A Linear Quadratic Control Problem for the Stochastic Heat Equation Driven by Q-Wiener Processes

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Abstract

We consider a control problem for the stochastic heat equation with Neumann boundary condition, where controls and noise terms are defined inside the domain as well as on the boundary. The noise terms are given by independent Q-Wiener processes. Under some assumptions, we derive necessary and sufficient optimality conditions stochastic controls have to satisfy. Using these optimality conditions, we establish explicit formulas with the result that stochastic optimal controls are given by feedback controls. This is an important conclusion to ensure that the controls are adapted to a certain filtration. Therefore, the state is an adapted process as well.

Keywords: Heat equation, Stochastic control, Neumann boundary condition, Q-Wiener process, Fractional power operator, Riccati equation

1. Introduction

The heat equation is a typical example of a parabolic partial differential equation describing the time-varying distribution of heat in a given region. It is well known that the heat equation can be reformulated as abstract Cauchy problem using Friedrichs extension of the Laplace operator, see [2, 34]. The abstract formulation has the advantage that the system is in many ways easier to handle than the specific partial differential equation and moreover, it provides a direct generalization of a finite dimensional system. The existence and uniqueness of a solution can be easily achieved using semigroup theory. As a consequence, one obtains that the semigroup property illustrates the irreversibility of heat conduction in nature, which gives us a physical motivation. Another aspect is that there may appear sources of heat inside the region as well as on the boundary. For heat sources inside the region, the system can be reformulated as a non-homogeneous linear evolution equation, which can be seen as a generalization of the abstract Cauchy problem described above. Using semigroup theory, one concludes that the solution is given in a mild sense, see [2, 3]. For heat sources defined on the boundary, the derivation of a mild solution requires the introduction of an operator mapping the boundary values inside the region, see [3] and the references therein. This operator is defined as the inverse of a trace operator and hence, the properties heavily depend on the type of the boundary condition, see [22]. For more general formulated systems, see [20, 21]. In the following, we will assume that we can influence the distribution of heat through these heat sources and therefore, we will call them controls.

Using stochastic processes, one can model random heating as well as cooling phenomenas. Here, we will assume that we can not influence these phenomenas and similar to the heat sources, one can include them as so called noise terms inside the region as well as on the boundary. This immediately leads us to the stochastic heat equation, which is a specific example of the wide class of linear stochastic partial differential equations, see [8].

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Stochastic partial differential equations belong to the modern research of infinite dimensional stochastic analysis. Such equations can be interpreted as stochastic evolution equations and the solutions are defined in a generalized sense. There exist different approaches on how to deal with these solutions. In [8, 10, 15, 28], the concept of weak solutions is introduced, where the construction in mainly based on inner products. Using Gelfand triples, another approach is given by variational solutions, see [28, 29]. For problems which contains a linear operator generating a semigroup on a Hilbert space, one can use mild solutions, see [8, 10, 15]. Mild solutions are considered as integral equation of Itô-Volterra type containing a stochastic convolution. All of these concepts are based on a given probability space and they are called (probabilistic) strong solutions. Solutions constructing the probability space are called (probabilistic) weak solutions or martingale solutions, see [8, 10]. In this paper, we will use the theory of mild solutions in order to cover all side effects, in particular, the non-homogeneous boundary condition.

Basically, there exist two different approaches on how to deal with stochastic optimal control problems. The dynamic programming considers a family of optimal control problems with different initial times and states. A relationship between these problems is given by the so called Hamilton-Jacobi-Bellman equation, which is a second order partial differential equation in the stochastic case. If this equation is solvable, then one can obtain an optimal feedback control, see [13, 31, 32]. Another method is given by the maximum principle. The basic idea is that the optimal control problem can be regarded as an optimization problem in infinite dimension. One derives necessary optimality conditions that must be satisfied by any optimal control. These necessary conditions become sufficient under certain convexity assumptions on the objective function. Regarding control problems the objective function is often called cost functional. Based on the optimality conditions, one can derive the adjoint system, which is a backward stochastic partial differential equation. Therefore, the solution of the stochastic control problem is the solution of a coupled system of forward and backward stochastic partial differential equations, see [18, 23, 32].

Let us recall some papers concerned with stochastic control problems. A derivation of optimal feedback controls, where the state satisfies a stochastic evolution equation is given in [1]. Stochastic maximum principles for the optimal control of stochastic partial differential equation involving nonlinear terms are considered in [14, 23, 37]. Even in the deterministic case, it is well known that including boundary conditions is a very difficult problem, see [3]. There are few papers covering the optimal control of the stochastic heat equation with Neumann boundary conditions. In [17], the system is completely linear. Systems including nonlinear terms are covered in [9, 16, 33]. For controlled stochastic partial differential equation with Dirichlet boundary conditions see [12, 24]. In [4], an approach with dynamical boundary conditions is given. A stochastic control problem for stochastic equations with delays is studied in [35]. In these papers, the region is one dimensional such that boundary noises are defined by one dimensional Brownian motions. For papers involving infinite dimensional noise terms, see [1, 14]. Controlled stochastic equations, where the time horizon is infinite, are analyzed in [36].

The shortcomings of these papers is the formulation of the stochastic heat equation with homogeneous boundary condition or the restriction to one or two-dimensional regions. We overcome these issues by considering a linear quadratic control problem for the stochastic heat equation with non-homogeneous Neumann boundary condition in more general regions without any restriction to the dimensionality. Here, the state fulfills a system, where the controls and the noise terms are defined inside the region as well as on the boundary. Due to the general formulation, it is necessary that the noise terms also depends on the spatial variable such that involving infinite dimensional stochastic processes is reasonable. We will assume that these stochastic processes are defined by Q-Wiener processes. The main difficulty is to include the non-homogeneous boundary condition. We introduce the so called Neumann operator and fractional power operators in order to obtain a well defined mild solution, which includes all side effects. The control problem is given by a so called tracking problem such that we can utilize the specific cost functional to derive necessary conditions stochastic optimal controls have to satisfy, which are also sufficient under certain requirements on the cost functional. Using these optimality conditions, we deduce explicit formulas of the stochastic optimal controls. Due to the presence of the non-homogeneous boundary condition, it is challenging to rewrite the formulas in order to obtain that the optimal controls fulfills a feedback law. However, we can define a suitable Riccati equation such that this result can be achieved. Therefore, we can conclude that the optimal controls are adapted to a certain filtration in order to ensure that the state is an adapted process as well.

The paper is organized as follows. In Section 2, we recall the main results known from the deterministic case. Some basic properties of operators used in the remaining sections are given here. Moreover, we motivate how to involve noise terms to partial differential equations. In Section 3, we specify the system considered in this paper and based on certain assumptions, we prove the existence of an unique solution. The optimization problem is determined in Section 4. Here, we consider a so called tracking problem. Using the special structure of the cost functional, we derive necessary and sufficient conditions stochastic optimal controls have to satisfy. Based on these optimality conditions, the explicit formulas of optimal controls are deduced. In Section 5, we rewrite the stochastic optimal controls such that they are given by feedback laws.

2. Preliminaries

First, let us recall some results known from the deterministic case. For more details, see [3, Part IV]. Let $G \subset \mathbb{R}^n$ be an open and bounded region with a smooth boundary ∂G and $T > 0$. We introduce the following controlled parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t,x) = \Delta y(t,x) + b(x)u(t,x) & (t,x) \in (0,T) \times G, \\ \quad y(0,x) = \xi(x) & x \in G, \\ \dfrac{\partial}{\partial \nu} y(t,x) = v(t,x) & (t,x) \in (0,T) \times \partial G, \end{cases} \tag{1}$$

where $\xi \in L^2(G)$ is the initial value, $u \in L^2([0,T] \times G)$ and $v \in L^2([0,T] \times \partial G)$ are the controls and $b \in L^\infty(G)$. The operator Δ is the Laplace operator in $L^2(G)$ and ν represents the outward normal to ∂G.

Let the space $H^2(G)$ denotes the Sobolev space restricted to square integrable functions with respect to the region G. In order to state the solution of system (1), we introduce the Neumann realization of the Laplace operator $A \colon D(A) \subset L^2(G) \to L^2(G)$ by

$$D(A) = \left\{ h \in H^2(G) : \frac{\partial}{\partial \nu} h = 0 \text{ on } \partial G \right\}, \quad \text{for } h \in D(A) : Ah = \Delta h.$$

Next, we define the Neumann operator $N \colon L^2(\partial G) \to L^2(G)$ by $g = Nh$ with

$$\Delta g(x) = \lambda g(x) \quad \text{for } x \in G, \quad \frac{\partial}{\partial \nu} g(x) = h(x) \quad \text{for } x \in \partial G,$$

where $\lambda > 0$. Before we specify the solution, let us give some remarks concerning the properties of the operators. Clearly, the domain $D(A)$ is dense in $L^2(G)$ and A is nonpositive and self adjoint in $L^2(G)$. Thus, the operator A is the infinitesimal generator of an analytic semigroup $\left(e^{At}\right)_{t \geq 0}$, see [11, Chapter II, Section 4]. Moreover by the Lumer-Phillips theorem, the analytic semigroup is a contraction semigroup. In [22, Chapter 1, Section 8 and 9], the result $N \in \mathcal{L}\left(L^2(\partial G); H^{3/2}(G)\right)$ was proven. Furthermore by [25], we have for all fixed $\lambda > 0$:

$$D((\lambda - A)^\alpha) = \begin{cases} H^{2\alpha} & \text{if } \alpha \in \left(0, \frac{3}{4}\right), \\ \left\{ h \in H^{2\alpha}(G) : \frac{\partial}{\partial \nu} h = 0 \text{ on } \partial G \right\} & \text{if } \alpha \in \left(\frac{3}{4}, 1\right), \end{cases}$$

where $(\lambda - A)^\alpha$ denotes the fractional power operator of $\lambda - A$. For a definition see [27, Section 2.6]. Therefore, we conclude $N \in \mathcal{L}(L^2(\partial G); D((\lambda - A)^\alpha))$ if $\alpha \in \left(0, \frac{3}{4}\right)$ and by the closed graph theorem, the operator $(\lambda - A)^\alpha N$ is linear and bounded. Based on [27, Section 2.6], we have for any $\beta \geq 0$ and any $\gamma, \delta \in \mathbb{R}$:

(a) the operator $(\lambda - A)^\beta$ is closed;

(b) $e^{At} \colon L^2(G) \to D\left((\lambda - A)^\beta\right)$ if $t > 0$;

(c) $D\left((\lambda - A)^{\beta}\right)$ is dense in $L^2(G)$;

(d) $(\lambda - A)^{\beta} e^{At} h = e^{At}(\lambda - A)^{\beta} h$ if $h \in D\left((\lambda - A)^{\beta}\right)$;

(e) if $t > 0$, the operator $(\lambda - A)^{\beta} e^{At}$ is bounded and we have for every $h \in L^2(G)$

$$\|(\lambda - A)^{\beta} e^{At} h\|_{L^2(G)} \leq \frac{M_{\beta}}{t^{\beta}} \|h\|_{L^2(G)};$$

(f) $(\lambda - A)^{\gamma+\delta} h = (\lambda - A)^{\gamma}(\lambda - A)^{\delta} h$ if $h \in D\left((\lambda - A)^{\varepsilon}\right)$, where $\varepsilon = \max\{\gamma, \delta, \gamma + \delta\}$.

