Big)\frac{\Sigma_{N-2-2m}}{4^m}\ . \end{aligned} \tag{4.32}$$

In dimensions $d>2$, one can see that Σ_N no longer drops out, and the spin sum of pure YM is a degree-N polynomial.

4.4.4. Implications for the power counting of loop momenta

The theory-dependent spin sums $\mathcal{G}^*_{I,\ldots,J}$ comprising N legs are accompanied by $n-N$ factors of $c_i(\ell)$ which are linear in ℓ and G_{ij} by (4.12). For half- or quarter-maximal SYM, $\mathcal{G}^{\frac{1}{2},\frac{1}{4}-\rm SYM}_{N=0}$ and $\mathcal{G}^{\frac{1}{2},\frac{1}{4}-\rm SYM}_{N=1}$ vanish by the cancellation of Σ_N, and $\mathcal{K}^{\frac{1}{2},\frac{1}{4}-\rm SYM}_n$ can be identified as polynomials of degree $n-2$ in ℓ and G_{ij}. Similarly, $\mathcal{K}^{d-\rm YM}_n$ of pure YM with $\mathcal{G}^{d-\rm YM}_N$ of degree N in G_{ij} are polynomials of degree n in ℓ and G_{ij}. For example, the four-point correlator is of degree four for the pure Yang–Mills case, of degree two for half- or quarter-maximal SYM and constant for maximal SYM.

Accordingly, by evaluation of the CHY integrals, an n-gon numerator in half- and quarter-maximal SYM can have a maximum power of $n-2$ loop momenta. Tadpole diagrams are suppressed by this power counting, and external bubbles cancel when combining the partial integrands to a color-ordered single-trace amplitude as in (3.28). In absence of supersymmetry, however, any n-gon diagram may appear with n loop momenta in the numerator.

4.5. The correlator in a basis of worldsheet functions

As we have shown, the one-loop correlators (4.26) with any amount of supersymmetry are written as a polynomial of G_{ij} with $\{i,j\}\in\{1,2,\ldots,n\}$. However, different monomials in these functions are not independent due to scattering equations and the Fay identity (3.11). In addition, higher-point correlators (4.26) also contain subcycles of propagators such as

G_{ij}^2 or $G_{ij}G_{jk}G_{ki}$. Such subcycles do not allow an immediate application of the methods in section 3.3, but we will see that they can always be eliminated from supersymmetric correlators via scattering equations, e.g.

$$G_{ij}^2 = \frac{G_{ij}}{s_{ij}} \sum_{k\neq i,j}^{n} s_{jk}G_{jk} + \frac{G_{ij}}{s_{ij}}\, \ell \cdot k_j \tag{4.33}$$

for a length-two cycle[4].

However, the elimination of length-m subcycles $G_{i_1i_2}G_{i_2i_3}\ldots G_{i_mi_1}$ generally introduces poles in the m-particle Mandelstam invariants $s_{i_1i_2\ldots i_m}$ which generalize the factor of s_{ij}^{-1} on the right hand side of (4.33). Hence, the treatment of length-m subcycles requires a momentum phase space of at least $m{+}2$ massless on-shell particles to keep $s_{i_1i_2\ldots i_m} \neq 0$ and avoid singularities. Given that supersymmetric n-point correlators involve a maximum of $n{-}2$ powers of G_{ij} by the discussion of section 4.4, their subcycles of maximum length $n{-}2$ are all compatible with this phase-space constraint. It remains to find a suitable treatment of the singularities from the length-n subcycles in the n-point correlator of pure Yang–Mills, possibly along the lines of [37].

In the appendix A we describe an algorithm to expand arbitrary polynomials in G_{ij} in a basis of worldsheet functions which do not depend on σ_1. A central role in our choice of basis is played by the following combinations of Green functions [46],

$$X_{a_1a_2} \equiv s_{a_1a_2}G_{a_1a_2}, \qquad X_{a_1a_2\cdots a_m} \equiv \prod_{p=2}^{m} \Big(\sum_{q=1}^{p-1} X_{a_qa_p} \Big) \ , \tag{4.34}$$

[4] In the corresponding superstring computation, such a double pole in the worldsheet variables appears in combination $\alpha' \partial^2 \log\theta_1(z_{ij},\tau) + s_{ij}(\partial \log\theta_1(z_{ij},\tau))^2$, see appendix B.1 of [77]. The result of integration by parts

$$\begin{aligned}\Big[\partial^2 \log\theta_1(z_{ij},\tau) + \alpha' s_{ij}(\partial \log\theta_1(z_{ij},\tau))^2\Big]\mathcal{I}_6 &= -\partial_j \Big[\partial \log\theta_1(z_{ij},\tau)\mathcal{I}_6\Big] \\ &+ \partial \log\theta_1(z_{ij},\tau) \sum_{k\neq i,j}^{n} \alpha' s_{jk} \partial \log\theta_1(z_{jk},\tau)\mathcal{I}_6\end{aligned}$$

with the six-point Koba–Nielsen factor $\mathcal{I}_6 = \prod_{p<q} |\theta_1(z_{pq},\tau)|^{2\alpha' s_{pq}}$ then degenerates to the right hand side of (4.33) and exemplifies that the correlators of the ambitwistor string and the superstring are identical after elimination of subcycles.

whose simplest instances at $m = 3, 4$ read

$$X_{234} \equiv X_{23}(X_{24} + X_{34}), \quad X_{2345} \equiv X_{23}(X_{24} + X_{34})(X_{25} + X_{35} + X_{45}) . \tag{4.35}$$

The choice of (4.34) is motivated by the simple action of scattering equations which can be used iteratively to eliminate any appearance of $a_i = 1$, see appendix A (in particular (A.4)) for further details. Moreover, their symmetry properties such as $X_{23} = -X_{32}$ as well as $X_{234} = -X_{324}$ and $X_{234} + \text{cyc}(2, 3, 4) = 0$ shared by nested commutators $[t^2, t^3]$ and $[[t^2, t^3], t^4]$ leave $(m-1)!$ independent permutations of $X_{a_1 a_2 \cdots a_m}$ [46].

When written in terms of a basis of functions $X_{a_1 a_2 \ldots a_m}$ with $a_i \in \{2, 3, 4, \ldots, n\}$, the one-loop correlators $\mathcal{K}_n^*$ for supersymmetric theories take the schematic form,

$$\begin{aligned} \mathcal{K}_n^* =& \mathcal{C}(\ell) + \sum_{2 \leq i < j} \mathcal{C}_{i,j}(\ell) X_{ij} + \sum_{2 \leq i < j,k} \mathcal{C}_{i,j,k}(\ell) X_{ijk} \\ &+ \sum_{2 \leq i < j} \sum_{i < k < l} \mathcal{C}_{i,j;k,l}(\ell) X_{ij} X_{kl} + \sum_{2 \leq i < j,k,l} \mathcal{C}_{i,j,k,l}(\ell) X_{ijkl} + \cdots , \end{aligned} \tag{4.36}$$

where the terms in the ellipsis involve at least three powers of G_{ij}. Since the worldsheet functions form a basis, the coefficients $\mathcal{C}(\ell)$ are gauge invariant kinematic factors. They build up the gauge invariant n-gon numerators $\mathcal{C}_{+|\rho(23\ldots n)|-}(\ell)$ in (3.38) and (3.39) through the dictionary $G_{ij} \rightarrow \frac{1}{2}\text{sgn}_{ij}^{\rho}$ of section 3.3.

Recall from (4.26) that $\mathcal{K}_n^*$ are polynomials in ℓ and G_{ij} of total degree $n-4$ for maximal supersymmetry, $n-2$ for reduced supersymmetry and n for zero supersymmetry. Accordingly, each accompanying factor of X_{ij} reduces the maximum power of ℓ in the kinematic factors $\mathcal{C}_{\ldots}$ of (4.36) by one. Given that the subleading symmetric polynomials Σ_k in the spin sums reduce the homogeneity degree in ℓ and G_{ij} by $2, 4, 6, \ldots$, the n-point kinematic factors along with p powers of X_{ij} have an expansion of the schematic form

$$\begin{aligned} (n-p) \text{ even}: \quad & \mathcal{C}_I(\ell) = C_I + \ell_m \ell_n \, C_I^{mn} + \ell_m \ell_n \ell_p \ell_q \, C_I^{mnpq} + \ldots , \\ (n-p) \text{ odd}: \quad & \mathcal{C}_I(\ell) = \ell_m \, C_I^{m} + \ell_m \ell_n \ell_p \, C_I^{mnp} + \ell_m \ell_n \ell_p \ell_q \ell_r \, C_I^{mnpqr} + \ldots . \end{aligned}$$

The subscript I collectively refers to the labels of the accompanying p factors of X_{ij}, and the highest powers of ℓ is $n-4-p$, $n-2-p$ or $n-p$ for maximal, reduced or zero supersymmetry.

$n \backslash k$	0	1	2	3	4	5	$\#\mathcal{C}_{\max}$	$\#\mathcal{C}_{\rm red}$
2	1						0	1
3	1	1					0	2
4	1	3	2				1	6
5	1	6	11	6			7	24
6	1	10	35	50	24		46	120
7	1	15	85	225	274	120	326	720

Table 1. The numbers $S_{n-1,n-k-1}$ of independent degree-k polynomials in G_{ij}. Summing over the ranges $0 \leq k \leq n{-}4$ and $0 \leq k \leq n{-}2$ admitted by maximal and reduced supersymmetry yields the tabulated numbers $\#\mathcal{C}_{\max}$ and $\#\mathcal{C}_{\rm red} = (n{-}1)!$ of worldsheet functions in (4.36), respectively.

Before ending, let's record the number of independent worldsheet functions in (4.36) for supersymmetric theories. The counting is governed by unsigned Stirling numbers $S_{N,r}$ of the first kind (see table 1) which count the number of ways to distribute N elements into r cycles. Scattering equations together the symmetry properties of $X_{a_1a_2\ldots a_m}$ leave $S_{n-1,n-k-1}$ independent polynomials in G_{ij} of degree k. Then, the range $0 \leq k \leq n{-}2$ for reduced supersymmetry yields a total of $\#\mathcal{C}_{\rm red} = \sum_{k=0}^{n-2} S_{n-1,n-k-1} = (n{-}1)!$ terms in (4.36). Maximal supersymmetry, however, only allows for $0 \leq k \leq n{-}4$, and the resulting numbers $\#\mathcal{C}_{\max} = \sum_{k=0}^{n-4} S_{n-1,n-k-1}$ of basis functions are gathered in table 1.

Even though the methods of this section only give access to their bosonic components, we will provide the maximally supersymmetric completions for the ($n{\leq}6$)-point kinematic factors $\mathcal{C}(\ell)$ in the next section. These kinematic factors in pure-spinor superspace are conveniently organized in terms of Berends–Giele currents, and we will spell out the corresponding Berends–Giele description of their gluon components which follows from the basis reduction described in this section.

4.6. Parity-odd contributions to RNS correlators

The above discussion has been tailored to the parity-even contributions to one-loop gauge-theory amplitudes. However, the running of chiral fermions in the loop yields additional parity-odd terms proportional to the d-dimensional Levi–Civita tensor ϵ_d. They arise from the single odd spin structure of the worldsheet fermions whose correlation functions in an ambitwistor setup have been described in [23]. By the integral over fermionic zero modes, these correlators are bound to vanish for multiplicities smaller than $\frac{d}{2}$.

As manifested by the expressions of [23] reviewed in appendix E, the parity-odd correlators are polynomials in ℓ and $\partial \log \theta_1(z_{ij},\tau)$ of degree $n{+}1{-}\frac{d}{2}$ after integration over fermionic zero modes. Hence, their degeneration (3.9) at $\tau \to i\infty$ is manifestly a polynomial in G_{ij} and ℓ, in complete analogy to the above parity-even results. However, from the additional zero modes in the ghost sector of this spin structure, a picture changing operator introduces a spurious dependence on its insertion point σ_0 via G_{0j}. Since BRST invariance of the RNS ambitwistor string guarantees that the final result is independent on σ_0, one can always eliminate any appearance of G_{0j} through a sequence of Fay identities and scattering equations.

The conclusion from the parity-even sector is therefore unchanged: The parity-odd contributions to the n-point correlator due to chiral fermions can be expressed as degree-$(n{+}1{-}\frac{d}{2})$ polynomials in ℓ and G_{ij} with $i,j\neq 0$. In particular, chiral theories in $d=10$ and $d=6$ dimensions lead to the degrees $n{-}4$ and $n{-}2$ familiar from the parity-even sectors with maximal and half-maximal supersymmetry, respectively.

