The Algebraic-Hyperbolic Approach to the Linearized Gravitational Constraints on a Minkowski Background

Jeffrey Winicour[1,2]

[1] Department of Physics and Astronomy
University of Pittsburgh, Pittsburgh, PA 15260, USA
[2] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, 14476 Golm, Germany

Abstract.

An algebraic-hyperbolic method for solving the Hamiltonian and momentum constraints has recently been shown to be well posed for general nonlinear perturbations of the initial data for a Schwarzschild black hole. This is a new approach to solving the constraints of Einstein's equations which does not involve elliptic equations and has potential importance for the construction of binary black hole data. In order to shed light on the underpinnings of this approach, we consider its application to obtain solutions of the constraints for linearized perturbations of Minkowski space. In that case, we find the surprising result that there are no suitable Cauchy hypersurfaces in Minkowski space for which the linearized algebraic-hyperbolic constraint problem is well posed.

PACS numbers: 04.20.-q, 04.20.Cv, 04.20.Ex, 04.25.D-, 04.30-w

1. Introduction

Physically realistic initial Cauchy data for Einstein's equations are a major ingredient for the numerical simulation of gravitational systems such as binary black holes. The formulation of such initial data is further complicated mathematically by the nonlinear constraint equations that they must satisfy. The traditional approach expresses the constraints in the form of elliptic equations, based upon the conformal treatment of the Hamiltonian constraint introduced by Lichnerowicz [1] and later extended by York [2, 3] to treat the momentum constraint. For reviews see [4, 5]. Recently, a new method of solving the constraints which only involves algebraic and hyperbolic equations [6] has been shown to lead to a well-posed constraint problem in the case of nonlinear perturbations of Schwarzschild black hole data [7]. A well-posed problem is a necessity for a stable numerical simulation.

The details of the gravitational waveform supplied by numerical simulation of the inspiral and merger of a binary black hole is a key input for interpreting the scientific content of the observed signal. It is thus important that the initial data does not introduce spurious effects into the waveform, in particular the "junk radiation" common to current elliptic formulations of the constraint problem. Elliptic equations require boundary data at inner boundaries in the strong field region surrounding the singularities inside the black holes, as well as at the outer boundary in the far field of the system. The treatment of the inner boundary is a potential source of junk radiation. The algebraic-hyperbolic method only requires data on the outer boundary, where the choice of boundary data can be guided by asymptotic flatness. The constraints are then satisfied by an inward "evolution" of the hyperbolic system along radial streamlines of the initial Cauchy hypersurface. The possibility of extending this method to binary black holes offers an alternative approach to the initialization problem that might prove to be more physically realistic.

In this note, in order to shed further light on the algebraic-hyperbolic formulation of the constraint system we investigate its application to linearized perturbations of initial data on a Minkowski background. We are led to the surprising result that there are no *useful* Cauchy hypersurfaces in Minkowski spacetime for which this linearized constraint problem is well posed.

By *useful*, we require that the Cauchy data be asymptoically flat so it can be expressed in non-singular form on a Cauchy hypersurface foliated by topological 2-spheres. The center of such a foliation has a singular limit where the area of the 2-sphere is zero. In the case of data for a perturbed Schwarzschild black hole, the singular point of the foliation agrees with the spacetime singularity and is therefore removed from the Cauchy hypersurface. In the perturbed Minkowski case, we require smoothness of the data at the center when referred to a Cartesian coordinate system. This leads to the requirements for a *useful* Cauchy hypersurface and foliation given in Sec. 2. Our results indicate that the algebraic-hyperbolic approach is applicable to asymptotically flat data only for spacetimes with singularities.

The initial data for solving Einstein's equations consist of a pair of symmetric tensor fields (h_{ab}, k_{ab}) on a smooth space-like three-dimensional manifold Σ, where h_{ab} is a Riemannian metric and k_{ab} is interpreted as the extrinsic curvature of Σ after its embedding in a 4-dimensional space-time. The constraints on initial data for a vacuum solution (see e.g. [9, 10]) consist of the Hamiltonian constraint

$$ {}^{(3)}R + (k^a{}_a)^2 - k_{ab}k^{ab} = 0 \,, \tag{1} $$

and the momentum constraints

$$D_b k^b{}_a - D_a k = 0\,, \quad k = k^b{}_b\,, \tag{2}$$

where ${}^{(3)}R$ and D_a denote the scalar curvature and the covariant derivative associated with h_{ab}.

