New superconformal multiplets and higher derivative invariants in six dimensions

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Abstract

Within the framework of six-dimensional $\mathcal{N} = (1, 0)$ conformal supergravity, we introduce new off-shell multiplets $\mathcal{O}^*(n)$, where $n = 3, 4, \ldots$, and use them to construct higher-rank extensions of the linear multiplet action. The $\mathcal{O}^*(n)$ multiplets may be viewed as being dual to well-known superconformal $\mathcal{O}(n)$ multiplets. We provide prepotential formulations for the $\mathcal{O}(n)$ and $\mathcal{O}^*(n)$ multiplets coupled to conformal supergravity. For every $\mathcal{O}^*(n)$ multiplet, we construct a higher derivative invariant which is superconformal on arbitrary superconformally flat backgrounds. We also show how our results can be used to construct new higher derivative actions in supergravity.

1. Introduction

Recently, there has been some interest in higher derivative superconformal invariants in six dimensions (6D) [1–3], e.g. in the context of general relations between the conformal and chiral anomaly coefficients in 6D $\mathcal{N} = (1, 0)$ superconformal theories. One example of such higher derivative models is the $\mathcal{N} = (1, 0)$ superconformal vector multiplet theory. It was formulated

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in Minkowski superspace in [4], and coupled to conformal supergravity in [5].[1] In this paper, we propose new off-shell superconformal multiplets and use them to construct higher derivative invariants that are quadratic in the fields of the new multiplets and naturally generalise the superconformal vector multiplet action.

We recall that the superconformal vector multiplet action is the supersymmetric extension of an $F \square F$ term, where F is the two-form field strength of a gauge one-form. Its construction starts with the supersymmetric BF invariant[2] which is, schematically, the product of a vector multiplet with a linear multiplet. It may be written as a full superspace integral in $\mathcal{N} = (1, 0)$ Minkowski superspace as follows[3]

$$I = \int \mathrm{d}^{6|8} z \, U_{ij} \, L^{ij} = \int \mathrm{d}^{6|8} z \, W^{\alpha i} \rho_{\alpha i} \,, \tag{1.1}$$

where the superfield $L^{ij} = L^{(ij)}$ satisfies the defining differential constraint for the linear multiplet [7,8],

$$D_\alpha^{(i} L^{jk)} = 0 \,. \tag{1.2}$$

The superfield $W^{\alpha i}$ satisfies the differential constraints

$$D_\alpha^{(i} W^{\beta j)} = \frac{1}{4} \delta_\alpha^\beta D_\gamma^{(i} W^{\gamma j)} \,, \quad D_{\alpha i} W^{\alpha i} = 0 \tag{1.3}$$

appropriate for a vector multiplet, see [8,9] and references therein. The real iso-triplet U_{ij} in (1.1) is the 6D counterpart of the Mezincescu prepotential [10] for the vector multiplet, and $\rho_{\alpha i}$ is the prepotential [11] for the linear multiplet. The prepotentials $\rho_{\alpha i}$ and U_{ij} play the role of gauge fields, while L^{ij} and $W^{\alpha i}$ are the gauge-invariant field strengths of the linear and vector multiplets, respectively. One can construct the supersymmetric $F \square F$ action by building a linear multiplet from the fields of the vector multiplet. The appropriate linear multiplet is

$$\boldsymbol{L}^{ij} = \mathrm{i} \square D_\gamma^{(i} W^{\gamma j)} \,, \quad \square := \partial^a \partial_a \,. \tag{1.4}$$

The $F \square F$ action is the BF invariant (1.1) with the replacement $L^{ij} \to \boldsymbol{L}^{ij}$:

$$S_{F \square F} = \int \mathrm{d}^{6|8} z \, U_{ij} \boldsymbol{L}^{ij} \,. \tag{1.5}$$

The above action turns out to be superconformal[4] and equivalent to the harmonic superspace action given in [4].

In order to formulate generalisations of the above construction, we will consider $\mathcal{O}(n)$ multiplets as higher-rank cousins of the linear multiplet, $\mathcal{O}(2)$, and uncover a new family of multiplets akin to the vector multiplet. These multiplets will allow us to generalise the BF action in such a way that it is now, schematically, the product of the $\mathcal{O}(n)$ multiplet with the new multiplet. Using this invariant we will show how to generate a whole family of superconformal higher derivative invariants bilinear in the new multiplets.

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[1] Its component Lagrangian in conformal supergravity was worked out in [6].

[2] The notation refers to the six-form which is the product of the field strength F and the four-form potential B contained in the linear multiplet. It is also known as the linear multiplet action.

[3] The second form of the invariant in (1.1) follows from the expressions for the covariant field strengths $W^{\alpha i}$ and L^{ij} in terms of their prepotentials.

[4] Unlike $D_\gamma^{(i} W^{\gamma j)}$, the higher-derivative linear multiplet (1.4) is a primary superfield.

In what follows, we will use the superspace formulation for conformal supergravity in [5] and refer the reader there for our notation and conventions. The results in Minkowski superspace are obtained by setting the super-Weyl tensor $W^{\alpha\beta}$ to zero and replacing the conformal superspace covariant derivatives with those of Minkowski superspace, $\nabla_A = (\nabla_a, \nabla_\alpha^i) \to (\partial_a, D_\alpha^i)$. In this paper, all (super)fields are chosen to transform in irreducible representations of the R-symmetry group SU(2) so that they are symmetric in their SU(2) spinor indices, for instance $L^{i_1 \cdots i_n} = L^{(i_1 \cdots i_n)}$.