One can show that system (1) has a unique solution $y \in C([0,T]; L^2(G))$ given by

$$y(t) = e^{At}\xi + \int_0^t e^{A(t-s)} Bu(s) ds + \int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N v(s) ds, \tag{2}$$

where $y(t)(x) = y(t,x)$, $Bu(t)(x) = b(x)u(t,x)$ and $v(t)(x) = v(t,x)$. For more details about abstract functions see [30, Section 3.4].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Equation (2) gives a motivation on how to involve noise terms inside the region G as well as on the boundary ∂G. Therefore, the process $(y(t))_{t\in[0,T]}$ satisfying for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$\begin{aligned} y(t) = e^{At}\xi &+ \int_0^t e^{A(t-s)} Bu(s) ds + \int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N v(s) ds + \int_0^t e^{A(t-s)} dW_1(s) \\ &+ \int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N dW_2(s) \end{aligned}$$

seems to be a good candidate for a solution of the heat equation affected by noise terms. We will assume, that the noise terms are given by Q-Wiener processes. Therefore, we recall briefly the definition in general Hilbert spaces. For more details, see [8, Part I, Chapter 4]. Let H be a separable Hilbert space and let $\mathcal{L}(H)$ be the space of linear and bounded operators defined on H. We assume that $Q \in \mathcal{L}(H)$ is a symmetric and nonnegative operator such that $\operatorname{Tr} Q < \infty$. Then, there exists a complete orthonormal system $(e_k)_{k\in\mathbb{N}}$ in H and a bounded sequence of nonnegative real numbers $(\mu_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$

$$Qe_k = \mu_k e_k.$$

We get the following definition.

Definition 1. *A stochastic process $(W(t))_{t\in[0,T]}$ with values in H is called a Q-Wiener process if*

- $W(0) = 0$*;*
- $(W(t))_{t\in[0,T]}$ *has continuous trajectories;*
- $(W(t))_{t\in[0,T]}$ *has independent increments;*
- *the distribution of $W(t) - W(s)$ is a Gaussian measure with mean 0 and covariance $(t-s)Q$ for $0 \leq s \leq t \leq T$.*

3. Properties of the solution of the state equation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We consider an $L^2(G)$-valued stochastic process $(y(t))_{t\in[0,T]}$, which fulfills the following SPDE:

$$\begin{cases} dy(t) = [Ay(t) + Bu(t) + (\lambda - A)Nv(t)]\,dt + dW_1(t) + (\lambda - A)NdW_2(t), \\ y(0) = \xi, \end{cases} \tag{3}$$

where the initial value $\xi \in L^2(\Omega; L^2(G))$ is an $\mathcal{F}_0$-measurable random variable. The stochastic processes $(W_1(t))_{t\in[0,T]}$ and $(W_2(t))_{t\in[0,T]}$ are Q-Wiener processes with values in $L^2(G)$ and $L^2(\partial G)$, respectively. Moreover, the processes are assumed to be independent and $\mathcal{F}_t$-adapted. Let $u \in U$ and $v \in V$ denote the controls, where

$$U = \left\{u \in L^2(\Omega; L^2([0,T]; L^2(G))) : (u(t))_{t\in[0,T]} \text{ is } \mathcal{F}_t\text{-adapted}\right\},$$
$$V = \left\{v \in L^2(\Omega; L^2([0,T]; L^2(\partial G))) : (v(t))_{t\in[0,T]} \text{ is } \mathcal{F}_t\text{-adapted}\right\}.$$

Furthermore, we assume that:

(i) the operator $A\colon D(A) \subset L^2(G) \to L^2(G)$ is the infinitesimal generator of an analytic semigroup $\left(e^{At}\right)_{t\geq 0}$ of contractions. Moreover, A is a self adjoint operator and $D(A)$ is dense in $L^2(G)$;

(ii) $B \in \mathcal{L}(L^2(G))$;

(iii) $N \in \mathcal{L}(L^2(\partial G); D((\lambda - A)^\alpha))$ if $\alpha \in \left(0, \frac{3}{4}\right)$ and $\lambda > 0$, where $(\lambda - A)^\alpha$ denotes the fractional power operator of $\lambda - A$.

Before we define a solution of system (3), a remark on these assumptions is in order.

Remark 1. *It seems to be a restriction, that the controls are $\mathcal{F}_t$-adapted. In the following, we will show that optimal controls satisfy this property. Therefore, this assumption is reasonable. Furthermore, the properties of the operators arise from the definitions in Section 2.*

Next, let $\mathcal{L}_2(K; H)$ be the space of Hilbert-Schmidt operators mapping from the Hilbert space K into another Hilbert space H. We denote the norm of $\mathcal{L}_2(K; H)$ by $\|\cdot\|_{\mathcal{L}_2(K;H)}$. Moreover, let the operators $Q_1 \in \mathcal{L}(L^2(G))$ and $Q_2 \in \mathcal{L}(L^2(\partial G))$ be the kernel covariance operators of the processes $(W_1(t))_{t\in[0,T]}$ or $(W_2(t))_{t\in[0,T]}$, respectively. Using [28, Proposition 2.3.4], there exist unique operators $Q_1^{1/2} \in \mathcal{L}(L^2(G))$ and $Q_2^{1/2} \in \mathcal{L}(L^2(\partial G))$ such that

$$Q_1^{1/2} \circ Q_1^{1/2} = Q_1, \qquad\qquad Q_2^{1/2} \circ Q_2^{1/2} = Q_2.$$

Therefore, we have the following definition.

Definition 2. *We say $(y(t))_{t\in[0,T]}$ is a mild solution of system (3) if*

$$\mathbb{P}\left(\int_0^T \left\|e^{A(T-s)}Bu(s) + (\lambda - A)^{1-\alpha}e^{A(T-s)}(\lambda - A)^\alpha Nv(s)\right\|_{L^2(G)} ds < \infty\right) = 1,$$

$$\int_0^T \left\|e^{A(T-s)}\right\|^2_{\mathcal{L}_2\left(Q_1^{1/2}(L^2(G)); L^2(G)\right)} ds < \infty,$$

$$\int_0^T \left\|(\lambda - A)^{1-\alpha}e^{A(T-s)}(\lambda - A)^\alpha N\right\|^2_{\mathcal{L}_2\left(Q_2^{1/2}(L^2(\partial G)); L^2(G)\right)} ds < \infty$$

and for arbitrary $t \in [0,T]$, we have $\mathbb{P}$-a.s.

$$y(t) = e^{At}\xi + \int\limits_0^t e^{A(t-s)}Bu(s)ds + \int\limits_0^t (\lambda - A)^{1-\alpha}e^{A(t-s)}(\lambda - A)^{\alpha}Nv(s)ds + \int\limits_0^t e^{A(t-s)}dW_1(s)$$
$$+ \int\limits_0^t (\lambda - A)^{1-\alpha}e^{A(t-s)}(\lambda - A)^{\alpha}NdW_2(s). \tag{4}$$

Although the following theorem was proven in the case of real valued Brownian motions, see [12], we show the existence and uniqueness of the mild solution of system (3) to obtain inequalities we use later.

Theorem 1. *Let $u \in U$ and $v \in V$ be fixed. If $\alpha \in \left(\frac{1}{2}, \frac{3}{4}\right)$, then there exists a unique mild solution of system (3) satisfying $y \in C([0,T]; L^2(\Omega; L^2(G)))$ for any $\xi \in L^2(\Omega; L^2(G))$. Furthermore, if $\beta \in \left[0, \alpha - \frac{1}{2}\right)$, then $\mathbb{P}$-a.s. and for arbitrary $t \in (0,T]$, we have $y(t) \in D\left((\lambda - A)^{\beta}\right)$.*

Proof. Since $\left(e^{At}\right)_{t\geq 0}$ is a contraction semigroup and $B \in \mathcal{L}(L^2(G))$, there exists a constant $C_1 > 0$ such that the following inequality holds $\mathbb{P}$-a.s.:

$$\int\limits_0^T \left\| e^{A(T-s)}Bu(s) \right\|_{L^2(G)} ds \leq C_1 \int\limits_0^T \|u(s)\|_{L^2(G)} ds.$$

By the Cauchy–Schwarz inequality and the properties of fractional power operators provided in Section 2, we obtain $\mathbb{P}$-a.s. and for any $\alpha \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$\left(\int\limits_0^T \left\| (\lambda - A)^{1-\alpha}e^{A(T-s)}(\lambda - A)^{\alpha}Nv(s) \right\|_{L^2(G)} ds \right)^2 \leq \frac{C_2 T^{2\alpha-1}}{2\alpha - 1} \int\limits_0^T \|v(s)\|^2_{L^2(\partial G)} ds,$$

where $C_2 > 0$ depends on α and the operator $(\lambda - A)^{\alpha}N$. Using again that $\left(e^{At}\right)_{t\geq 0}$ is a contraction semigroup and the properties of fractional power operators, we get for any $\alpha \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$\int\limits_0^T \left\| e^{A(T-s)} \right\|^2_{\mathcal{L}_2\left(Q_1^{1/2}(L^2(G)); L^2(G)\right)} ds \leq \widetilde{C}_1 T,$$

$$\int\limits_0^T \left\| (\lambda - A)^{1-\alpha}e^{A(T-s)}(\lambda - A)^{\alpha}N \right\|^2_{\mathcal{L}_2\left(Q_2^{1/2}(L^2(\partial G)); L^2(G)\right)} ds \leq \frac{\widetilde{C}_2 T^{2\alpha-1}}{2\alpha - 1},$$

where $\widetilde{C}_1 > 0$ depends on the kernel covariance operator Q_1 and $\widetilde{C}_2 > 0$ depends on the kernel covariance operator Q_2 and the operator $(\lambda - A)^{\alpha}N$. Therefore, the process $(y(t))_{t\in[0,T]}$ given by (4) is well defined.

Next, we show that $y(t)$ is square integrable for arbitrary $t \in [0,T]$. By the Cauchy–Schwarz inequality and properties of Bochner integrals and stochastic integrals, we get similarly to the inequalities above for arbitrary $t \in [0,T]$ and any $\alpha \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$\mathbb{E}\|e^{At}\xi\|^2_{L^2(G)} \leq \mathbb{E}\|\xi\|^2_{L^2(G)},$$

$$\mathbb{E}\left\| \int\limits_0^t e^{A(t-s)}Bu(s)ds \right\|^2_{L^2(G)} \leq C_1^2 T \,\mathbb{E} \int\limits_0^T \|u(s)\|^2_{L^2(G)} ds,$$

$$\mathbb{E}\left\|\int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N v(s) ds\right\|_{L^2(G)}^2 \leq \frac{C_2 T^{2\alpha-1}}{2\alpha - 1} \mathbb{E} \int_0^T \|v(s)\|_{L^2(\partial G)}^2 \, ds,$$

$$\mathbb{E}\left\|\int_0^t e^{A(t-s)} dW_1(s)\right\|_{L^2(G)}^2 \leq \widetilde{C}_1 T,$$

$$\mathbb{E}\left\|\int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N dW_2(s)\right\|_{L^2(G)}^2 \leq \frac{\widetilde{C}_2 T^{2\alpha-1}}{2\alpha - 1}.$$

Hence, there exists a constant $K > 0$ such that for arbitrary $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\|y(t)\|_{L^2(G)}^2 &\leq 5\mathbb{E}\|e^{At}\xi\|_{L^2(G)}^2 + 5\mathbb{E}\left\|\int_0^t e^{A(t-s)} Bu(s) ds\right\|_{L^2(G)}^2 \\
&\quad + 5\mathbb{E}\left\|\int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N v(s) ds\right\|_{L^2(G)}^2 \\
&\quad + 5\mathbb{E}\left\|\int_0^t e^{A(t-s)} dW_1(s)\right\|_{L^2(G)}^2 + 5\mathbb{E}\left\|\int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N dW_2(s)\right\|_{L^2(G)}^2 \\
&\leq K\left(1 + \mathbb{E}\|\xi\|_{L^2(G)}^2 + \mathbb{E}\int_0^T \|u(s)\|_{L^2(G)}^2 \, ds + \mathbb{E}\int_0^T \|v(s)\|_{L^2(\partial G)}^2 \, ds\right).
\end{aligned}$$

Therefore, we have $y(t) \in L^2(\Omega; L^2(G))$ for arbitrary $t \in [0, T]$.