Since analogous statements hold for the RNS superstring, we will translate results for string correlators with all dependence on σ_0 eliminated to the ambitwistor setup. In case of ten-dimensional SYM, the contributions from chiral fermions vanishes below five points[5]

$$\mathcal{K}^{\epsilon_{10}}_{n\leq 4}=0\ ,\qquad \mathcal{K}^{\epsilon_{10}}_{5}=i\ell_m\epsilon^m_{10}(e_1,k_2,e_2,k_3,e_3,k_4,e_4,k_5,e_5)\ , \tag{4.38}$$

where the shorthand $\epsilon^m_{10}(e_1,k_2,e_2,k_3,e_3,k_4,e_4,k_5,e_5) = \epsilon_{10}^{mnpqrsabcd}e_1^n k_2^p e_2^q k_3^r e_3^s k_4^a e_4^b k_5^c e_5^d$ avoids proliferation of indices. While the five-point correlator does not allow any contribution with G_{0j} on kinematic grounds, a long calculation is needed to demonstrate the disappearance of G_{0j} from the six-point correlator. The manifestly σ_0-independent superstring correlators of [78,77] then degenerate into

$$\begin{aligned}\mathcal{K}^{\epsilon_{10}}_{6} &= i\big[(\ell\cdot e_2)\epsilon_{10}(\ell,e_1,k_3,e_3,k_4,e_4,k_5,e_5,k_6,e_6)+(2\leftrightarrow 3,4,5,6)\big] \\ &\quad + i\big[G_{12}\ell_m E^m_{12|3,4,5,6}+(2\leftrightarrow 3,4,5,6)\big]+i\big[G_{23}\ell_m E^m_{1|23,4,5,6}+(2,3|2,3,4,5,6)\big]\ ,\end{aligned} \tag{4.39}$$

[5] The factor of i reflects our conventions $\epsilon_d^{m_1 m_2\ldots m_d}\epsilon_d^{m_1 m_2\ldots m_d}=+d!$ for the normalization of the Levi–Civita tensor.

with kinematic factors

$$E^m_{12|3,4,5,6} = (e_1 \cdot k_2)\epsilon^m_{10}(e_2, k_3, e_3, \ldots, k_6, e_6) - (e_2 \cdot k_1)\epsilon^m_{10}(e_1, k_3, e_3, \ldots, k_6, e_6)$$
$$- (e_1 \cdot e_2)\epsilon^m_{10}(k_2, k_3, e_3, \ldots, k_6, e_6) \quad (4.40)$$
$$E^m_{1|23,4,5,6} = (e_2 \cdot k_3)\epsilon^m_{10}(e_1, k_{23}, e_3, \ldots, k_6, e_6) - (e_3 \cdot k_2)\epsilon^m_{10}(e_1, k_{23}, e_2, \ldots, k_6, e_6)$$
$$- (e_2 \cdot e_3)\epsilon^m_{10}(e_1, k_2, k_3, \ldots, k_6, e_6) - (k_2 \cdot k_3)\epsilon^m_{10}(e_1, e_2, e_3, \ldots, k_6, e_6) \quad (4.41)$$

and k_4, e_4, k_5, e_5 in the ellipsis. The notation $(i_1, \ldots, i_p|i_1, \ldots, i_q)$ on the right hand side of (4.39) with $q > p$ instructs to sum over all possibilities to choose p elements $i_1, \ldots, i_p$ out of the larger set $\{i_1, \ldots, i_q\}$, for a total of $\binom{q}{p}$ terms.

For chiral SYM in six dimensions, the counting of minimum multiplicities is shifted by two such that

$$\mathcal{K}^{\epsilon_6}_{n\leq 2} = 0 \ , \qquad \mathcal{K}^{\epsilon_6}_3 = i\ell_m \epsilon^m_6(e_1, k_2, e_2, k_3, e_3) \ . \quad (4.42)$$

The above expressions for $\mathcal{K}^{\epsilon_{10}}_6$ including the mechanisms for the decoupling of G_{0j} have been generalized to arbitrary even dimensions in [49]. Accordingly, the six-dimensional four-point correlator

$$\mathcal{K}^{\epsilon_6}_4 = i\big[(\ell \cdot e_2)\epsilon_6(\ell, e_1, k_3, e_3, k_4, e_4) + (2 \leftrightarrow 3, 4)\big] \quad (4.43)$$
$$+ i\big[G_{12}\ell_m E^m_{12|3,4} + (2 \leftrightarrow 3, 4)\big] + i\big[G_{23}\ell_m E^m_{1|23,4} + (2, 3|2, 3, 4)\big] \ ,$$

follows the structure of $\mathcal{K}^{\epsilon_{10}}_6$ with kinematic factors resembling (4.40) and (4.41),

$$E^m_{12|3,4} = (e_1 \cdot k_2)\epsilon^m_6(e_2, k_3, e_3, k_4, e_4) - (e_2 \cdot k_1)\epsilon^m_6(e_1, k_3, e_3, k_4, e_4)$$
$$- (e_1 \cdot e_2)\epsilon^m_6(k_2, k_3, e_3, k_4, e_4) \quad (4.44)$$
$$E^m_{1|23,4} = (e_2 \cdot k_3)\epsilon^m_6(e_1, k_{23}, e_3, k_4, e_4) - (e_3 \cdot k_2)\epsilon^m_6(e_1, k_{23}, e_2, k_4, e_4)$$
$$- (e_2 \cdot e_3)\epsilon^m_6(e_1, k_2, k_3, k_4, e_4) - (k_2 \cdot k_3)\epsilon^m_6(e_1, e_2, e_3, k_4, e_4) \ . \quad (4.45)$$

The pure-spinor superspace expressions for the ten-dimensional correlators to be discussed in the following section automatically combine both the parity-even and the parity-odd components. The BCJ master numerators in $n \leq 4$-point amplitudes of chiral SYM in six dimensions will be given in section 6.

5. Pure-spinor representations for the gauge multiplet

In this section we present the field-theory limit of the superstring one-loop correlators that have been computed using the pure-spinor formalism [34,79]. By the arguments of [29], identical results are obtained when performing the computation with the loop-level version [26] of the pure-spinor ambitwistor string [25].

5.1. Review of pure-spinor superspace

Supersymmetric scattering amplitudes in ten dimensions admit compact representations in the language of pure-spinor superspace. This new type of superspace arises naturally within the pure-spinor formalism of the superstring and its properties played an important role in recent advances in the computation of string scattering amplitudes.

A super-Poincaré invariant description of ten-dimensional SYM theory uses four types of superfields

$$A_\alpha(x,\theta),\ A_m(x,\theta),\ W^\alpha(x,\theta),\ F_{mn}(x,\theta) \tag{5.1}$$

that depend on the superspace coordinates x^m, θ^α with vector indices $m = 0, \ldots, 9$ and spinor indices $\alpha = 1, \ldots, 16$ of the ten-dimensional Lorentz-group. They satisfy the following (linearized) equations of motion [57]

$$\begin{aligned} D_{(\alpha} A_{\beta)} &= \gamma^m_{\alpha\beta} A_m, & D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn} \\ D_\alpha A_m &= (\gamma_m W)_\alpha + \partial_m A_\alpha, & D_\alpha F_{mn} &= \partial_{[m}(\gamma_{n]} W)_\alpha \,, \end{aligned} \tag{5.2}$$

where $D_\alpha \equiv \partial_\alpha + \frac{1}{2}\partial_m(\gamma^m\theta)_\alpha$ is the supersymmetric covariant derivative and $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$ denote 16×16 Pauli matrices[6]. The θ-expansions of the superfields (5.1) are written[7] in terms of gluon polarizations e_m, gluino wavefunctions χ^α as well as the field-strength $f_{mn} = 2k_{[m}e_{n]}$ [80]:

$$\begin{aligned} A_\alpha(x,\theta) &= \Big(\frac{1}{2}e_m(\gamma^m\theta)_\alpha - \frac{1}{3}(\chi\gamma_m\theta)(\gamma^m\theta)_\alpha - \frac{1}{32}f_{mn}(\gamma_p\theta)_\alpha(\theta\gamma^{mnp}\theta) + \cdots\Big)\mathrm{e}^{k\cdot x} \\ A_m(x,\theta) &= \Big(e_m - (\chi\gamma_m\theta) - \frac{1}{8}(\theta\gamma_m\gamma^{pq}\theta)f_{pq} + \frac{1}{12}(\theta\gamma_m\gamma^{pq}\theta)k_p(\chi\gamma_q\theta) + \cdots\Big)\mathrm{e}^{k\cdot x} \\ W^\alpha(x,\theta) &= \Big(\chi^\alpha - \frac{1}{4}(\gamma^{mn}\theta)^\alpha f_{mn} + \frac{1}{4}(\gamma^{mn}\theta)^\alpha k_m(\chi\gamma_n\theta) + \cdots\Big)\mathrm{e}^{k\cdot x} \\ F_{mn}(x,\theta) &= \Big(f_{mn} - 2k_{[m}(\chi\gamma_{n]}\theta) + \frac{1}{4}(\theta\gamma_{[m}\gamma^{pq}\theta)k_{n]}f_{pq} + \cdots\Big)\mathrm{e}^{k\cdot x} \;. \end{aligned} \tag{5.3}$$

[6] They often appear in antisymmetrized combinations subject to $\gamma^{mnp}_{\alpha\beta} = -\gamma^{mnp}_{\beta\alpha}$ and $\gamma^{mnpqr}_{\alpha\beta} = \gamma^{mnpqr}_{\beta\alpha}$ with normalization conventions such as $\gamma^{mn}{}_\alpha{}^\beta \equiv \frac{1}{2}(\gamma^m\gamma^n - \gamma^n\gamma^m)_\alpha^\beta$.

[7] For historical reasons, we omit the factor of i in the plane wave expansion.

Pure-spinor superspace expressions are defined as expansions of the form [34]

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta) \rangle, \tag{5.4}$$

where $f_{\alpha\beta\gamma}(\theta)$ denotes an arbitrary function of the superfields (5.1) and encodes the information about the polarizations of the particles participating in the scattering. For example, the three-particle scattering of SYM states is described by $f_{\alpha\beta\gamma}(\theta) = A^1_\alpha(\theta) A^2_\beta(\theta) A^3_\gamma(\theta)$. In the above definition (5.4), the variables λ^α are the zero-modes of a pure spinor subject to $(\lambda\gamma^m\lambda) = 0$, and the angular bracket is defined by [34,81]

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta) \rangle = 2880\,, \tag{5.5}$$

while expressions of different degrees $\lambda^{\neq 3}$ or $\theta^{\neq 5}$ yield a vanishing bracket. The prescription (5.5) is motivated by supersymmetry and the cohomology of the BRST operator $Q = \lambda^\alpha D_\alpha$: BRST invariant superfields $Q(\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)) = 0$ are mapped to supersymmetric and gauge invariant components $\langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(\theta)\rangle$. BRST exact superfields, on the other hand, are annihilated, i.e. $\langle Q(\lambda^\alpha\lambda^\beta g_{\alpha\beta}(\theta))\rangle = 0$ for any choice of $g_{\alpha\beta}(\theta)$.

5.2. Review of one-loop building blocks

The superspace description of SYM theory can be generalized to a multiparticle setup which is convenient to describe the scattering of a high number of external particles. The so-called *multiparticle superfields* have been defined using recursion relations both in local and non-local forms [44,45]. For example, in a notation where uppercase latin letters $P = 123\ldots p$ encompass the labels of p external legs, the non-local recursion relations are given by

$$\mathcal{A}^P_\alpha = \frac{1}{2s_P} \sum_{XY=P} \big[\mathcal{A}^Y_\alpha (k^Y \cdot \mathcal{A}^X) + \mathcal{A}^Y_m (\gamma^m \mathcal{W}^X)_\alpha - (X \leftrightarrow Y) \big] \tag{5.6}$$

$$\mathcal{A}^P_m = \frac{1}{2s_P} \sum_{XY=P} \big[\mathcal{A}^Y_m (k^Y \cdot \mathcal{A}^X) + \mathcal{A}^Y_n \mathcal{F}^X_{mn} + (\mathcal{W}^X \gamma_m \mathcal{W}^Y) - (X \leftrightarrow Y) \big] \tag{5.7}$$

$$\mathcal{W}^\alpha_P = \frac{1}{2s_P} k^m_P \gamma^{\alpha\beta}_m \sum_{XY=P} \big[\mathcal{A}^n_X (\gamma_n \mathcal{W}_Y)_\beta - (X \leftrightarrow Y) \big] \tag{5.8}$$

$$\mathcal{F}^{mn}_P = k^m_P \mathcal{A}^n_P - k^n_P \mathcal{A}^m_P - \sum_{XY=P} \big(\mathcal{A}^m_X \mathcal{A}^n_Y - \mathcal{A}^n_X \mathcal{A}^m_Y \big)\,, \tag{5.9}$$

and they give rise to a supersymmetric generalization of the Berends–Giele currents [82]. In the above formulae, the summation over $XY = P$ denotes a sum over the deconcatenations of $P = 123\ldots p$ into into non-empty words $X = 12\ldots j$ and $Y = j{+}1\ldots p$ with

$j = 1, 2, \ldots, p-1$. The propagators $1/s_P$ in the above recursions identify the tree-level subdiagrams described by the currents and characterize their non-local nature.

The above Berends–Giele supercurrents constitute the fundamental building blocks for kinematic factors in ten-dimensional one-loop superstring amplitudes. They have been systematically assembled in [83] by closely following the zero-mode saturation rules in the pure-spinor formalism [79]. For example, from the definitions

$$\begin{aligned} M_{A,B,C} &\equiv \frac{1}{3}(\lambda\gamma^m \mathcal{W}_A)(\lambda\gamma^n \mathcal{W}_B)\mathcal{F}_C^{mn} + (A \leftrightarrow B, C)\,, \\ \mathcal{W}^m_{A,B,C,D} &\equiv \frac{1}{12}(\lambda\gamma_n \mathcal{W}_A)(\lambda\gamma_p \mathcal{W}_B)(\mathcal{W}_C\gamma^{mnp}\mathcal{W}_D) + (A, B|A, B, C, D)\,, \end{aligned} \tag{5.10}$$

it follows that

$$\begin{aligned} M^m_{A,B,C,D} &\equiv \big[\mathcal{A}^m_A M_{B,C,D} + (A \leftrightarrow B, C, D)\big] + \mathcal{W}^m_{A,B,C,D} \\ M^{mn}_{A,B,C,D,E} &\equiv \mathcal{A}^n_A M^m_{B,C,D,E} + \mathcal{A}^m_A \mathcal{W}^n_{B,C,D,E} + (A \leftrightarrow B, C, D, E) \end{aligned} \tag{5.11}$$

exhibit covariant BRST transformations, and they naturally appear in string scattering computations at one loop.