2. The $\mathcal{O}(n)$ multiplets and their prepotentials

Given an integer $n \geq 1$, the $\mathcal{O}(n)$ multiplet is a primary superfield $L^{i_1 \cdots i_n}$ of dimension $2n$ that satisfies the differential constraint[5]

$$\nabla_\alpha^{(i_1} L^{i_2 \cdots i_{n+1})} = 0 \,. \tag{2.1}$$

The constraint (2.1) is primary since the S-supersymmetry generator S_j^β annihilates the left hand side (without requiring the constraint). This condition actually fixes the dimension of $L^{i_1 \cdots i_n}$. The complex conjugate of $L^{i_1 \cdots i_n}$, which is defined by

$$\overline{L}_{i_1 \ldots i_n} := \overline{L^{i_1 \ldots i_n}} \,, \tag{2.2}$$

is also an $\mathcal{O}(n)$ multiplet. If n is even, $n = 2m$, one can consistently define real $\mathcal{O}(2m)$ multiplets by imposing the condition $\overline{L}_{i_1 \ldots i_{2m}} = \varepsilon_{i_1 j_1} \ldots \varepsilon_{i_{2m} j_{2m}} L^{j_1 \ldots j_{2m}} \equiv L_{i_1 \ldots i_{2m}}$.

The $n = 1$ case corresponds to the 6D version of the Fayet–Sohnius hypermultiplet [22], which is necessarily on-shell in six dimensions (the constraint $\nabla_\alpha^{(i} L^{j)} = 0$ implies $\Box L^i = 0$) and, therefore, it will not be considered in what follows. The case $n = 2$ is the linear multiplet, which can be formulated in conformal superspace in terms of an unconstrained prepotential ρ_α^i as follows[6]

$$L^{ij} = \nabla^{(ijkl}\nabla_{klp}^\alpha \rho_\alpha^{p)} - \frac{12\mathrm{i}}{5}\nabla^{ijkl}\left(\nabla_{\alpha k}\nabla^{\alpha\beta}\rho_{\beta l} + 4\nabla_{\alpha k}(W^{\alpha\beta}\rho_{\beta l})\right) \,, \tag{2.3}$$

where we have introduced $\nabla^{\alpha ijk} := \frac{1}{3!}\varepsilon^{\alpha\beta\gamma\delta}\nabla_\beta^{(i}\nabla_\gamma^j\nabla_\delta^{k)}$ and the covariant projection operator[7]

$$\nabla^{ijkl} := -\frac{1}{96}\varepsilon^{\alpha\beta\gamma\delta}\nabla_\alpha^{(i}\nabla_\beta^j\nabla_\gamma^k\nabla_\delta^{l)} \,. \tag{2.4}$$

The prepotential is defined up to the following gauge transformations

$$\delta\rho_\alpha^i = \nabla_\alpha^i \tau + \nabla_{\beta j}\tau_\alpha{}^{\beta ij} \,, \quad \tau_\alpha{}^{\alpha ij} = 0 \,, \tag{2.5}$$

where τ and $\tau_\alpha{}^{\beta ij}$ are dimensionless primary superfields. These gauge transformations leave L^{ij} invariant. In the flat-superspace limit, eq. (2.3) reduces to the one given in [11].

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[5] The $\mathcal{O}(n)$ multiplets are well known in the literature on supersymmetric field theories with eight supercharges in diverse dimensions. For 4D $\mathcal{N} = 2$ Poincaré supersymmetry, general $\mathcal{O}(n)$ multiplets, with $n > 2$, were introduced in [12–14]. The case $n = 4$ was first studied in [15]. The terminology "$\mathcal{O}(n)$ multiplet" was coined in [16]. As 6D $\mathcal{N} = (1, 0)$ superconformal multiplets, their complete description was given in [17] following the earlier approaches in four and five dimensions [18–21].

[6] The derivation of (2.3) is rather technical and will be given elsewhere.

[7] In SU(2) superspace, it coincides with the one in [17].

Table 1
Field content of the $\mathcal{O}(n)$ supermultiplet.

Component field	Dimension
$\varphi^{i_1\cdots i_n}$	$2n$
$\psi_\alpha{}^{i_1\cdots i_{n-1}}$	$2n + \frac{1}{2}$
$G_{[\alpha\beta]}{}^{i_1\cdots i_{n-2}}$	$2n + 1$
$\psi_{[\alpha\beta\gamma]}{}^{i_1\cdots i_{n-3}}$	$2n + \frac{3}{2}$
$\varphi^{i_1\cdots i_{n-4}}$	$2n + 2$

