

Curvature of closed subsets of Euclidean space and minimal submanifolds of arbitrary codimension

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Abstract

A second fundamental form is introduced for arbitrary closed subsets of Euclidean space, extending the same notion introduced by J. Fu for sets of positive reach in [Fu89]. We extend well known integral-geometric formulas to this general setting and we provide a structural result in terms of second fundamental forms of submanifolds of class 2 that is new even for sets of positive reach. In the case of a large class of minimal submanifolds, which include viscosity solutions of the minimal surface system and rectifiable stationary varifolds of arbitrary codimension and higher multiplicities, we prove the area formula for the generalized Gauss map in terms of the discriminant of the second fundamental form and, adapting techniques from the theory of viscosity solutions of elliptic equations to our geometric setting, we conclude a natural second-order-differentiability property almost everywhere. Moreover the trace of the second fundamental form is proved to be zero for stationary integral varifolds.

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1 Introduction *(For notation and terminology, see 2.)*

Motivation

In the attempt to introduce models to efficiently study the severe singularities emerging from many mathematical problems in the Euclidean space, classical smooth submanifolds have been generalized in different ways, and many classes of *generalized submanifolds* are currently studied for both theoretical and practical reasons. Since the concept of curvature has a fundamental role in the study of smooth submanifolds, it is natural to try to introduce a similar concept for such generalized submanifolds and this has been actually done in various cases. Among all of them, we recall here the class of (certain union of) *sets of positive reach* in [Fed59], [Zäh86], [Fu89] and [RZ01], for which it is possible to extend classical facts of differential and convex geometry as the Steiner formula, the principal kinematic formula, the existence of the normal cycle and the

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Morse theory. Furthermore we recall the class of *curvature varifolds* in [Hut86] and [Man96], that is designed for applications in variational problems involving curvature, because of the existence of a second fundamental form satisfying a classical integration-by-parts identity and good compactness and semicontinuity properties. However, no concept of curvature has been developed yet, that could be used to study the geometric and regularity properties of generalized submanifolds arising as solutions of very classical variational problems, as the critical points of the area functional modelled by *stationary integral varifolds*. On the other hand, a concept of curvature for arbitrary closed sets has been introduced in [Sta79] and [HLW04] to obtain a very general Steiner formula; it agrees with the aforementioned case of sets of positive reach and have found applications in stochastic geometry, see [HLW04]. Therefore it is a natural question to understand if this concept of curvature can be used to describe the geometric features of generalized minimal submanifolds and we address this question in the present paper.

Results of the present paper

The aim of this paper is twofold. In the first part (sections 3–6) we study curvature properties of closed subsets of Euclidean space. Then, in section 7, the results of the first part are applied to very general notions of minimal submanifolds of arbitrary codimension, represented by either *viscosity minimal sets* (a concept recently introduced in [Sav17]) or by *stationary varifolds*, and new results on their geometric and regularity properties are proved.

We now provide a brief outline of the paper.

Second fundamental form of arbitrary closed subsets of Euclidean space: sections 3–5. If $A \subseteq \mathbf{R}^n$ is closed and $N(A)$ is the unit normal bundle considered in [Sta79] and [HLW04], i.e.

$$N(A) = (A \times \mathbf{S}^{n-1}) \cap \{(a, u) : \text{distance}(A, a + su) = s \text{ for some } s > 0\},$$

(whose fiber at a is denoted by $N(A, a)$, see 4.1), we introduce in 4.11 *the second fundamental form* Q_A of A as a function on $N(A)$. This second fundamental form extends the analogous concept introduced in [Fu89] for sets of positive reach (see 4.15), and its principal curvatures, defined as in 4.14 and denoted by $-\infty < \kappa_1 \leq \dots \leq \kappa_{n-1} \leq \infty$, coincide up to an exceptional set of measure zero with the *generalized principal curvatures* of [HLW04] (see 4.16). We extend well known integral-geometric formulas, previously obtained for various special classes, to the general setting of arbitrary closed sets (see 4.14(3), 5.6 and 5.7). Furthermore we introduce the natural stratification of a closed set $A \subseteq \mathbf{R}^n$ (see 5.1–5.3) given by the subsets

$$A^{(m)} = A \cap \{a : 0 < \mathcal{H}^{n-m-1}(N(A, a)) < \infty\} \quad \text{for } m = 0, \dots, n-1,$$

and we prove the following structural result, that is new even in the case of sets of positive reach.

Structural theorem on the second fundamental form (see 5.9). *If A is a closed subset of \mathbf{R}^n and $0 \leq m \leq n-1$, then there exists a Borel set $R \subseteq A^{(m)}$ such that $\mathcal{H}^m(A^{(m)} \setminus R) = 0$ and the following two conditions hold.*

Lusin condition (N): if $S \subseteq R$ such that $\mathcal{H}^m(S) = 0$, then

$$\mathcal{H}^{n-1}(N(A) \cap \{(a, u) : a \in S\}) = 0.$$

Locality of the second fundamental form : if M is an m dimensional submanifold of class 2 with second fundamental form \mathbf{b}_M , then

$$Q_A(a, u) = -\mathbf{b}_M(a) \bullet u$$

for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A) \cap \{(a, u) : a \in R \cap M\}$.

The main result of [MS17] proves that $A^{(m)}$ can be \mathcal{H}^m almost covered by the union of a countable collection of m dimensional submanifolds of class 2 of \mathbf{R}^n . Combining this result with the locality property, we conclude that the second fundamental form Q_A can be described on $A^{(m)}$, outside an *exceptional subset* of $N(A) \cap \{(a, u) : a \in A^{(m)}\}$, in terms of classical second fundamental forms. However the exceptional subset, though projects onto a set of \mathcal{H}^m measure zero, may have positive \mathcal{H}^{n-1} measure even in the case of convex sets, as the example in 5.11 shows. All the results described so far are based on the study carried in section 3 of the approximate differentiability properties of the nearest point projection onto an arbitrary closed subset of Euclidean space.

A Lusin condition (N) for the normal bundle: section 6. A special class of closed subsets of Euclidean space can be introduced by ruling out the exceptional subsets of positive \mathcal{H}^{n-1} measure in 5.9. More precisely, for a closed set A , we say that $N(A)$ satisfies *the m dimensional Lusin condition (N)* if

$$\mathcal{H}^{n-1}(N(A) \cap \{(a, u) : a \in S\}) = 0, \quad \text{if } S \subseteq A \text{ such that } \mathcal{H}^m(A^{(m)} \cap S) = 0.$$

This property of $N(A)$ is not shared by all convex sets, as the example in 5.11 shows. On the other hand the main goal of section 7 is to prove that this property holds for a large class of generalized minimal submanifolds (see paragraph below). An immediate consequence of this condition is the area formula for the generalized Gauss map in 6.6, where the discriminant of the second fundamental form plays the same role of the Jacobian in the classical area formula for functions. The following theorem is the central result of this section.