5.2.1. BRST invariant combinations

Generalizing the above structures paves the way for the definition of kinematic BRST invariants and so-called pseudo-invariants of arbitrary tensor rank [83]. For example, using $M_P \equiv \lambda^\alpha \mathcal{A}^P_\alpha$ one can recursively[8] define scalar BRST invariants such as:

$$\begin{aligned} C_{1|2,3,4} &\equiv M_1 M_{2,3,4} \\ C_{1|23,4,5} &\equiv M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}\,, \\ C_{1|234,5,6} &\equiv M_1 M_{234,5,6} + M_{12} M_{34,5,6} + M_{123} M_{4,5,6} - M_{124} M_{3,5,6} \\ &\quad - M_{14} M_{23,5,6} - M_{142} M_{3,5,6} + M_{143} M_{2,5,6}\,, \\ C_{1|23,45,6} &\equiv M_1 M_{23,45,6} + M_{12} M_{45,3,6} - M_{13} M_{45,2,6} + M_{14} M_{23,5,6} - M_{15} M_{23,4,6} \\ &\quad + \big[M_{124} M_{3,5,6} - M_{134} M_{2,5,6} + M_{142} M_{3,5,6} - M_{143} M_{2,5,6} - (4 \leftrightarrow 5)\big]\,. \end{aligned} \tag{5.12}$$

[8] See [83] for the explicit form of the recursion and associated definitions. To keep the presentation short, here we chose to write down a few examples of their outcome.

In addition, one can define pseudo-invariants[9] of arbitrary tensor ranks and multiplicity. In this paper we will be concerned with explicit amplitudes up to multiplicity six, for which the following definitions suffice

$$
\begin{aligned}
C^m_{1|2,3,4,5} &\equiv M_1 M^m_{2,3,4,5} + \big[k_2^m M_{12} M_{3,4,5} + (2 \leftrightarrow 3,4,5)\big]\,, \qquad (5.13)\\
C^m_{1|23,4,5,6} &\equiv M_1 M^m_{23,4,5,6} + M_{12} M^m_{3,4,5,6} - M_{13} M^m_{2,4,5,6}\\
&\quad + \big[k_3^m M_{123} M_{4,5,6} + (3 \leftrightarrow 4,5,6)\big] - \big[k_2^m M_{132} M_{4,5,6} + (2 \leftrightarrow 4,5,6)\big]\\
&\quad + \big[k_4^m M_{14} M_{23,5,6} + k_4^m M_{142} M_{3,5,6} - k_4^m M_{143} M_{2,5,6} + (4 \leftrightarrow 5,6)\big]\,,\\
C^{mn}_{1|2,3,4,5,6} &\equiv M_1 M^{mn}_{2,3,4,5,6} + 2\big[k_2^{(m} M_{12} M^{n)}_{3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big]\\
&\quad + 2\big[k_2^{(m} k_3^{n)} (M_{123} + M_{132}) M_{4,5,6} + (2,3|2,3,4,5,6)\big].
\end{aligned}
$$

5.2.2. Pure-spinor superspace versus gluon components[10]

The above kinematic expressions are written in pure-spinor superspace. While compact superspace expressions suffice for most purposes, one might still want to obtain the explicit component form of the amplitudes written in terms of the physical gluon and gluino polarizations. Fortunately, the properties of the pure-spinor superspace measure (5.5) can be exploited to easily automate this task. In addition, with the techniques advanced in [45] the results take an elegant and compact form even at the level of components. To see this one defines Berends–Giele currents for the gluon polarization and field-strength in component form starting with the single-particle cases $\mathfrak{e}_i^m = e_i^m$ and $\mathfrak{f}_i^{mn} = k_i^m e_i^n - k_i^n e_i^m$:

$$
\mathfrak{e}_P^m \equiv \frac{1}{2s_P} \sum_{XY=P} \big[\mathfrak{e}_Y^m (k^Y \cdot \mathfrak{e}^X) + \mathfrak{e}_n^Y \mathfrak{f}_X^{mn} - (X \leftrightarrow Y)\big] \qquad (5.14)
$$

$$
\mathfrak{f}_P^{mn} \equiv k_P^m \mathfrak{e}_P^n - k_P^n \mathfrak{e}_P^m - \sum_{XY=P} \big(\mathfrak{e}_X^m \mathfrak{e}_Y^n - \mathfrak{e}_X^n \mathfrak{e}_Y^m\big)\ . \qquad (5.15)
$$

This can be viewed as a truncation of (5.7) and (5.9) where the fermionic variables are suppressed, and it is straightforward to generalize (5.14) and (5.15) to include gluino polarizations. Using the multiparticle Harnad–Shnider gauge introduced in [45], one can

[9] *Pseudo*-invariants are defined to be expressions whose BRST variation, instead of vanishing, gives rise to *anomalous* superfields [83] that carry the fingerprints of the hexagon anomaly of ten-dimensional SYM. For a prominent example of their use, see [77].

[10] This subsection was written by Carlos Mafra with the aid of [84].

show that the gluon components of the above BRST pseudo-invariants can be compactly written in terms of the t_8-tensor [85]:

$$
\begin{aligned}
t_{A,B,C,D} &\equiv \mathfrak{f}_A^{mn}\mathfrak{f}_B^{np}\mathfrak{f}_C^{pq}\mathfrak{f}_D^{qm} - \frac{1}{4}(\mathfrak{f}_A^{mn}\mathfrak{f}_B^{nm})(\mathfrak{f}_C^{pq}\mathfrak{f}_D^{qp}) + \mathrm{cyc}(B,C,D) = t_8(\mathfrak{f}_A,\mathfrak{f}_B,\mathfrak{f}_C,\mathfrak{f}_D) \\
t^m_{A,B,C,D,E} &\equiv \big[\mathfrak{e}_A^m t_{B,C,D,E} + (A\leftrightarrow B,C,D,E)\big] + \frac{i}{2}\epsilon_{10}^m(\mathfrak{e}_A,\mathfrak{f}_B,\mathfrak{f}_C,\mathfrak{f}_D,\mathfrak{f}_E) \\
t^{mn}_{A,B,C,D,E,F} &\equiv 2\big[\mathfrak{e}_A^{(m}\mathfrak{e}_B^{n)} t_{C,D,E,F} + (A,B|A,B,C,D,E,F)\big] \\
&\quad + i\big[\mathfrak{e}_B^{(m}\epsilon_{10}^{n)}(\mathfrak{e}_A,\mathfrak{f}_C,\mathfrak{f}_D,\mathfrak{f}_E,\mathfrak{f}_F) + (B\leftrightarrow C,D,E,F)\big]\,.
\end{aligned}
\tag{5.16}
$$

Moreover, we have used the shorthand $\epsilon_{10}^m(\mathfrak{e}_A,\mathfrak{f}_B,\mathfrak{f}_C,\mathfrak{f}_D,\mathfrak{f}_E) \equiv \epsilon_{10}^{mnpqrsabcd}\mathfrak{e}_A^n\mathfrak{f}_B^{pq}\mathfrak{f}_C^{rs}\mathfrak{f}_D^{ab}\mathfrak{f}_E^{cd}$ to avoid proliferation of indices in the parity-odd contributions from section 4.6. Motivated by the simple examples

$$
\begin{aligned}
-16\,\langle C_{1|2,3,4}\rangle &= t_{1,2,3,4} \\
-16\,\langle C_{1|23,4,5}\rangle &= t_{12,3,4,5} + t_{1,23,4,5} - t_{13,2,4,5}\,, \\
-16\,\langle C^m_{1|2,3,4,5}\rangle &= t^m_{1,2,3,4,5} + \big(k_2^m t_{12,3,4,5} + 2\leftrightarrow 3,4,5\big) \\
-16\,\langle C^{mn}_{1|2,3,4,5,6}\rangle &= t^{mn}_{1,2,3,4,5,6} + \big(2k_2^{(m}t^{n)}_{12,3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big) \\
&\quad - \big(2k_2^{(m}k_3^{n)}t_{213,4,5,6} + (2,3|2,3,4,5,6)\big)\,,
\end{aligned}
\tag{5.17}
$$

one can verify that the translation from pseudo-BRST invariants (5.12) and (5.13) to their gluonic components can be obtained as follows[11]

$$
\begin{aligned}
-16\,\langle M_A M_{B,C,D}\rangle &\to t_{A,B,C,D} \\
-16\,\langle M_A M^m_{B,C,D,E}\rangle &\to t^m_{A,B,C,D,E} \\
-16\,\langle M_A M^{mn}_{B,C,D,E,F}\rangle &\to t^{mn}_{A,B,C,D,E,F}\,.
\end{aligned}
\tag{5.18}
$$

5.3. Maximally supersymmetric one-loop correlators from string theory

In string theory, the supersymmetric one-loop integrands at four, five and six points have been computed using the pure-spinor formalism in [79,46,77]. In the field-theory limit they

[11] It is important to stress that the validity of the map (5.18) is checked *within* BRST (pseudo-)invariants as its contact-term mismatch cancels in such cases.

can be written as[12],

$$
\begin{aligned}
\mathcal{K}_4 &= \langle C_{1|2,3,4}\rangle &(5.19)\\
\mathcal{K}_5 &= \langle \ell_m C^m_{1|2,3,4,5} + [X_{23}C_{1|23,4,5} + (2,3|2,3,4,5)]\rangle &(5.20)\\
\mathcal{K}_6 &= \langle \frac{1}{2}\ell_m\ell_n C^{mn}_{1|2,3,4,5,6} + \ell_m\big[X_{23}C^m_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big]\\
&\quad + \big[X_{23}\,X_{34}C_{1|234,5,6} - X_{23}\,X_{24}C_{1|324,5,6} - X_{24}\,X_{34}C_{1|243,5,6} + (2,3,4|2,3,4,5,6)\big]\\
&\quad + \big[X_{23}\,X_{45}C_{1|23,45,6} + (2,3|4,5|2,3,4,5,6)\big] - \frac{1}{4}k_1^m k_1^n C^{mn}_{1|2,3,4,5,6}\rangle\ , &(5.21)
\end{aligned}
$$

see the previous subsection for their gluon component expansions. We note that the last term in the six-point correlator without any accompanying factors of X_{ij} or ℓ is permutation symmetric and in fact proportional to the six-point tree-level amplitude of Born–Infeld theory[13]:

$$
\begin{aligned}
-\frac{1}{4}k_1^m k_1^n\langle C^{mn}_{1|2,3,4,5,6}\rangle &= M^{\rm tree}_{\rm BI}(1,2,3,4,5,6) &(5.22)\\
&= \sum_{\rho\in S_4} s_{1\rho(2)}(s_{1\rho(3)} + s_{\rho(23)})(s_{\rho(45)} + s_{\rho(4)6})s_{\rho(5)6}A^{\rm tree}(1,\rho(2,3,4,5),6)
\end{aligned}
$$

The representation of the Born–Infeld amplitude is based on its double-copy structure [20] involving gauge-theory trees and the BCJ master numerators for the NLSM of [30,32].

5.4. Pure-spinor representations of BCJ numerators and partial integrands

As discussed in section 3.4, one can identify BCJ master numerators at one loop by rewriting the correlator in terms of Parke–Taylor factors. Applying the dictionary (3.17) to the pure-spinor correlators (5.19) to (5.21) and exploiting the absence of G_{1j} in our basis of functions leads to a manifestly supersymmetric CHY integrand of the form (3.38). We obtain the following supersymmetric BCJ master numerators $\mathcal{C}_{+|\rho(2,\ldots,n)|-}(\ell)$ for the n-gon diagrams:

$$
\mathcal{C}_{+|\rho(2,3,4)|-}(\ell) = \langle C_{1|2,3,4}\rangle = s_{12}s_{23}A^{\rm tree}(1,2,3,4) \tag{5.23}
$$

---

[12] This particular representation using the explicit loop momentum ℓ^m is based on unpublished work [86].

[13] We thank Carlos Mafra for several discussions on finding a compact representation for the LHS of (5.22).

$$\mathcal{C}_{+|\rho(2,\ldots,5)|-}(\ell) = \ell_m \langle C^m_{1|2,3,4,5}\rangle + \frac{1}{2}\Big[s_{23}\text{sgn}^\rho_{23}\langle C_{1|23,4,5}\rangle + (2,3|2,3,4,5)\Big] \tag{5.24}$$

$$\begin{aligned}\mathcal{C}_{+|\rho(2,\ldots,6)|-}(\ell) = &\frac{1}{2}\ell_m\ell_n\langle C^{mn}_{1|2,3,4,5,6}\rangle + \frac{1}{2}\ell_m\Big[s_{23}\text{sgn}^\rho_{23}\langle C^m_{1|23,4,5,6}\rangle + (2,3|2,3,4,5,6)\Big]\\ &+\frac{1}{4}\Big[s_{23}s_{45}\text{sgn}^\rho_{23}\text{sgn}^\rho_{45}\langle C_{1|23,45,6}\rangle + (2,3|4,5|2,3,4,5,6)\Big]\\ &+\frac{1}{4}\Big[s_{23}s_{34}\text{sgn}^\rho_{23}\text{sgn}^\rho_{34}\langle C_{1|234,5,6}\rangle - s_{23}s_{24}\text{sgn}^\rho_{23}\text{sgn}^\rho_{24}\langle C_{1|324,5,6}\rangle\\ &\quad - s_{24}s_{34}\text{sgn}^\rho_{24}\text{sgn}^\rho_{34}\langle C_{1|243,5,6}\rangle + (2,3,4|2,3,4,5,6)\Big]\\ &-\frac{1}{4}k^1_m k^1_n\langle C^{mn}_{1|2,3,4,5,6}\rangle\ .\end{aligned} \tag{5.25}$$

An explicit form of the bosonic components in terms of recursive Berends–Giele currents is readily obtained via (5.18). The notation $+(2,3|4,5|2,3,4,5,6)$ in the second line of (5.25) means a sum over all pairs $\{i,j\}$ and $\{p,q\}$ such that $i,j,p,q\in\{2,3,4,5,6\}$ and $i<j$, $p<q$ and $i<p$. In absence of factors of sgn^ρ_{1j}, the above n-gon numerators do not depend on the position of leg 1, and by the kinematic Jacobi identities, numerators with leg 1 involved in a tree-level subdiagram vanish.

