Supplementary material

K. C. van Kruining, A. G. Hayrapetyan & J. B. Götte

Obtaining the exact solutions of the Dirac equation in a magnetic field

The solutions of te squared Dirac equation in a constant magnetic field ($\mathbf{A} = \frac{1}{2}B[-y,x,0]$) are, using the rescaled coordinate $\tilde{r} = \sqrt{\frac{|e|B}{2}}r$ and taking l positive

$$\Psi = e^{i(kz-\mathcal{E}t\pm l\phi)}\tilde{r}^l e^{-\frac{\tilde{r}^2}{2}} L_p^l(\tilde{r}^2)\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \vee \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right).$$

The exact solutions pf the first order Dirac equation can be obtained by applying $\not{P}+m$ to the solutions of the squared Dirac equation. Using

$$(\not{P}+m)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}i\partial_t + m\\0\\-i\partial_z\\-i\partial_x+\partial_y+\frac{|e|B}{2}(ix-y)\end{bmatrix}, \qquad (\not{P}+m)\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\i\partial_t + m\\-i\partial_x-\partial_y-\frac{|e|B}{2}(ix+y)\\-i\partial_z\end{bmatrix},$$

the de derivatives with respect to t and z are easy to compute and give resp. $\mathcal{E}$ and k. For the transverse derivatives, one can use the rescaled coordinates $\tilde{x} = \sqrt{\frac{|e|B}{2}}x$, $\tilde{y} = \sqrt{\frac{|e|B}{2}}y$ to rewrite them as

$$(\not{P}+m)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}m+\mathcal{E}\\0\\k\\\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}}+\partial_{\tilde{y}}+(i\tilde{x}-\tilde{y}))\end{bmatrix}, \qquad (\not{P}+m)\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\m+\mathcal{E}\\\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}}-\partial_{\tilde{y}}-(i\tilde{x}+\tilde{y}))\\k\end{bmatrix}.$$

The components $\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}}+\partial_{\tilde{y}}+(i\tilde{x}-\tilde{y}))$ and $\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}}-\partial_{\tilde{y}}-(i\tilde{x}+\tilde{y}))$ give rise to the spin-orbit mixing terms, whose explicit computation is rather lengthy. The following three identities will be of use

$$\begin{aligned}(\partial_{\tilde{x}} \pm i\partial_{\tilde{y}})\tilde{r}^n &= n\tilde{r}^{n-1}e^{\pm i\phi},\\ (\partial_{\tilde{x}} \pm i\partial_{\tilde{y}})\tilde{r}^{|n|}e^{\pm i|n|\phi} &= 0,\\ (\partial_{\tilde{x}} \pm i\partial_{\tilde{y}})\tilde{r}^{|n|}e^{\mp i|n|\phi} &= 2|n|\tilde{r}^{|n|-1}e^{\pm i(|n|-1)\phi}.\end{aligned}$$

The form of the spin-orbit term depends on the signs of the spin and orbital angular momentum. For spin and orbital angular momentum positive, one has

$$\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}}+\partial_{\tilde{y}}+(i\tilde{x}-\tilde{y}))e^{i(kz-\mathcal{E}t+l\phi)}\tilde{r}^l e^{-\frac{\tilde{r}^2}{2}}L_p^l(\tilde{r}^2) = i\sqrt{2|e|B}\left(e^{i(kz-\mathcal{E}t+(l+1)\phi)}\tilde{r}^{l+1}e^{-\frac{\tilde{r}^2}{2}}\left(L_p^l(\tilde{r}^2)-L'^l_p(\tilde{r}^2)\right)\right).$$