A general criterion for second-order-differentiability (see 6.10). *Suppose $1 \leq m < n$ are integers, $A \subseteq \mathbf{R}^n$ is a closed set with locally finite \mathcal{H}^m measure, $N(A)$ satisfies the m dimensional condition (N), for \mathcal{H}^m a.e. $a \in A$ there exists $v \in \mathbf{R}^n \sim \{0\}$ such that $\lim_{r \rightarrow 0} r^{-1} \sup\{v \bullet (x - a) : x \in \mathbf{B}(a, r) \cap A\} = 0$, and there exists a nonnegative \mathcal{H}^{n-1} measurable function f on $N(A)$ such that*

$$\text{trace } Q_A(a, u) \leq f(a, u) \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(A),$$

$$\int_{K \cap A} \int_{\{z\} \times N(A, z)} f^m d\mathcal{H}^{n-m-1} d\mathcal{H}^m z < \infty,$$

whenever $K \subseteq \mathbf{R}^n \times \mathbf{S}^{n-1}$ is compact. Then $\mathcal{H}^m(A \sim A^{(m)}) = 0$.

This result can be naturally compared with the classical result in [Tru89, Theorem 1] (see also [CCKS96, 3.5]) asserting the twice super(sub)differentiability

of viscosity sub(super)solutions of certain second order elliptic operators. For continuous scalar functions the notion of twice super(sub)differentiability and the associated notion of super(sub)differential have been introduced in [Tru89] (see also [CCKS96, p. 381]). In our geometric setting, noting that no natural distinction exists between the two cases, the subset $A^{(m)}$ can be interpreted to be the set of points where A is “one-sided second-order differentiable” and the role of the second order super(sub)differential belongs to $N(A)$. Therefore the conclusion “ $\mathcal{H}^m(A \sim A^{(m)}) = 0$ ” is analogous to prove that a continuous scalar function is twice sub(super)differentiable almost everywhere. However it is remarkable that our result holds for non-graphical sets of arbitrary codimension. The proof is based on the aforementioned area formula for the generalized Gauss map and we refer to 6.11 for further comments.

Regularity of generalized minimal submanifolds: section 7. In this final section we deal with applications to minimal submanifolds of arbitrary codimension in Euclidean space. They are represented by the viscosity minimal sets recently introduced in [Sav17] as a notion of viscosity solution for the minimal surface system, see 7.1. They can be intuitively described as the largest class of closed subsets including all minimal submanifolds of class 2 of arbitrary codimension, for which the weak maximum principle holds; see [Sav17, p. 2] for further comments. Employing the weak maximum principle in [Whi10], it is not difficult to see that the support of a general stationary varifold is a viscosity minimal set, see 7.10(1). The main result of this section is contained in 7.5, where the Lusin condition (N) is verified for the normal bundle of a very large class of viscosity minimal sets (which include every stationary varifold with a uniform lower bound on the density). Then, the general criterion of section 6 can be applied to deduce second order differentiability properties of viscosity minimal sets in 7.8, providing a natural extension of classical results for viscosity solutions of elliptic equations in this geometric setting; see 6.11 and 7.9. In the special case of stationary varifolds, our main regularity result can be summarized as follows.

Rectifiable stationary varifolds of arbitrary codimension (see 7.11). *If $0 < d < \infty$, $1 \leq m \leq n - 1$ are integers and V is an m dimensional stationary varifold in \mathbf{R}^n such that $\Theta^m(\|V\|, x) \geq d$ for $\|V\|$ a.e. $x \in \mathbf{R}^n$, then the following three statements hold.*

- (1) $N(\text{spt } \|V\|)$ satisfies the m dimensional condition (N) and

$$\begin{aligned} & \int_{\mathbf{S}^{n-1}} \mathcal{H}^0\{a : (a, u) \in B\} d\mathcal{H}^{n-1}u \\ &= \int_{\text{spt } \|V\|} \int_{\{z\} \times \{v : (z, v) \in B\}} |\text{discr } Q_{\text{spt } \|V\|}| d\mathcal{H}^{n-m-1}z d\mathcal{H}^m z, \end{aligned}$$

for every \mathcal{H}^{n-1} measurable set $B \subseteq N(\text{spt } \|V\|)$.

- (2) $\text{trace } Q_{\text{spt } \|V\|}(a, u) \leq 0$ for \mathcal{H}^{n-1} a.e. $(a, u) \in N(\text{spt } \|V\|)$; in case V is integral, “ \leq ” can be replaced by “ $=$ ”.
- (3) $\mathcal{H}^m((\text{spt } \|V\|) \sim (\text{spt } \|V\|)^{(m)}) = 0$; in particular $\text{spt } \|V\|$ can be \mathcal{H}^m almost covered by countably many m dimensional submanifolds of class 2.

We refer to 7.12–7.15 for comments within the theory of varifolds.

Lines of further studies

Once the Lusin condition (N) introduced in 6.1 has been verified for the minimal submanifolds considered in section 7, the regularity results are consequences of the abstract theory established in section 6. In particular the general criterion for second-order-differentiability is applied in the special case $f = 0$.

The general setting of section 6, as well as the summability condition of f in the general criterion, suggest other possible applications, than the case considered here. In particular, it is natural to ask if the hypothesis “ $\delta V = 0$ ” can be replaced by “ $\|\delta V\|_{\text{sing}} = 0$ and $\mathbf{h}(V, \cdot) \in \mathbf{L}_m^{\text{loc}}(\|V\|, \mathbf{R}^n)$ ” in the regularity result for varifolds in section 7. These varifolds naturally arise in variational problems involving the Willmore functional (see [Sch09]). The key step would be to establish the Lusin condition (N) in this more general setting and, in this regard, the announcement in 7.13 is surely encouraging.

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2 Notation and preliminaries

Notation and terminology

The notation and the terminology used without comments agree with [Fed69, pp. 669–676]. Additionally,

if $A \subseteq \mathbf{R}^n$, then $\delta_A : \mathbf{R}^n \rightarrow \mathbf{R}$ is the distance function to A ;

the symbol \bullet denotes the standard inner product of \mathbf{R}^n ;

if T is a linear subspace of \mathbf{R}^n , then $T_{\perp} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is the orthogonal projection onto T and $T^{\perp} = \mathbf{R}^n \cap \{v : v \bullet u = 0\}$;

if X is a set, $f : X \rightarrow \mathbf{R}^n$ is a function and $u \in \mathbf{R}^n$, then $f \bullet u$ denotes the scalar function given by $(f \bullet u)(x) = f(x) \bullet u$ whenever $x \in X$;

if X and Y are sets, $Z \subseteq X \times Y$ and $S \subseteq X$, then $Z|S = Z \cap \{(x, y) : x \in S\}$;

if M is an m dimensional submanifold of class 2, then \mathbf{b}_M is the second fundamental form;

the maps $\mathbf{p}, \mathbf{q} : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ are given by $\mathbf{p}(x, v) = x$ and $\mathbf{q}(x, v) = v$, whenever $(x, v) \in \mathbf{R}^n \times \mathbf{R}^n$.

if $A \subseteq \mathbf{R}^n$ and $m \geq 1$ is an integer, we say that A has locally finite \mathcal{H}^m measure if $\mathcal{H}^m(A \cap K) < \infty$ whenever $K \subseteq \mathbf{R}^n$ is compact;

if $A \subseteq \mathbf{R}^n$ and $m \geq 1$ is an integer, we say that A is countably (\mathcal{H}^m, m) rectifiable of class 2 if A can be \mathcal{H}^m almost covered by the union of countably many m dimensional submanifolds of class 2 of \mathbf{R}^n ; we omit the prefix “countably” when $\mathcal{H}^m(A) < \infty$;

if X and Y are metric spaces and $f : X \rightarrow Y$ is a function such that f and f^{-1} are Lipschitzian functions, then we say that f is a bi-Lipschitzian homeomorphism;

Preliminaries

Basic preliminaries for the present paper are the classical methods and results of Geometric Measure Theory for which we refer to [Fed69]. Further preliminaries are collected here.