Obviously, the process $(z(t))_{t\in[0,T]}$ satisfying for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$z(t) = e^{At}\xi + \int_0^t e^{A(t-s)} Bu(s) ds + \int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N v(s) ds$$

is continuous in mean square. Based on the results in [3, Part IV], the process $(z(t))_{t\in[0,T]}$ has continuous trajectories. To prove the continuity of the stochastic convolutions, we first note that

$$\int_0^T s^{-\gamma} \left\|e^{As}\right\|_{\mathcal{L}_2\left(Q_1^{1/2}(L^2(G)); L^2(G)\right)}^2 ds + \int_0^T s^{-\gamma} \left\|(\lambda - A)^{1-\alpha} e^{As} (\lambda - A)^{\alpha} N\right\|_{\mathcal{L}_2\left(Q_2^{1/2}(L^2(\partial G)); L^2(G)\right)}^2 ds < \infty$$

hold for $\gamma \in (0, 2\alpha - 1)$. By [8, Chapter 5], the process $(I_1(t))_{t\in[0,T]}$ satisfying for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$I_1(t) = \int_0^t e^{A(t-s)} dW_1(s)$$

is continuous in mean square and has a continuous version. The continuity of the process $(I_2(t))_{t\in[0,T]}$ satisfying for all $t \in [0, T]$ and $\mathbb{P}$-a.s.

$$I_2(t) = \int_0^t (\lambda - A)^{1-\alpha} e^{A(t-s)} (\lambda - A)^{\alpha} N dW_2(s)$$

results from the argumentation as in [6, Theorem 2.3]. Thus, the process $(y(t))_{t\in[0,T]}$ is continuous in mean square and has continuous trajectories.

Finally, we prove that for any $\beta \in \left[0, \alpha - \frac{1}{2}\right)$, all $t \in (0,T]$ and $\mathbb{P}$-a.s. $y(t) \in D\left((\lambda - A)^{\beta}\right)$. By the properties of the fractional power operators and the assumptions on the operators B and N, we obtain for arbitrary $t \in [0,T]$

$$\mathbb{E}\int_0^t \left\|(\lambda - A)^{\beta} e^{A(t-s)} B u(s)\right\|_{L^2(G)} ds + \mathbb{E}\int_0^t \left\|(\lambda - A)^{\beta}(\lambda - A)^{1-\alpha} e^{A(t-s)}(\lambda - A)^{\alpha} N v(s)\right\|_{L^2(G)} ds < \infty,$$

$$\int_0^t \left\|(\lambda - A)^{\beta} e^{A(t-s)}\right\|^2_{\mathcal{L}_2\left(Q_1^{1/2}(L^2(G)); L^2(G)\right)} ds < \infty,$$

$$\int_0^t \left\|(\lambda - A)^{\beta}(\lambda - A)^{1-\alpha} e^{A(t-s)}(\lambda - A)^{\alpha} N\right\|^2_{\mathcal{L}_2\left(Q_2^{1/2}(L^2(\partial G)); L^2(G)\right)} ds < \infty.$$

Moreover, we have $\mathbb{P}$-a.s. $e^{At}\xi \in D\left((\lambda - A)^{\beta}\right)$ for all $t \in (0,T]$. Since the operator $(\lambda - A)^{\beta}$ is closed, we get $\mathbb{P}$-a.s. $y(t) \in D\left((\lambda - A)^{\beta}\right)$ for all $t \in (0,T]$. □

In the following corollary, we give properties of the mild solution of system (3), which we will use in the next section.

Corollary 1. *Let $y(\cdot;\xi,u,v) \in C([0,T]; L^2(\Omega; L^2(G)))$ be given by (4). Then we have for all $t \in [0,T]$ and $\mathbb{P}$-a.s.:*

(a) *$y(t;\xi,u,v)$ is affine linear in both u and v;*

(b) *$\mathbb{E}\|y(t;\xi_1,u,v) - y(t;\xi_2,u,v)\|^2_{L^2(G)} \le \mathbb{E}\|\xi_1 - \xi_2\|^2_{L^2(G)}$ for every $\xi_1, \xi_2 \in L^2(\Omega; L^2(G))$, which are $\mathcal{F}_0$-measurable;*

(c) *$\mathbb{E}\|y(t;\xi,u_1,v) - y(t;\xi,u_2,v)\|^2_{L^2(G)} \le C_1^2 T \mathbb{E}\|u_1 - u_2\|^2_{L^2([0,T];L^2(G))}$ for every $u_1, u_2 \in U$;*

(d) *$\mathbb{E}\|y(t;\xi,u,v_1) - y(t;\xi,u,v_2)\|^2_{L^2(G)} \le \frac{C_2 T^{2\alpha-1}}{2\alpha-1}\, \mathbb{E}\|v_1 - v_2\|^2_{L^2([0,T];L^2(\partial G))}$ for every $v_1, v_2 \in V$.*

Proof. The proof of this corollary is easily done by using inequalities we have proved in the previous theorem. □

4. Derivation of stochastic optimal controls

First, we introduce the cost functional as follows:

$$J(\xi,u,v) = \frac{1}{2}\mathbb{E}\|y(T) - \hat{y}\|^2_{L^2(G)} + \frac{\kappa_1}{2}\mathbb{E}\|u\|^2_{L^2([0,T];L^2(G))} + \frac{\kappa_2}{2}\mathbb{E}\|v\|^2_{L^2([0,T];L^2(\partial G))}, \tag{5}$$

where $(y(t))_{t\in[0,T]}$ is the mild solution of system (3) and $(u,v) \in U \times V$. Moreover, the function $\hat{y} \in L^2(G)$ is called the target function and $\kappa_1, \kappa_2 \ge 0$ are weights. Using Corollary 1, we conclude that for fixed ξ the functional J is convex, continuous and coercive in both u and v. Hence, the cost functional is bounded from below and attains a minimizer for fixed ξ. This minimizer is unique if and only if J is strictly convex (if and only if $\kappa_1, \kappa_2 > 0$). Therefore, we can introduce the value function

$$\psi(\xi) = \inf_{u\in U, v\in V} J(\xi,u,v) \tag{6}$$

and we have the following definition.

Definition 3. *The control* $(\overline{u},\overline{v}) \in U \times V$ *is called an optimal control if the infimum in (6) is attained, i.e.,* $\overline{u}$ *and* $\overline{v}$ *satisfy*

$$\psi(\xi) = J(\xi,\overline{u},\overline{v}).$$

The state corresponding to the optimal controls $\overline{u}$ *and* $\overline{v}$ *is called the optimal state and will be denoted by* $\overline{y} \in C([0,T];L^2(\Omega;L^2(G)))$.

Since the cost functional J is a sum of squared norms, it is also Fréchet differentiable. Thus, we can state necessary optimality conditions by calculating the Fréchet derivatives of J with respect to u and v denoted by d_uJ or d_vJ respectively. Using the chain rule, we get

$$d_uJ(\xi,u,v)[h_1] = \mathbb{E}\left\langle y(T)-\hat{y}, \int_0^T e^{A(T-t)}Bh_1(t)dt \right\rangle_{L^2(G)} + \kappa_1 \mathbb{E}\left\langle u,h_1\right\rangle_{L^2([0,T];L^2(G))}, \tag{7}$$

$$\begin{aligned} d_vJ(\xi,u,v)[h_2] =& \mathbb{E}\left\langle y(T)-\hat{y}, \int_0^T (\lambda-A)^{1-\alpha}e^{A(T-t)}(\lambda-A)^{\alpha}Nh_2(t)dt \right\rangle_{L^2(G)} \\ &+ \kappa_2 \mathbb{E}\left\langle v,h_2\right\rangle_{L^2([0,T];L^2(\partial G))}, \end{aligned} \tag{8}$$

where $h_1 \in U$ and $h_2 \in V$. As the controls are not constrained, the necessary optimality conditions are

$$d_uJ(\xi,\overline{u},\overline{v})[h_1] = 0, \tag{9}$$

$$d_vJ(\xi,\overline{u},\overline{v})[h_2] = 0 \tag{10}$$

for every $h_1 \in U$ and every $h_2 \in V$. These optimality conditions are sufficient if $\kappa_1,\kappa_2 > 0$. For more details about optimization problems in Hilbert spaces see [19].

Remark 2. *The cost functional given by (5) is a generalization of the following two cases:*

- *If* $\kappa_1 > 0$ *and* $\kappa_2 = 0$, *then we consider a control problem with distributed controls.*
- *If* $\kappa_1 = 0$ *and* $\kappa_2 > 0$, *then we consider a control problem with boundary controls.*

Note also that the properties of the cost functional heavily depend on the choice of the parameters κ_1 *and* κ_2. *In the following, we assume* $\kappa_1,\kappa_2 > 0$ *in order to obtain that the optimal controls are unique. Moreover, we get that the optimality conditions (9) and (10) are necessary and sufficient.*

Using equations (9) and (10), we can derive explicit formulas optimal controls have to satisfy.

Theorem 2. *Let the cost functional be given by (5). Then the optimal control inside the region satisfies a.e. on* $[0,T]$ *and* $\mathbb{P}$*-a.s.*

$$\overline{u}(t) = -\frac{1}{\kappa_1}B^* e^{A(T-t)}\left(\mathbb{E}\left[\overline{y}(T)|\mathcal{F}_t\right]-\hat{y}\right), \tag{11}$$

where $B^* \in \mathcal{L}(L^2(G))$ *denotes the adjoint operator of* B.

Proof. Using Fubini's theorem and the fact that the operator A is self adjoint, we obtain for every $h_1 \in U$

$$\begin{aligned}
\mathbb{E}\left\langle y(T)-\hat{y}, \int_0^T e^{A(T-t)}Bh_1(t)dt\right\rangle_{L^2(G)} &= \mathbb{E}\left[\int_G (y(T,x)-\hat{y}(x))\int_0^T e^{A(T-t)}Bh_1(t)(x)dt\,dx\right] \\
&= \int_0^T\int_G \mathbb{E}\left[\mathbb{E}\left[(y(T,x)-\hat{y}(x))e^{A(T-t)}Bh_1(t)(x)\Big|\mathcal{F}_t\right]\right]dx\,dt \\
&= \mathbb{E}\int_0^T\int_G \left(\mathbb{E}\left[y(T,x)|\mathcal{F}_t\right]-\hat{y}(x)\right)e^{A(T-t)}Bh_1(t)(x)dx\,dt \\
&= \mathbb{E}\int_0^T \left\langle \mathbb{E}\left[y(T)|\mathcal{F}_t\right]-\hat{y}, e^{A(T-t)}Bh_1(t)\right\rangle_{L^2(G)} dt \\
&= \mathbb{E}\int_0^T \left\langle B^*e^{A(T-t)}\left(\mathbb{E}\left[y(T)|\mathcal{F}_t\right]-\hat{y}\right), h_1(t)\right\rangle_{L^2(G)} dt.
\end{aligned}$$

By equation (7), we get

$$d_uJ(\xi,u,v)[h_1] = \mathbb{E}\int_0^T \left\langle B^*e^{A(T-t)}\left(\mathbb{E}\left[y(T)\big|\mathcal{F}_t\right]-\hat{y}\right)+\kappa_1 u(t), h_1(t)\right\rangle_{L^2(G)} dt.$$

Using condition (9), the optimal control inside the region satisfies a.e. on $[0,T]$ and $\mathbb{P}$-a.s.

$$\overline{u}(t) = -\frac{1}{\kappa_1}B^*e^{A(T-t)}\left(\mathbb{E}\left[\overline{y}(T)\big|\mathcal{F}_t\right]-\hat{y}\right).$$

□

To prove an explicit formula the optimal control on the boundary satisfies, we need the following lemma.