Note that the above BCJ numerators yield the following partial integrands

$$a(1,2,3,4,-,+) = \frac{\langle C_{1|2,3,4}\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} \tag{5.26}$$

$$\begin{aligned}a(1,2,\ldots,5,-,+) = &\frac{\langle \ell_m C^m_{1|2,3,4,5} - \frac{1}{2}[s_{23}C_{1|23,4,5} + (2,3|2,3,4,5)]\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{1234,\ell}}\\ &-\frac{\langle C_{1|23,4,5}\rangle}{s_{1,\ell}s_{123,\ell}s_{1234,\ell}} - \frac{\langle C_{1|34,2,5}\rangle}{s_{1,\ell}s_{12,\ell}s_{1234,\ell}} - \frac{\langle C_{1|45,2,3}\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}}\end{aligned} \tag{5.27}$$

$$\begin{aligned}a(1,2,\ldots,6,-,+) = &\frac{\frac{1}{2}\langle \ell_m\ell_n C^{mn}_{1|2,3,4,5,6} - \ell_m[s_{23}C^m_{1|23,4,5,6} + (2,3|2,3,4,5,6)]\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{1234,\ell}s_{12345,\ell}}\\ &+\frac{\langle C_{1|2;3;4;5;6}\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{1234,\ell}s_{12345,\ell}} + \frac{\langle C_{1|23;4;5;6} - \ell_m C^m_{1|23,4,5,6}\rangle}{s_{1,\ell}s_{123,\ell}s_{1234,\ell}s_{12345,\ell}} + \frac{\langle C_{1|2;34;5;6} - \ell_m C^m_{1|2,34,5,6}\rangle}{s_{1,\ell}s_{12,\ell}s_{1234,\ell}s_{12345,\ell}}\\ &+\frac{\langle C_{1|2;3;45;6} - \ell_m C^m_{1|2,3,45,6}\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{12345,\ell}} + \frac{\langle C_{1|2;3;4;56} - \ell_m C^m_{1|2,3,4,56}\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{1234,\ell}} + \frac{\langle C_{1|234,5,6}\rangle}{s_{1,\ell}s_{1234,\ell}s_{12345,\ell}}\\ &+\frac{\langle C_{1|2,345,6}\rangle}{s_{1,\ell}s_{12,\ell}s_{12345,\ell}} + \frac{\langle C_{1|2,3,456}\rangle}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} + \frac{\langle C_{1|23,45,6}\rangle}{s_{1,\ell}s_{123,\ell}s_{12345,\ell}} + \frac{\langle C_{1|23,4,56}\rangle}{s_{1,\ell}s_{123,\ell}s_{1234,\ell}} + \frac{\langle C_{1|2,34,56}\rangle}{s_{1,\ell}s_{12,\ell}s_{1234,\ell}}\end{aligned} \tag{5.28}$$

with the scalar hexagon numerator

$$\begin{aligned}4C_{1|2;3;4;5;6} = &-k^1_m k^1_n C^{mn}_{1|2,3,4,5,6} + \big[s_{23}s_{45}C_{1|23,45,6} + (2,3|4,5|2,3,4,5,6)\big]\\ &+\big[s_{23}s_{34}C_{1|234,5,6} - s_{23}s_{24}C_{1|324,5,6} - s_{24}s_{34}C_{1|243,5,6} + (2,3,4|2,3,4,5,6)\big]\end{aligned} \tag{5.29}$$

and scalar pentagons such as

$$2C_{1|23;4;5;6} = s_{45}C_{1|23,45,6} + s_{46}C_{1|23,46,5} + s_{56}C_{1|23,4,56} \\ + \big[s_{34}C_{1|234,5,6} - s_{24}C_{1|324,5,6} + (4 \leftrightarrow 5,6)\big] \ . \tag{5.30}$$

The partial integrands (5.26) to (5.28) have been checked to follow from the partial-fraction decomposition of the Feynman integrals in the color-ordered amplitudes of [9]. Given that the scalar invariants $\langle C_{1|A,B,C}\rangle$ can be expanded in a BCJ basis of SYM tree amplitudes $A^{\rm tree}(\ldots)$ [46,44], the five-point kinematic factors $\langle \ell_m C^m_{1|2,3,4,5}\rangle$ and $\langle C_{1|ij,k,l}\rangle$ allow for three linearly independent permutations of the partial integrand (5.27). This is another example of how maximal supersymmetry introduces extra degeneracies beyond the upper bound of $(n-1)!$ linearly independent n-point partial integrands.

5.5. Reconciling the hexagon anomaly with the BCJ duality

5.5.1. Deviation from BCJ relations in the literature

Among the one-loop integrands constructed in [9] from BRST invariance and locality, only the five-point numerators were found to obey the BCJ duality. It deserves clarification why the six-point amplitude of [9] incorporated deviations from the BCJ duality even though it gives rise to the same partial integrand (5.28) as the BCJ master numerators (5.25). Generally speaking, the puzzle is resolved by the different bookkeeping of cubic diagrams resulting from the new representation of Feynman integrals reviewed in section 2.5. As explained in subsection 3.4, this leaves more flexibility to tune the numerators such as to satisfy the kinematic Jacobi relations.

An example for a kinematic Jacobi relation which has been violated in the six-point amplitude representation of [9] is depicted in Fig. 5: Since each cyclically inequivalent pentagon in the reference is associated with a single numerator, diagrams with different positions of ℓ among the internal edges are interlocked through shifts such as $\ell \rightarrow \ell - k_{23}$. The resulting numerator for the rightmost diagram in Fig. 5 was found to violate the depicted Jacobi relation [9].

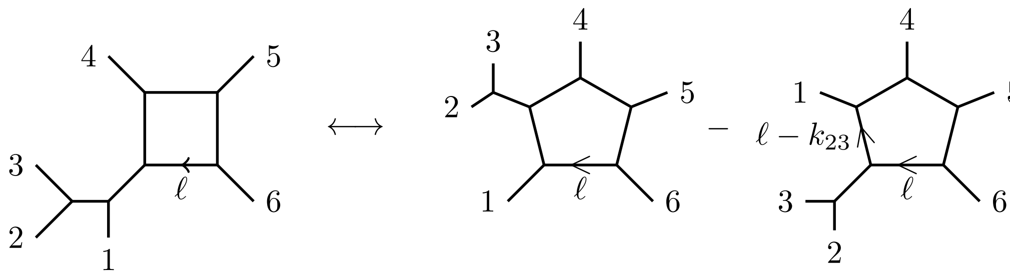

Fig. 5 Counterexample for kinematic Jacobi relations at six points.

In the present context with most propagators linear in ℓ, however, the numerators for the pentagon diagrams on the right hand side are both given by differences $\mathcal{C}_{+|23456|-}(\ell) - \mathcal{C}_{+|32456|-}(\ell)$ of hexagon numerators (5.25), without any relation to $\mathcal{C}_{+|45623|-}(\ell-k_{23}) - \mathcal{C}_{+|45632|-}(\ell-k_{23})$. This relies on the fact that the hexagon numerators (5.25) do not depend on the position of leg 1 in the diagram, and the box numerator on the left hand side of Fig. 5 with leg 1 in a massive corner vanishes accordingly.

5.5.2. The hexagon anomaly from a partial integrand

It has been speculated in [9] that the deviations from six-point kinematic Jacobi relations in the representation of the reference are related to the hexagon anomaly of ten-dimensional SYM. We will propose a treatment of the anomaly which preserves both the BCJ duality and the KLT relations for supergravity amplitudes.

At the level of the partial integrand (5.28), the hexagon anomaly can be seen from the tensor hexagon numerator $\frac{1}{2}\ell_m\ell_n C^{mn}_{1|2,3,4,5,6}$ whose non-zero BRST variation $\sim \ell_m\ell_n\eta^{mn}$ [83] signals a breakdown of linearized gauge invariance. The gauge variations [77]

$$a(1,2,\ldots,6,-,+)\big|_{e_1\to k_1} = \frac{\ell_m\ell_n\eta^{mn} i\epsilon_{10}(k_2,e_2,k_3,e_3,\ldots,k_6,e_6)}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{1234,\ell}s_{12345,\ell}} \tag{5.31}$$

of the partial integrands combine to a rational term in the ten-dimensional color-stripped single-trace amplitude after undoing the partial-fraction rearrangement of the hexagon:

$$\begin{aligned} A(1,2,\ldots,6)\big|_{e_1\to k_1} &= i\epsilon_{10}(k_2,e_2,k_3,e_3,\ldots,k_6,e_6)\int\frac{\mathrm{d}^{10}\ell}{\ell^2} \\ &\quad\times\left\{\frac{\ell_m\ell_n\eta^{mn}}{s_{1,\ell}s_{12,\ell}s_{123,\ell}s_{1234,\ell}s_{12345,\ell}} + \mathrm{cyc}(1,2,3,4,5,6)\right\} \\ &= 32 i\epsilon_{10}(k_2,e_2,k_3,e_3,\ldots,k_6,e_6)\int \mathrm{d}^{10}\ell \\ &\quad\left\{\frac{\ell_m\ell_n\eta^{mn}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2\ldots(\ell{+}k_{12345})^2} - \frac{1}{(\ell{+}k_1)^2(\ell{+}k_{12})^2\ldots(\ell{+}k_{12345})^2}\right\} \\ &= i\epsilon_{10}(k_2,e_2,k_3,e_3,\ldots,k_6,e_6)\frac{(2\pi)^5}{5!} \end{aligned} \tag{5.32}$$

In dimensional regularization with $\mathrm{d}^{10}\ell\to\mathrm{d}^{10-2\varepsilon}\ell$, the rational result can be understood from the difference between the ten-dimensional components $\ell_m\ell_n\eta^{mn}$ in the numerator and the $(10{-}2\varepsilon)$-dimensional loop momenta in the propagators [87].

5.5.3. BCJ and KLT relations in presence of anomalies

Given that the partial integrand (5.28) only exhibits a gauge variation (5.31) in the first leg but stays invariant under $e_j \to k_j$ for the remaining ones $j = 2, 3, \ldots, 6$, it cannot stem from a permutation invariant CHY integrand. Indeed, the breakdown of permutation symmetry in the string-theory correlator underlying (5.21) has been identified as a boundary term in moduli space [77] which translates into

$$\mathcal{K}_6 - (\mathcal{K}_6\big|_{1\leftrightarrow 2}) = \ell_m \ell_n \eta^{mn} i\epsilon_{10}(e_1, e_2, k_3, e_3, k_4, e_4, k_5, e_5, k_6, e_6) \ . \tag{5.33}$$

Strictly speaking, one-loop ($n \geq 6$)-point correlators single out one external leg[14] which carries the violation of linearized gauge invariance by a rational term (5.32) [88,77]. Keeping track of the singled-out leg j in the correlator through an additional superscript $\mathcal{K}_n^{(j)}$ (and $\mathcal{I}^{(j)}_{\rm SYM}$ according to (3.17)), partial integrands also need to be defined with a reference leg,

$$a^{(j)}(\tau(1,2,\ldots,n,+,-)) = \int {\rm d}\mu^{\rm tree}_{n+2} \ {\rm PT}(\tau(1,2,\ldots,n,+,-))\, \mathcal{I}^{(j)}_{\rm SYM}(\ell) \ , \tag{5.34}$$

which is taken to be $j = 1$ in the above expressions. However, this dependence on the reference leg j does not alter the BCJ relations (3.35) and (3.36) among $a^{(j)}(\tau(1,2,\ldots,n,+,-))$ with different permutations τ, provided that j is the same for each term in the BCJ relations: They are a sole consequence of the scattering equations relating the Parke–Taylor factors in (5.34), regardless of the permutation properties of the accompanying $\mathcal{I}^{(j)}_{\rm SYM}(\ell)$. By a similar argument, kinematic Jacobi relations are not affected by the dependence (5.33) of the underlying correlators on the reference leg.

Accordingly, there is no obstruction in constructing six-point integrands for ten-dimensional supergravity from the double-copy of the BCJ numerators (5.25) or from the partial integrands (5.28) along with the one-loop KLT relations. Regardless of the relative chirality of the fermions in the two gauge-theory copies, the resulting supergravity is known to have no hexagon anomaly [89].

In the context of the double-copy approach, anomaly cancellation suggests that the integrated supergravity amplitude

$$M_6 = \int \frac{{\rm d}^{10}\ell}{\ell^2} \sum_{\rho,\tau\in S_5} a^{(j)}(+,\rho(2,3,4,5,6),1,-)\, S[\rho|\tau]_\ell\, \tilde a^{(j)}(+,\tau(2,3,4,5,6),-,1) \tag{5.35}$$

does not depend on the choice of the reference leg j in the SYM constituents. It would be interesting to verify this by explicitly integrating the hexagon contributions along the lines of (5.32) which carry the spurious sensitivity to j.

[14] In the opening line for the computation of the superstring amplitudes, one leg enters through the unintegrated vertex operator in the pure-spinor formalism or in the -1 superghost picture in the parity-odd sector of the RNS setup.

6. Bosonic correlators with reduced supersymmetry

This section is devoted to explicit and simplified representations of the CHY correlators for half-maximal and parity-even parts of quarter-maximal SYM. The three- and four-point results of this section coincide with the field-theory limits of superstring one-loop correlators with reduced supersymmetry, with the results of [49,50] as a starting point.