Notions of differentiability In the present paper we employ the concept of approximate differentiability for functions and sets, for which we refer to [San17]. Moreover we use the classical notion of pointwise differentiability for functions defined as in [San17, p. 4]. Finally, the concept of pointwise differentiability for sets introduced in [Men16] is employed in 2.10 and 3.10(3).

For reader's convenience, we collect here some additional (basic) facts on approximately differentiable functions.

2.1 Lemma. *Suppose $n \geq 1$ is an integer, $B \subseteq A \subseteq \mathbf{R}^n$, $a \in A$ and $f : A \rightarrow \mathbf{R}$ are such that f is approximately differentiable at a , $\Theta^{*n}(\mathcal{L}^n \llcorner B, a) = 1$ and $f(x) \leq f(a)$ for every $x \in B$.*

Then $\text{apD} f(a) = 0$.

Proof. Assume $a = 0$ and $f(0) = 0$. If $\text{apD} f(0) \neq 0$ then there would be $\epsilon > 0$ and a non empty open cone C such that $\text{apD} f(0)(x) \geq 2\epsilon|x|$ for every $x \in C$. Therefore $f(x) - \text{apD} f(0)(x) \leq -2\epsilon|x|$ for every $x \in C \cap B$ and

$$\begin{aligned} \Theta^{*n}(\mathcal{L}^n \llcorner B \sim C, 0) &< 1, & \Theta^{*n}(\mathcal{L}^n \llcorner B \cap C, 0) &> 0, \\ \Theta^{*n}(\mathcal{L}^n \llcorner \mathbf{R}^n \sim \{x : |f(x) - \text{apD} f(0)(x)| \leq \epsilon|x|\}, 0) &> 0. \end{aligned}$$

This would be a contradiction.

2.2 Remark. We observe that a similar argument proves that *if f is approximately differentiable of order 2 at a then $\text{apD}^2 f(a) \leq 0$* . However, this fact will not be used in the sequel.

2.3 Lemma. *Suppose $n \geq 1$ and $\nu \geq 1$ are integers, $B \subseteq A \subseteq \mathbf{R}^n$, $a \in B$ and $f : A \rightarrow \mathbf{R}^\nu$ are such that f is approximately differentiable at a , $f|_B$ is a bi-Lipschitzian homeomorphism and $\Theta^n(\mathcal{L}^n \llcorner \mathbf{R}^n \sim B, a) = 0$.*

Then $\ker \text{apD} f(a) = \{0\}$.

Proof. If $\Gamma = (1/2)(\text{Lip}(f|_B)^{-1})^{-1}$ then $|f(y) - f(x)| \geq 2\Gamma|y - x|$ whenever $y, x \in B$. If there was $v \in \mathbf{R}^n \sim \{0\}$ such that $\text{apD} f(a)(v) = 0$, then there would exist a non empty open cone C such that

$$|\text{apD} f(a)(u)| \leq \Gamma|u| \quad \text{whenever } u \in C.$$

Choosing $0 < \epsilon < \Gamma$ and letting $D = \{u + a : u \in C\}$ and

$$E = A \cap \{x : |f(x) - f(a) - \text{apD} f(a)(x - a)| \leq \epsilon|x - a|\},$$

we would notice that $\Theta^n(\mathcal{L}^n \llcorner \mathbf{R}^n \sim E, a) = 0$ and $B \cap D \cap E = \emptyset$ and we would get a contradiction.

2.4 Lemma. *If m, n, ν are positive integers, $D \subseteq \mathbf{R}^m$, $U \subseteq \mathbf{R}^n$ is open, $f : D \rightarrow \mathbf{R}^n$, $g : U \rightarrow \mathbf{R}^\nu$, $x \in D$, $f(x) \in U$, f is approximately differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is approximately differentiable at x with*

$$\text{apD}(g \circ f)(x) = \text{D}g(f(x)) \circ \text{apD} f(x).$$

Proof. Combine [San17, 2.8] and [Fed69, 3.1.1(2)].

2.5 Lemma. *If $n, \nu \geq 1$ are integers, $D \subseteq \mathbf{R}^n$, $z \in D$ and $g : \mathbf{R}^n \rightarrow \mathbf{R}^\nu$ is a Lipschitzian function such that $g|_D$ is approximately differentiable at z , then g is differentiable at z with $\text{apD}(g|_D)(z) = \text{D}g(z)$.*

Proof. This is proved in [Fed69, 3.1.5].

Level sets of distance function We recall classical facts on the structure of the level sets of δ_A for an arbitrary closed set A . Besides the well known rectifiability property that is an immediate consequence of Coarea formula (see 2.7), we state the structural result in [GP72] in a slightly different way (see 2.9).

2.6. It follows mechanically from the definitions that

$$\delta_{\delta_A^{-1}\{s\}}(y) = \delta_A(y) - s \quad \text{whenever } \delta_A(y) \geq s \text{ and } s > 0.$$

2.7. Since $\text{Lip } \delta_A \leq 1$ (see [Fed59, 4.8(1)]), it is a consequence of [Fed69, 3.2.15, 3.2.11] that $\delta_A^{-1}\{r\}$ is countably ($\mathcal{H}^{n-1}, n-1$) rectifiable with locally finite \mathcal{H}^{n-1} measure, for \mathcal{L}^1 a.e. $r > 0$.

2.8 Lemma. *If $T \in \mathbf{G}(n, n-1)$, $f : T \rightarrow T^\perp$, $\alpha \in T$, $a = \alpha + f(\alpha)$ and $A = \{\chi + f(\chi) : \chi \in T\}$ are such that $\text{Tan}(A, a) \subseteq T$, then f is differentiable at α with $\text{D}f(\alpha) = 0$.*

Proof. It follows immediately from the definition of $\text{Tan}(A, a)$.

2.9 Theorem. *If A is a closed subset of \mathbf{R}^n , then δ_A is differentiable at \mathcal{H}^{n-1} a.e. $x \in \delta_A^{-1}\{r\}$, for \mathcal{L}^1 a.e. $r > 0$. Moreover if $r > 0$, $x \in \delta_A^{-1}\{r\}$ is such that δ_A is differentiable at x and $T = \{v : v \bullet \text{grad } \delta_A(x) = 0\}$, then there exists an open neighborhood V of x and a Lipschitzian function $f : T \rightarrow T^\perp$ differentiable at $T_x(x)$ such that $\text{D}f(T_x(x)) = 0$ and $V \cap \delta_A^{-1}\{r\} = V \cap \{\chi + f(\chi) : \chi \in T\}$.*

Proof. The first part evidently follows from [Fed69, 3.1.6, 3.2.11].