The field content of the $\mathcal{O}(n)$ multiplets follows from taking spinor covariant derivatives of the superfield $L^{i_1\cdots i_n}$ and analysing the consequences of the constraint (2.1). The independent component fields are summarised in Table 1 from which it can be seen that the number of off-shell degrees of freedom is $8(n-1) + 8(n-1)$. The cases $n \leq 3$ are special since the component fields for which the number of SU(2) indices become negative are truncated away. Note that, as the Dirac matrices are anti-symmetric, two antisymmetric Lorentz spinor indices are equivalent to a vector index, while three antisymmetric lower Lorentz spinor indices are equivalent to a single raised Lorentz spinor index by making use of the Levi-Civita symbol $\varepsilon^{\alpha\beta\gamma\delta}$. The $n = 2$ case is exceptional in the sense that the linear multiplet is a gauge multiplet because $G_a = \frac{1}{4}(\tilde{\gamma}_a)^{\alpha\beta}G_{\alpha\beta}$ may be identified with the dual of the field strength of a four-form gauge field since G_a is divergenceless. For $n \geq 3$ the supermultiplets do not possess similar restrictions at the component level and are therefore not gauge multiplets.

Since we will be concerned with the cases for which prepotential formulations exist, we restrict ourselves to the class of multiplets with $n \geq 2$. Indeed, in addition to $n = 2$, which was discussed above, prepotential formulations can also be given for $n \geq 3$. As $n = 3$ and $n > 3$ need separate treatment, we will discuss them in turn.

2.1. $n = 3$

We can solve the defining constraint for the $\mathcal{O}(3)$ multiplet in terms of an unconstrained prepotential as follows

$$L^{ijk} = \nabla^{ijkl}\nabla_{\alpha l}V^\alpha , \qquad (2.6)$$

where the prepotential V^α is defined up to the gauge transformations

$$\delta V^\alpha = \nabla_{\beta j}\zeta^{\alpha\beta j} , \quad \zeta^{\alpha\beta i} = \zeta^{(\alpha\beta)i} . \qquad (2.7)$$

One can check that L^{ijk} is primary using the fact that V^α is a primary superfield of dimension 7/2.

2.2. $n \geq 4$

For these cases, the solution of (2.1) is

$$L^{i_1\cdots i_n} = \nabla^{(i_1\cdots i_4}V^{i_5\cdots i_n)} , \qquad (2.8)$$

which is invariant under the gauge transformations of the unconstrained prepotential

$$\delta V^{i_1\cdots i_{n-4}} = \nabla_\alpha^{(i_1}\zeta^{\alpha i_2\cdots i_{n-4})} . \qquad (2.9)$$

Table 2
Field content of the vector multiplet.

Component field	Dimension
$V_{[\alpha\beta]}$	1
$\lambda^{\alpha i}$	3/2
Y^{ij}	2

The $\mathcal{O}(n)$ multiplet (2.8) is primary. This is a consequence of the fact that the prepotential $V^{i_1 \cdots i_{n-4}}$ is a primary superfield of dimension $2n - 2$. The analogue of the above prepotential formulation in 5D appeared in appendix G of [23].

3. New superconformal multiplets and their gauge prepotentials

We will now introduce new multiplets that generalise the vector multiplet. To do this we focus on a key property of the vector multiplet, which is that an invariant can be constructed by multiplying its prepotential with an $\mathcal{O}(2)$ multiplet as in eq. (1.1). This implies that the index structure of the prepotential for the vector multiplet is such that it can be contracted with L^{ij} to form an $SU(2)$ singlet. In generalising this property we seek to find multiplets that have prepotentials of the form $U_{i_1 \cdots i_n}$ that can be used to build a gauge invariant expression with an $\mathcal{O}(n)$ multiplet as an integral over full superspace. In this way the new multiplets will be 'dual' to the $\mathcal{O}(n)$ multiplets.

The field content of the vector multiplet is summarised in Table 2, where $V_a = \frac{1}{4}(\tilde{\gamma}_a)^{\alpha\beta} V_{\alpha\beta}$ is a gauge field. Comparison with Table 1 shows that every component field of the vector multiplet can be contracted with a component field of the $\mathcal{O}(2)$ multiplet to give a scalar of dimension 6. The analogous property also holds for the component fields of the new multiplets. We have to distinguish three cases, depending on the number of $SU(2)$ indices of the prepotential. Note that while the vector multiplet is a gauge multiplet, the new multiplets are not.

3.1. n = 3

Starting from a primary superfield T_α of dimension $-3/2$ we impose the differential constraint

$$\nabla^i_{(\alpha} T_{\beta)} = 0 . \tag{3.1}$$

This constraint can be solved in terms of a primary dimension -4 unconstrained prepotential U_{ijk} as follows

$$T_\alpha = \nabla_{\alpha l} \nabla^{ijkl} U_{ijk} , \tag{3.2}$$

where the prepotential is defined up to the gauge transformations

$$\delta U_{ijk} = \nabla^l_\alpha \xi^\alpha{}_{ijkl} . \tag{3.3}$$

The field content of this new multiplet is summarised in Table 3.

3.2. n = 4

We can define another supermultiplet by a primary dimension -4 scalar superfield T satisfying the constraint

Table 3
Field content of the supermultiplet described by T_α.

Component field	Dimension
λ_α	$-3/2$
$V_{[\alpha\beta]}{}^i$	-1
$\lambda_{[\alpha\beta\gamma]}{}^{ij}$	$-1/2$
Y^{ijk}	0

Table 4
Field content of the $\mathcal{O}^*(n)$ multiplet.