Using the recurrence relations for Laguerre polynomials $L'^l_p(\tilde{r}^2 = -L_{p-1}^{l+1}(\tilde{r}^2)$ (prime denotes differentiation with respect to $\tilde{r}^2$) and $L_p^l(\tilde{r}^2) = L_p^{l+1}(\tilde{r}^2) - L_{p-1}^{l+1}(\tilde{r}^2)$, the Laguerre polynomials in the brackets become simply $L_p^{l+1}(\tilde{r}^2)$, thus the overall solution of the first order Dirac equation in this case becomes

$$\Psi = e^{i(kz-\mathcal{E}t+l\phi)-\frac{\tilde{r}^2}{2}} \left(\tilde{r}^l L_p^l(\tilde{r}^2) \begin{bmatrix} m+\mathcal{E} \\ 0 \\ k \\ 0 \end{bmatrix} + \sqrt{2} i e^{i\phi} \tilde{r}^{l+1} L_p^{l+1}(\tilde{r}^2) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{B|e|} \end{bmatrix} \right).$$

Now for positive orbital angular momentum and negative spin, the spin-orbit term is

$$\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}} - \partial_{\tilde{y}} - (i\tilde{x}+\tilde{y}))e^{i(kz-\mathcal{E}t+l\phi)}\tilde{r}^l e^{-\frac{\tilde{r}^2}{2}} L_p^l(\tilde{r}^2) =$$
$$i\sqrt{2|e|B}\left(e^{i(kz-\mathcal{E}t+(l-1)\phi)}\tilde{r}^{l-1}e^{-\frac{\tilde{r}^2}{2}}\left(-lL_p^l(\tilde{r}^2) - \tilde{r}^2 {L'}_p^l(\tilde{r}^2)\right)\right).$$

Using the recurrence relation

$$\tilde{r}^2 {L'}_p^l(\tilde{r}^2) = pL_p^l(\tilde{r}^2) - (p+l)L_{p-1}^l(\tilde{r}^2)$$

to rewrite the derivatives of the Laguerre polynomial, one gets

$$-\left(lL_p^l(\tilde{r}^2) + pL_p^l(\tilde{r}^2) - (p+l)L_{p-1}^l(\tilde{r}^2)\right).$$

With the relation $L_p^{l-1}(\tilde{r}^2) = L_p^l(\tilde{r}^2) - L_{p-1}^l(\tilde{r}^2)$ this expression simplifies to $-(p+l)L_p^{l-1}(\tilde{r}^2)$ and the solution of the first order Dirac equation for positive orbital angular momentum and negative spin becomes

$$\Psi = e^{i(kz-\mathcal{E}t+l\phi)-\frac{\tilde{r}^2}{2}} \left(\tilde{r}^l L_p^l(\tilde{r}^2) \begin{bmatrix} 0 \\ m+\mathcal{E} \\ 0 \\ -k \end{bmatrix} - \sqrt{2}(p+l) i e^{-i\phi} \tilde{r}^{l-1} L_p^{l-1}(\tilde{r}^2) \begin{bmatrix} 0 \\ 0 \\ \sqrt{B|e|} \\ 0 \end{bmatrix} \right).$$

For negative orbital angular momentum and positive spin, one has

$$\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}} + \partial_{\tilde{y}} + (i\tilde{x}-\tilde{y}))e^{i(kz-\mathcal{E}t-l\phi)}\tilde{r}^l e^{-\frac{\tilde{r}^2}{2}} L_p^l(\tilde{r}^2) =$$
$$i\sqrt{2|e|B}\left(e^{i(kz-\mathcal{E}t+(l+1)\phi)}\tilde{r}^{l-1}e^{-\frac{\tilde{r}^2}{2}}\left(\tilde{r}^2 L_p^l(\tilde{r}^2) - lL_p^l(\tilde{r}^2) - \tilde{r}^2 {L'}_p^l(\tilde{r}^2)\right)\right).$$

Using $L_p^l(\tilde{r}^2) = -{L'}_{p-1}^{l+1}(\tilde{r}^2)$, where the prime denotes differentiation with respect to $\tilde{r}^2$, one can rewrite the first Laguerre polynomial:

$$-\left(r^2 {L'}_{p+1}^{l-1}(\tilde{r}^2) + lL_p^l(\tilde{r}^2) + \tilde{r}^2 {L'}_p^l(\tilde{r}^2)\right).$$