If $r > 0$, δ_A is differentiable at $x \in \delta_A^{-1}\{r\}$ and $T = \{v : v \bullet \text{grad } \delta_A(x) = 0\}$, then, employing [Fed59, 4.8(3)], the proof of [GP72, Theorem 1] reveals that there exist an open neighborhood V of x and a Lipschitzian function $f : T \rightarrow T^\perp$ such that $V \cap \delta_A^{-1}\{r\} = V \cap \{\chi + f(\chi) : \chi \in T\}$. Moreover we observe that $\text{Tan}(\delta_A^{-1}\{r\}, x) \subseteq T$. Therefore the conclusion comes from 2.8.

2.10 Remark. Employing [Men16, 3.14] we conclude that $\delta_A^{-1}\{r\}$ is pointwise differentiable of order 1 with $\text{Tan}(\delta_A^{-1}\{r\}, x) = T$ at \mathcal{H}^{n-1} a.e. $x \in \delta_A^{-1}\{r\}$ and for \mathcal{L}^1 a.e. $r > 0$.

Approximate tangent cone of a measure The concept of approximate tangent vector to a measure is introduced in [Fed69, 3.2.16]. Besides the fundamental results given in [Fed69, 3.2.16–3.2.22, 3.3.18], we prove here some additional facts (see 2.12 and 2.14) that directly follows from them.

First, the following elementary inequality is useful here and elsewhere.

2.11 Lemma. *If X and Y are metric spaces, $m \geq 1$ is an integer, $\theta(x) \geq 0$ for \mathcal{H}^m a.e. $x \in X$, $0 \leq \gamma < \infty$ and $f : X \rightarrow Y$ is an univalent Lipschitzian map onto Y such that γ is a Lipschitz constant for f^{-1} , then*

$$\int_X^* \theta d\mathcal{H}^m \leq \gamma^m \int_Y^* \theta \circ f^{-1} d\mathcal{H}^m.$$

Proof. We assume $\int_Y^* \theta \circ f^{-1} d\mathcal{H}^m < \infty$. Then the conclusion easily follows from the definition of upper integral in [Fed69, 2.4.2], using approximation by upper functions.

2.12 Lemma. *Suppose X and Y are normed vector spaces, $P \subseteq X$, $m \geq 1$ is an integer, $\theta(x) \geq 0$ for \mathcal{H}^m a.e. $x \in P$, $a \in P$ and $f : X \rightarrow Y$ is a function differentiable at a such that $f|P$ is a bi-Lipschitzian homeomorphism. Additionally, we define the measures*

$$\psi(A) = \int_{A \cap P}^* \theta d\mathcal{H}^m, \quad \mu(B) = \int_{B \cap f|P}^* \theta \circ (f|P)^{-1} d\mathcal{H}^m,$$

whenever $A \subseteq X$ and $B \subseteq Y$.

Then $Df(a)[\text{Tan}^m(\psi, a)] \subseteq \text{Tan}^m(\mu, f(a))$.

Proof. Firstly we prove that $\Theta^m(\psi \llcorner X \sim f^{-1}[T], a) = 0$, whenever $T \subseteq Y$ such that $\Theta^m(\mu \llcorner Y \sim T, f(a)) = 0$. In fact, for such a subset T , if $S = f^{-1}[T]$, γ is a Lipschitz constant for $f|P$ and $(f|P)^{-1}$ and $r > 0$, we observe that

$$f[(P \sim S) \cap \mathbf{B}(a, r)] \subseteq (f|P) \sim T \cap \mathbf{B}(f(a), \gamma r),$$

and we employ 2.11 to get that $\psi(\mathbf{B}(a, r) \sim S) \leq \gamma^m \mu(\mathbf{B}(f(a), \gamma r) \sim T)$. Therefore $Df(a)[\text{Tan}^m(\psi, a)] \subseteq \text{Tan}^m(\mu, f(a))$ by [Fed69, 3.1.21, p. 234] and [Fed69, 3.2.16, p. 252].

2.13 Remark. If θ is the characteristic function of P then, by [Fed69, 2.4.5], we have that $\psi = \mathcal{H}^m \llcorner P$ and $\mu = \mathcal{H}^m \llcorner f|P$.

2.14 Lemma. *Suppose $1 \leq k \leq \nu$ are integers, $E \subseteq \mathbf{R}^\nu$ is countably (\mathcal{H}^k, k) rectifiable and \mathcal{H}^k measurable, θ is a $\mathcal{H}^k \llcorner E$ measurable \mathbf{R} valued map such that*

$$\int_E \theta d\mathcal{H}^k < \infty, \quad \theta(z) > 0 \text{ for } \mathcal{H}^k \text{ a.e. } z \in E,$$

and ψ is the measure over \mathbf{R}^ν given by

$$\psi(S) = \int_{E \cap S}^* \theta d\mathcal{H}^k \quad \text{whenever } S \subseteq \mathbf{R}^\nu.$$

Then $\text{Tan}^k(\psi, z)$ is a k dimensional plane contained in $\text{Tan}^k(\mathcal{H}^k \llcorner E, z)$ for \mathcal{H}^k a.e. $z \in E$ and

$$\text{Tan}^k(\mathcal{H}^k \llcorner F, z) \subseteq \text{Tan}^k(\psi, z) \quad \text{for } \mathcal{H}^k \text{ a.e. } z \in F,$$

whenever $F \subseteq E$ is \mathcal{H}^k measurable such that $\mathcal{H}^k(F) < \infty$.

Proof. Firstly we observe that $\psi(S) = 0$ if and only if $\mathcal{H}^k(S) = 0$. Therefore \mathbf{R}^ν is (ψ, k) rectifiable and, employing [Fed69, 2.4.10, 2.10.19(3)],

$$\Theta^{*k}(\psi, z) < \infty \quad \text{for } \psi \text{ a.e. } z \in \mathbf{R}^\nu.$$

We apply [Fed69, 3.3.18] to conclude that $\text{Tan}^k(\psi, z) \in \mathbf{G}(n, k)$ for \mathcal{H}^k a.e. $z \in E$. If $F \subseteq E$ is \mathcal{H}^k measurable and $\mathcal{H}^k(F) < \infty$, we define

$$F_i = F \cap \{z : \theta(z) \geq i^{-1}\} \quad \text{for every integer } i \geq 1,$$

we observe that $\text{Tan}^k(\mathcal{H}^k \llcorner F, z) = \text{Tan}^k(\mathcal{H}^k \llcorner F_i, z)$ for \mathcal{H}^k a.e. $z \in F_i$ by [Fed69, 2.10.19(4)], and we use [Fed69, 3.2.16] to conclude

$$\text{Tan}^k(\mathcal{H}^k \llcorner F, z) \subseteq \text{Tan}^k(\psi, z) \quad \text{for } \mathcal{H}^k \text{ a.e. } z \in F.$$

Since by [Fed69, 3.2.14] the set E can be \mathcal{H}^k almost covered by countably many \mathcal{H}^k measurable k rectifiable subsets of \mathbf{R}^ν , we may apply [Fed69, 3.2.19] to conclude that $\text{Tan}^k(\psi, z) \subseteq \text{Tan}^k(\mathcal{H}^k \llcorner E, z)$ for \mathcal{H}^k a.e. $z \in E$.

Normal bundle of smooth submanifolds We recall the basic structural result on the normal bundle of a submanifold of class 2 of Euclidean space.