Most of the subsequent expressions are tailored to chiral SYM in six dimensions with eight supercharges. Their dimensional reductions and quarter-maximally supersymmetric counterparts in four dimensions are straightforwardly obtained by dropping the parity-odd contributions $\sim \epsilon_6$ and rescaling the scalar box numerator in the four-point correlator of section 6.2.

6.1. Review of Minahaning

As a consequence of the spin sums (4.29) and (4.30), n-point CHY correlators of half-maximal and quarter-maximal SYM are polynomials in ℓ and G_{ij} of degree $n-2$. The symmetry properties of the resulting BCJ master numerators (3.18) give rise to triangle and bubble diagrams in the partial integrands (3.31). This includes bubbles in the external legs as depicted in figure Fig. 6, where one of the propagator $\sim s^{-1}_{12\ldots n-1} = k_n^{-2}$ formally diverges in the phase space of n massless particles.

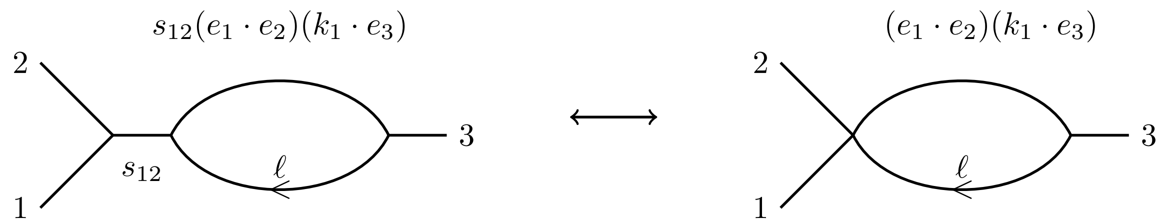

Fig. 6 The divergent propagator s_{ij}^{-1} in external bubbles is cancelled by a formally vanishing factor of s_{ij} in the kinematic numerator.

The external-bubble numerators derived from the CHY- or superstring correlators turn out to vanish with $s_{12\ldots n-1}$. The resulting "0/0" indeterminate can be regularized by relaxing momentum conservation in intermediate steps, following the proposal of Minahan in 1987 [68] and the recent four-point implementation in [49,50]. The idea is to use no relation among Mandelstam invariants other than $\sum_{1\leq i<j}^n s_{ij} = 0$ which amounts to a lightlike deformation of momentum conservation

$$k_1 + k_2 + \ldots + k_n = p \ , \qquad p^2 = 0 \ , \qquad (e_i \cdot p) = 0 \ . \tag{6.1}$$

In this regularization scheme for infrared divergences, the three-point correlator (4.15) with the spin sums (4.29), (4.30) and parity-odd part (4.42) is evaluated as

$$\begin{aligned}\mathcal{K}_3^{1/2} &= \ell_m\big[e_1^m(e_2\cdot k_3)(e_3\cdot k_2)+(1\leftrightarrow 2,3)\big]+i\epsilon_6(\ell,e_1,k_2,e_2,k_3,e_3)\\ &\quad+\big[G_{12}s_{12}(e_1\cdot e_2)(k_1\cdot e_3)+\text{cyc}(1,2,3)\big]\ ,\end{aligned}\tag{6.2}$$

see [68,49,50] for the superstring ancestors. The deformation (6.1) temporarily assigns nonzero values such as $s_{12}=\frac{1}{2}(k_1+k_2)^2=\frac{1}{2}(k_3+p)^2=(k_3\cdot p)$ to the three-particle Mandelstam invariants, and the resulting triangle numerators (3.18) are given by

$$\begin{aligned}N_{+|123|-}(\ell) &= \ell_m\big[e_1^m(e_2\cdot k_3)(e_3\cdot k_2)+(1\leftrightarrow 2,3)\big]+i\epsilon_6(\ell,e_1,k_2,e_2,k_3,e_3)\\ &\quad-\frac{1}{2}\big[s_{12}(e_1\cdot e_2)(k_1\cdot e_3)+s_{13}(e_1\cdot e_3)(k_1\cdot e_2)+s_{23}(e_2\cdot e_3)(k_2\cdot e_1)\big]\ .\end{aligned}\tag{6.3}$$

After dressing with the doubly-partial amplitudes (3.24), all potential divergences from propagators s_{ij}^{-1} are compensated by the numerator factors of $\sim s_{ij}$ in second line. In other words, the limit $p\to 0$ and thereby $s_{ij}\to 0$ is taken in the last step of

$$\begin{aligned}a^{1/2}(1,2,3,-,+) &= \lim_{s_{ij}\to 0}\lim_{k_\pm\to\pm\ell}\sum_{\rho\in S_3} m^{\text{tree}}[+,1,2,3,-|+,\rho(1,2,3),-]\,N_{+|\rho(123)|-}(\ell)\\ &= \lim_{s_{ij}\to 0}\Big\{-\frac{1}{2}\Big[\frac{2}{s_{12}s_{12,\ell}}+\frac{1}{s_{1,\ell}s_{12,\ell}}\Big]s_{12}(e_1\cdot e_2)(k_1\cdot e_3)-\frac{1}{2}\frac{1}{s_{1,\ell}s_{12,\ell}}s_{13}(e_1\cdot e_3)(k_1\cdot e_2)\\ &\qquad-\frac{1}{2}\Big[\frac{2}{s_{23}s_{1,\ell}}+\frac{1}{s_{1,\ell}s_{12,\ell}}\Big]s_{23}(e_2\cdot e_3)(k_2\cdot e_1)+\frac{\ell_m N^m_{1,2,3}}{s_{1,\ell}s_{12,\ell}}\Big\}\\ &= \frac{\ell_m N^m_{1,2,3}}{s_{1,\ell}s_{12,\ell}}-\frac{(e_1\cdot e_2)(k_1\cdot e_3)}{s_{12,\ell}}-\frac{(e_2\cdot e_3)(k_2\cdot e_1)}{s_{1,\ell}}\ ,\end{aligned}\tag{6.4}$$

see (3.24) for the doubly-partial amplitudes. For the external bubble adjacent to leg 3, the cubic-diagram numerator $N_{+|123|-}(\ell)-N_{+|213|-}(\ell)=-s_{12}(e_1\cdot e_2)(k_1\cdot e_3)$ cancels the divergent propagator[15] s_{12}^{-1}, and the situation is depicted in Fig. 6. The vector triangle contribution $\ell_m N^m_{1,2,3}$ refers to the first line of (6.3) which is unaffected by the limit $s_{ij}\to 0$.

The analogous discussion with propagators quadratic in ℓ can be found in [50], and in both the reference and in (6.4), gauge invariance of the integrands relies on the bubble contributions. Although the partial-fraction representation of the external bubbles manifest

[15] A similar interplay between divergent propagators and vanishing numerators has been observed in the four-point four-loop amplitude of maximal SYM [3]. Their finite net contribution plays an important role to obtain the expected UV divergence.

that they integrate to zero in color-stripped single-trace amplitudes (3.28), it would obscure gauge invariance to drop them in (6.4).

Before discussing the cancellation of divergent propagators in four-point partial integrands analogous to (6.4), we describe the correlator (6.2) in a Berends–Giele framework and set the stage for kinematic factors at higher multiplicity.

6.2. Berends–Giele representation of reduced-supersymmetry correlators

Kinematic factors in maximally supersymmetric correlators are conveniently expressed in terms of the Berends–Giele currents in (5.16) and their supersymmetrizations. In the same way, the following building blocks are tailored to describe the polarization dependence in gluonic one-loop amplitudes with reduced supersymmetry [49,50],

$$
\begin{aligned}
t_{A,B} &\equiv \frac{1}{2}\mathfrak{f}_A^{mn}\mathfrak{f}_B^{nm} \\
t^m_{A,B,C} &\equiv \big[\mathfrak{e}_A^m t_{B,C} + (A \leftrightarrow B,C)\big] + \frac{i}{4}\epsilon_6^m(\mathfrak{e}_A,\mathfrak{f}_B,\mathfrak{f}_C) \\
t^{mn}_{A,B,C,D} &\equiv 2\big[\mathfrak{e}_A^{(m}\mathfrak{e}_B^{n)} t_{C,D} + (A,B|A,B,C,D)\big] + \frac{i}{2}\big[\mathfrak{e}_B^{(m}\epsilon_6^{n)}(\mathfrak{e}_A,\mathfrak{f}_C,\mathfrak{f}_D) + (B\leftrightarrow C,D)\big]\,.
\end{aligned}
\tag{6.5}
$$

By inserting the recursive definitions (5.14) and (5.15) of the Berends–Giele currents $\mathfrak{e}_A^m$ and $\mathfrak{f}_B^{mn}$, the kinematic factor of the external bubble in Fig. 6 is reproduced by

$$
t_{12,3} = (e_1\cdot e_2)(k_1\cdot e_3)\ . \tag{6.6}
$$

The cancellation of the pole $\mathfrak{f}_{12}^{mn} \sim s_{12}^{-1}$ in $t_{12,3}$ follows from the infrared regularization scheme in (6.1). One can analogously show that the four-point scalars $t_{12,34}$ and $t_{123,4}$ only have simple poles in s_{ij} [50] in spite of the spurious pole structure $\sim (s_{ij}s_{123})^{-1}$ of $\mathfrak{f}_{123}^{mn}$.

In complete analogy to their maximally supersymmetric counterparts (5.12) and (5.13), the bosonic building blocks (6.5) enter one-loop amplitudes in their gauge invariant combinations [49,50]

$$
\begin{aligned}
C^{1/2}_{1|23} &\equiv t_{1,23} + t_{12,3} - t_{13,2}\ , \\
C^{1/2}_{1|234} &\equiv t_{1,234} + t_{12,34} + t_{123,4} - t_{124,3} - t_{14,23} - t_{142,3} + t_{143,2} \\
C^{m,1/2}_{1|2,3} &\equiv t^m_{1,2,3} + k_2^m t_{12,3} + k_3^m t_{13,2}\ , \\
C^{m,1/2}_{1|23,4} &\equiv t^m_{1,23,4} + t^m_{12,3,4} - t^m_{13,2,4} + k_3^m t_{123,4} - k_2^m t_{132,4} + k_4^m\big[t_{14,23} - t_{214,3} + t_{314,2}\big]\ .
\end{aligned}
\tag{6.7}
$$

The parity-odd part of the tensorial generalization

$$C^{mn,1/2}_{1|2,3,4} \equiv t^{mn}_{1,2,3,4} + 2\big[k_2^{(m} t^{n)}_{12,3,4} + (2 \leftrightarrow 3,4)\big] - 2\big[k_2^{(m} k_3^{n)} t_{213,4} + (2,3|2,3,4)\big] \tag{6.8}$$

which will appear in a box numerator gives rise to an anomalous gauge variation

$$C^{mn,1/2}_{1|2,3,4}\big|_{e_1\to k_1} = 2i\eta^{mn}\epsilon_6(k_2,e_2,k_3,e_3,k_4,e_4) \tag{6.9}$$

analogous to (5.31) due to the ten-dimensional tensor hexagon in pure-spinor superspace. The kinematic factors (6.7) and (6.8) have been noticed in the simplification of the superstring correlators [49] as well as the resulting field-theory limits [50], and the scalar instances coincide with the tree-level amplitudes, $C^{1/2}_{1|23\ldots n} = A^{\rm tree}(1,2,3,\ldots,n)$.

In terms of the kinematic variables in (6.5), the three-point correlator (6.2) and its four-point counterpart are given by

$$\mathcal{K}_3^{1/2} = \ell_m t^m_{1,2,3} + [X_{12} t_{12,3} + {\rm cyc}(1,2,3)] \tag{6.10}$$

$$\begin{aligned}\mathcal{K}_4^{1/2} &= \frac{1}{2}\ell_m\ell_n t^{mn}_{1,2,3,4} + \ell_m[X_{12} t^m_{12,3,4} + (1,2|1,2,3,4)] \\ &\quad + [X_{12}(X_{13}+X_{23})t_{123,4} + X_{13}(X_{12}+X_{32})t_{132,4} + (4\leftrightarrow 3,2,1)] \\ &\quad + [X_{12}X_{34}t_{12,34} + {\rm cyc}(2,3,4)] + \frac{1}{4}t_8(1,2,3,4)\ ,\end{aligned} \tag{6.11}$$

see [49] for the superstring antecedent of the latter with the loop momentum integrated out. Following the spin sums in (4.29) and (4.31), the relative factor between the last term $t_8(1,2,3,4)$ and the remaining correlator depends on the particle content, also see section 5.2 of [50] for a discussion in a string-theory context. The use of scattering equations explained in section 4.5 leads to the manifestly gauge invariant rewritings

$$\mathcal{K}_3^{1/2} = \ell_m C^{m,1/2}_{1|2,3} + X_{23} C^{1/2}_{1|23} \tag{6.12}$$

$$\begin{aligned}\mathcal{K}_4^{1/2} &= \frac{1}{2}\ell_m\ell_n C^{mn,1/2}_{1|2,3,4} + \ell_m[X_{23} C^{m,1/2}_{1|23,4} + {\rm cyc}(2,3,4)] \\ &\quad + [X_{23}X_{34}C^{1/2}_{1|234} - X_{23}X_{24}C^{1/2}_{1|324} - X_{24}X_{34}C^{1/2}_{1|243}] + \frac{1}{4}t_8(1,2,3,4)\ ,\end{aligned} \tag{6.13}$$

and the manipulations in section 4.6 and appendix A allow to express higher-multiplicity correlators in a similar basis of functions. Note the close structural similarity to the maximally supersymmetric five- and six-point correlators in (5.20) and (5.21).