Using again $\tilde{r}^2 {L'}_p^l(\tilde{r}^2) = pL_p^l(\tilde{r}^2) - (p+l)L_{p-1}^l(\tilde{r}^2)$ this expression becomes

$$-\left((p+1)L_{p+1}^{l-1}(\tilde{r}^2) - (p+l)L_p^{l-1}(\tilde{r}^2) + lL_p^l(\tilde{r}^2) + pL_p^l(\tilde{r}^2) - (p+l)L_{p-1}^l(\tilde{r}^2)\right).$$

Because of $L_p^{l-1}(\tilde{r}^2) = L_p^l(\tilde{r}^2) - L_{p-1}^l(\tilde{r}^2)$, everything but the first term cancels and the solution for the first order Dirac equation for negative orbital angular momentum and positive spin becomes

$$\Psi = e^{i(kz-\mathcal{E}t-l\phi)-\frac{\tilde{r}^2}{2}} \left(\tilde{r}^l L_p^l(\tilde{r}^2) \begin{bmatrix} m+\mathcal{E} \\ 0 \\ k \\ 0 \end{bmatrix} - \sqrt{2}(p+1) i e^{i\phi} \tilde{r}^{l-1} L_{p+1}^{l-1}(\tilde{r}^2) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{B|e|} \end{bmatrix} \right)$$

The case of negative orbital angular momentum and spin is simple, one has

$$\sqrt{\frac{|e|B}{2}}(-i\partial_{\tilde{x}} - \partial_{\tilde{y}} - (i\tilde{x}+\tilde{y}))e^{i(kz-\mathcal{E}t+l\phi)}\tilde{r}^l e^{-\frac{\tilde{r}^2}{2}} L_p^l(\tilde{r}^2) = -i\sqrt{2|e|B}\left(e^{i(kz-\mathcal{E}t+(l-1)\phi)}\tilde{r}^{l-1}e^{-\frac{\tilde{r}^2}{2}}\tilde{r}^2 {L'}_p^l(\tilde{r}^2)\right),$$

and using again $L_p^l(\tilde{r}^2) = -L'^{l+1}_{p-1}(\tilde{r}^2)$, the overall solution of the first order Dirac equation becomes

$$\Psi = e^{i(kz-\mathcal{E}t-l\phi)-\frac{\tilde{r}^2}{2}} \left(\tilde{r}^l L_p^l(\tilde{r}^2) \begin{bmatrix} 0 \\ m+\mathcal{E} \\ 0 \\ -k \end{bmatrix} + \sqrt{2} i e^{-i\phi} \tilde{r}^{l+1} L_{p-1}^{l+1}(\tilde{r}^2) \begin{bmatrix} 0 \\ 0 \\ \sqrt{B|e|} \\ 0 \end{bmatrix} \right)$$

Explicit forms of the radial and azimuthal gamma and spin matrices

$$\gamma^r = \cos\phi\,\gamma^x + \sin\phi\,\gamma^y = \begin{bmatrix} 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & -e^{-i\phi} & 0 & 0 \\ -e^{i\phi} & 0 & 0 & 0 \end{bmatrix},$$

$$\gamma^\phi = -\sin\phi\,\gamma^x + \cos\phi\,\gamma^y = \begin{bmatrix} 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \\ 0 & ie^{-i\phi} & 0 & 0 \\ -ie^{i\phi} & 0 & 0 & 0 \end{bmatrix},$$

$$\Sigma_r = \cos\phi\,\Sigma_x + \sin\phi\,\Sigma_y = \begin{bmatrix} 0 & e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \end{bmatrix},$$

$$\Sigma_\phi = -\sin\phi\,\Sigma_x + \cos\phi\,\Sigma_y = \begin{bmatrix} 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \end{bmatrix}.$$

Explicit evaluation of $\frac{\int \Psi^\dagger \tilde{r}^2 \Psi}{\int \Psi^\dagger \Psi}$.