Component field	Dimension
$\phi^{i_1\cdots i_{n-4}}$	$4 - 2n$
$\lambda_\alpha{}^{i_1\cdots i_{n-3}}$	$\frac{9}{2} - 2n$
$V_{[\alpha\beta]}{}^{i_1\cdots i_{n-2}}$	$5 - 2n$
$\lambda_{[\alpha\beta\gamma]}{}^{i_1\cdots i_{n-1}}$	$\frac{11}{2} - 2n$
$Y^{i_1\cdots i_n}$	$6 - 2n$

$$\nabla_{(\alpha}^k \nabla_{\beta)k} T = 0 \Longrightarrow \nabla_{(\alpha}^i \nabla_{\beta)}^j T = 0 . \tag{3.4}$$

One can solve this in terms of an unconstrained prepotential U_{ijkl} as follows[8]

$$T = \nabla^{ijkl} U_{ijkl} , \tag{3.5}$$

where U_{ijkl} is primary of dimension -6. It is straightforward to check that T is primary given U_{ijkl} is primary. One can also check that U_{ijkl} is defined up to the gauge transformations

$$\delta U_{ijkl} = \nabla_\alpha^m \xi^\alpha{}_{ijklm} . \tag{3.6}$$

To check the invariance of T, the following identities are useful:

$$\nabla_\alpha^{(m} \nabla^{ijkl)} = \nabla^{(ijkl} \nabla_\alpha^{m)} = \nabla_{(\alpha}^m \nabla_{\beta)m} \nabla^{ijkl} = \nabla^{ijkl} \nabla_{(\alpha}^m \nabla_{\beta)m} = 0 . \tag{3.7}$$

The field content of this multiplet is summarised in Table 4 with $n = 4$.

3.3. $n \geq 5$

As a generalisation of the scalar superfield we used in the previous subsection, we can introduce superfields $T_{i_1\cdots i_{n-4}}$ of dimension $4 - 2n$ satisfying the constraint[9]

$$\nabla_\alpha^j T_{i_1\cdots i_{n-5}j} = 0 \quad \Longrightarrow \quad \nabla_{(\alpha}^j \nabla_{\beta)j} T_{i_1\cdots i_{n-4}} = 0 . \tag{3.8}$$

Its solution in terms of an unconstrained prepotential is

$$T_{i_1\cdots i_{n-4}} = \nabla^{i_{n-3}\cdots i_n} U_{i_1\cdots i_n} , \tag{3.9}$$

where $U_{i_1\cdots i_n}$ is defined up to the gauge transformations

$$\delta U_{i_1\cdots i_n} = \nabla_\alpha^j \xi^\alpha{}_{i_1\cdots i_n j} . \tag{3.10}$$

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[8] This multiplet and its prepotential description first appeared in Minkowski superspace in [8].
[9] Similar constraints were considered in [24].

The field content of these multiplets is summarized in Table 4. The form of the constraints (3.8) is 'transverse-like' as opposed to the 'longitudinal-like' constraints (2.1) of the $\mathcal{O}(n)$ multiplets and for this reason we may refer to the new multiplets as transverse multiplets. Although in this sense it might be natural to refer to the $\mathcal{O}(n)$ multiplets as longitudinal multiplets, we will stick to the well-established nomenclature for them. As for the transverse multiplets, we will denote them by $\mathcal{O}^*(n)$ as a reminder that they were constructed to be 'dual' to the $\mathcal{O}(n)$ multiplets in the sense that one will be able to use them to construct an invariant as a product of it with an $\mathcal{O}(n)$ multiplet. For convenience, we also define $\mathcal{O}^*(2)$ to be the vector multiplet, $\mathcal{O}^*(3)$ to be the multiplet described by the superfield T_α and $\mathcal{O}^*(4)$ to be the multiplet described by the scalar superfield T. We have not defined $\mathcal{O}^*(1)$ here but we will introduce it in section 6.

4. New superconformal invariants

We will now use the new multiplets and their prepotentials to generalise the BF invariant (1.1).

4.1. Generalised BF invariants

With the help of the results of the previous sections, one can immediately write down a supersymmetric invariant

$$I_{(n)} = \int d^{6|8}z\, E\, U_{i_1\cdots i_n}\, L^{i_1\cdots i_n}\ , \tag{4.1}$$

where $E = \mathrm{Ber}(E_M{}^A)$ and, as usual, $E^A = dz^M E_M{}^A$ is the supervielbein. Taking into account the differential constraint on the $\mathcal{O}(n)$ multiplet, it is straightforward to check that $I_{(n)}$ is invariant with respect to the gauge transformations of the prepotential $U_{i_1\cdots i_n}$. In the $n = 2$ case the invariant (4.1) corresponds to the standard BF invariant (1.1), while for $n \geq 3$ we obtain new generalisations.[10] In the flat case, the supersymmetric invariant (4.1) can be seen to naturally originate in harmonic superspace, see Appendix A.