The correlators (6.12) and (6.13) translate into the following gauge invariant BCJ master numerators

$$\mathcal{C}^{1/2}_{+|\rho(2,3)|-}(\ell) = \ell_m C^{m,1/2}_{1|2,3} + \frac{1}{2} s_{23}\text{sgn}^{\rho}_{23} C^{1/2}_{1|23} \tag{6.14}$$

$$\begin{aligned}\mathcal{C}^{1/2}_{+|\rho(2,3,4)|-}(\ell) &= \frac{1}{2}\ell_m \ell_n C^{mn,1/2}_{1|2,3,4} + \frac{1}{2}\ell_m\big[s_{23}\text{sgn}^{\rho}_{23}C^{m,1/2}_{1|23,4} + \text{cyc}(2,3,4)\big] + \frac{1}{4}t_8(1,2,3,4) \\ &+ \frac{1}{4}\big[s_{23}\text{sgn}^{\rho}_{23}s_{34}\text{sgn}^{\rho}_{34}C^{1/2}_{1|234} - s_{23}\text{sgn}^{\rho}_{23}s_{24}\text{sgn}^{\rho}_{24}C^{1/2}_{1|324} - s_{24}\text{sgn}^{\rho}_{24}s_{34}\text{sgn}^{\rho}_{34}C^{1/2}_{1|243}\big]\end{aligned} \tag{6.15}$$

for triangle- and box diagrams, respectively. These numerators result in the following expressions for the three- and four-point partial integrands [13]

$$\begin{aligned} a^{1/2}(1,2,3,-,+) &= \frac{\ell_m C^{m,1/2}_{1|2,3}}{s_{1,\ell}s_{12,\ell}} - \frac{C^{1/2}_{1|23}}{s_{1,\ell}} \\ &= \frac{\ell_m\big[e_1^m(k_2\cdot e_3)(k_3\cdot e_2) + (1\leftrightarrow 2,3)\big] + \big[(\ell\cdot k_2)(e_1\cdot e_2)(k_1\cdot e_3) + (2\leftrightarrow 3)\big]}{s_{1,\ell}s_{12,\ell}} \\ &+ \frac{i\epsilon_6(\ell,e_1,k_2,e_2,k_3,e_3)}{s_{1,\ell}s_{12,\ell}} + \frac{(e_1\cdot e_2)(k_1\cdot e_3) + (e_2\cdot e_3)(k_2\cdot e_1) + (e_1\cdot e_3)(k_3\cdot e_2)}{s_{1,\ell}} \end{aligned} \tag{6.16}$$

$$\begin{aligned} a^{1/2}(1,2,3,4,-,+) &= \frac{C^{1/2}_{1|234}}{s_{1,\ell}} - \frac{\ell_m C^{m,1/2}_{1|23,4}}{s_{1,\ell}s_{123,\ell}} - \frac{\ell_m C^{m,1/2}_{1|34,2}}{s_{1,\ell}s_{12,\ell}} + \frac{t_8(1,2,3,4) - s_{23}s_{34}C^{1/2}_{1|234}}{4s_{1,\ell}s_{12,\ell}s_{123,\ell}} \\ &+ \frac{\ell_m\ell_n C^{mn,1/2}_{1|2,3,4} - \ell_m\big[s_{23}C^{m,1/2}_{1|23,4} + s_{24}C^{m,1/2}_{1|24,3} + s_{34}C^{m,1/2}_{1|34,2}\big]}{2s_{1,\ell}s_{12,\ell}s_{123,\ell}} \end{aligned} \tag{6.17}$$

cf. (6.4) for the three-point case. In identifying the scalar box numerator in the first line of (6.17), we have used that $s_{23}s_{34}C^{1/2}_{1|234}$ is permutation symmetric in $2,3,4$.

Note that Kleiss–Kuijf relations (3.30) imply the vanishing of non-planar partial integrands at three points,

$$a^{\beta-\text{SYM}}(1,2,-,3,+) = 0\ , \qquad \beta = 1, \frac{1}{2}, \frac{1}{4}\ . \tag{6.18}$$

Accordingly, the three-point supergravity integrand from the KLT formula (3.43) involving at least one supersymmetric gauge-theory copy $\beta = 1, \frac{1}{2}, \frac{1}{4}$ is identically zero,

$$m_3^{\text{(S)YM}\otimes\beta-\text{SYM}}(\ell) = \sum_{\rho,\tau\in S_2} a^{\text{(S)YM}}(+,\rho(2,3),1,-)\, S[\rho|\tau]_\ell\, \tilde{a}^{\beta-\text{SYM}}(+,\tau(2,3),-,1) = 0\ . \tag{6.19}$$

At four points, the anomalous gauge variation (6.9) of the tensor building block yields

$$a^{1/2}(1,2,3,4,-,+)\,\big|_{e_1\to k_1} = \frac{\ell_m \ell_n \eta^{mn} i\epsilon_6(k_2,e_2,k_3,e_3,k_4,e_4)}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} \tag{6.20}$$

in analogy to (5.31). For a six-dimensional color-stripped single-trace amplitude, one can follow the manipulations of (5.32) to undo the partial-fraction rearrangement of the box and to identify the anomaly as a purely rational term:

$$A^{1/2}(1,2,3,4)\,\big|_{e_1\to k_1} = i\epsilon_6(k_2,e_2,k_3,e_3,k_4,e_4)\frac{(2\pi)^3}{3!}\ . \tag{6.21}$$

The representation of $A^{1/2}(1,2,3,4)$ constructed in [50] from gauge invariance and locality is equivalent to the partial integrand (6.17), but it was observed in the reference to deviate from the BCJ duality. By the arguments of section 5.5, organizing the loop integrand in terms of cubic diagrams with propagators linear in ℓ (cf. section 3.4) alleviates the task of finding BCJ numerators. Hence, there is no contradiction in presenting BCJ master numerators (6.15) in terms of the same building blocks seen in the BCJ violating setup of [50] since the cubic diagrams in the reference were tailored to propagators quadratic in ℓ.

Similar to the maximally supersymmetric six-point correlator, the anomalous four-point correlator (6.13) also violates permutation invariance, cf. (5.33). Following the reasoning around (5.34), a fully accurate labelling of the partial integrand (6.17) would involve an additional superscript $a^{1/2}(1,2,3,4,-,+) \to a^{1/2,(j=1)}(1,2,3,4,-,+)$ to indicate that linearized gauge invariance is violated in the j^{th} leg. Finally, the dependence on j is expected to disappear after integrating the supergravity amplitude from the one-loop KLT formula (3.43) over ℓ.

7. Conclusions

In this paper we studied new BCJ representations of one-loop scattering amplitudes in supersymmetric gauge-theory and gravity amplitudes, which are largely inspired by both the CHY/ambitwistor-string formulation and superstring theory. Based on the CHY-inspired representation for supersymmetric amplitudes, we give a general proof of one-loop BCJ and KLT relations for the *partial integrands* proposed in [13]. In the RNS incarnation of this new representation, we bring one-loop correlators on a nodal Riemann sphere into a form which makes BCJ numerators accessible for all multiplicities. The method works for external bosons in presence of any amount of supersymmetry and for both parity-even and parity-odd sectors. Moreover, from the field-theory limit of pure-spinor superstrings, we supersymmetrized the ($n \leq 6$)-point BCJ numerators to include external fermions as well.

We would like to highlight three intriguing features of our results. First, the manifestly gauge- and diffeomorphism-invariant BCJ and KLT relations can be proved solely based on structural results on one-loop CHY formulae, without referring to the explicit form of the BCJ numerators. Second, correlators with maximal and reduced supersymmetry are shown to be degree-($n{-}4$) and degree-($n{-}2$) polynomials in loop momentum ℓ and the Green function on the nodal sphere, manifesting the powercounting of ℓ including the no-triangle property for maximal supersymmetry. Last but not least, since we naturally obtain one-loop amplitudes with linear propagators, our BCJ numerators satisfy the color-kinematics duality in a slightly different organization scheme of cubic diagrams as compared to its original loop-level formulation [2], see section 3.4. However, to our best knowledge, this is the first D-dimensional, all-multiplicity control of one-loop BCJ numerators which can be directly double copied to give supergravity integrands.

Although we have only considered supersymmetric gauge theories and gravity, we expect our results to hold for non-supersymmetric theories as well. Besides, our main results naturally apply to other theories as well: the one-loop KLT formula with the NLSM and (super-)Yang–Mills theory yields integrands of Born–Infeld theory along with supersymmetric extensions to Dirac–Born–Infeld–Volkov–Akulov theories. As will be elaborated elsewhere, the one-loop amplitude relations for EYM partial integrands [13] can be proved using CHY representations. Using explicit results for the correlators, one can obtain BCJ numerators for one-loop amplitudes of the NLSM and for EYM in a similar way.

There are several directions to investigate in the future. Already at one loop, it would be highly desirable to determine higher-point supersymmetric correlators from field-theory limit of the pure-spinor formalism. We expect the results to be expanded in a basis of worldsheet functions as explained in section 4, with coefficients given by BRST pseudo-invariants, which have been studied in [83]. Moreover, it would be interesting to incorporate α'-corrections of the superstring using the same approach and to study one-loop BCJ numerators and KLT relations for amplitudes from higher-dimensional operators as well as those in Z-theory [30,31,32].

A particularly exciting direction is to generalize the new BCJ representations and their applications to higher loops. For example, a natural follow-up question is how to construct BCJ numerators and derive KLT formulae at higher loops. We expect that a strategic path forward is to again organize g-loop correlators on the nodal Riemann spheres in terms of

Parke–Taylor factors with g pairs of double points $\sigma_\pm$. Although a systematic study such higher-loop correlators, KLT relations and BCJ numerators will be given in the future, we would like to display the two-loop four-point correlator as an encouraging example.

7.1. Preview example: The two-loop four-point correlator on the nodal sphere

A central ingredient of genus-g correlators are the global holomorphic one-forms ω_J with $J = 1, 2, \ldots, g$ which degenerate as follows on nodal Riemann spheres:

$$\omega_J(\sigma_i) = \frac{(\sigma_{J+} - \sigma_{J-})\, \mathrm{d}\sigma_i}{(\sigma_i - \sigma_{J+})(\sigma_i - \sigma_{J-})} \ . \tag{7.1}$$

They enter the genus-two superstring correlators of [90] through the antisymmetric combinations

$$\Delta_{i,j} \equiv \omega_1(\sigma_i)\omega_2(\sigma_j) - \omega_2(\sigma_i)\omega_1(\sigma_j) = \varepsilon^{IJ}\omega_I(\sigma_i)\omega_J(\sigma_j) \ , \tag{7.2}$$

in lines with modular invariance. Moreover, the moduli-space measure introduces differences of the double points $\sigma_{1\pm}$ and $\sigma_{2\pm}$ into the correlator on the nodal sphere [40],

$$\prod_{j=1}^{4} \mathrm{d}\sigma_j \, \mathcal{I}_4^{2-\text{loop}} = \frac{s_{12}\Delta_{4,1}\Delta_{2,3} + s_{23}\Delta_{1,2}\Delta_{3,4}}{(\sigma_{1+}{-}\sigma_{2+})(\sigma_{1+}{-}\sigma_{2-})(\sigma_{1-}{-}\sigma_{2+})(\sigma_{1-}{-}\sigma_{2-})} \ , \tag{7.3}$$

where the overall kinematic factor $t_8(f_1, f_2, f_3, f_4)$ is suppressed. This result can be expanded in terms of eight-point Parke Taylor factors involving $\sigma_{5,6} \equiv \sigma_{1\pm}$ and $\sigma_{7,8} \equiv \sigma_{2\pm}$:

$$\begin{aligned} \mathcal{I}_4^{2-\text{loop}} &= s_{12}\big[\mathrm{PT}(7,1,2,5,3,4,6,8) + \mathrm{PT}(7,1,2,6,3,4,5,8)\big] \\ &\quad + s_{23}\big[\mathrm{PT}(7,1,5,2,3,6,4,8) + \mathrm{PT}(7,1,6,2,3,5,4,8)\big] \\ &\quad + s_{12}\big[\mathrm{PT}(7,5,1,2,6,3,4,8) + \mathrm{PT}(7,6,1,2,5,3,4,8)\big] + \mathrm{perm}(1,2,3,4) \ . \end{aligned} \tag{7.4}$$

Based on this 144-term sum, it would be very interesting to study two-loop KLT formulae as well as BCJ numerators, for maximally supersymmetric Yang-Mills and gravity amplitudes. Of course, more work is needed to obtain the parental string correlators at higher multiplicity and loop order as well as reduced supersymmetry for generic points in the moduli space of the relevant Riemann surface.

Acknowledgements: We are indebted to Carlos Mafra for a variety of enlightening discussions, his participation in parts of the project – in particular his proof of equation (3.15)

– and valuable comments on a draft. We would like to thank Nima Arkani-Hamed, Marcus Berg, Freddy Cachazo, Hao Fu, Xiangrui Gao, Yvonne Geyer, Ricardo Monteiro, Ellis Yuan and Minshan Zheng for combinations of useful discussions and helpful comments on a draft. This research was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915, and we are grateful to the KITP Santa Barbara as well as the organizers of the workshop "Scattering Amplitudes and Beyond" for providing stimulating atmosphere, support and hospitality. S.H.'s research is supported in part by the Thousand Young Talents program and the Key Research Program of Frontier Sciences of CAS. Y.Z.'s research is partly supported by NSFC Grants No. 11375026 and 11235003.