Lemma 1. *Let $A\colon D(A)\subset L^2(G)\to L^2(G)$ be a self adjoint operator and an infinitesimal generator of an analytic semigroup. Moreover, let $\gamma\in(0,1)$ be fixed, $\beta\in(\gamma,1)$ and $\lambda>0$. If either $f\in D(A)$, $g\in D\left((\lambda-A)^\beta\right)$ or $f\in D\left((\lambda-A)^\beta\right)$, $g\in D(A)$, then we have*

$$\langle(\lambda-A)^\gamma f, g\rangle_{L^2(G)} = \langle f, (\lambda-A)^\gamma g\rangle_{L^2(G)}.$$

Proof. If $f\in D(A)$ and $g\in D\left((\lambda-A)^\beta\right)$, then by [27, Section 2.6, Theorem 6.9], we have the following formula:

$$(\lambda-A)^\gamma f = \frac{\sin(\pi\gamma)}{\pi}\int_0^\infty z^{\gamma-1}(\lambda-A)(z+\lambda-A)^{-1}f\,dz. \tag{12}$$

We set $R(\mu:A)=(\mu-A)^{-1}$, where μ is an element of the resolvent set of A. The operator $R(\mu:A)$ is called the resolvent of A. Based on the properties of the resolvent, we have

$$R(\mu:A)A = AR(\mu:A), \tag{13}$$

$$R(\mu:A)^* = R(\mu:A^*) = R(\mu:A). \tag{14}$$

Thus, the resolvent of a self-adjoint operator is again self-adjoint. For more details about the resolvent see [11]. By equations (12) and (13), we find for every $f \in D(A)$

$$(\lambda - A)^{\gamma} f = \frac{\sin(\pi\gamma)}{\pi} \int_0^{\infty} z^{\gamma-1} R(z+\lambda : A)(\lambda - A) f \, dz. \tag{15}$$

Using this representation and the properties of the Bochner integral, we obtain

$$\langle (\lambda - A)^{\gamma} f, g \rangle_{L^2(G)} = \int_0^{\infty} \left\langle \frac{\sin(\pi\gamma)}{\pi} z^{\gamma-1} R(z+\lambda : A)(\lambda - A) f, g \right\rangle_{L^2(G)} dz. \tag{16}$$

Since the operators are self adjoint, we get for all $z \geq 0$

$$\left\langle \frac{\sin(\pi\gamma)}{\pi} z^{\gamma-1} R(z+\lambda : A)(\lambda - A) f, g \right\rangle_{L^2(G)} = \left\langle f, \frac{\sin(\pi\gamma)}{\pi} z^{\gamma-1} (\lambda - A) R(z+\lambda : A) g \right\rangle_{L^2(G)}.$$

Substituting this equality in (16) and using equation (12), we have

$$\langle (\lambda - A)^{\gamma} f, g \rangle_{L^2(G)} = \langle f, (\lambda - A)^{\gamma} g \rangle_{L^2(G)}.$$

If $f \in D\left((\lambda - A)^{\beta}\right)$ and $g \in D(A)$, we get similarly

$$\langle (\lambda - A)^{\gamma} f, g \rangle_{L^2(G)} = \left\langle f, \int_0^{\infty} \frac{\sin(\pi\gamma)}{\pi} z^{\gamma-1} R(z+\lambda : A)(\lambda - A) g \, dz \right\rangle_{L^2(G)}.$$

By equation (15), we find

$$\langle (\lambda - A)^{\gamma} f, g \rangle_{L^2(G)} = \langle f, (\lambda - A)^{\gamma} g \rangle_{L^2(G)}.$$

□

We are now able to state an analytical expression of the boundary control.

Theorem 3. *Let the cost functional be given by (5). Then the optimal control on the boundary satisfies a.e. on* $[0,T]$ *and* $\mathbb{P}$*-a.s.*

$$\overline{v}(t) = -\frac{1}{\kappa_2} \mathcal{G}^* (\lambda - A)^{1-\alpha} e^{A(T-t)} \left(\mathbb{E}\left[\overline{y}(T) | \mathcal{F}_t \right] - \hat{y} \right), \tag{17}$$

where $\mathcal{G}^* \in \mathcal{L}(L^2(G); L^2(\partial G))$ *denotes the adjoint operator of* $\mathcal{G} = (\lambda - A)^{\alpha} N$.

Proof. First, we prove the existence of an approximating sequence $(\tilde{y}_i(T))_{i \in \mathbb{N}} \subset L^2(\Omega; D(A))$ for the random variable $y(T) - \hat{y} \in L^2(\Omega; L^2(G))$. Let z be a $L^2(G)$-valued simple random variable, i.e., there exist functions $f_j \in L^2(G)$ for $j = 1, 2, ..., N$ such that $\mathbb{P}$-a.s.

$$z = \sum_{j=1}^{N} f_j \mathbb{1}_{\mathcal{A}_j},$$

where $\mathbb{1}_{\mathcal{A}_j}$ denotes the indicator function of $\mathcal{A}_j \in \mathcal{F}$. Since $D(A)$ is dense in $L^2(G)$, there exists a sequence $\left(f_j^i\right)_{i \in \mathbb{N}} \subset D(A)$ for every $j \in \{1, 2, ..., N\}$ such that

$$\left\| f_j - f_j^i \right\|_{L^2(G)} \to 0 \quad \text{as } i \to \infty.$$

We set $\mathbb{P}$-a.s. $z_i = \sum_{j=1}^{N} f_j^i \mathbb{1}_{\mathcal{A}_j}$. Then we obtain

$$\mathbb{E} \left\| z - z_i \right\|^2_{L^2(G)} \to 0 \quad \text{as } i \to \infty.$$

Furthermore, it is well known that every random variable with values in $L^2(G)$ can be approximated by a sequence of $L^2(G)$-valued simple random variables, see [8, Lemma 1.3]. Therefore, we conclude that for $y(T) - \hat{y} \in L^2(\Omega; L^2(G))$ there exists a sequence $(\tilde{y}_i(T))_{i \in \mathbb{N}} \subset L^2(\Omega; D(A))$ such that

$$\mathbb{E}\| y(T) - \hat{y} - \tilde{y}_i(T) \|^2_{L^2(G)} \to 0 \quad \text{as } i \to \infty.$$

Using Fubini's theorem and Lemma 1, we have for every $\tilde{y}_i(T) \in L^2(\Omega; D(A))$ and every $h_2 \in V$

$$\begin{aligned}
&\mathbb{E} \left\langle \tilde{y}_i(T), \int_0^T (\lambda - A)^{1-\alpha} e^{A(T-t)} (\lambda - A)^{\alpha} N h_2(t) dt \right\rangle_{L^2(G)} \\
&= \int_0^T \int_G \mathbb{E} \left[\tilde{y}_i(T,x) (\lambda - A)^{1-\alpha} e^{A(T-t)} (\lambda - A)^{\alpha} N h_2(t)(x) \right] dx\, dt \\
&= \int_0^T \int_G \mathbb{E} \left[\mathbb{E} \left[\tilde{y}_i(T,x) (\lambda - A)^{1-\alpha} e^{A(T-t)} (\lambda - A)^{\alpha} N h_2(t)(x) \middle| \mathcal{F}_t \right] \right] dx\, dt \\
&= \mathbb{E} \int_0^T \left\langle \mathbb{E} \left[\tilde{y}_i(T) | \mathcal{F}_t \right], (\lambda - A)^{1-\alpha} e^{A(T-t)} (\lambda - A)^{\alpha} N h_2(t) \right\rangle_{L^2(G)} dt \\
&= \mathbb{E} \int_0^T \left\langle \mathcal{G}^* (\lambda - A)^{1-\alpha} e^{A(T-t)} \mathbb{E} \left[\tilde{y}_i(T) | \mathcal{F}_t \right], h_2(t) \right\rangle_{L^2(\partial G)} dt.
\end{aligned} \tag{18}$$

Next, let the operator $\mathcal{M}(t) \colon L^2(G) \to L^2(\partial G)$ be defined by $\mathcal{M}(t) = \mathcal{G}^*(\lambda - A)^{1-\alpha} e^{A(T-t)}$ for all $t \in (0,T]$. Since the operator $\mathcal{G}^*$ is linear and bounded and using the properties of the fractional power operator provided in Section 2, the operator $\mathcal{M}(t)$ is linear and there exists a constant $C > 0$ such that for all $t \in (0,T]$ and every $g \in L^2(G)$

$$\| \mathcal{M}(t) g \|_{L^2(\partial G)} \leq C M_{1-\alpha} (T-t)^{\alpha - 1} \| g \|_{L^2(G)}. \tag{19}$$

By inequality (19) and Fubini's theorem, we obtain

$$\begin{aligned}
&\mathbb{E} \int_0^T \left\| \mathcal{M}(t) \left(\mathbb{E}\left[y(T) | \mathcal{F}_t \right] - \hat{y} \right) - \mathcal{M}(t) \mathbb{E} \left[\tilde{y}_i(T) | \mathcal{F}_t \right] \right\|^2_{L^2(\partial G)} dt \\
&\leq C^2 M_{1-\alpha}^2 \mathbb{E} \int_0^T (T-t)^{2\alpha - 2} \left\| \left(\mathbb{E}\left[y(T) | \mathcal{F}_t \right] - \hat{y} - \mathbb{E}\left[\tilde{y}_i(T) | \mathcal{F}_t \right] \right) \right\|^2_{L^2(G)} dt \\
&\leq C^2 M_{1-\alpha}^2 \int_0^T (T-t)^{2\alpha - 2} \mathbb{E} \left[\mathbb{E} \left[\| y(T) - \hat{y} - \tilde{y}_i(T) \|^2_{L^2(G)} \middle| \mathcal{F}_t \right] \right] dt \\
&= \frac{C^2 M_{1-\alpha}^2 T^{2\alpha - 1}}{2\alpha - 1} \mathbb{E} \left\| y(T) - \hat{y} - \tilde{y}_i(T) \right\|^2_{L^2(G)}.
\end{aligned}$$

Since $\tilde{y}_i(T) \to y(T) - \hat{y}$ in $L^2(\Omega; L^2(G))$ as $i \to \infty$, we conclude

$$\mathcal{M}(\cdot)\mathbb{E}\left[\tilde{y}_i(T)|\mathcal{F}_\cdot\right] \to \mathcal{M}(\cdot)\left(\mathbb{E}\left[y(T)|\mathcal{F}_\cdot\right] - \hat{y}\right)$$

in $L^2(\Omega; L^2([0,T]; L^2(\partial G)))$ as $i \to \infty$. Therefore, we have for every $h_2 \in V$

$$\begin{aligned}\mathbb{E}\left\langle y(T) - \hat{y}, \int_0^T (\lambda - A)^{1-\alpha} e^{A(T-t)} (\lambda - A)^\alpha N h_2(t) dt \right\rangle_{L^2(G)} &= \lim_{i\to\infty} \mathbb{E} \int_0^T \left\langle \mathcal{M}(t)\mathbb{E}\left[\tilde{y}_i|\mathcal{F}_t\right], h_2(t)\right\rangle_{L^2(\partial G)} dt \\ &= \mathbb{E} \int_0^T \left\langle \mathcal{M}(t)\left(\mathbb{E}\left[y(T)|\mathcal{F}_t\right] - \hat{y}\right), h_2(t)\right\rangle_{L^2(\partial G)} dt.\end{aligned}$$

Using equation (8), we find for every $h_2 \in V$

$$d_v J(\xi, u, v)[h_2] = \mathbb{E} \int_0^T \left\langle \mathcal{M}(t)\left(\mathbb{E}\left[y(T)|\mathcal{F}_t\right] - \hat{y}\right) + \kappa_2 v(t), h_2(t)\right\rangle_{L^2(\partial G)} dt.$$

Applying condition (10), we infer that the optimal control on the boundary satisfies a.e. on $[0,T]$ and $\mathbb{P}$-a.s.

$$\overline{v}(t) = -\frac{1}{\kappa_2}\mathcal{M}(t)\left(\mathbb{E}[\overline{y}(T)|\mathcal{F}_t] - \hat{y}\right).$$

This implies (17) and proves the theorem. □

5. Stochastic optimal controls as feedback controls

Based on Theorems 2 and 3, the optimal controls can be determined by calculating $\mathbb{E}\left[\overline{y}(T)|\mathcal{F}_t\right]$. Since this leads to serious problems in applications, we avoid the calculation of the conditional expectation by using a martingale representation theorem according to the results given in [15, Section 2.2.5].