Note that one can equivalently define the generalised BF invariants by explicitly using the prepotential for the $\mathcal{O}(n)$ multiplets. For $n = 3$ eq. (4.1) is equivalent to

$$I_{(3)} = \int d^{6|8}z\, E\, V^\alpha T_\alpha\ , \tag{4.2}$$

while for $n \geq 4$ it is equivalent to

$$I_{(n)} = \int d^{6|8}z\, E\, V_{i_1\cdots i_{n-4}} T^{i_1\cdots i_{n-4}}\ . \tag{4.3}$$

However, as one can see from the above, the fact that the prepotential obtains a Lorentz index for the $n \leq 3$ cases means that we cannot treat all cases ubiquitously in this form.

4.2. Superform realisation of the invariants

It is worth mentioning that a *manifestly gauge-invariant* form of the invariant (4.1) for $n \geq 3$ can be given by making use of the superform construction in [24,5]. It is based on the existence

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[10] The invariant $I_{(4)}$ is similar to the one used in the description of the relaxed hypermultiplet action [8] in Minkowski superspace.

of a closed six-form in superspace that describes a locally superconformal invariant. It is expressed entirely in terms of a basic primary superfield $A_\alpha{}^{ijk}$ of dimension 9/2 which satisfies the differential constraint $\nabla^{(i}_{(\alpha} A_{\beta)}{}^{jkl)} = 0$. The superfield $A_\alpha{}^{ijk}$ plays the role of a supersymmetric Lagrangian similar to the chiral Lagrangian in 4D. Its component structure was elaborated in [6].

The invariant (4.1) can be equivalently described by a composite superfield $A_\alpha{}^{ijk}$ which is, roughly, the product of an $\mathcal{O}^*(n)$ multiplet and the corresponding $\mathcal{O}(n)$ multiplet. More specifically, for $n = 3$ one has

$$A_\alpha{}^{ijk} \propto T_\alpha L^{ijk} \,, \tag{4.4}$$

while for $n \geq 4$, one takes

$$A_\alpha{}^{ijk} \propto (n-3) T_{i_1 \cdots i_{n-4}} \nabla_{\alpha l} L^{ijkl i_1 \cdots i_{n-4}} + (n+1)(\nabla_{\alpha l} T_{i_1 \cdots i_{n-4}}) L^{ijkl i_1 \cdots i_{n-4}} \,. \tag{4.5}$$

The above representations are democratic in the sense that they put both multiplets on the same footing.[11]

The above results are useful in elaborating the component structure since they make the connection with the component action more direct through the results of [5,6]. Note, however, that while a manifestly gauge-invariant description exists for $n \geq 3$, an analogous description in terms of the superfield $A_\alpha{}^{ijk}$ does not exist for $n = 2$.

5. Higher derivative superconformal invariants

With the results obtained so far, it is possible to construct composite $\mathcal{O}(n)$ multiplets and use them to construct higher derivative actions for the $\mathcal{O}^*(n)$ multiplets. In this section we will restrict ourselves to a superconformally flat geometry, where the super-Weyl tensor vanishes, $W^{\alpha\beta} = 0$.

One can construct an $\mathcal{O}(3)$ multiplet out of the superfield T_α as follows

$$\boldsymbol{L}^{ijk} = \square^3 \nabla^{\alpha ijk} T_\alpha \,. \tag{5.1}$$

The power of the d'Alembertian operator, $\square := \nabla^a \nabla_a$, is chosen to ensure that $\boldsymbol{L}^{ijk}$ has the right dimension and is primary in the superconformally flat geometry. To check that it is primary, the following identities are useful:

$$\left[S^\alpha_i, \square \right] = -2\mathrm{i}\nabla^{\alpha\beta}\nabla_{\beta i} + \mathrm{i}[\nabla^{\alpha\beta}, \nabla_{\beta i}] \,, \tag{5.2a}$$

$$\{S^\alpha_l, \nabla^{\beta ijk}\} = 8\delta^{(i}_l \nabla^{\alpha\beta jk)} + \nabla^{\beta\gamma(ij}\{S^\alpha_l, \nabla^{l)}_\gamma\} \,. \tag{5.2b}$$

A higher derivative invariant can now be described by plugging in $\overline{\boldsymbol{L}}^{ijk}$ into the generalised BF invariant (4.1), giving

$$J_{(3)} = \int \mathrm{d}^{6|8}z \, E \, U_{ijk} \overline{\boldsymbol{L}}^{ijk} \,. \tag{5.3}$$

This invariant is the supersymmetric extension of a $\overline{\lambda}_\alpha \square^4 \partial^{\alpha\beta} \lambda_\beta$ term.

We can also describe higher derivative invariants for the $\mathcal{O}^*(n)$ multiplets with $n \geq 4$. We begin by building an $\mathcal{O}(n)$ multiplet from the superfield $T_{i_1 \cdots i_{n-4}}$:

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[11] Any full superspace invariant $\int \mathrm{d}^{6|8}z \, E \, \mathcal{L}$, with the Lagrangian $\mathcal{L}$ being a primary scalar superfield of dimension $+2$, can equivalently be described by (4.5) with a composite $\mathcal{O}(4)$ multiplet $\mathbb{L}^{ijkl} = \nabla^{ijkl}(T^{-1}\mathcal{L})$, where T is chosen to be nowhere vanishing. Its dependence on T is artificial since the invariant is independent of T modulo a total derivative.

$$L^{i_1 \cdots i_n} = \Box^{2n-3} \nabla^{(i_1 \cdots i_4} T^{i_5 \cdots i_n)} . \tag{5.4}$$