The quantity $\frac{\int \Psi^\dagger \tilde{r}^2 \Psi}{\int \Psi^\dagger \Psi}$ appears in the computation of the gauge covariant angular momentum of the electron vortex states. For the four different combinations of positive and negative orbital angular momentum and spin (order the same as in the main text) $\int \Psi^\dagger \tilde{r}^2 \Psi$ can be shown to be resp.

$$\int \Psi^\dagger \tilde{r}^2 \Psi = 2\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) \tilde{r}^{2l+2} {L_p^l}^2(\tilde{r}^2) e^{-\tilde{r}^2} + 2B|e| \tilde{r}^{2l+4} {L_p^{l+1}}^2(\tilde{r}^2) e^{-\tilde{r}^2} d\tilde{r},$$

$$\int \Psi^\dagger \tilde{r}^2 \Psi = 2\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) \tilde{r}^{2l+2} {L_p^l}^2(\tilde{r}^2) e^{-\tilde{r}^2} + 2B|e|(p+l)^2 \tilde{r}^{2l} {L_p^{l-1}}^2(\tilde{r}^2) e^{-\tilde{r}^2} d\tilde{r},$$

$$\int \Psi^\dagger \tilde{r}^2 \Psi = 2\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) \tilde{r}^{2l+2} {L_p^l}^2(\tilde{r}^2) e^{-\tilde{r}^2} + 2B|e|(p+1)^2 \tilde{r}^{2l} {L_{p+1}^{l-1}}^2(\tilde{r}^2) e^{-\tilde{r}^2} d\tilde{r}$$

$$\int \Psi^\dagger \tilde{r}^2 \Psi = 2\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) \tilde{r}^{2l+2} {L_p^l}^2(\tilde{r}^2) e^{-\tilde{r}^2} + 2B|e| \tilde{r}^{2l+4} {L_{p-1}^{l+1}}^2(\tilde{r}^2) e^{-\tilde{r}^2} d\tilde{r}.$$

Substituting $x = \tilde{r}^2$ turns these integrals to

$$\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) x^{l+1} {L_p^l}^2(x) e^{-x} + 2B|e| x^{l+2} {L_p^{l+1}}^2(x) e^{-x} dx,$$

$$\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) x^{l+1} {L_p^l}^2(x) e^{-x} + 2B|e|(p+l)^2 x^l {L_p^{l-1}}^2(x) e^{-x} dx,$$

$$\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) x^{l+1} {L_p^l}^2(x) e^{-x} + 2B|e|(p+1)^2 x^l {L_{p+1}^{l-1}}^2(x) e^{-x} dx$$

$$\pi \int_0^\infty \left((\mathcal{E}+m)^2 + k^2\right) x^{l+1} {L_p^l}^2(x) e^{-x} + 2B|e| x^{l+2} {L_{p-1}^{l+1}}^2(x) e^{-x} dx.$$

For Laguerre polynomials, we have the orthogonality relation $\int_0^\infty L_p^l(x)L_{p'}^l(x)e^{-x}dx = \frac{(l+p)!}{p!}\delta_{pp'}$. Using this relation and $L_p^{l-1}(x) = L_p^l(x) - L_{p-1}^l(x)$, we obtain the following integral identity

$$\int_0^\infty x^{l+1}{L_p^l}^2(x)e^{-x}dx = \int_0^\infty x^{l+1}\left(L_p^{l+1}(x) - L_{p-1}^{l+1}(x)\right)^2 e^{-x}dx = \frac{(l+p+1)!}{p!} + \frac{l+p!}{(p-1)!} = \frac{l+p!}{p!}(2p+l+1),$$

which can be used to evaluate all the integals and obtain

$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(\left((\mathcal{E}+m)^2+k^2\right)(2p+l+1) + 2B|e|(p+l+1)(2p+l+2)\right),$$
$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(\left((\mathcal{E}+m)^2+k^2\right)(2p+l+1) + 2B|e|(p+l)(2p+l)\right),$$
$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(\left((\mathcal{E}+m)^2+k^2\right)(2p+l+1) + 2B|e|(p+1)(2p+l+2)\right),$$
$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(\left((\mathcal{E}+m)^2+k^2\right)(2p+l+1) + 2B|e|p(2p+l)\right).$$

By noting that $\mathcal{E}_L^2 + \mathcal{E}_Z^2$ is $2B|e|$ times resp. $p+l+1$, $p+l$, $p+1$ and p and rearranging, one gets