Appendix A. One-loop basis of worldsheet functions

The goal of this appendix is to arrive at a basis of worldsheet functions for field-theory amplitudes at one loop. We will describe how to achieve this in two steps:

1. Eliminating all subcycles of propagators $G_{a_1a_2}G_{a_2a_3}\ldots G_{a_na_1}$
2. Eliminating the dependence on the position of leg 1 from any G_{ij}

A.1. Eliminating subcycles of propagators

Since multiple subcycles can be recursively reduced to cases with fewer subcycles, it is sufficient to consider the case with one subcycle, say $G_{a_1a_2}G_{a_2a_3}\cdots G_{a_ma_1}$. The algorithm to break it open selects a subset of its propagators (therefore this is not a cycle by itself) and rewrites it in a basis of "IBP functions" $X_{a_1a_2\ldots a_m}$ defined in (4.34). For example,

$$G_{12}G_{13} = \frac{1}{s_{123}}\left\{\frac{s_{23}}{4} + \frac{X_{123}}{s_{12}} + \frac{X_{132}}{s_{13}}\right\}, \tag{A.1}$$

$$\begin{aligned} G_{12}G_{13}G_{14} = \frac{1}{s_{1234}}\Bigg\{ &\Big[\frac{X_{1234}}{s_{12}s_{123}} + \text{symm}(2,3,4)\Big] \\ &+ \Big[\frac{s_{34}}{4}\Big(\frac{1}{s_{12}} + \frac{1}{s_{134}}\Big)X_{12} + \frac{1}{4}\Big(\frac{s_{24}}{s_{124}} - \frac{s_{34}}{s_{134}}\Big)X_{23} + \text{cyc}(2,3,4)\Big]\Bigg\}, \end{aligned} \tag{A.2}$$

and such inverse relations exist for any monomial of propagators without subcycles. One can check these relations by plugging in (4.34) and by using the Fay identity (3.11). Alternatively, we will sketch how to derive such relations below.

For example, to break the subcyle $G_{12}G_{23}G_{13}$ we rewrite $G_{12}G_{13}$ in terms of IBP functions as shown in (A.1). Since both labels 2 and 3 appear in X_{123} and X_{132}, one uses an IBP relation to rewrite $X_{123} = X_{12}(X_{34} + X_{35} + \cdots X_{3n} + \ell \cdot k_3)$ and similarly, $X_{132} = X_{13}(X_{24} + X_{25} + \cdots X_{2n} + \ell \cdot k_2)$. Then we have no subcycles left:

$$G_{12}G_{23}G_{13} = \frac{1}{s_{123}}\Big\{\frac{s_{23}G_{23}}{4} + G_{12}G_{23}\Big(\sum_{p=4}^{n} s_{3p}G_{3p} + \ell \cdot k_3\Big) + G_{13}G_{23}\Big(\sum_{p=4}^{n} s_{2p}G_{2p} + \ell \cdot k_2\Big)\Big\}\,, \tag{A.3}$$

and the same idea can be applied to any other subcycle. Apart from loop momenta and terms with fewer powers of G_{ij} (such as the contribution of $\frac{s_{23}G_{23}}{4}$ in (A.3)) which are intrinsic to genus one, the elimination of $G_{a_1a_2}G_{a_2a_3}\cdots G_{a_ma_1}$ largely follows the tree-level techniques to address products of Parke–Taylor factors (see e.g. [67]).

After eliminating all subcycles, we are left with products of X functions with overlapping labels, such as $X_{\cdots a_1 \cdots}G_{a_ma_1}$ and $G_{12}G_{23}G_{3p}$ with $p = 4, \cdots, n$ above. By using (A.1), (A.2) and generalizations, we can again rewrite them in terms of products of functions without overlapping labels, which are suitable for integration by parts. One therefore obtains a polynomial of IBP functions $X_{a_1a_2\ldots a_m}$ where in every monomial each particle label appears at most once as a subscript.

A.2. Eliminating the dependence on σ_1

After eliminating subcycles, the resulting X functions are not yet linearly independent; it is straightforward to see that one can still eliminate their dependence on particle 1 using scattering equations,

$$X_{123} = X_{12}(X_{13} + X_{23}) = (X_{23} + \ldots + X_{2n} + k_2 \cdot \ell)(X_{34} + \ldots + X_{3n} + k_3 \cdot \ell)\,, \tag{A.4}$$

where, as we mentioned, no subcycle will appear and again we recast e.g. $X_{23}X_{34}$ into X_{234} and X_{243} using e.g. (A.1). By repeating this process we obtain a basis of $X_{a_1a_2\ldots a_m}$ functions where particle 1 is eliminated. Moreover, one can always fix the first subscript of X to be the smallest[16], for example $X_{342} = -X_{234} + X_{243}$.

There is a straightforward way to count the degree directly in terms of X functions. It is convenient to introduce $X_i \equiv 1$ $(i \neq 1)$ for labels that did not appear in a monomial such that after inserting them, each label $2, \ldots, n$ appears exactly once. For example, we write the identity $1 = \prod_{i=2}^{n} X_i$, $X_{23}X_{45} = X_{23}X_{45}X_6$ for $n = 6$, and $X_{234}X_{56} = X_{234}X_{56}X_7X_8$ for $n = 8$ etc.. After inserting these identities, we see that the degree of $G_{i,j}$'s is given

[16] This follows from the fact that (4.34) satisfies Lie symmetries [46].

by $n-1$ minus the total number of X functions. In the examples above, the degree is 0, $5-3=2$ and $7-4=3$, respectively.

Now we are ready to count the number of basis elements of IBP friendly functions, for n points with a given degree in G_{ij}'s. Since label 1 is eliminated, the number of independent monomials in X functions with degree $0 \leq k \leq n-2$ is given by the number of ways to distribute $n-1$ labels into $n-k-1$ disjoint, non-empty sets, where labels in each set form a cycle (including length-1 cycles). The solution to this counting problem is known as the Stirling number of the first kind, $S_{n-1,n-k-1}$, see table 1. For example, for $k=0$, $S_{n-1,n-1}=1$ corresponds to the identity 1. For $k=1$, choosing $n=4$ and $n=5$ allows for 3 and 6 elements X_{23}, X_{24}, X_{34} and $X_{23}, X_{24}, X_{25}, \ldots, X_{45}$, respectively. Finally, $k=2$, $n=4$ gives rise to the 2 basis elements X_{234} and X_{243}.

For correlators with reduced supersymmetry, the degree of the polynomial in ℓ and G_{ij} is $n-2$, thus the total number of basis elements for n points is given by $\sum_{k=0}^{n-2} S_{n-1,n-k-1} = (n-1)!$ (see table 1). For example, for $n=5$, in addition to the elements with $k=0,1$ above, we have 11 elements for $k=2$: $X_{23}X_{45}, X_{24}X_{35}, X_{25}X_{34}$ and X_{234}, X_{243} along with their images under $\mathrm{cyc}(2,3,4,5)$. Finally, $n=5$ and $k=3$ introduces the six permutations of X_{2345} in $3,4,5$ which altogether yields $1+6+11+6=24$ basis elements at $n=5$.

Maximal supersymmetry allows for maximum degree $n-4$ in X functions, thus the total number of basis elements is $\sum_{k=0}^{n-4} S_{n-1,n-k-1} \equiv a_n$ (see table 1). Here a_n counts the number of $(n-1)$-permutations with at least 3 cycles (sequence **A067318** of [91]), e.g. $a_4=1$, $a_5=7$, $a_6=46$. For example, the 7-element basis for $n=5$ consists of 1 (along with ℓ) as well as $X_{23}, X_{24}, X_{25}, X_{34}, X_{35}, X_{45}$. For $n=6$, we have 1 (along with ℓ^2), $X_{23}, X_{24}, \ldots, X_{56}$ (along with ℓ) as well as $X_{23}X_{45}, X_{24}X_{35}, X_{25}X_{34}$ plus $(2345 \leftrightarrow 2346, 2356, 2456, 3456)$ and X_{234}, X_{243} plus $(234 \leftrightarrow 235, 236, \ldots, 456)$, altogether 46 elements.

Appendix B. Combinatoric proof of the formula (3.15).

In this appendix[17] we prove the formula (3.15), namely

$$G_{i_1 i_2} \ldots G_{i_{2p-1} i_{2p}} \prod_{j=1}^{n} \frac{1}{\sigma_j} = \frac{(-1)^n}{2^p} \sum_{\rho \in S_n} \mathrm{sgn}^{\rho}_{i_1 i_2} \ldots \mathrm{sgn}^{\rho}_{i_{2p-1} i_{2p}} \mathcal{Z}_{0\rho(1,2,\ldots,n)} \,, \tag{B.1}$$

[17] This appendix was written by Carlos Mafra.

where $\sigma_0 \equiv \sigma_+ \equiv 0$, and $\mathcal{Z}_P$ was defined in (3.12). For convenience, define the shorthands

$$\Sigma_{123\ldots n} \equiv \frac{1}{\sigma_1\sigma_2\ldots\sigma_n}, \qquad \Sigma^i_{123\ldots n} \equiv \sigma_i \Sigma_{123\ldots n} = \frac{1}{\sigma_1\sigma_2\ldots\widehat{\sigma}_i\ldots\sigma_n}\ , \tag{B.2}$$

where $\hat{\sigma}_i$ denotes the absence of σ_i, and the generalization to multiparticle Σ^Q_P is obvious. Note that Σ^Q_P is totally symmetric in P and Q. Recalling the auxiliary variable $\sigma_0 = 0$ and denoting a sum over permutations of the indices in P by (P) one can show that[18],

$$\Sigma^Q_P = (-1)^{|P|-|Q|}\mathcal{Z}_{0(P\backslash Q)}\,, \tag{B.3}$$

$$\mathcal{Z}_{0(Q)}\Sigma^Q_P = (-1)^{|P|-|Q|}\mathcal{Z}_{0(P)} \tag{B.4}$$

$$\mathcal{Z}_{0(P)}\mathcal{Z}_{0(Q)} = \mathcal{Z}_{0(PQ)} \tag{B.5}$$

$$\mathcal{Z}_{0(Pj)}(2\mathcal{Z}_{jk} + \mathcal{Z}_{k0}) = \mathcal{Z}_{0(jkP)}\text{sign}_{jk}, \quad \text{if } k \cap P = \emptyset\,, \tag{B.6}$$

where we used that $\mathcal{Z}_{PiQ} = (-1)^{|P|}\mathcal{Z}_{i\tilde{P}\sqcup Q}$. Note that (B.3) is also valid when $Q = \emptyset$, i.e., $\Sigma_{123\ldots n} = (-1)^n\mathcal{Z}_{0\{1\sqcup 2\sqcup 3\sqcup\ldots\sqcup n\}}$.

Now let us consider products of $G_{ij}\Sigma^Q_P$ with a single or no overlap between ij and Q:

$$G_{12}\Sigma_P = G_{12}\mathcal{Z}_{10}\mathcal{Z}_{20}\Sigma^{12}_P = -\frac{1}{2}(\mathcal{Z}_{120} - \mathcal{Z}_{210})\Sigma^{12}_P = \frac{1}{2}\text{sign}_{12}\mathcal{Z}_{0(12)}\Sigma^{12}_P \tag{B.7}$$

$$G_{23}\Sigma^2_P = G_{23}\mathcal{Z}_{30}\Sigma^{23}_P = -\frac{1}{2}(2\mathcal{Z}_{23} + \mathcal{Z}_{30})\Sigma^{23}_P\ , \tag{B.8}$$

where we used $\Sigma_{123\ldots n} = \mathcal{Z}_{i0}\Sigma^i_{123\ldots n} = \mathcal{Z}_{i0}\mathcal{Z}_{j0}\Sigma^{ij}_{123\ldots n} = \ldots$ etc as well as

$$G_{12}\mathcal{Z}_{10}\mathcal{Z}_{20} = -\frac{1}{2}\left(\frac{\sigma_{10}}{\sigma_{12}} + \frac{\sigma_{20}}{\sigma_{12}}\right)\frac{1}{\sigma_{10}\sigma_{20}} = -\frac{1}{2}\left(\frac{1}{\sigma_{12}\sigma_{20}} + \frac{1}{\sigma_{12}\sigma_{10}}\right) = -\frac{1}{2}(\mathcal{Z}_{120} - \mathcal{Z}_{210})$$

$$G_{23}\mathcal{Z}_{30} = -\frac{1}{2}\left(\frac{\sigma_{20}}{\sigma_{23}} + \frac{\sigma_{30}}{\sigma_{23}}\right)\frac{1}{\sigma_{30}} = -\frac{1}{2}\left(\frac{2}{\sigma_{23}} + \frac{1}{\sigma_{30}}\right) = -\frac{1}{2}(2\mathcal{Z}_{23} + \mathcal{Z}_{30})\,. \tag{B.9}$$

The general case $G_{i_1 i_2}\ldots G_{i_{2p-1}i_{2p}}\Sigma_P$ in (B.1) can be proven by induction using (B.7), (B.8), (B.5), (B.6) and starting with (B.7)

$$G_{12}\Sigma_P = \frac{1}{2}\text{sgn}_{12}\mathcal{Z}_{0(12)}\Sigma^{12}_P = \frac{1}{2}\text{sgn}_{12}\mathcal{Z}_{0(P)}\,. \tag{B.10}$$

[18] The proof (B.6) is as follows: $\mathcal{Z}_{0(Pj)}(2\mathcal{Z}_{jk} + \mathcal{Z}_{k0}) = 2\mathcal{Z}_{kj0(P)} + \mathcal{Z}_{k0(Pj)} = 2\mathcal{Z}_{0\{jk\sqcup(P)\}} - \mathcal{Z}_{0\{k\sqcup(jP)\}} = \mathcal{Z}_{0\{jk\sqcup(P)\}} - \mathcal{Z}_{0\{kj\sqcup(P)\}} = \text{sgn}_{jk}\mathcal{Z}_{0(jkP)}$, since one factor of $\mathcal{Z}_{0\{jk\sqcup(P)\}}$ is cancelled by the permutations in $-\mathcal{Z}_{0\{k\sqcup(jP)\}}$ in which the labels j and k are in the same order as jk. Also note that $\mathcal{Z}_{0(Pj)} = -\mathcal{Z}_{j0(P)}$ was used in the first equality above.