2.15 Lemma. *Let $M \subseteq \mathbf{R}^n$ be an m dimensional submanifold of class 2 and let $N = \text{Nor}(M) \cap (M \times \mathbf{S}^{n-1})$.*

Then N is a $n - 1$ dimensional submanifold of class 1 of $\mathbf{R}^n \times \mathbf{R}^n$ and, if $(a, u) \in N$ then $\text{Tan}(N, (a, u))$ is the set of $(\tau, v + \text{D}\nu(a)(\tau))$ such that $\tau \in \text{Tan}(M, a)$, $v \in \text{Nor}(M, a)$ is orthogonal to u and ν is a unit normal vector field of class 1 on an open neighborhood of a such that $\nu(a) = u$.

Proof. The conclusion is a direct consequence of the fact that, using a normal frame of M in an open neighborhood Z of a , we can locally parametrize N at (a, u) using the product manifold $(M \cap Z) \times \mathbf{S}^{n-m-1}$.

2.16 Remark. If $(a, u) \in N$, $\tau \in \text{Tan}(M, a)$, $\tau_1 \in \text{Tan}(M, a)$ and $\sigma_1 \in \mathbf{R}^n$ is such that $(\tau_1, \sigma_1) \in \text{Tan}(N, (a, u))$, then

$$\tau \bullet \sigma_1 = -\mathbf{b}_M(a)(\tau, \tau_1) \bullet u.$$

3 Approximate differentiability of the nearest point projection

In this section we provide the basic technical tools that are used in the subsequent sections. In particular we study the approximate differentiability properties of the nearest point projection onto an arbitrary closed set A in 3.6, and the second-order differentiability properties of certain subsets of the level sets of δ_A in 3.10.

3.1 Definition. Suppose $A \subseteq \mathbf{R}^n$ is closed and U is the set of all $x \in \mathbf{R}^n$ such that there exists a unique $a \in A$ with $|x - a| = \delta_A(x)$. The *nearest point projection onto A* is the map ξ_A characterised by the requirement

$$|x - \xi_A(x)| = \delta_A(x) \quad \text{for } x \in U.$$

We let ν_A and ψ_A to be the functions on $U \sim A$ such that

$$\nu_A(z) = \delta_A(z)^{-1}(z - \xi_A(z)) \quad \text{and} \quad \psi_A(z) = (\xi_A(z), \nu_A(z)),$$

whenever $z \in U \sim A$.

3.2 Remark. The notation agree with [Fed59, 4.1].

It is proved in [Fed59, 4.8(4)], [MS17, 3.5] and [Fed59, 4.8(2)] that ξ_A is continuous, $\text{dmn } \xi_A$ is a Borel subset of \mathbf{R}^n and $\xi_A^{-1}\{a\}$ is a convex subset of \mathbf{R}^n whenever $a \in A$.

3.3 Remark. If $U = (\text{dmn } \xi_A) \sim A$, then, noting 3.2, we readily infer that for every $0 < r < \infty$ the map $\psi_A|U \cap \delta_A^{-1}\{r\}$ is an homeomorphism with

$$(\psi_A|U \cap \delta_A^{-1}\{r\})^{-1}(a, u) = a + ru \quad \text{whenever } (a, u) \in \psi_A[U \cap \delta_A^{-1}\{r\}].$$

3.4 Remark. If $v \in \mathbf{R}^n \sim \{0\}$, $a \in A$ and $|v| = \delta_A(a + v)$, then $a + tv \in (\text{dmn } \xi_A) \sim A$ and $\xi_A(a + tv) = a$ whenever $0 < t < 1$.

3.5 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is closed, $x \in (\text{dmn } \xi_A) \sim A$, ξ_A is approximately differentiable at x and $T = \mathbf{R}^n \cap \{v : v \bullet \nu_A(x) = 0\}$.*

Then δ_A is differentiable at x , ν_A is approximately differentiable at x ,

$$\text{apD } \xi_A(x) \bullet \nu_A(x) = 0 \quad \text{and} \quad \text{apD } \nu_A(x) = |x - \xi_A(x)|^{-1}(T_{\mathfrak{h}} - \text{apD } \xi_A(x)).$$

In particular $\ker \text{apD } \psi_A(x) \subseteq T^\perp$.

Proof. Since $\delta_A(y) = |y - \xi_A(y)|$ for $y \in \text{dmn } \xi_A$, we use 2.4 and 2.5 to deduce that δ_A is differentiable at x . Moreover ν_A is approximately differentiable at x by 2.4. Let $r = |x - \xi_A(x)|$ and we use the continuity of ξ_A (see 3.2) to select $0 < \delta < r$ such that $|\xi_A(z) - \xi_A(x)| < r$ whenever $z \in \mathbf{U}(x, \delta) \cap \text{dmn } \xi_A$. Since $|\xi_A(z) - x| \geq r$ whenever $z \in \text{dmn } \xi_A$, we compute

$$\begin{aligned} (\xi_A(z) - x) \bullet \nu_A(x) &= (\xi_A(z) - \xi_A(x)) \bullet \nu_A(x) - r < 0, \\ |(\xi_A(z) - x) \bullet \nu_A(x)|^2 &= |\xi_A(z) - x|^2 - |T_{\mathfrak{h}}(\xi_A(z) - \xi_A(x))|^2 \\ &\geq r^2 - |T_{\mathfrak{h}}(\xi_A(z) - \xi_A(x))|^2, \\ (\xi_A(z) - \xi_A(x)) \bullet \nu_A(x) &\leq r - (r^2 - |T_{\mathfrak{h}}(\xi_A(z) - \xi_A(x))|^2)^{1/2}, \end{aligned}$$

whenever $z \in \mathbf{U}(x, \delta) \cap \text{dmn } \xi_A$. Therefore $\text{apD } \xi_A(x) \bullet \nu_A(x) = 0$ by 2.1 and 2.4. Using 2.4 we can now easily compute the desired formula for $\text{apD } \nu_A(x)$, whence we deduce the postscript.

3.6 Theorem. *If A is a closed subset of \mathbf{R}^n and if we define:*

the sets A_λ corresponding to $1 < \lambda < \infty$, given by $x \in (\text{dmn } \xi_A) \sim A$ such that $\delta_A(\xi_A(x) + \lambda(x - \xi_A(x))) = \lambda\delta_A(x)$,

the sets D_λ corresponding to $1 < \lambda < \infty$, given by $x \in A_\lambda$ such that $\Theta^n(\mathcal{L}^n \llcorner \mathbf{R}^n \sim A_\lambda, x) = 0$ and ξ_A is approximately differentiable at x ,

the maps h_t on $(\text{dmn } \xi_A) \sim A$ corresponding to $0 < t < \infty$, given by

$$h_t(z) = \xi_A(z) + t(z - \xi_A(z)) \quad \text{whenever } z \in (\text{dmn } \xi_A) \sim A,$$

then the following four statements hold for $1 < \lambda < \infty$ and $0 < t < \lambda$.