$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(2\mathcal{E}(\mathcal{E}+m)(2p+l+1) + \mathcal{E}_L^2 + \mathcal{E}_Z^2\right),$$
$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(2\mathcal{E}(\mathcal{E}+m)(2p+l+1) - \mathcal{E}_L^2 - \mathcal{E}_Z^2\right),$$
$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(2\mathcal{E}(\mathcal{E}+m)(2p+l+1) + \mathcal{E}_L^2 + \mathcal{E}_Z^2\right),$$
$$\int \Psi^\dagger \tilde{r}^2 \Psi = \pi \frac{(l+p)!}{p!}\left(2\mathcal{E}(\mathcal{E}+m)(2p+l+1) - \mathcal{E}_L^2 - \mathcal{E}_Z^2\right).$$

Dividing by $\int \Psi^\dagger \Psi = \int j_0 = 2\pi\mathcal{E}(\mathcal{E}+m)\frac{(p+l)!}{p!}$ and adding the canonical angular momentum, resp. $l+\frac{1}{2}$, $l-\frac{1}{2}$, $-l+\frac{1}{2}$ and $-l-\frac{1}{2}$, one gets for the gauge covariant angular momentum

$$\mathcal{J}_z = \quad 2p + 2l + \frac{3}{2} + \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2\mathcal{E}(\mathcal{E}+m)},$$
$$\mathcal{J}_z = \quad 2p + 2l + \frac{1}{2} - \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2\mathcal{E}(\mathcal{E}+m)},$$
$$\mathcal{J}_z = \quad 2p + \frac{3}{2} + \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2\mathcal{E}(\mathcal{E}+m)},$$
$$\mathcal{J}_z = \quad 2p + \frac{1}{2} - \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2\mathcal{E}(\mathcal{E}+m)}.$$

Computation of the magnetic moment

Using the explicit form of the azimuthal Dirac matrix, it is easy to see that only the crossterms between the main and spin-orbit parts contribute to the azimuthal current and these can be computed to be

$$j_\phi = \quad 2\sqrt{2}\tilde{r}^{2l+1}e^{-\tilde{r}^2}L_p^l(\tilde{r}^2)L_p^{l+1}(\tilde{r}^2)(\mathcal{E}+m)\sqrt{B|e|},$$
$$j_\phi = \quad 2\sqrt{2}(p+l)\tilde{r}^{2l-1}e^{-\tilde{r}^2}L_p^l(\tilde{r}^2)L_p^{l-1}(\tilde{r}^2)(\mathcal{E}+m)\sqrt{B|e|},$$
$$j_\phi = \quad -2\sqrt{2}(p+1)\tilde{r}^{2l-1}e^{-\tilde{r}^2}L_p^l(\tilde{r}^2)L_{p+1}^{l-1}(\tilde{r}^2)(\mathcal{E}+m)\sqrt{B|e|},$$
$$j_\phi = \quad -2\sqrt{2}\tilde{r}^{2l+1}e^{-\tilde{r}^2}L_p^l(\tilde{r}^2)L_{p-1}^{l+1}(\tilde{r}^2)(\mathcal{E}+m)\sqrt{B|e|}.$$

Now $M_z = \int \frac{e}{2} r j_\phi \tilde{r} d\phi d\tilde{r} = \sqrt{\frac{2}{B|e|}} \int \frac{e}{2} \tilde{r} j_\phi r d\phi d\tilde{r}$. Substituting the explicit currents into this integral, using $x = \tilde{r}^2$ and performing the angular integration yields

$$M_z = -2\pi(\mathcal{E}+m)|e| \int_0^\infty x^{l+1} L_p^l(x) L_p^{l+1}(x) e^{-x} dx,$$
$$M_z = -2\pi(\mathcal{E}+m)|e|(p+l) \int_0^\infty x^{l} L_p^l(x) L_p^{l-1}(x) e^{-x} dx,$$
$$M_z = 2\pi(\mathcal{E}+m)|e|(p+1) \int_0^\infty x^{l} L_p^l(x) L_{p+1}^{l-1}(x) e^{-x} dx,$$
$$M_z = 2\pi(\mathcal{E}+m)|e| \int_0^\infty x^{l+1} L_p^l(x) L_{p-1}^{l+1}(x) e^{-x} dx.$$