First, we apply [8, Proposition 4.3] to obtain a series expansion of the Q-Wiener processes $(W_1(t))_{t\in[0,T]}$ and $(W_2(t))_{t\in[0,T]}$. For $i = 1, 2$ and arbitrary $t \in [0,T]$, we have

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{\mu_{i,k}}\, w_{i,k}(t) e_{i,k},$$

where $(w_{i,k}(t))_{t\in[0,T]}$, $k \in \mathbb{N}$, are independent real valued Brownian motions. Moreover, the sequences $(e_{1,k})_{k\in\mathbb{N}}$ and $(e_{2,k})_{k\in\mathbb{N}}$ are complete orthonormal systems in $L^2(G)$ and $L^2(\partial G)$, respectively. For $i = 1, 2$, the sequence $(\mu_{i,k})_{k\in\mathbb{N}}$ is a bounded sequence of nonnegative real numbers such that $Q_i e_{i,k} = \mu_{i,k} e_{i,k}$ for all $k \in \mathbb{N}$. Let us denote $\mathcal{F}_t^i = \sigma\left\{\bigcup_{k=1}^{\infty} \sigma\{w_{i,k}(s) : 0 \le s \le t\}\right\}$ for $i = 1, 2$ and all $t \in [0,T]$, where $\sigma\{\cdot\}$ denotes the σ-algebra. Next, we define the filtration $\mathcal{F}_t = \sigma\left\{\mathcal{F}_t^1 \cup \mathcal{F}_t^2\right\}$ for all $t \in [0,T]$ and we set $\mathcal{F} = \mathcal{F}_T$. The process $(\mathbb{E}\left[\overline{y}(T)|\mathcal{F}_t\right])_{t\in[0,T]}$ is a square integrable $\mathcal{F}_t$-martingale, which is continuous in time. Thus, there exist predictable processes $(\Phi_1(t))_{t\in[0,T]}$ and $(\Phi_2(t))_{t\in[0,T]}$ with values in the spaces $\mathcal{L}_2\left(Q_1^{1/2}(L^2(G)); L^2(G)\right)$ and $\mathcal{L}_2\left(Q_2^{1/2}(L^2(\partial G)); L^2(G)\right)$, respectively, such that

$$\mathbb{E}\int_0^T \|\Phi_1(t)\|^2_{\mathcal{L}_2\left(Q_1^{1/2}(L^2(G)); L^2(G)\right)} dt < \infty,$$

$$\mathbb{E}\int_0^T \|\Phi_2(t)\|^2_{\mathcal{L}_2\left(Q_2^{1/2}(L^2(\partial G)); L^2(G)\right)} dt < \infty$$

and for arbitrary $t \in [0,T]$, we have $\mathbb{P}$-a.s.

$$\mathbb{E}\left[\overline{y}(T)|\mathcal{F}_t\right] = \mathbb{E}[\overline{y}(T)] + \int_0^t \Phi_1(s)dW_1(s) + \int_0^t \Phi_2(s)dW_2(s). \tag{20}$$

Let the process $(q(t))_{t\in[0,T]}$ satisfy for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$q(t) = e^{A(T-t)}(\mathbb{E}\left[\overline{y}(T)|\mathcal{F}_t\right] - \hat{y}). \tag{21}$$

Then by equation (20), we have for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$q(t) = e^{A(T-t)}\left(\mathbb{E}[\overline{y}(T)] - \hat{y} + \int_0^t \Phi_1(s)dW_1(s) + \int_0^t \Phi_2(s)dW_2(s)\right).$$

Next, we introduce the adjoint state $(p(t))_{t\in[0,T]}$ satisfying for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$p(t) = e^{A(T-t)}(\overline{y}(T) - \hat{y}) - \int_t^T e^{A(s-t)}\Phi_1^{(T)}(s)dW_1(s) - \int_t^T e^{A(s-t)}\Phi_2^{(T)}(s)dW_2(s),$$

where $\Phi_1^{(T)}(s) = e^{A(T-s)}\Phi_1(s)$ and $\Phi_2^{(T)}(s) = e^{A(T-s)}\Phi_2(s)$. Using equation (20), we obtain for all $t \in [0,T]$ and $\mathbb{P}$-a.s.

$$q(t) = \mathbb{E}\left[p(t)|\mathcal{F}_t\right] \tag{22}$$

and by equations (11) and (17), the stochastic optimal controls satisfy a.e. on $[0,T]$ and $\mathbb{P}$-a.s.

$$\overline{u}(t) = -\frac{1}{\kappa_1}B^*q(t), \tag{23}$$

$$\overline{v}(t) = -\frac{1}{\kappa_2}\mathcal{G}^*(\lambda - A)^{1-\alpha}q(t). \tag{24}$$

In the following, we derive a representation for the process $(q(t))_{t\in[0,T]}$ without calculating Φ_1 and Φ_2 explicitly. Therefore, we introduce the operator valued process $(\mathcal{P}(t))_{t\in[0,T]}$, which fulfills the following Riccati equation:

$$\begin{cases} \mathcal{P}'(t) = A\mathcal{P}(t) + \mathcal{P}(t)A - \frac{1}{\kappa_1}\mathcal{P}(t)BB^*\mathcal{P}(t) - \frac{1}{\kappa_2}\mathcal{H}^*(t)\mathcal{G}\mathcal{G}^*\mathcal{H}(t), \\ \mathcal{P}(T) = I, \end{cases} \tag{25}$$

where $\mathcal{H}(t) = (\lambda - A)^{1-\alpha}\mathcal{P}(t)$ and I is the identity operator in $L^2(G)$.

Definition 4. *We say $(\mathcal{P}(t))_{t\in[0,T]}$ is a mild solution of (25) if for arbitrary $t \in [0,T]$ and any $h \in L^2(G)$*

$$\begin{aligned}\mathcal{P}(t)h = e^{A(T-t)}e^{A(T-t)}h - \frac{1}{\kappa_1}\int_t^T e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)e^{A(s-t)}hds \\ - \frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}hds.\end{aligned} \tag{26}$$

Remark 3. *In [3, Part IV], distributed and boundary controls are considered separately. The existence and uniqueness of the mild solution for the corresponding Riccati equations are proved. Since equation (25) is a generalization of these special cases, an existence and uniqueness result can be easily obtained.*

In the following remark, we recall some important properties of the operator $\mathcal{P}(t)$.

Remark 4. *For $\alpha > \frac{1}{2}$, we have*

- $\mathcal{P}(t)h \in D((\lambda - A)^{1-\alpha})$ *for any* $h \in L^2(G)$ *and all* $t \in [0,T)$*;*
- $\mathcal{H} = (\lambda - A)^{1-\alpha}\mathcal{P} \in C([0,T); \mathcal{L}(L^2(G)))$*;*
- $\mathcal{P}(t) \in \mathcal{L}(L^2(G))$ *is self adjoint for all* $t \in [0,T]$.

To prove the representation of optimal controls as feedback controls, we need the following lemma.

Lemma 2. *Let $A\colon D(A) \subset L^2(G) \to L^2(G)$ be a self adjoint operator and an infinitesimal generator of an analytic semigroup. Moreover, let the process $(\mathcal{P}(t))_{t\in[0,T]}$ be the mild solution of system (25) and $\lambda > 0$. If $z \in D(A)$, then we have for all $t \in [0,T]$*

$$\mathcal{P}(t)(\lambda - A)^{1-\alpha}z = (\lambda - A)^{1-\alpha}\mathcal{P}(t)z. \tag{27}$$

Proof. Due to the fact that $\mathcal{P}(T) = I$, equation (27) holds obviously for $t = T$ and every $z \in D(A)$. In the following let $t \in [0,T)$. Furthermore, let $(h_i)_{i\in\mathbb{N}} \subset D(A)$ be the eigenfunctions of the operator A. Then $(h_i)_{i\in\mathbb{N}}$ is a complete orthonormal system of $L^2(G)$. We define $z_m = \sum_{i=1}^{m} \mu_i h_i$, where $\mu_1, ..., \mu_m \in \mathbb{R}$ such that $z_m \to z$ in $L^2(G)$ as $m \to \infty$. Since $\mathcal{P}(t)$ is self adjoint and by Lemma 1, we find for every $m \in \mathbb{N}$

$$\left\langle \mathcal{P}(t)(\lambda - A)^{1-\alpha}z_m, z_m \right\rangle_{L^2(G)} = \left\langle (\lambda - A)^{1-\alpha}\mathcal{P}(t)z_m, z_m \right\rangle_{L^2(G)}.$$

Thus, we get for every $m \in \mathbb{N}$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \left\langle \left(\mathcal{P}(t)(\lambda - A)^{1-\alpha} - (\lambda - A)^{1-\alpha}\mathcal{P}(t)\right) h_i, h_j \right\rangle_{L^2(G)} \mu_i \mu_j = 0.$$

Hence, we conclude for every $i, j = 1, ..., m$

$$\left\langle \left(\mathcal{P}(t)(\lambda - A)^{1-\alpha} - (\lambda - A)^{1-\alpha}\mathcal{P}(t)\right) h_i, h_j \right\rangle_{L^2(G)} = 0.$$

By Parseval's identity, we obtain for every $i = 1, ..., m$

$$\begin{aligned}
&\left\| \left(\mathcal{P}(t)(\lambda - A)^{1-\alpha} - (\lambda - A)^{1-\alpha}\mathcal{P}(t)\right) h_i \right\|_{L^2(G)} \\
&= \sum_{j=1}^{\infty} \left\langle \left(\mathcal{P}(t)(\lambda - A)^{1-\alpha} - (\lambda - A)^{1-\alpha}\mathcal{P}(t)\right) h_i, h_j \right\rangle_{L^2(G)}^2 = 0.
\end{aligned}$$

Thus, we have for every $i = 1, ..., m$

$$\mathcal{P}(t)(\lambda - A)^{1-\alpha}h_i = (\lambda - A)^{1-\alpha}\mathcal{P}(t)h_i.$$