The induction step leads to two cases for an additional propagator G_{ij} multiplying the left-hand side of (B.10). When there is no overlap between G_{ij} and the previous propagators,

$$\begin{aligned} G_{12}G_{34}\Sigma_P &= \frac{1}{2}\mathrm{sgn}_{12}\mathcal{Z}_{0(12)}\left(G_{34}\Sigma_P^{12}\right) = \frac{1}{4}\mathrm{sgn}_{12}\mathrm{sgn}_{34}\mathcal{Z}_{0(12)}\mathcal{Z}_{0(34)}\Sigma_P^{1234} \\ &= \frac{1}{4}\mathrm{sgn}_{12}\mathrm{sgn}_{34}\mathcal{Z}_{0(1234)}\Sigma_P^{1234} = \frac{1}{4}\mathrm{sgn}_{12}\mathrm{sgn}_{34}\mathcal{Z}_{0(P)}\,, \end{aligned} \tag{B.11}$$

where we used (B.5) and (B.4) on the last line. If there is an overlap with the previous propagators one gets instead,

$$\begin{aligned} G_{12}G_{23}\Sigma_P &= \frac{1}{2}\mathrm{sgn}_{12}\mathcal{Z}_{0(12)}\left(G_{23}\Sigma_P^{12}\right) = -\frac{1}{4}\mathrm{sgn}_{12}\mathcal{Z}_{0(12)}(2\mathcal{Z}_{23}+\mathcal{Z}_{30})\Sigma_P^{123} \\ &= -\frac{1}{4}\mathrm{sgn}_{12}\mathrm{sgn}_{23}\mathcal{Z}_{0(123)}\Sigma_P^{123} = \frac{1}{4}\mathrm{sgn}_{12}\mathrm{sgn}_{23}\mathcal{Z}_{0(P)}\,, \end{aligned} \tag{B.12}$$

where we used (B.6) to arrive at the second line. Since these steps can be freely iterated, it is now easy to see that each additional propagator G_{ij} leads to a factor of $\frac{1}{2}\mathrm{sgn}_{ij}$ on the right-hand side of (B.1), finishing its proof. ☐

Appendix C. Theta functions and q series

Even Jacobi theta functions are defined by

$$\begin{aligned} \theta_2(z,\tau) &= 2q^{1/8}\cos(\pi z)\prod_{j=1}^{\infty}(1-q^j)(1+e^{2\pi iz}q^j)(1+e^{-2\pi iz}q^j) \\ \theta_3(z,\tau) &= \prod_{j=1}^{\infty}(1-q^j)(1+e^{2\pi iz}q^{j-1/2})(1+e^{-2\pi iz}q^{j-1/2}) \\ \theta_4(z,\tau) &= \prod_{j=1}^{\infty}(1-q^j)(1-e^{2\pi iz}q^{j-1/2})(1-e^{-2\pi iz}q^{j-1/2})\,. \end{aligned} \tag{C.1}$$

These definitions yield the following leading q-orders for the Szegö kernel (4.5):

$$\begin{aligned} S_2(z_{ij},\tau)|_{q^0} &= i\pi\frac{\sigma_i+\sigma_j}{\sigma_i-\sigma_j} \\ S_3(z_{ij},\tau)|_{q^0} &= 2\pi i\frac{\sqrt{\sigma_i\sigma_j}}{\sigma_i-\sigma_j} \\ S_3(z_{ij},\tau)|_{q^{1/2}} &= 2\pi i\frac{\sigma_i-\sigma_j}{\sqrt{\sigma_i\sigma_j}}\,. \end{aligned} \tag{C.2}$$

The contributions to the $\tau \to i\infty$ limit of the ambitwistor-string and superstring correlators are selected by the partition functions

$$\left[\frac{\theta_2(0,\tau)}{\theta_1'(0,\tau)}\right]^4 = \frac{1}{(2\pi i)^4}\big[16+\mathcal{O}(q)\big] \ , \qquad \left[\frac{\theta_{3,4}(0,\tau)}{\theta_1'(0,\tau)}\right]^4 = \frac{1}{(2\pi i)^4}\left[\frac{1}{\sqrt{q}} \pm 8 + \mathcal{O}(q^{1/2})\right] \ . \quad \text{(C.3)}$$

Appendix D. Five-point example with maximal supersymmetry

This appendix is devoted to a maximally supersymmetric five-point example to illustrate the procedure of section 4 to express one-loop CHY integrands as a polynomial in ℓ and $G_{ij} = \frac{\sigma_i+\sigma_j}{2\sigma_{ij}}$. The starting point is the 5!-term expansion (4.10) of the five-point correlator,

$$\begin{aligned}\mathcal{K}_5 = c_1c_2c_3c_4c_5\mathcal{G}_\emptyset &+ \big(c_1c_2c_3\text{tr}(f_{(45)})\mathcal{G}_{(45)} + 9 \text{ more}\big) + \big(c_1c_2\text{tr}(f_{(345)})\mathcal{G}_{(345)} + 19 \text{ more}\big)\\ &+ \big(c_1\text{tr}(f_{(2345)})\mathcal{G}_{(2345)} + 29 \text{ more}\big) + \big(c_1\text{tr}(f_{(23)})\text{tr}(f_{(45)})\mathcal{G}_{(23)(45)} + 14 \text{ more}\big) \quad \text{(D.1)}\\ &+ \big(\text{tr}(f_{(12345)})\mathcal{G}_{(12345)} + 23 \text{ more}\big) + \big(\text{tr}(f_{(12)})\text{tr}(f_{(345)})\mathcal{G}_{(12)(345)} + 19 \text{ more}\big) \ ,\end{aligned}$$

see (4.4) and (4.12) for the polarization-dependent ingredients $\text{tr}(f_I)$ and $c_i(\ell)$. In case of maximal supersymmetry, spin sums $\mathcal{G}_{I,J}$ with three or fewer particles in the union of the cycles I,J vanish, see section 4.3. Their four-point instances in turn are given by $\mathcal{G}_{(ij)(kl)} = \mathcal{G}_{(ijkl)} = 1$, and five-point cases (4.19) give rise to linear functions in G_{ij}. Hence, one can collect the coefficients of ℓ and G_{ij} in (D.1):

$$\mathcal{K}_5 = \ell_m T^m_{1,2,3,4,5} + \big[G_{12}T_{12,3,4,5} + (1,2|1,2,3,4,5)\big] \quad \text{(D.2)}$$

$$T^m_{1,2,3,4,5} = e_1^m\Big(\frac{1}{4}\text{tr}(f_2f_3)\text{tr}(f_4f_5) - \text{tr}(f_2f_3f_4f_5) + \text{cyc}(3,4,5)\Big) + (1 \leftrightarrow 2,3,4,5) \quad \text{(D.3)}$$

$$\begin{aligned}T_{12,3,4,5} = (e_1\cdot k_2)&\Big(\frac{1}{4}\text{tr}(f_2f_3)\text{tr}(f_4f_5) - \text{tr}(f_2f_3f_4f_5) + \text{cyc}(3,4,5)\Big) \quad \text{(D.4)}\\ - (e_2\cdot k_1)&\Big(\frac{1}{4}\text{tr}(f_1f_3)\text{tr}(f_4f_5) - \text{tr}(f_1f_3f_4f_5) + \text{cyc}(3,4,5)\Big)\\ + \Big(\frac{1}{2}&\text{tr}(f_1f_2f_3)\text{tr}(f_4f_5) + \text{cyc}(3,4,5)\Big) - \Big(\text{tr}(f_1f_2f_3f_4f_5) + \text{perm}(3,4,5)\Big) \ .\end{aligned}$$

These expressions can be streamlined using the two-particle field-strength

$$\begin{aligned}s_{12}\mathfrak{f}_{12}^{mn} &= e_1\cdot e_2(k_2^mk_1^n - k_1^mk_2^n) + s_{12}(e_2^me_1^n - e_1^me_2^n)\\ &\quad + \Big(k_2\cdot e_1(k_2^me_2^n - e_2^mk_2^n + k_1^me_2^n - e_2^mk_1^n) - (1\leftrightarrow 2)\Big) \quad \text{(D.5)}\end{aligned}$$

obtained as a special case of (5.15) as well as the definition t_8-tensor in (5.16):

$$T^m_{1,2,3,4,5} = e^m_1 t_8(\mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4, \mathfrak{f}_5) + (1 \leftrightarrow 2, 3, 4, 5) \tag{D.6}$$

$$T_{12,3,4,5} = s_{12} t_8(\mathfrak{f}_{12}, \mathfrak{f}_3, \mathfrak{f}_4, \mathfrak{f}_5) = s_{12} t_{12,3,4,5} \ . \tag{D.7}$$

The dictionary (3.18) then implies the pentagon numerator

$$\begin{aligned} N_{+|12345|-} = \ell_m T^m_{1,2,3,4,5} - \frac{1}{2}(&T_{12,3,4,5} + T_{13,2,4,5} + T_{14,2,3,5} + T_{15,2,3,4} \\ &+ T_{23,1,4,5} + T_{24,1,3,5} + T_{25,1,3,4} + T_{34,1,2,5} + T_{35,1,2,4} + T_{45,1,2,3}) \ , \end{aligned} \tag{D.8}$$

and the corresponding box numerators determined by the BCJ duality collapse to

$$N^{\rm box}_{12} = N_{+|12345|-} - N_{+|21345|-} = -T_{12,3,4,5} \tag{D.9}$$

for the box diagram with legs 1 and 2 in a massive corner. The resulting partial integrand can be assembled via (3.31) and comprises four box diagrams, see example C of [13]:

$$\begin{aligned} a(1,2,3,4,5,-,+) = &\frac{N_{+|12345|-}}{s_{1,\ell} s_{12,\ell} s_{123,\ell} s_{1234,\ell}} - \frac{T_{12,3,4,5}}{s_{12} s_{12,\ell} s_{123,\ell} s_{1234,\ell}} - \frac{T_{1,23,4,5}}{s_{23} s_{1,\ell} s_{123,\ell} s_{1234,\ell}} \\ &- \frac{T_{1,2,34,5}}{s_{34} s_{1,\ell} s_{12,\ell} s_{1234,\ell}} - \frac{T_{1,2,3,45}}{s_{45} s_{1,\ell} s_{12,\ell} s_{123,\ell}} \end{aligned} \tag{D.10}$$

After eliminating any G_{1j} via scattering equations, the functions in (D.2) are converted to a basis. Their coefficients are then gauge invariant and match the bosonic components (5.17) of the five-point correlator (5.20) in pure-spinor superspace. Hence, the same conclusions can be obtained by taking the $\tau \to i\infty$ limit of superstring correlators in the RNS formalism [51,52,53] or the pure-spinor formalism [46].

Appendix E. Parity-odd correlators from the ambitwistor string

As explained in [23], the parity-odd contributions to the d-dimensional RNS correlators of section 4.6 can be represented as a Pfaffian

$$\mathcal{K}^{\epsilon_d}_n = i \int {\rm d}^d \Psi_0 \ {\rm Pf} \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \ . \tag{E.1}$$

The Grassmann integral requires the saturation of all the d zero-mode components Ψ^m_0 of the worldsheet fermions in their odd spin structure,

$$\int {\rm d}^d \Psi_0 \ \Psi^{m_1}_0 \Psi^{m_2}_0 \ldots \Psi^{m_d}_0 = \epsilon_d^{m_1 m_2 \ldots m_d} \ . \tag{E.2}$$

In the $\tau \to i\infty$ limit, the entries of the $n \times n$ blocks A, B and C in (E.1) are given as follows: In the off-diagonal cases with $i \neq j$, we have

$$\begin{aligned} A_{i\,j} &= k_i \cdot k_j \, G_{ij} + k_i \cdot \Psi_0 \, k_j \cdot \Psi_0 \,, \quad \text{for } i, j \neq 1 \\ B_{i\,j} &= e_i \cdot e_j \, G_{ij} + e_i \cdot \Psi_0 \, e_j \cdot \Psi_0 \,, \\ C_{i\,j} &= e_i \cdot k_j \, G_{ij} + e_i \cdot \Psi_0 \, k_j \cdot \Psi_0 \,, \quad \text{for } i \neq 1 \,, \end{aligned} \tag{E.3}$$

while the diagonal entries are given by

$$\begin{aligned} A_{i\,i} &= B_{i\,i} = 0 \,, \\ C_{i\,i} &= -e_i \cdot \ell - \sum_{j \neq i}^{n} e_i \cdot k_j \, G_{ij} - e_i \cdot \Psi_0 \, k_i \cdot \Psi_0 \,, \quad \text{for } i \neq 1 \,. \end{aligned} \tag{E.4}$$

In the first row or column with $i = 1$, the entries of A and C are modified to

$$\begin{aligned} A_{1\,j} &= P(\sigma_0) \cdot k_j \, G_{0j} + P(\sigma_0) \cdot \Psi_0 \, k_i \cdot \Psi_0 \,, \quad \text{for } j \neq 1 \\ C_{j\,1} &= e_j \cdot P(\sigma_0) \, G_{j0} + e_j \cdot \Psi_0 \, P(\sigma_0) \cdot \Psi_0 \,, \end{aligned} \tag{E.5}$$

where the picture changing operator of the RNS string contributes a factor of

$$P^m(\sigma_0) = \ell^m + \sum_{j=1}^{n} k_j^m G_{0j} \,. \tag{E.6}$$

Since the Pfaffian in (E.1) is a polynomial of degree $n{+}1$ in G_{ij}, ℓ and $(\Psi_0 \Psi_0)$, the zero-mode integral (E.2) leaves a polynomial of degree $n{+}1 - \frac{d}{2}$ in G_{ij} and ℓ. Note that the correlator $\mathbf{K}_n^{\epsilon_d}(\tau)$ at finite values of τ can be easily obtained from (E.4) and (E.5) by replacing $G_{ij} \to \partial \log \theta_1(z_{ij})$.

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