Now one can again use $L_p^l(x) = L_p^{l+1}(x) - L_{p-1}^{l+1}(x)$ and the orthogonality relation of associated Laguerre polynomials to evaluate these integrals

$$M_z = -2\pi(\mathcal{E}+m)|e| \frac{(l+p+1)!}{p!} = -\frac{\int j_0}{\mathcal{E}} \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2B},$$
$$M_z = -2\pi(\mathcal{E}+m)|e|(p+l) \frac{(p+l)!}{p!} = -\frac{\int j_0}{\mathcal{E}} \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2B},$$
$$M_z = -2\pi(\mathcal{E}+m)|e|(p+1) \frac{(l+p)!}{p!} = -\frac{\int j_0}{\mathcal{E}} \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2B},$$
$$M_z = -2\pi(\mathcal{E}+m)|e| \frac{(l+p)!}{(p-1)!} = -\frac{\int j_0}{\mathcal{E}} \frac{\mathcal{E}_L^2 + \mathcal{E}_Z^2}{2B}.$$

Commutator identities for the gauge covariant angular momentum operators

For this section we write the angular omentum operators in antisymmetric tensor form. The gauge covariant angular momentum can be split in a spin and an orbital part like

$$\mathcal{J}_{\mu\nu} = \mathcal{L}_{\mu\nu} + \frac{i}{2}\sigma_{\mu\nu}, \qquad \text{with } \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \text{ and } \mathcal{L}_{\mu\nu} = x_{[\mu} P_{\nu]} \equiv x_\mu P_\nu - x_\nu P_\mu.$$

Because $\mathcal{L}_{\mu\nu}$ contains no Dirac matrices, one obviously has $[\mathcal{L}_{\mu\nu}, \sigma_{\rho\sigma}] = 0$, so $[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = [\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] - \frac{1}{4}[\sigma_{\mu\nu}, \sigma_{\rho\sigma}]$. Writing out the commutator for the σ-tensor gives

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = \frac{1}{4}\left((\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho) - (\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\right).$$

One can check that if all four indices are different, this commutator is zero. If two indices are the same one can eliminate the identical Dirac matrices and obtain after some algebra

$$[\sigma_{\mu\nu}, \sigma_{\rho\sigma}] = 2(-\eta_{\mu\rho}\sigma_{\nu\sigma} + \eta_{\mu\sigma}\sigma_{\nu\rho} + \eta_{\nu\rho}\sigma_{\mu\sigma} - \eta_{\nu\sigma}\sigma_{\mu\rho}).$$

For the orbital part, we need the commutation relations of the gauge covariant momentum $P_\mu = i\partial_\mu - eA_\mu$. It is easy to check that

$$[ix_{[\mu}\partial_{\nu]}, ix_{[\rho}\partial_{\sigma]}] = -\eta_{\mu\rho} x_{[\nu}\partial_{\sigma]} + \eta_{\mu\sigma} x_{[\nu}\partial_{\rho]} + \eta_{\nu\rho} x_{[\mu}\partial_{\sigma]} - \eta_{\nu\sigma} x_{[\mu}\partial_{\rho]}.$$

To get the commutators for the gauge coveriant orbital angular momenta, we need to add $[ix_{[\mu}\partial_{\nu]}, -ex_{[\rho}A_{\sigma]}] + [-ex_{[\mu}A_{\nu]}, ix_{[\rho}\partial_{\sigma]}] = [ix_{[\mu}\partial_{\nu]}, -ex_{[\rho}A_{\sigma]}] - [ix_{[\rho}\partial_{\sigma]}, -ex_{[\mu}A_{\nu]}]$ (the vector potentias commute with each other). Using that both terms are the same up to the index swap $\mu \leftrightarrow \rho$, $\nu \leftrightarrow \sigma$ and using $\partial_\mu A_\rho - (\mu \leftrightarrow \rho) = F_{\mu\rho}$ these terms can be evaluated to be