The approach to solving the constraints depends upon the choice of which metric and extrinsic curvature variables are prescribed freely. In the algebraic-hyperbolic approach, the free variables consist of the initial 3-metric h_{ab} and four components of the extrinsic curvature which are picked out by a foliation of Σ by topological 2-spheres. In the case where Σ is spherical symmetric with foliation by metric 2-spheres of surface area $4\pi r^2$, so that

$$h_{ab} = R_a R_b + Q_{ab}, \quad Q_{ab} = r^2 q_{ab}, \quad R_a = D_a r (h^{bc} D_b r D_c r)^{-1/2}, \tag{3}$$

where q_{ab} is the unit sphere metric, the freely prescribed components of the extrinsic curvature are the transverse-tracefree components

$$K^{(TT)}_{ab} = Q^c_a Q^d_b k_{cd} - \frac{1}{2} Q_{ab} Q^{cd} k_{cd}. \tag{4}$$

The remaining 4 components of the extrinsic curvature are then determined by the constraints. The component

$$\kappa = R^c R^d k_{cd} \tag{5}$$

is determined algebraically from the Hamiltonian constraint and the components

$$K_a = R^b k_{ab} - \kappa R_a \quad \text{and} \quad K = Q^{ab} k_{ab} \tag{6}$$

are then determined by the momentum constraints. In this process, the momentum constraints form a well-posed symmetric hyperbolic system for (K, K_a) provided that the inequality

$$\kappa K < 0 \tag{7}$$

is satisfied. For general nonlinear perturbations of a Schwarzschild spacetime, it has been shown that this inequality is satisfied [7] and a pseudo-code for numerical implementation of the algebraic-hyperbolic constraint system has been formulated [8].

This algebraic-hyperbolic approach was introduced by Rácz along with other possibilities for solving the constraints without elliptic equations [6, 11]. One of those possibilities is a parabolic-hyperbolic formulation, in which the Hamiltonian constraint is reduced to a parabolic equation. In subsequent work, Rácz has shown that the parabolic-hyperbolic system has attractive features for constructing binary black hole data [12]. It has also been show that the parabolic-hyperbolic constraint problem is well posed for nonlinear perturbations of Minkowski space, and current work has successfully carried out a numerical implementation of this problem [13]. Here we investigate the application of the algebraic-hyperbolic approach to perturbed Minkowski data.

2. Linearized algebraic-hyperbolic constraint system on a Minkowski background

Consider the linearized perturbation $(\delta h_{ab}, \delta k_{ab})$ of data for Minkowski space, with background metric $\eta_{ab} = diag(-,+,+,+)$ in standard Cartesian inertial coordinates $x^a = (t, x^i) = (t, x, y, z)$. For the perturbed algebraic-hyperbolic constraint system to be well posed the inequality (7) must be satisfied. However, for a Cauchy hypersurface

Σ based on the standard inertial time slicing, the extrinsic curvature k_{ab} of the $t = constant$ hypersurfaces vanishes so that (7) cannot be satisfied to linearized order. Moreover, the Hamiltonian constraint (1) can be re-expressed in the form

$$2\kappa K = 2K^a K_a - \frac{1}{2}K^2 - {}^{(3)}R + K^{(TT)ab}K^{(TT)}_{ab}, \tag{8}$$

where the components $(\kappa, K, K_a, K^{(TT)}_{ab})$ of the extrinsic curvature k_{ab} are defined with respect to the foliation of Σ in a manner analogous to (4) – (6) for the case of a spherically symmetric metric. For a generic perturbation of the Minkowski background $\delta^{(3)}R \neq 0$ so that $K\delta\kappa + \kappa\delta K \neq 0$ to linear order. where K and κ refer to the background. Thus the stability inequality (7) cannot be satisfied to linear order unless the Cauchy hypersurface for the Minkowski background be chosen to have nonzero extrinsic curvature.

This leads to the question whether there are Cauchy hypersurfaces in Minkowski space $t - f(x^i) = 0$ for which the perturbed data satisfies the stability requirement (7). We now show that there are no such useful Cauchy hypersurfaces. By *useful*, we require:

- That Σ be spherically symmetric, i.e. $f(x^i) = F(r)$, where $r^2 = x^2 + y^2 + z^2$. Although this requirement does not immediately rule out more general foliations which might satisfy (7), no essential generality is lost by requiring spherically symmetry. For the purpose of exploring techniques for numerical implementation of the algebraic-hyperbolic approach, there would be little utility in an asymmetric Cauchy hypersurface for simulating asymptotically flat perturbations of Minkowski space.
- That Σ be smooth. Smoothness at the origin requires a local Taylor series expansion $F(r) = F(0) + \frac{r^2}{2}F''(0) + \ldots$, where we use the notation $F' = \partial_r F$. For example, the spacelike hypersurface $t = Cr$, $0 < C < 1$, has a conical singularity at the origin whereas the hyperboloidal hypersurface $t = F(r) = \sqrt{C^2 + r^2}$ is smooth.
- That Σ have a complete future domain of dependence, i.e. that it must extend from the origin $r = 0$ to either spacelike infinity or future null infinity. However, as seen below, our main result does not depend upon this assumption.