Furthermore, since $(\lambda - A)^{1-\alpha}\mathcal{P}(t)$ is linear and bounded for all $t \in [0,T)$, we get $(\lambda - A)^{1-\alpha}\mathcal{P}(t)z_m \to (\lambda - A)^{1-\alpha}\mathcal{P}(t)z$ in $L^2(G)$ as $m \to \infty$. Next, using equation (26) with $h = (\lambda - A)^{1-\alpha}(z - z_m)$, we find

$$\begin{aligned}
&\left\|\mathcal{P}(t)(\lambda - A)^{1-\alpha}z - \mathcal{P}(t)(\lambda - A)^{1-\alpha}z_m\right\|_{L^2(G)} \\
&\leq \left\| e^{A(T-t)}(\lambda - A)^{1-\alpha}e^{A(T-t)}(z - z_m) \right\|_{L^2(G)} \\
&\quad + \frac{1}{\kappa_1}\int_t^T \left\| e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)(\lambda - A)^{1-\alpha}e^{A(s-t)}(z - z_m) \right\|_{L^2(G)} ds \\
&\quad + \frac{1}{\kappa_2}\int_t^T \left\| e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)(\lambda - A)^{1-\alpha}e^{A(s-t)}(z - z_m) \right\|_{L^2(G)} ds \\
&\leq C \left\| z - z_m \right\|_{L^2(G)},
\end{aligned}$$

where the constant $C > 0$ depends on T and α. Hence, we have

$$\mathcal{P}(t)(\lambda - A)^{1-\alpha} z_m \to \mathcal{P}(t)(\lambda - A)^{1-\alpha} z$$

in $L^2(G)$ as $m \to \infty$. Therefore, we obtain

$$\begin{aligned}\mathcal{P}(t)(\lambda - A)^{1-\alpha} z &= \lim_{m\to\infty} \mathcal{P}(t)(\lambda - A)^{1-\alpha} z_m = \lim_{m\to\infty} \sum_{i=1}^{m} \mu_i \mathcal{P}(t)(\lambda - A)^{1-\alpha} h_i \\ &= \lim_{m\to\infty} \sum_{i=1}^{m} \mu_i (\lambda - A)^{1-\alpha} \mathcal{P}(t) h_i = \lim_{m\to\infty} (\lambda - A)^{1-\alpha} \mathcal{P}(t) z_m = (\lambda - A)^{1-\alpha} \mathcal{P}(t) z.\end{aligned}$$

□

This enables us to prove the following theorem, where we closely follow the proof of [5, Theorem 7.8].

Theorem 4. *Let the process $(q(t))_{t\in[0,T]}$ be given by (22). Then for all $t \in [0,T]$ and $\mathbb{P}$-a.s., we have*

$$q(t) = \mathcal{P}(t)\overline{y}(t) + a(t), \tag{28}$$

where the process $(\mathcal{P}(t))_{t\in[0,T]}$ is the mild solution of system (25) and the process $(a(t))_{t\in[0,T]}$ is the unique solution of the following deterministic backward integral equation:

$$a(t) = \int_t^T e^{A(s-t)} \left(-\frac{1}{\kappa_1} \mathcal{P}(s) BB^* - \frac{1}{\kappa_2} \mathcal{H}^*(s) \mathcal{G}\mathcal{G}^* (\lambda - A)^{1-\alpha} \right) a(s) ds - e^{A(T-t)} \hat{y}.$$

Proof. Let $t \in [0,T]$. Substituting equations (23) and (24) in (4), we find for any $r \in [t,T]$ and $\mathbb{P}$-a.s.

$$\begin{aligned}\overline{y}(r) = e^{A(r-t)}\overline{y}(t) &- \frac{1}{\kappa_1} \int_t^r e^{A(r-s)} BB^* q(s) ds - \frac{1}{\kappa_2} \int_t^r (\lambda - A)^{1-\alpha} e^{A(r-s)} \mathcal{G}\mathcal{G}^* (\lambda - A)^{1-\alpha} q(s) ds \\ &+ \int_t^r e^{A(r-s)} dW_1(s) + \int_t^r (\lambda - A)^{1-\alpha} e^{A(r-s)} (\lambda - A)^{\alpha} N dW_2(s).\end{aligned}$$

Next, we define for any $s \in [t,r]$ and $\mathbb{P}$-a.s.

$$\tilde{q}(s) = \mathbb{E}[p(s)|\mathcal{F}_t].$$

Then by equation (22), we get $\mathbb{P}$-a.s. $q(t) = \tilde{q}(t)$ and

$$\mathbb{E}[q(s)|\mathcal{F}_t] = \mathbb{E}\left[\mathbb{E}[p(s)|\mathcal{F}_s]\middle|\mathcal{F}_t\right] = \mathbb{E}[p(s)|\mathcal{F}_t] = \tilde{q}(s).$$

Thus, we have for all $r \in [t,T]$ and $\mathbb{P}$-a.s.

$$\mathbb{E}[\overline{y}(r)|\mathcal{F}_t] = e^{A(r-t)}\overline{y}(t) - \frac{1}{\kappa_1} \int_t^r e^{A(r-s)} BB^* \tilde{q}(s) ds - \frac{1}{\kappa_2} \int_t^r (\lambda - A)^{1-\alpha} e^{A(r-s)} \mathcal{G}\mathcal{G}^* (\lambda - A)^{1-\alpha} \tilde{q}(s) ds. \tag{29}$$

Using equation (21), we obtain $\mathbb{P}$-a.s.

$$\begin{aligned}q(t) = e^{A(T-t)} e^{A(T-t)} \overline{y}(t) &- \frac{1}{\kappa_1} \int_t^T e^{A(T-t)} e^{A(T-s)} BB^* \tilde{q}(s) ds \\ &- \frac{1}{\kappa_2} \int_t^T (\lambda - A)^{1-\alpha} e^{A(T-t)} e^{A(T-s)} \mathcal{G}\mathcal{G}^* (\lambda - A)^{1-\alpha} \tilde{q}(s) ds - e^{A(T-t)} \hat{y}\end{aligned}$$

By equation (26) with $h = \overline{y}(t)$, we find $\mathbb{P}$-a.s.

$$
\begin{aligned}
q(t) &= \mathcal{P}(t)\overline{y}(t) - e^{A(T-t)}\hat{y} + \frac{1}{\kappa_1}\int_t^T \left[e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)e^{A(s-t)}\overline{y}(t) - e^{A(T-t)}e^{A(T-s)}BB^*\tilde{q}(s)\right] ds \\
&\quad + \frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}\overline{y}(t)ds - \frac{1}{\kappa_2}\int_t^T (\lambda - A)^{1-\alpha}e^{A(T-t)}e^{A(T-s)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)ds \\
&= \mathcal{P}(t)\overline{y}(t) - e^{A(T-t)}\hat{y} + \mathcal{I}_1(t) + \mathcal{I}_2(t), \qquad (30)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{I}_1(t) &= \frac{1}{\kappa_1}\int_t^T \left[e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)e^{A(s-t)}\overline{y}(t) - e^{A(T-t)}e^{A(T-s)}BB^*\tilde{q}(s)\right] ds, \\
\mathcal{I}_2(t) &= \frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}\overline{y}(t)ds - \frac{1}{\kappa_2}\int_t^T (\lambda - A)^{1-\alpha}e^{A(T-t)}e^{A(T-s)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)ds.
\end{aligned}
$$

Using again equation (26) with $h = BB^*\tilde{q}(s)$, we get $\mathbb{P}$-a.s.

$$
\begin{aligned}
\mathcal{I}_1(t) = \frac{1}{\kappa_1}\int_t^T & e^{A(s-t)}\mathcal{P}(s)BB^*\left[\mathcal{P}(s)e^{A(s-t)}\overline{y}(t) - \tilde{q}(s)\right] ds \\
&- \frac{1}{\kappa_1^2}\int_t^T\int_s^T e^{A(r-t)}\mathcal{P}(r)BB^*\mathcal{P}(r)e^{A(r-s)}BB^*\tilde{q}(s)drds \\
&- \frac{1}{\kappa_1\kappa_2}\int_t^T\int_s^T e^{A(r-t)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)e^{A(r-s)}BB^*\tilde{q}(s)drds.
\end{aligned}
$$

By Fubini's theorem, we have $\mathbb{P}$-a.s.

$$
\begin{aligned}
\mathcal{I}_1(t) = \frac{1}{\kappa_1}\int_t^T & e^{A(s-t)}\mathcal{P}(s)BB^*\left[\mathcal{P}(s)e^{A(s-t)}\overline{y}(t) - \tilde{q}(s)\right] ds \\
&- \frac{1}{\kappa_1^2}\int_t^T\int_t^r e^{A(r-t)}\mathcal{P}(r)BB^*\mathcal{P}(r)e^{A(r-s)}BB^*\tilde{q}(s)dsdr \\
&- \frac{1}{\kappa_1\kappa_2}\int_t^T\int_t^r e^{A(r-t)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)e^{A(r-s)}BB^*\tilde{q}(s)dsdr.
\end{aligned}
$$

Through interchanging the integration variables in the last two integrals, we find $\mathbb{P}$-a.s.

$$\begin{aligned}\mathcal{I}_1(t) = &\frac{1}{\kappa_1}\int\limits_t^T e^{A(s-t)}\mathcal{P}(s)BB^*\left[\mathcal{P}(s)e^{A(s-t)}\overline{y}(t) - \tilde{q}(s)\right]ds \\ &- \frac{1}{\kappa_1^2}\int\limits_t^T\int\limits_t^s e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)e^{A(s-r)}BB^*\tilde{q}(r)drds \\ &- \frac{1}{\kappa_1\kappa_2}\int\limits_t^T\int\limits_t^s e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-r)}BB^*\tilde{q}(r)drds. \end{aligned} \tag{31}$$

To reformulate $\mathcal{I}_2(t)$, we need the following result. Equation (26) with $h = \tilde{z}$ for an arbitrary $\tilde{z} \in D(A)$ yields for all $s \in [t, T]$

$$\begin{aligned}(\lambda - A)^{1-\alpha}e^{A(T-s)}e^{A(T-s)}\tilde{z} = (\lambda - A)^{1-\alpha}\mathcal{P}(s)\tilde{z} + \frac{1}{\kappa_1}\int\limits_s^T(\lambda - A)^{1-\alpha}e^{A(r-s)}\mathcal{P}(r)BB^*\mathcal{P}(r)e^{A(r-s)}\tilde{z}dr \\ + \frac{1}{\kappa_2}\int\limits_s^T(\lambda - A)^{1-\alpha}e^{A(r-s)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)e^{A(r-s)}\tilde{z}dr.\end{aligned}$$

Similarly, by using additionally Lemma 2, we obtain for all $s \in [t, T]$

$$\begin{aligned}e^{A(T-s)}e^{A(T-s)}(\lambda - A)^{1-\alpha}\tilde{z} = (\lambda - A)^{1-\alpha}\mathcal{P}(s)\tilde{z} + \frac{1}{\kappa_1}\int\limits_s^T e^{A(r-s)}\mathcal{P}(r)BB^*\mathcal{P}(r)(\lambda - A)^{1-\alpha}e^{A(r-s)}\tilde{z}dr \\ + \frac{1}{\kappa_2}\int\limits_s^T e^{A(r-s)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)(\lambda - A)^{1-\alpha}e^{A(r-s)}\tilde{z}dr.\end{aligned}$$

Since $(\lambda - A)^{1-\alpha}e^{A(T-s)}e^{A(T-s)}\tilde{z} = e^{A(T-s)}e^{A(T-s)}(\lambda - A)^{1-\alpha}\tilde{z}$ for every $\tilde{z} \in D(A)$, we conclude for all $s \in [t, T]$

$$\begin{aligned}&\frac{1}{\kappa_1}\int\limits_s^T(\lambda - A)^{1-\alpha}e^{A(r-s)}\mathcal{P}(r)BB^*\mathcal{P}(r)e^{A(r-s)}\tilde{z}dr + \frac{1}{\kappa_2}\int\limits_s^T(\lambda - A)^{1-\alpha}e^{A(r-s)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)e^{A(r-s)}\tilde{z}dr \\ &= \frac{1}{\kappa_1}\int\limits_s^T e^{A(r-s)}\mathcal{P}(r)BB^*\mathcal{P}(r)(\lambda - A)^{1-\alpha}e^{A(r-s)}\tilde{z}dr \\ &\quad + \frac{1}{\kappa_2}\int\limits_s^T e^{A(r-s)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)(\lambda - A)^{1-\alpha}e^{A(r-s)}\tilde{z}dr.\end{aligned} \tag{32}$$