$$[ix_{[\mu}\partial_{\nu]}, -ex_{[\rho}A_{\sigma]}] + [-ex_{[\mu}A_{\nu]}, ix_{[\rho}\partial_{\sigma]}] = -ie(\eta_{\mu\rho} x_{[\nu}A_{\sigma]} - \eta_{\nu\rho} x_{[\mu}A_{\sigma]} - \eta_{\mu\sigma} x_{[\nu}A_{\rho]} + \eta_{\nu\sigma} x_{[\mu}A_{\rho]})$$
$$- ie(x_\mu x_\rho F_{\nu\sigma} - x_\mu x_\sigma F_{\nu\rho} - x_\nu x_\rho F_{\mu\sigma} + x_\nu x_\sigma F_{\mu\rho}).$$

Using $-x_{[\mu}\partial_{\rho]} - iex_{[\mu}A_{\rho]} = ix_{[\mu}P_{\rho]}$, and putting things together gives

$$-i[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] = \eta_{\mu\rho}\mathcal{L}_{\nu\sigma} - \eta_{\mu\sigma}\mathcal{L}_{\nu\rho} - \eta_{\nu\rho}\mathcal{L}_{\mu\sigma} + \eta_{\nu\sigma}\mathcal{L}_{\mu\rho} + e(x_\mu x_\rho F_{\nu\sigma} - x_\mu x_\sigma F_{\nu\rho} - x_\nu x_\rho F_{\mu\sigma} + x_\nu x_\sigma F_{\mu\rho}),$$
$$-i[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\rho\sigma}] = \eta_{\mu\rho}\mathcal{J}_{\nu\sigma} - \eta_{\mu\sigma}\mathcal{J}_{\nu\rho} - \eta_{\nu\rho}\mathcal{J}_{\mu\sigma} + \eta_{\nu\sigma}\mathcal{J}_{\mu\rho} + e(x_\mu x_\rho F_{\nu\sigma} - x_\mu x_\sigma F_{\nu\rho} - x_\nu x_\rho F_{\mu\sigma} + x_\nu x_\sigma F_{\mu\rho}).$$

Then using $\mathcal{J}_x = \mathcal{J}_{23}$, $\mathcal{J}_y = \mathcal{J}_{31}$ and $\mathcal{J}_z = \mathcal{J}_{12}$, one gets

$$[\mathcal{J}_j, \mathcal{J}_k] = -i\varepsilon_{jkl}(\mathcal{J}_l - x_l \mathbf{x}\cdot\mathbf{B}).$$

For the commutator $[\not{P} - m, J_{\mu\nu}]$, one can first note that m commutes with any operator. Again using $\mathcal{J}_{\mu\nu} = \mathcal{L}_{\mu\nu} + \frac{i}{2}\sigma_{\mu\nu}$, one can compute the commutators of the spin and orbital parts seperately using $[P_\mu, P_\nu] = -ieF_{\mu\nu}$:

$$[\not{P}, \sigma_{\mu\nu}] = \frac{1}{2}P^\lambda\left[\gamma_\lambda, [\gamma_\mu, \gamma_\nu]\right] = 2P^\lambda \eta_{\lambda[\mu}\gamma_{\nu]} = 2P_{[\mu}\gamma_{\nu]} = -2\gamma_{[\mu}P_{\nu]},$$
$$\left[\gamma^\lambda P_\lambda, x_{[\mu}P_{\nu]}\right] = i\gamma^\lambda \eta_{\lambda[\mu}P_{\nu]} + iex_{[\mu}F_{\nu]\lambda}\gamma^\lambda = i\gamma_{[\mu}P_{\nu]} + iex_{[\mu}F_{\nu]\lambda}\gamma^\lambda,$$
$$\left[\gamma^\lambda P_\lambda, x_{[\mu}P_{\nu]} + \frac{i}{2}\sigma_{\mu\nu}\right] = [\not{P}, J_{\mu\nu}] = iex_{[\mu}F_{\nu]\lambda}\gamma^\lambda.$$