- (1) A_λ is a Borel subset of \mathbf{R}^n , $\text{Lip}(\xi_A|A_\lambda) \leq \lambda(\lambda - 1)^{-1}$ and $h_t|A_\lambda$ is a bi-Lipschitzian homeomorphism onto $A_{\lambda/t}$ with $(h_t|A_\lambda)^{-1} = h_{t^{-1}}|A_{\lambda/t}$.
- (2) $\mathcal{L}^n(A_\lambda \sim D_\lambda) = 0$, the map $\psi_A|A_\lambda$ has an extension $\Psi : \mathbf{R}^n \rightarrow \mathbf{R}^n \times \mathbf{R}^n$ such that Ψ is differentiable at every $x \in D_\lambda$ with $\text{D}\Psi(x) = \text{apD } \psi_A(x)$, and $\ker \text{apD } \psi_A(x) = \{s\nu_A(x) : s \in \mathbf{R}\}$ whenever $x \in D_\lambda$.

(3) If $x \in D_\lambda$, then $h_t(x) \in D_{\lambda/t}$, the map $h_{t^{-1}}$ is approximately differentiable at $h_t(x)$ with $\text{ap D } h_{t^{-1}}(h_t(x)) = \text{ap D } h_t(x)^{-1}$ and

$$\text{ap D } \psi_A(x) = \text{ap D } \psi_A(h_t(x)) \circ \text{ap D } h_t(x).$$

(4) If $x \in D_\lambda$, the eigenvalues of $\text{ap D } \xi_A(x)$ and $\text{ap D } \nu_A(x)$ belong to the intervals $0 \leq s \leq \lambda(\lambda - 1)^{-1}$ and $(1 - \lambda)^{-1}\delta_A(x)^{-1} \leq s \leq \delta_A(x)^{-1}$, respectively. In case $\text{ap D } \xi_A(x)$ is a symmetric endomorphism, so are $\text{ap D } \xi_A(h_t(x))$ and $\text{ap D } \nu_A(h_t(x))$.

Proof of (1). By 3.2 the set A_λ is a Borel subset of \mathbf{R}^n . If $x \in A_\lambda$ and $y \in A_\lambda$, then we apply [MS17, 4.7(1)] with q , a , b and v replaced by $\lambda|x - \xi_A(x)|$, $\xi_A(x)$, $\xi_A(y)$ and $x - \xi_A(x)$ respectively, to infer that

$$(\xi_A(y) - \xi_A(x)) \bullet (x - \xi_A(x)) \leq (2\lambda)^{-1} |\xi_A(x) - \xi_A(y)|^2,$$

and symmetrically,

$$(\xi_A(x) - \xi_A(y)) \bullet (y - \xi_A(y)) \leq (2\lambda)^{-1} |\xi_A(x) - \xi_A(y)|^2.$$

Combining the two equations we get

$$|\xi_A(x) - \xi_A(y)| |x - y| \geq (\xi_A(x) - \xi_A(y)) \bullet (x - y) \geq \lambda^{-1}(\lambda - 1) |\xi_A(x) - \xi_A(y)|^2.$$

By 3.4, one infers $\xi_A(h_t(x)) = \xi_A(x)$ and $h_{t^{-1}}(h_t(x)) = x$ whenever $x \in A_\lambda$, and $h_t[A_\lambda] \subseteq A_{\lambda/t}$. Since $0 < t^{-1} < \lambda/t$, the same conclusions hold with λ and t replaced by λ/t and t^{-1} respectively, and (1) is proved.

Proof of (2). By (1), [Fed69, 2.10.19(4), 2.10.35] and [San17, 2.11(1)], we conclude that $\mathcal{L}^n(A_\lambda \sim D_\lambda) = 0$. By (1) and [Fed69, 2.10.43], the map $\xi_A|_{A_\lambda}$ has a Lipschitzian extension $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$. Then, by 2.5, the map F is differentiable at every $x \in D_\lambda$ with

$$D F(x) = \text{ap D } \xi_A(x).$$

Defining $G : \mathbf{R}^n \sim A \rightarrow \mathbf{R}^n$ as $G(x) = \delta_A(x)^{-1}(x - F(x))$ for $x \in \mathbf{R}^n \sim A$, we notice that G is differentiable at every $x \in D_\lambda$ with $D G(x) = \text{ap D } \nu_A(x)$ by 3.5 and [Fed59, 4.8(3)] and the existence of the extension of $\psi_A|_{A_\lambda}$ is proved.

Finally, if $x \in D_\lambda$, we notice that $D F(x)(\nu_A(x)) = 0$, since $F(x + s\nu_A(x)) = \xi_A(x)$ for every $-\delta_A(x) < s < (\lambda - 1)\delta_A(x)$. Then, using 3.5, we get the remaining part of (2).

Proof of (3). If $y = h_t(x)$, we notice that h_t is approximately differentiable at x and $h_{t^{-1}}(y) = x$ by (1). Therefore, by 2.3, (1) and [Buc92, Theorem 1], we infer that

$\text{ap D } h_t(x)$ is an isomorphism of \mathbf{R}^n ,

$$\Theta^n(\mathcal{L}^n \llcorner \mathbf{R}^n \sim A_{\lambda/t}, y) = 0,$$

$$\Theta^n(\mathcal{L}^n \llcorner A_{\lambda/t} \cap \{z : |h_{t^{-1}}(z) - x - \text{ap D } h_t(x)^{-1}(z - y)| > \epsilon|z - y|\}, y) = 0$$

whenever $\epsilon > 0$. Therefore $h_{t^{-1}}$ is approximately differentiable at y with

$$\text{ap D } h_{t^{-1}}(y) = \text{ap D } h_t(x)^{-1}.$$

Let Ψ be an extension of $\psi_A|_{A_\lambda}$ given by (2). Since $\psi_A(z) = (\Psi \circ h_{t^{-1}})(z)$ whenever $z \in A_{\lambda/t}$ by 3.4, we use 2.4 to infer that ψ_A is approximately differentiable at y with

$$\text{ap D } \psi_A(y) = \text{ap D } \psi_A(x) \circ \text{ap D } h_{t^{-1}}(y).$$

Proof of (4). If $\mu \in \mathbf{R}$, $v \in \mathbf{S}^{n-1}$ and $\text{ap D } \xi_A(x)(v) = \mu v$, then $(1-s)\mu + s \neq 0$ whenever $0 < s < \lambda$ and $0 \leq \mu \leq \lambda(\lambda-1)^{-1}$, since $\text{ap D } h_s(x)$ is injective by (3). If $\mu \neq 0$, $v \in \mathbf{S}^{n-1}$ and $\text{ap D } \nu_A(x)(v) = \mu v$ then $v \bullet \nu_A(x) = 0$, $\text{ap D } \xi_A(x)(v) = (1 - \delta_A(x)\mu)v$ by 3.5, and $(1-\lambda)^{-1}\delta_A(x)^{-1} \leq \mu \leq \delta_A(x)^{-1}$.

If $\text{ap D } \xi_A(x)$ is symmetric, then there exists an orthonormal basis v_1, \dots, v_n of \mathbf{R}^n and $0 \leq \mu_1 \leq \dots \leq \mu_n$ such that $\text{ap D } \xi_A(x)(v_i) = \mu_i v_i$ whenever $i = 1, \dots, n$, by [Fed69, 1.7.3], and

$$\text{ap D } \xi_A(h_t(x))(v_i) = \mu_i((1-t)\mu_i + t)^{-1}v_i \quad \text{whenever } i = 1, \dots, n,$$

by (3). Therefore $\text{ap D } \xi_A(h_t(x))$ is symmetric and so is $\text{ap D } \nu_A(h_t(x))$, by 3.5.