One can show that in a superconformally flat geometry $L^{i_1 \cdots i_n}$ is primary, using again eqs. (5.2) and the following identities:

$$\left[S_m^\alpha, \nabla^{ijkl} \right] = 3\delta_m^{(i} \nabla^{\alpha jkl)} - \frac{1}{4} \nabla^{\beta(ijk} \{S_m^\alpha, \nabla_\beta^{l)}\} , \tag{5.5a}$$

$$\nabla^{\alpha\beta} \nabla_{\beta\gamma} = -\delta_\gamma^\alpha \Box + \frac{1}{2} [\nabla^{\alpha\beta}, \nabla_{\beta\gamma}] , \tag{5.5b}$$

$$\nabla_{\alpha j} \nabla^{(ji_1 i_2 i_3} T^{i_4 \cdots i_{n-1})} = -\frac{2(n+1)\mathrm{i}}{n} \nabla_{\beta\gamma} \nabla^{\gamma(i_1 i_2 i_3} T^{i_4 \cdots i_{n-1})} + (W^{\alpha\beta} \text{ terms}) . \tag{5.5c}$$

The higher derivative invariants are now described by the generalised BF invariant (4.1) with $\overline{L}^{i_1 \cdots i_n}$, i.e.

$$J_{(n)} = \int \mathrm{d}^{6|8} z \, E \, U_{i_1 \cdots i_n} \overline{L}^{i_1 \cdots i_n} = \int \mathrm{d}^{6|8} z \, E \, T_{i_1 \cdots i_n} \Box^{2n-3} \overline{T}^{i_1 \cdots i_n} . \tag{5.6}$$

$J_{(n)}$ contains a term of the form $\overline{\phi}^{i_1 \cdots i_{n-4}} \Box^{2n-1} \phi_{i_1 \cdots i_{n-4}}$ at the component level.

6. Discussion

A family of higher derivative superconformal actions was proposed in [2,3]. Upon appropriate identification of the fields, one can see that the actions sketched in [2,3] correspond to the higher derivative superconformal invariants presented in this paper. Note that analogous higher derivative invariants realised in terms of $\mathcal{O}(n)$ multiplets is not possible since all its fields have dimension $2n$ or higher. As a result, the only case when a local superconformal action, that is quadratic in the fields, could exist is $n = 1$, but this corresponds to an on-shell hypermultiplet. This tells us that such higher derivative invariants require the use of the $\mathcal{O}^*(n)$ multiplets.

The question whether the higher derivative invariants of the previous section can be lifted to conformal supergravity is a separate problem. As a matter of fact, it appears not to be the case. The point is that (5.6) should include the term $\bar{\phi} \Box^{2n-1} \phi$ once one truncates to $\mathcal{N} = 0$ and it is known that there does not exist a locally conformal completion for $n \geq 3$ [25]. The same is expected for the higher derivative invariant (5.3). In conformally flat backgrounds the obstruction for their existence is, however, absent [26].

The results in this paper may have interesting consequences in the construction of new higher derivative supergravity theories. To construct Poincaré supergravity theories within a superconformal framework, one must use a conformal compensating multiplet, which is a real scalar superfield Φ that we choose, without loss of generality, to have dimension one. In this framework it becomes apparent that the construction of composite $\mathcal{O}(n)$ multiplets from the $\mathcal{O}^*(n)$ multiplets can be made in many more ways using the results of this paper. For instance, we can construct $\mathcal{O}(n)$ multiplets with $n \geq 4$ as follows

$$L^{i_1 \cdots i_n} = \nabla^{(i_1 \cdots i_4} (T^{i_5 \cdots i_n)} \Phi^{4n-6}) . \tag{6.1}$$

Furthermore, we can choose the compensator as $\Phi^{-4(n-2)} = T^{i_1 \cdots i_{n-4}} \overline{T}_{i_1 \cdots i_{n-4}}$ leading to a four-derivative invariant for the $\mathcal{O}^*(n)$ multiplet once one plugs the composite into the generalised BF invariant (4.1). We will explore the supercurrents of such theories in a forthcoming paper [27].

It is interesting to note that supersymmetric invariants built out of $\mathcal{O}^*(n)$ for $n \geq 4$ may be mapped to invariants of the $\mathcal{O}(2)$ multiplet for even $n = 2m$. One only needs to make use of

an $\mathcal{O}(2)$ multiplet L^{ij} such that $L^2 := L^{ij}L_{ij}$ is nowhere vanishing. Given the $\mathcal{O}(2)$ multiplet, we can choose the $\mathcal{O}^*(2m)$ multiplet to be[12]

$$\boldsymbol{T}^{i_1 \cdots i_{2m}} = L^{-(2m+1)} L^{(i_1 i_2} \cdots L^{i_{2m-1} i_{2m})} . \tag{6.2}$$

In this way the resulting invariant will be expressed entirely in terms of the $\mathcal{O}(2)$ multiplet.