The metric of such a hypersurface $t = F(r)$ in Minkowski space has the $2 + 1$ decomposition (3), with line element in spherical coordinates given by

$$d\ell^2 = A^{-2}dr^2 + r^2 q_{AB}dx^A dx^B\,, \quad A = \frac{1}{\sqrt{1 - F'^2}}, \tag{9}$$

where $x^A = (\theta, \phi)$ and q_{AB} is the unit sphere metric. The corresponding curvature scalar is

$${}^{(3)}R = -\frac{2A^2F'^2}{r^2} - \frac{4A^4F'F''}{r}.$$

The future-directed unit normal to Σ is $n_b = -A\nabla_b\big(t - F(r)\big)$ and the extrinsic curvature $k_{ab} = h^c_a\nabla_c n_b$, where ∇_c is the covariant derivative associated with η_{ab}, is given by

$$k_{ab} = A^3F''R_aR_b + AF'rq_{ab}.$$

Here $R_a = A(\nabla_\alpha r - F'\nabla_\alpha t)$ is the unit normal to the $r = const$ foliation of Σ. The background components corresponding to (4) – (6) are $K^{(TT)}_{ab} = K_a = 0$ and

$$\kappa = A^3F''\,, \quad K = \frac{2AF'}{r}. \tag{10}$$

Now consider the product

$$\kappa K = \frac{2A^4 F' F"}{r}. \tag{11}$$

The stability inequality (7) requires

$$2F'F'' = (F'^2)' < 0. \tag{12}$$

But smoothness at the origin requires $F'(0) = 0$ and since $F'^2 \geq 0$ the inequality cannot be satisfied. Hence there is no suitable Cauchy hypersurface in Minkowski space on which to base a well-posed algebraic-hyperbolic linearized constraint problem.

3. Discussion

We have shown that there is no Cauchy hypersurface in Minkowski space which is useful for numerical investigation of the algebraic-hyperbolic formulation of the constraint problem for non-singular asymptotically flat data. This is at first surprising since such hypersurfaces exist in a Schwarzschild spacetime, namely the $t = constant$ hypersurfaces in the ingoing Kerr-Schild form $g_{ab} = \eta_{ab} + (2m/r)\ell_a \ell_b$, where t is the inertial time for the Minkowski background metric η_{ab}. In that case, the stability inequality (7) holds everywhere outside the singularity at $r = 0$. It appears that some singular structure of the Cauchy hypersurface is necessary for a well-posed algebraic-hyperbolic constraint problem. In fact, for the pure Minkowski background, the hypersurface

$$t = \frac{Cr}{C + r}, \quad C > 0,$$

has extrinsic curvature components

$$\kappa = -\frac{2A^3 C^2}{(C + r)^3}, \quad K = \frac{2AC^2}{r(C + r)^2}, \quad A = \frac{(C + r)^2}{\sqrt{(C + r)^4 - C^4}}, \tag{13}$$

which does satisfy the stability inequality for $r > 0$. But this hypersurface has a conical singularity at $r = 0$.

Acknowledgments

This research was supported by NSF grant PHY-1505965 to the University of Pittsburgh. I thank I. Rácz and C. Schell for useful communication. The problem was initiated by discussions with L. Lehner while a guest at the Perimeter Institute, which is supported by the Government of Canada through NSERC and by the Province of Ontario through MEDT.

References

[1] Lichnerowicz A: *L'integration des Equations de la Gravitation Relativiste et le Probleme des n Corps*, J. Math. Pures Appl. **23** 39 (1944)
[2] York J W: *Role of conformal three-geometry in the dynamics of gravitation*, Phys. Rev. Letters **28** 1082 (1972)
[3] York J W: *Covariant decompositions of symmetric tensors in the theory of gravitation*, Ann. Inst. Henri Poincaré A **21** 319 (1974)
[4] Cook G B: *Initial data for numerical relativity*, Living Rev. Relativity **3** 5 (2000)
[5] Gourgoulhon E: *Construction of initial data for 3+1 numerical relativity*, J. Phys.: Conf. Ser. **91** 012001 (2007)

[6] Rácz I *Constraints as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)
[7] Rácz I and Winicour J: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
[8] Rácz I and Winicour J *On solving the constraints by integrating a strongly hyperbolic system*, arXiv:gr-qc/1601.05386
[9] Choquet-Bruhat Y: *General relativity and Einstein's equations*, Oxford University Press Inc., New York (2009)
[10] Wald R M: *General relativity*, University of Chicago Press, Chicago (1984)
[11] Rácz I: *Constraints as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)
[12] Rácz I: *A simple method of constructing binary black hole initial data*, arXiv:gr-qc/1605.01669
[13] Schell C: private communication