Due to the fact that $D(A)$ is dense in $L^2(G)$, the previous equation holds for every $\tilde{z} \in L^2(G)$. Next, applying equation (26) with $h = \mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)$, we get $\mathbb{P}$-a.s.

$$
\begin{aligned}
\mathcal{I}_2(t) = &\frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\left[\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}\overline{y}(t) - \mathcal{H}(s)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)\right]ds \\
&- \frac{1}{\kappa_1\kappa_2}\int_t^T\int_s^T (\lambda - A)^{1-\alpha}e^{A(r-t)}\mathcal{P}(r)BB^*\mathcal{P}(r)e^{A(r-s)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)drds \\
&- \frac{1}{\kappa_2^2}\int_t^T\int_s^T (\lambda - A)^{1-\alpha}e^{A(r-t)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)e^{A(r-s)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)drds.
\end{aligned}
$$

Using equation (32) and Fubini's theorem, we have $\mathbb{P}$-a.s.

$$
\begin{aligned}
\mathcal{I}_2(t) = &\frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\left[\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}\overline{y}(t) - \mathcal{H}(s)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)\right]ds \\
&- \frac{1}{\kappa_1\kappa_2}\int_t^T\int_t^r e^{A(r-t)}\mathcal{P}(r)BB^*\mathcal{P}(r)(\lambda - A)^{1-\alpha}e^{A(r-s)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)dsdr \\
&- \frac{1}{\kappa_2^2}\int_t^T\int_t^r e^{A(r-t)}\mathcal{H}^*(r)\mathcal{G}\mathcal{G}^*\mathcal{H}(r)(\lambda - A)^{1-\alpha}e^{A(r-s)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)dsdr.
\end{aligned}
$$

Through interchanging the integration variables in the last two integrals, we find $\mathbb{P}$-a.s.

$$
\begin{aligned}
\mathcal{I}_2(t) = &\frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\left[\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}\overline{y}(t) - \mathcal{H}(s)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)\right]ds \\
&- \frac{1}{\kappa_1\kappa_2}\int_t^T\int_t^s e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)(\lambda - A)^{1-\alpha}e^{A(s-r)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(r)drds \\
&- \frac{1}{\kappa_2^2}\int_t^T\int_t^s e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)(\lambda - A)^{1-\alpha}e^{A(s-r)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(r)drds. \qquad (33)
\end{aligned}
$$

Using equations (31) and (33), we obtain $\mathbb{P}$-a.s.

$$
\begin{aligned}
&\mathcal{I}_1(t) + \mathcal{I}_2(t) \\
&= \frac{1}{\kappa_1}\int_t^T e^{A(s-t)}\mathcal{P}(s)BB^*\Bigg[\mathcal{P}(s)e^{A(s-t)}\overline{y}(t) - \tilde{q}(s) - \frac{1}{\kappa_1}\mathcal{P}(s)\int_t^s e^{A(s-r)}BB^*\tilde{q}(r)dr \\
&\quad - \frac{1}{\kappa_2}\mathcal{P}(s)\int_t^s (\lambda - A)^{1-\alpha}e^{A(s-r)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(r)dr\Bigg]ds \\
&\quad + \frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\mathcal{P}(s)\Bigg[e^{A(s-t)}\overline{y}(t) - \frac{1}{\kappa_1}\int_t^s e^{A(s-r)}BB^*\tilde{q}(r)dr \\
&\quad - \frac{1}{\kappa_2}\int_t^s (\lambda - A)^{1-\alpha}e^{A(s-r)}\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(r)dr\Bigg]ds - \frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\mathcal{H}(s)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\tilde{q}(s)ds.
\end{aligned}
$$

By equation (29), we get $\mathbb{P}$-a.s.

$$\mathcal{I}_1(t)+\mathcal{I}_2(t)=\frac{1}{\kappa_1}\int\limits_t^T e^{A(s-t)}\mathcal{P}(s)BB^*\left[\mathcal{P}(s)\mathbb{E}[\overline{y}(s)|\mathcal{F}_t]-\tilde{q}(s)\right]ds$$
$$+\frac{1}{\kappa_2}\int\limits_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\left[\mathcal{P}(s)\mathbb{E}[\overline{y}(s)|\mathcal{F}_t]-\tilde{q}(s)\right]ds$$
$$+\frac{1}{\kappa_2}\int\limits_t^T e^{A(s-t)}\left(\mathcal{H}^*(s)-\mathcal{H}(s)\right)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\tilde{q}(s)ds.$$

Using Lemma 1 and Lemma 2, we have for every $\tilde{z}\in L^2(G)$ and $\mathbb{P}$-a.s.

$$\left\langle\int\limits_t^T e^{A(s-t)}\left(\mathcal{H}^*(s)-\mathcal{H}(s)\right)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\tilde{q}(s)ds,\tilde{z}\right\rangle_{L^2(G)}$$
$$=\int\limits_t^T\left\langle\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\tilde{q}(s),(\lambda-A)^{1-\alpha}\mathcal{P}(s)e^{A(s-t)}\tilde{z}\right\rangle_{L^2(G)}ds$$
$$-\int\limits_t^T\left\langle\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\tilde{q}(s),\mathcal{P}(s)(\lambda-A)^{1-\alpha}e^{A(s-t)}\tilde{z}\right\rangle_{L^2(G)}ds=0.$$

Hence, we conclude that $\mathbb{P}$-a.s.

$$\int\limits_t^T e^{A(s-t)}\left(\mathcal{H}^*(s)-\mathcal{H}(s)\right)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\tilde{q}(s)ds=0.$$

Therefore, we have $\mathbb{P}$-a.s.

$$\mathcal{I}_1(t)+\mathcal{I}_2(t)=\frac{1}{\kappa_1}\int\limits_t^T e^{A(s-t)}\mathcal{P}(s)BB^*\left[\mathcal{P}(s)\mathbb{E}[\overline{y}(s)|\mathcal{F}_t]-\tilde{q}(s)\right]ds$$
$$+\frac{1}{\kappa_2}\int\limits_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\left[\mathcal{P}(s)\mathbb{E}[\overline{y}(s)|\mathcal{F}_t]-\tilde{q}(s)\right]ds.$$

Next, we define $\mathbb{P}$-a.s. $z(t)=-e^{A(T-t)}\hat{y}+\mathcal{I}_1(t)+\mathcal{I}_2(t)$. Then by equation (30), we obtain (28). Moreover, we get $\mathbb{P}$-a.s.

$$z(t)=\int\limits_t^T e^{A(s-t)}\left(-\frac{1}{\kappa_1}\mathcal{P}(s)BB^*-\frac{1}{\kappa_2}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\right)a(s)ds-e^{A(T-t)}\hat{y},$$

where $a(s)=\tilde{q}(s)-\mathcal{P}(s)\mathbb{E}[\overline{y}(s)|\mathcal{F}_t]$. Using equation (28), we obtain $a(t)=z(t)$. Thus, the function $a(t)$ satisfies the following deterministic backward integral equation:

$$a(t)=\int\limits_t^T e^{A(s-t)}\left(-\frac{1}{\kappa_1}\mathcal{P}(s)BB^*-\frac{1}{\kappa_2}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*(\lambda-A)^{1-\alpha}\right)a(s)ds-e^{A(T-t)}\hat{y}.$$

In the following, we show regularity results of the function $a(t)$. First, let $\mathcal{M}(t)\colon D\left((\lambda - A)^{1-\alpha}\right) \to L^2(G)$ be defined by

$$\mathcal{M}(t) = -\frac{1}{\kappa_1}\mathcal{P}(t)BB^* - \frac{1}{\kappa_2}\mathcal{H}^*(t)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}.$$

Then clearly $\mathcal{M}(t)$ is linear and closed. Using additionally the properties of fractional power operators provided in Section 2, we obtain

$$\int_0^T \left\|(\lambda - A)^{1-\alpha}e^{At}\mathcal{M}(t)a(t)\right\|_{L^2(G)} dt < \infty.$$

Since the operator $(\lambda - A)^{1-\alpha}$ is closed, we have $\int_t^T e^{A(s-t)}\mathcal{M}(s)a(s)ds \in D\left((\lambda - A)^{1-\alpha}\right)$. Moreover, we have $e^{A(T-t)}\hat{y} \in D\left((\lambda - A)^{1-\alpha}\right)$ for $t \in [0,T)$. Hence, we conclude $a(t) \in D\left((\lambda - A)^{1-\alpha}\right)$ for $t \in [0,T)$. Let $t_0 \in [0,T]$, then we get (w.l.o.g. let $t \geq t_0$)

$$\begin{aligned}\|a(t) - a(t_0)\|_{L^2(G)} \leq &\int_t^T \left\|\left(I - e^{A(t-t_0)}\right)\mathcal{M}(s)a(s)\right\|_{L^2(G)} ds + \int_{t_0}^t \|\mathcal{M}(s)a(s)\|_{L^2(G)} ds \\ &+ \left\|\left(e^{A(t-t_0)} - I\right)\hat{y}\right\|_{L^2(G)} \to 0 \quad \text{as } t \to t_0.\end{aligned}$$

Hence, we conclude $a \in C([0,T];L^2(G))$. We recall that for every $f \in L^2(G)$ there exists a sequence $(g_i)_{i\in\mathbb{N}} \subset D(A)$ such that $\|f - g_i\|_{L^2(G)} \to 0$ as $i \to \infty$. Thus, by Lemma 1 and the Cauchy-Schwarz inequality, we get for all $t_0 \in [0,T)$

$$\begin{aligned}&\left\langle(\lambda - A)^{1-\alpha}a(t) - (\lambda - A)^{1-\alpha}a(t_0), f\right\rangle_{L^2(G)} \\ &= \|a(t) - a(t_0)\|_{L^2(G)} \lim_{i\to\infty} \left\|(\lambda - A)^{1-\alpha}g_i\right\|_{L^2(G)} \to 0 \quad \text{as } t \to t_0.\end{aligned}$$

Let $(f_i)_{i\in\mathbb{N}} \subset L^2(G)$ be an orthonormal system. Then using Parseval's identity, we obtain for all $t_0 \in [0,T)$

$$\begin{aligned}&\left\|(\lambda - A)^{1-\alpha}a(t) - (\lambda - A)^{1-\alpha}a(t_0)\right\|_{L^2(G)} \\ &= \left(\sum_{i=1}^{\infty} \left\langle(\lambda - A)^{1-\alpha}a(t) - (\lambda - A)^{1-\alpha}a(t_0), f_i\right\rangle^2_{L^2(G)}\right)^{1/2} \to 0 \quad \text{as } t \to t_0.\end{aligned}$$

Therefore, we conclude that $a \in C\left([0,T); D\left((\lambda - A)^{1-\alpha}\right)\right)$. Finally, we show the uniqueness of the solution. Let $a_1(t)$ and $a_2(t)$ be solutions of the deterministic backward integral equation. If $t = T$, then it is obvious that $a_1(T) = a_2(T)$ holds. In the following, let $t \in [0,T)$. Since $(\lambda - A)^{1-\alpha}a_1(t)$ and $(\lambda - A)^{1-\alpha}a_2(t)$ are continuous in $L^2(G)$, we get

$$\int_0^T \left\|(\lambda - A)^{1-\alpha}(a_1(t) - a_2(t))\right\|^2_{L^2(G)} dt < \infty.$$