It is also possible to construct composite $\mathcal{O}^*(n)$ multiplets out of $\mathcal{O}(n)$ multiplets leading to superconformal invariants for the $\mathcal{O}(n)$ multiplets. For instance, for $n \geq 4$ we can take the composite

$$\boldsymbol{T}_{i_1 \cdots i_{n-4}} = \nabla^{i_{n-3} \cdots i_n} (L_{i_1 \cdots i_n} \Phi^{-4n+2}) . \tag{6.3}$$

We can choose the compensator to be $\Phi^{4n} = L^{i_1 \cdots i_n} \overline{L}_{i_1 \cdots i_n}$ and then plugging the result into the invariant (4.3) gives a four-derivative invariant for the $\mathcal{O}(n)$ multiplet. Many more invariants can be constructed by iterating eqs. (6.1) and (6.3) in various ways, and using the fact that there is an obvious multiplication defined on the space of $\mathcal{O}(n)$ multiplets and the space of $\mathcal{O}^*(n)$ multiplets. Similar constructions appeared in 4D for the $\mathcal{O}(2)$ multiplet in [28].

In this paper, we have defined the $\mathcal{O}^*(n)$ multiplets for $n \geq 2$. The $\mathcal{O}^*(n)$ multiplet has the same number of degrees of freedom as the corresponding $\mathcal{O}(n)$ multiplet as follows from the generalised BF invariant (4.1). We have ignored the $n = 1$ case since the $\mathcal{O}(1)$ multiplet is on-shell and does not have a prepotential formulation. However, it is interesting to ask whether there exists another on-shell multiplet possessing the same number of degrees of freedom as that of the $\mathcal{O}(1)$ multiplet that one could naturally add to the class of $\mathcal{O}^*(n)$ multiplets. Such a superconformal multiplet is described by a primary superfield T^α of dimension $-1/2$ satisfying the constraint

$$\nabla_\alpha^i T^\beta = \frac{1}{4} \delta_\alpha^\beta \nabla_\gamma^i T^\gamma . \tag{6.4}$$

It might be appropriate to define this multiplet as the $\mathcal{O}^*(1)$ multiplet.

The multiplets described in this paper certainly do not exhaust all possible superconformal multiplets nor those that permit a prepotential formulation. To illustrate this we introduce a supermultiplet described by a superfield $M_{\alpha_1 \cdots \alpha_n} = M_{(\alpha_1 \cdots \alpha_n)}$ with $n \geq 1$ and subject to the constraint[13]

$$\nabla_{(\alpha_1}^i M_{\alpha_2 \cdots \alpha_{n+1})} = 0 , \tag{6.5}$$

which describes a superconformal multiplet when $\mathbb{D}M_{\alpha_1 \cdots \alpha_n} = -\frac{3n}{2} M_{\alpha_1 \cdots \alpha_n}$. For $n > 2$ the differential constraint can be solved in terms of a prepotential as

$$M_{\alpha_1 \cdots \alpha_n} = \nabla_{(\alpha_1}^k \nabla_{\alpha_2 k} V_{\alpha_3 \cdots \alpha_n)} , \tag{6.6}$$

where the prepotential $V_{\alpha_1 \cdots \alpha_{n-2}} = V_{(\alpha_1 \cdots \alpha_{n-2})}$ possesses the gauge transformations

$$\delta V_{\alpha_1 \cdots \alpha_{n-2}} = \nabla_{(\alpha_1}^k \xi_{\alpha_2 \cdots \alpha_{n-2})k} . \tag{6.7}$$

Interestingly, upon introducing a 'dual' multiplet described by a superfield $N^{\alpha_1 \cdots \alpha_{n-2}} = N^{(\alpha_1 \cdots \alpha_{n-2})}$ that is constrained by $\nabla_\beta^k N^{\alpha_1 \cdots \alpha_{n-3}\beta} = 0$, one can write down a gauge invariant

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[12] It is a straightforward exercise to check that the composite multiplet satisfies the appropriate differential constraint.

[13] These superconformal constraints appeared in Minkowski superspace in [29].

$$\int \mathrm{d}z^{6|8}\, E\, N^{\alpha_1\cdots\alpha_{n-2}} V_{\alpha_1\cdots\alpha_{n-2}} \;. \tag{6.8}$$

However, it does not appear that such an invariant can be used to describe higher derivative superconformal actions.

It is worthwhile mentioning that the new multiplets and constructions introduced in this paper imply the existence of analogues in lower spacetime dimensions. We leave their exploration for future work.

Acknowledgements

Appendix A. Harmonic superspace construction

In this appendix we present a harmonic superspace origin of the supersymmetric invariant (4.1) in the flat case. Our construction is an extension of the known procedure to read off the Mezincescu prepotential from the analytic prepotential [30,31].