3.7 Remark. Combining 3.5 and 3.6(3), if $1 < \lambda < \infty$, $0 < t < \lambda$, $x \in D_\lambda$ and $T = \mathbf{R}^n \cap \{v : v \bullet \nu_A(x) = 0\}$, then

$$\text{im ap D } \xi_A(h_t(x)) = \text{im ap D } \xi_A(x) \subseteq T,$$

$$\text{im ap D } \nu_A(h_t(x)) = \text{im ap D } \nu_A(x) \subseteq T.$$

3.8 Remark. If $0 < R = \text{reach}(A)$, $0 < r < R$ and $0 < \delta_A(x) \leq r$ then, by [Fed59, 4.8(6)],

$$\sup\{t : \xi_A(\xi_A(x) + t(x - \xi_A(x))) = \xi_A(x)\} \geq R/r;$$

in particular, $\mathbf{R}^n \cap \{x : 0 < \delta_A(x) \leq r\} \subseteq A_{R/r}$.

3.9 Remark. In case A is convex, the map h_t is called ‘‘dilation with center A ’’ in [Wal76, §3].

3.10 Theorem. *Suppose $1 < \lambda < \infty$, A_λ and D_λ are as in 3.6,*

$$M_r = \delta_A^{-1}\{r\} \quad \text{and} \quad N_r = M_r \cap A_\lambda \quad \text{for every } 0 < r < \infty.$$

Then the following four statements hold.

(1) *For every $r > 0$, $\psi_A|_{N_r}$ is a bi-Lipschitzian homeomorphism and $\psi_A[N_r]$ is a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable closed subset of $\mathbf{R}^n \times \mathbf{S}^{n-1}$ with locally finite \mathcal{H}^{n-1} measure.*

(2) *For every $r > 0$, N_r is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable of class 2 with locally finite \mathcal{H}^{n-1} measure, $\text{ap Tan}(N_r, x) \in \mathbf{G}(n, n-1)$ and*

$$-|v|^2 \leq \text{ap D}^2 N_r(x)(v, v) \bullet (x - \xi_A(x)) \leq (\lambda-1)^{-1}|v|^2,$$

whenever $v \in \text{ap Tan}(N_r, x)$, for \mathcal{H}^{n-1} a.e. $x \in N_r$.

(3) *For \mathcal{L}^1 a.e. $r > 0$ and for \mathcal{H}^{n-1} a.e. $x \in N_r$,*

$$\text{ap Tan}(N_r, x) = \text{Tan}(M_r, x) = \mathbf{R}^n \cap \{v : v \bullet \nu_A(x) = 0\}$$

and there exist an open neighborhood V of x and a Lipschitzian function $f : \text{Tan}(M_r, x) \rightarrow \text{Nor}(M_r, x)$, pointwise differentiable of order 2 at $\text{Tan}(M_r, x)_\natural(x)$, such that

$$V \cap M_r = V \cap \{\chi + f(\chi) : \chi \in \text{Tan}(M_r, x)\}, \quad \text{D}f(\text{Tan}(M_r, x)_\natural(x)) = 0,$$

$$\text{pt D}^2 f(\text{Tan}(M_r, x)_\natural(x)) \circ \odot_2 \text{Tan}(M_r, x)_\natural = \text{ap D}^2 N_r(x).$$

(4) For \mathcal{L}^1 a.e. $r > 0$, $\mathcal{H}^{n-1}(N_r \sim D_\lambda) = 0$,

$$(\mathcal{H}^{n-1} \llcorner N_r, n-1) \text{apD} \psi_A(x) = \text{apD} \psi_A(x) | \text{apTan}(N_r, x),$$

$$\text{apD}^2 N_r(x)(u, v) \bullet \nu_A(x) = - \text{apD} \nu_A(x)(u) \bullet v,$$

whenever $u, v \in \text{apTan}(N_r, x)$ and for \mathcal{H}^{n-1} a.e. $x \in N_r$.

Proof of (1). Firstly we notice that whenever $0 < r < \infty$ the map $\psi_A|_{N_r}$ is a bi-Lipschitzian homeomorphism by 3.3 and 3.6(1) and

$$\psi_A[N_r] = (A \times \mathbf{S}^{n-1}) \cap \{(a, u) : \delta_A(a + \lambda ru) = \lambda r\}.$$

Therefore $\psi_A[N_r]$ is closed and $\psi_A[N_r] \subseteq \psi_A[N_s]$ whenever $0 < s < r < \infty$. Moreover $\psi_A[N_r] \cap (\mathbf{B}(0, t) \times \mathbf{S}^{n-1}) \subseteq \psi_A[N_r \cap \mathbf{B}(0, t+r)]$ whenever $r > 0$ and $t > 0$. Therefore (1) follows from 2.7.

Proof of (2). Let $r > 0$. By 3.6(1),

$$\psi_A[N_r \cap \mathbf{B}(x, t)] \subseteq \psi_A[N_r] \cap (\mathbf{B}(\xi_A(x), \lambda(\lambda-1)^{-1}t) \times \mathbf{S}^{n-1})$$

whenever $t > 0$ and $x \in N_r$, whence we deduce that N_r has locally finite \mathcal{H}^{n-1} measure by (1). Evidently $\mathbf{U}(\xi_A(x), r) \cap M_r = \emptyset$ whenever $x \in M_r$, and by 2.6

$$\mathbf{U}(\xi_A(x) + \lambda(x - \xi_A(x)), (\lambda-1)r) \cap M_r = \emptyset \quad \text{whenever } x \in N_r.$$

Noting that $\xi_A(x) = x - r\nu_A(x)$ and $\xi_A(x) + \lambda(x - \xi_A(x)) = x + (\lambda-1)r\nu_A(x)$ for $x \in N_r$, we conclude that

$$(*) \quad \limsup_{t \rightarrow 0} t^{-2} \sup\{\delta_T(z-x) : z \in \mathbf{U}(x, t) \cap M_r\} < \infty,$$

whenever $x \in N_r$ and $T = \{v : v \bullet \nu_A(x) = 0\}$. Therefore, applying [San17, 5.4] with $k = 1$ and $\alpha = 1$, [Fed69, 3.1.15] and [San17, 3.23, 4.12], we conclude that N_r is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable of class 2, N_r is approximately differentiable of order 2 with $\text{apTan}(N_r, x) \in \mathbf{G}(n, n-1)$ for \mathcal{H}^{n-1} a.e. $x \in N_r$ and the desired estimate for $\text{apD}^2 N_r(x)$ holds at \mathcal{H}^{n-1} a.e. $x \in N_r$.