Using [27, Section 2, Corollary 6.11.], we find

$$\left\|(\lambda - A)^{1-\alpha}(a_1(t) - a_2(t))\right\|^2_{L^2(G)} \leq C \int_t^T \left\|(\lambda - A)^{1-\alpha}(a_1(s) - a_2(s))\right\|^2_{L^2(G)} ds,$$

where $C > 0$ depends on T and α. By a backward type of Gronwall's inequality, see [26, Chapter 6, Corollary 6.62], we have

$$\left\|(\lambda - A)^{1-\alpha}(a_1(t) - a_2(t))\right\|^2_{L^2(G)} = 0.$$

Using again [27, Section 2, Corollary 6.11.], we obtain

$$\|a_1(t) - a_2(t)\|^2_{L^2(G)} \le \widetilde{C} \left\|(\lambda - A)^{1-\alpha}(a_1(t) - a_2(t))\right\|^2_{L^2(G)} = 0,$$

where $\widetilde{C} > 0$ is a constant. Therefore, we obtain $a_1(t) = a_2(t)$ in $D\left((\lambda - A)^{1-\alpha}\right)$ for all $t \in [0, T)$. Moreover, we have $a_1(t) = a_2(t)$ in $L^2(G)$ for all $t \in [0, T]$. □

6. Example

In this section, we demonstrate the applicability of the abstract setting presented in this paper by considering a concrete control problem. In [7], the region $G = (0, \pi)^n$ is considered and properties of the Laplace operator as well as of the Neumann operator are given. For the sake of simplicity, we will deal with the case $n = 1$. Similar to Section 2, we first consider the following deterministic system:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t,x) = \dfrac{\partial^2}{\partial x^2} y(t,x) + b(x)u(t,x) & (t,x) \in (0,T) \times (0,\pi), \\ y(0,x) = \xi(x) & x \in (0,\pi), \\ \dfrac{\partial}{\partial x} y(t,0) = v_1(t), \quad \dfrac{\partial}{\partial x} y(t,\pi) = v_2(t) & t \in (0,T), \end{cases} \tag{34}$$

where $y(t,x)$ describes the heat distribution, $u(t,x)$ is the distributed control active on a part of $(0,\pi)$ specified by $b(x)$. The boundary controls are denoted by $v_1(t)$ and $v_2(t)$.

Let $A \colon D(A) \subset L^2(0,\pi) \to L^2(0,\pi)$ be the Neumann realization of the Laplace operator, where the domain is given by $D(A) = \left\{h \in H^2(0,\pi) : \frac{\partial}{\partial x} h(0) = \frac{\partial}{\partial x} h(\pi) = 0\right\}$ and $Ah = \frac{\partial^2}{\partial x^2} h$ for $h \in D(A)$. It is well known that the operator A is the generator of an analytic semigroup $(e^{At})_{t \ge 0}$. Moreover, the eigenfunctions of A are given by

$$g_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } k = 0 \\ \sqrt{\frac{2}{\pi}} \cos(kx) & \text{if } k \in \mathbb{N}\backslash\{0\} \end{cases}$$

with corresponding eigenvalues $\mu_k = -k^2$ for $k \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, the sequence $(g_k)_{k \in \mathbb{N}_0}$ is an orthonormal basis of $L^2(0,\pi)$ and we have for all $h \in L^2(0,\pi)$

$$e^{At} h = \sum_{k=0}^{\infty} e^{-tk^2} \langle h, g_k \rangle_{L^2(0,\pi)} g_k.$$

Next, we define for fixed $\lambda > 0$ and all $x \in [0,\pi]$

$$z_1(x) = -\frac{\cosh(\sqrt{\lambda}(\pi - x))}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)}, \quad z_2(x) = \frac{\cosh(\sqrt{\lambda}x)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)}.$$

Note that z_1, z_2 satisfy the following Neumann problems:

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} z_i(x) = \lambda z_i(x), & x \in (0,\pi), i = 1,2, \\ \dfrac{\partial}{\partial x} z_1(0) = 1, \quad \dfrac{\partial}{\partial x} z_1(\pi) = 0, \\ \dfrac{\partial}{\partial x} z_2(0) = 0, \quad \dfrac{\partial}{\partial x} z_2(\pi) = 1. \end{cases}$$

Thus, we have $z_i \in D((\lambda - A)^\alpha)$ for $i = 1,2$ and $\alpha \in \left(0, \frac{3}{4}\right)$. We set $v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$ for $t \in [0,T]$. Hence, system (34) can be reformulated as the following abstract system in $L^2(0,\pi)$:

$$\begin{cases} \dfrac{\partial}{\partial t} y(t) = Ay(t) + Bu(t) + (\lambda - A)Nv(t) \\ y(0) = \xi, \end{cases} \tag{35}$$

where $y(t)(x) = y(t,x)$, $Bu(t)(x) = b(x)u(t,x)$, $Nv(t)(x) = z_1(x)v_1(t) + z_2(x)v_2(t)$. Then, system (35) has a unique solution $y \in C([0,T]; L^2(0,\pi))$ given by equation (2), see [3].

This motivates how to involve noise terms inside the region as well as on the boundary. In general, noise enters the system potentially due to random environments, imperfect insulation and other uncertain heating or cooling phenomenas. For a mathematical description we will again consider the one-dimensional case. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We extend system (34) by noise terms as follows:

$$\begin{cases} \frac{\partial}{\partial t} y(t,x) = \frac{\partial^2}{\partial x^2} y(t,x) + b(x)u(t,x) + w_1(t,x) & (t,x) \in (0,T) \times (0,\pi), \\ y(0,x) = \xi(x) & x \in (0,\pi), \\ \frac{\partial}{\partial x} y(t,0) = v_1(t) + w_2^1(t), \quad \frac{\partial}{\partial x} y(t,\pi) = v_2(t) + w_2^2(t) & t \in (0,T), \end{cases} \tag{36}$$

where $w_1(t,x)$ describes the noise inside the region. The noise on the boundary is described by $w_2^1(t)$ and $w_2^2(t)$. Note that stochastic processes are not differentiable with respect to the time variable in general. Hence, system (36) is only a symbolic description in order to illustrate the setting. Using the operators defined above, we rewrite system (36) in abstract form, which is more common in the literature, but still formal. We introduce the following system:

$$\begin{cases} dy(t) = [Ay(t) + Bu(t) + (\lambda - A)Nv(t)]\, dt + dW_1(t) + (\lambda - A)N dW_2(t), \\ y(0) = \xi. \end{cases} \tag{37}$$

The process $(W_1(t))_{t\in[0,T]}$ is assumed to be a Q-Wiener process with values in $L^2(0,\pi)$. A concrete representation of this process heavily depends on the covariance operator, see for instance [8, Example 4.9]. Furthermore, the process $(W_2(t))_{t\in[0,T]}$ is a two-dimensional Brownian motion. We assume that the processes $(W_1(t))_{t\in[0,T]}$ and $(W_2(t))_{t\in[0,T]}$ are independent. Moreover, let $u \in U$ and $v \in V$, where

$$\begin{aligned} U &= \left\{ u \in L^2(\Omega; L^2([0,T]; L^2(0,\pi))) : (u(t))_{t\in[0,T]} \text{ is } \mathcal{F}_t\text{-adapted} \right\}, \\ V &= \left\{ v \in L^2(\Omega; L^2([0,T]; \mathbb{R}^2)) : (v(t))_{t\in[0,T]} \text{ is } \mathcal{F}_t\text{-adapted} \right\}. \end{aligned}$$

Using Theorem 1, we get the existence and uniqueness of a mild solution $y \in C([0,T]; L^2(\Omega; L^2(0,\pi)))$ of system (37).

Next, we consider the cost functional given by (5). For $\kappa_1, \kappa_2 > 0$, we get the existence and uniqueness of an optimal control $(\overline{u}, \overline{v}) \in U \times V$ such that

$$J(\xi, \overline{u}, \overline{v}) = \inf_{u \in U, v \in V} J(\xi, u, v).$$

By Theorems 2 and 3 the optimal controls satisfy $\mathbb{P}$-a.s. and a.e. on $[0,T]$

$$\begin{aligned} \overline{u}(t) &= -\frac{1}{\kappa_1} B^* e^{A(T-t)} (\mathbb{E}\left[\overline{y}(T) | \mathcal{F}_t\right] - \hat{y}), \\ \overline{v}(t) &= -\frac{1}{\kappa_2} \mathcal{G}^* (\lambda - A)^{1-\alpha} e^{A(T-t)} (\mathbb{E}\left[\overline{y}(T) | \mathcal{F}_t\right] - \hat{y}). \end{aligned}$$

Let the process $(q(t))_{t\in[0,T]}$ be defined by

$$q(t) = e^{A(T-t)} (\mathbb{E}\left[\overline{y}(T) | \mathcal{F}_t\right] - \hat{y}).$$

Using Theorem 4, we obtain $\mathbb{P}$-a.s. and for all $t \in [0,T]$

$$q(t) = \mathcal{P}(t)\overline{y}(t) + a(t).$$

The process $(\mathcal{P}(t))_{t\in[0,T]}$ is given by

$$\mathcal{P}(t)h = e^{A(T-t)}e^{A(T-t)}h - \frac{1}{\kappa_1}\int_t^T e^{A(s-t)}\mathcal{P}(s)BB^*\mathcal{P}(s)e^{A(s-t)}hds - \frac{1}{\kappa_2}\int_t^T e^{A(s-t)}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*\mathcal{H}(s)e^{A(s-t)}hds$$

and $(a(t))_{t\in[0,T]}$ satisfies

$$a(t) = \int_t^T e^{A(s-t)}\left(-\frac{1}{\kappa_1}\mathcal{P}(s)BB^* - \frac{1}{\kappa_2}\mathcal{H}^*(s)\mathcal{G}\mathcal{G}^*(\lambda - A)^{1-\alpha}\right)a(s)ds - e^{A(T-t)}\hat{y}.$$

We obtain that the optimal controls satisfy $\mathbb{P}$-a.s. and a.e. on $[0,T]$

$$\overline{u}(t) = -\frac{1}{\kappa_1}B^*(\mathcal{P}(t)\overline{y}(t) + a(t)),$$
$$\overline{v}(t) = -\frac{1}{\kappa_2}\mathcal{G}^*(\lambda - A)^{1-\alpha}(\mathcal{P}(t)\overline{y}(t) + a(t)).$$

Note that the purpose of this example is the illustration of the theoretical approach developed in this paper. Further analysis requires a discretization method due to the fact that especially the solution of the Riccati equation (25) is used to determine the optimal controls.

7. Conclusion

We have considered the stochastic heat equation, where the region is a general n-dimensional subspace such that including infinite dimensional noise terms is reasonable. A stochastic control problem given by a tracking problem was analyzed. Based on the specific cost functional, we derived explicit formulas for stochastic optimal controls inside the region as well as on the boundary. Finally, we have rewritten the formulas to obtain that the stochastic optimal controls are given in feedback form.

Several directions of future research are possible. To generalize the state equation, we can define the noise terms by more general stochastic processes such as Lévy processes or fractional Brownian motions. Furthermore, it might be possible that the noise term also depends on the state. This leads us to linear equations with multiplicative noise, which are not covered in our approach. Another aspect is that a target function only defined at the end of the time interval results in inactivity of the controls during nearly the whole period. This might be undesirable in many applications, as the strong control activity at the end of the time interval might harness the physical system. Therefore, it is necessary to extend the cost functional by including another target function defined over the whole time interval.

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