Given an $\mathcal{O}(n)$ multiplet $L^{i_1\cdots i_n}(z)$ in Minkowski superspace, with $n \geq 2$,

$$D_\alpha^{(i_1} L^{i_2\cdots i_{n+1})} = 0 \;, \tag{A.1}$$

we associate with it a harmonic superfield

$$L^{(+n)}(z, u) = L^{i_1\cdots i_n}(z) u_{i_1}^+ \ldots u_{i_n}^+ \;, \tag{A.2}$$

which is analytic

$$D_\alpha^+ L^{(+n)} = 0 \;, \qquad D_\alpha^\pm := u_i^\pm D_\alpha^i \tag{A.3a}$$

and obeys the harmonic shortness constraint

$$D^{++} L^{(+n)} = 0 \;, \tag{A.3b}$$

where the harmonic derivative D^{++} is defined as usual [30,31]. Then we can define a supersymmetric invariant

$$\hat{I}_{(n)} = \int \mathrm{d}\zeta^{(-4)}\, \Omega^{(4-n)} L^{(+n)} \;, \qquad D_\alpha^+ \Omega^{(4-n)} = 0 \;, \tag{A.4}$$

in which the integrand involves an analytic potential $\Omega^{(4-n)}$. The integration in (A.4) is carried out over the analytic subspace of the harmonic superspace. It is defined by

$$\int \mathrm{d}\zeta^{(-4)}\, \mathcal{L}^{(4)} = \int \mathrm{d}^6x \int \mathrm{d}u\, (D^-)^4 \mathcal{L}^{(4)} \;, \qquad D_\alpha^+ \mathcal{L}^{(4)} = 0 \;,$$
$$(D^-)^4 := -\frac{1}{96} \varepsilon^{\alpha\beta\gamma\delta} D_\alpha^- D_\beta^- D_\gamma^- D_\delta^- \;, \tag{A.5}$$

for any analytic Lagrangian $\mathcal{L}^{(4)}$. Here the u-integral denotes the integration over the group manifold SU(2) defined as in [30,31]. The functional (A.4) is invariant under gauge transformations of the form

$$\delta\Omega^{(4-n)} = D^{++}\Lambda^{(2-n)} , \qquad D_\alpha^+ \Lambda^{(2-n)} = 0 , \tag{A.6}$$

with the gauge parameter $\Lambda^{(2-n)}$ being an analytic superfield of U(1) charge $(2-n)$.

The analyticity constraint on $\Omega^{(4-n)}$ is solved by

$$\Omega^{(4-n)} = (D^+)^4 U^{(-n)} , \qquad (D^+)^4 := -\frac{1}{96}\varepsilon^{\alpha\beta\gamma\delta} D_\alpha^+ D_\beta^+ D_\gamma^+ D_\delta^+ . \tag{A.7}$$

Here the prepotential $U^{(-n)}(z, u)$ is an unconstrained harmonic superfield which is defined modulo gauge transformations

$$\delta U^{(-n)} = D_\alpha^+ \xi^{(-n-1)\,\alpha} , \tag{A.8}$$

with the gauge parameter $\xi^{(-n-1)\,\alpha}(z, u)$ being an unconstrained harmonic superfield.

The analyticity constraint on the gauge parameter in (A.6) can be solved similarly to the representation (A.7) for $\Omega^{(4-n)}$,

$$\Lambda^{(2-n)} = (D^+)^4 \rho^{(-n-2)} . \tag{A.9}$$

Then the gauge transformation becomes equivalent to

$$\delta U^{(-n)} = D^{++} \rho^{(-n-2)} . \tag{A.10}$$

The unconstrained harmonic superfields $U^{(-n)}$ and $\rho^{(-n-2)}$ can be represented by convergent harmonic series

$$U^{(-n)}(z, u) = U^{i_1 \dots i_n}(z) u_{i_1}^- \dots u_{i_n}^-$$
$$+ \sum_{m=1}^{\infty} U^{(i_1 \dots i_{n+2m})}(z) u_{i_1}^+ \dots u_{i_m}^+ u_{i_{m+1}}^- \dots u_{i_{n+2m}}^- , \tag{A.11a}$$

$$\rho^{(-n-2)}(z, u) = \sum_{m=0}^{\infty} \rho^{(i_1 \dots i_{n+2+2m})}(z) u_{i_1}^+ \dots u_{i_m}^+ u_{i_{m+1}}^- \dots u_{i_{n+2+2m}}^- . \tag{A.11b}$$

It follows from these expressions that the gauge symmetry (A.10) allows us to impose a gauge condition

$$U^{(-n)}(z, u) = U^{i_1 \dots i_n}(z) u_{i_1}^- \dots u_{i_n}^- , \tag{A.12}$$

which completely fixes the gauge freedom (A.10). In this gauge, we still have the freedom to perform those gauge transformations (A.8) which are generated by parameters of the form[14]

$$\xi^{(-n-1)\,\alpha}(z, u) = \xi^{i_1 \dots i_{n+1}\,\alpha}(z) u_{i_1}^- \dots u_{i_{n+1}}^- . \tag{A.13}$$

The resulting gauge transformation of the prepotential $U^{i_1 \dots i_n}(z)$ coincides with the flat-superspace version of (3.10).

In the gauge (A.12), the supersymmetric invariant (A.4) can be represented as an integral over Minkowski superspace by making use of the identity

$$\int d\zeta^{(-4)} (D^+)^4 \mathcal{L}(z, u) = \int d^{6|8} z \int du \, \mathcal{L}(z, u) , \tag{A.14}$$

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[14] Such a transformation should be accompanied by a compensating ρ-transformation (A.10) which is required to preserve the gauge condition (A.12).

and computing the relevant harmonic integral. This reduces the supersymmetric invariant (A.4) to the flat-superspace version of (4.1) modulo a numerical factor.

In the $n = 2$ case, the supersymmetric invariant (A.4) is the harmonic superspace realisation of the linear multiplet action.

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