Proof of (3). First, we notice $\text{grad} \delta_A(x) = \nu_A(x)$ for \mathcal{H}^{n-1} a.e. $x \in M_r$ and for \mathcal{L}^1 a.e. $r > 0$ by 2.9 and [Fed59, 4.8(3)]. Second, we apply 2.10, (*) and [Men16, 5.7(3)] to infer that M_r is pointwise differentiable of order 2 with

$$\text{Tan}(M_r, x) = \mathbf{R}^n \cap \{v : v \bullet \nu_A(x) = 0\},$$

at \mathcal{H}^{n-1} a.e. $x \in N_r$ and for \mathcal{L}^1 a.e. $r > 0$. Third, M_r and N_r are approximately differentiable of order 2 with $\text{apTan}(N_r, x) = \text{apTan}(M_r, x) = \text{Tan}(M_r, x)$ and

$$\text{apD}^2 N_r(x) = \text{apD}^2 M_r(x) = \text{ptD}^2 M_r(x, \text{Tan}(M_r, x)),$$

at \mathcal{H}^{n-1} a.e. $x \in N_r$ and for \mathcal{L}^1 a.e. $r > 0$, as may be inferred from (2), [San17, 3.23, 3.22] and [Fed69, 2.10.19(4)]. Fourth, if we fix $r > 0$ and $x \in N_r$ such that the statements above hold and δ_A is differentiable at x , then we may combine 2.9 and [Men16, 3.14] to infer the existence of V and f as required by (3).

Proof of (4). By 3.6(2) and [Fed69, 3.2.11] we get that $\mathcal{H}^{n-1}(N_r \sim D_\lambda) = 0$ for \mathcal{L}^1 a.e. $r > 0$. Therefore for such a number $r > 0$, by 3.6(2) and [Fed69, 3.2.16], the map ψ_A is $(\mathcal{H}^{n-1} \llcorner N_r, n-1)$ approximately differentiable at \mathcal{H}^{n-1} a.e. $x \in N_r$ with

$$(\mathcal{H}^{n-1} \llcorner N_r, n-1) \text{apD } \psi_A(x) = \text{apD } \psi_A(x) | \text{apTan}(N_r, x).$$

Therefore, by [San17, 3.25], (1), (2) and (3),

$$\text{apD}^2 N_r(x)(u, v) \bullet \nu_A(x) = - \text{apD } \nu_A(x)(u) \bullet v,$$

for \mathcal{L}^1 a.e. $r > 0$, \mathcal{H}^{n-1} a.e. $x \in N_r$ and whenever $u, v \in \text{apTan}(N_r, x)$.

4 Second fundamental form

For an arbitrary closed set we introduce the normal bundle in 4.1, the second fundamental form in 4.11 and the associated principal curvatures in 4.14. Classical facts (including a well known integral geometric formula), which have been previously proved for various special classes of closed subsets, are extended to arbitrary closed sets in 4.14. Moreover, a basic estimate is proved in 4.12 and we compute in 4.19 the second fundamental form of the image of an arbitrary closed set under smooth diffeomorphisms.

4.1 Definition. Suppose A is a closed subset of \mathbf{R}^n . We define

$$N(A) = (A \times \mathbf{S}^{n-1}) \cap \{(a, u) : \delta_A(a + su) = s \text{ for some } s > 0\}.$$

Moreover we let $N(A, a) = \{v : (a, v) \in N(A)\}$ for $a \in A$.

4.2 Remark. We notice that $N(A)$ coincides with the *normal bundle of A* introduced in [HLW04, §2.1]. Moreover, we let $\text{Dis}(A)$ to be the *distance bundle of A* introduced in [MS17, 4.1] and we recall from [MS17, 4.2, 4.6] that $\text{Dis}(A, a)$ is a closed convex subset of $\text{Nor}(A, a)$ and

$$N(A) = \{(a, |v|^{-1}v) : 0 \neq v \in \text{Dis}(A, a)\}.$$

4.3 Remark. Using the notation of 3.6 and noting 3.4, we observe that

$$N(A) = \psi_A[(\text{dmn } \xi_A) \sim A] = \psi_A[A_\lambda] = \bigcup_{r>0} \psi_A[A_\lambda \cap \delta_A^{-1}\{r\}],$$

whenever $1 < \lambda < \infty$. Therefore, by 3.10(1), it follows that $N(A)$ is a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable Borel subset of $\mathbf{R}^n \times \mathbf{S}^{n-1}$. There are closed sets A for which $N(A)$ does not have locally finite \mathcal{H}^{n-1} measure.

4.4 Remark. Suppose $\text{reach}(A) = R > 0$, $1 < \lambda < \infty$ and $0 < r < R\lambda^{-1}$. Since $A_{R/r} \subseteq A_\lambda$, we deduce by 3.8, 4.2, 4.3, [Fed59, 4.8(12)] and the displayed equation in the proof of 3.10(1), that

$$\delta_A^{-1}\{r\} = A_\lambda \cap \delta_A^{-1}\{r\}, \quad \psi_A[\delta_A^{-1}\{r\}] = \text{Nor}(A) \cap (A \times \mathbf{S}^{n-1}).$$

Therefore $N(A) = \text{Nor}(A) \cap (A \times \mathbf{S}^{n-1})$ and $N(A)$ has locally finite \mathcal{H}^{n-1} measure by 3.10(1).

In 4.5–4.10 we provide the necessary preliminaries for 4.11.

4.5 Definition. If A is a closed subset of \mathbf{R}^n , we define

$$\rho(A, x) = \sup\{t : \delta_A(\xi_A(x) + t(x - \xi_A(x))) = t\delta_A(x)\},$$

whenever $x \in (\text{dmn } \xi_A) \sim A$.

4.6 Remark. Evidently $1 \leq \rho(A, x) \leq \infty$ whenever $x \in (\text{dmn } \xi_A) \sim A$. Moreover (see 3.6) $A_\lambda = \{y : \rho(A, y) \geq \lambda\}$ whenever $\lambda > 1$; whence we deduce that $\rho(A, \cdot)$ is a Borel map whose domain is a Borel subset of \mathbf{R}^n by 3.2 and 3.6(1).

4.7 Definition. If f is a function mapping a subset of \mathbf{R}^n into $\overline{\mathbf{R}}$ and $a \in \mathbf{R}^n$, we define

$$\text{aplim inf}_{x \rightarrow a} f(x) = \sup\{s : \Theta^n(\mathcal{L}^n \llcorner \{x : f(x) < s\}, a) = 0\}.$$

4.8 Definition. Suppose A is a closed subset of \mathbf{R}^n and $x \in (\text{dmn } \xi_A) \sim A$. We say that x is a *smooth point of ξ_A* if and only if

- (1) $\text{aplim inf}_{y \rightarrow x} \rho(A, y) \geq \rho(A, x) > 1$,
- (2) ξ_A is approximately differentiable at x ,
- (3) $\text{apD } \xi_A(x)$ is a symmetric endomorphism of \mathbf{R}^n .

We say that a point $(a, u) \in N(A)$ is a *smooth point of $N(A)$* if and only if $(a, u) = \psi_A(x)$ for some smooth point x of ξ_A .

4.9 Lemma. *If $A \subseteq \mathbf{R}^n$ is closed, then the following three statements hold.*

- (1) *If x is a smooth point of ξ_A , then $\xi_A(x) + t(x - \xi_A(x))$ is a smooth point of ξ_A whenever $0 < t < \rho(A, x)$.*
- (2) *\mathcal{L}^n a.e. $x \in (\text{dmn } \xi_A) \sim A$ such that $\rho(A, x) > 1$ is a smooth point of ξ_A .*
- (3) *\mathcal{H}^{n-1} a.e. $(a, u) \in N(A)$ is a smooth point of $N(A)$.*

Proof. We define h_t as in 3.6 for every $0 < t < \infty$. If x is a smooth point of ξ_A and $0 < t < \rho(A, x)$, then $\rho(A, h_t(x)) = t^{-1}\rho(A, x)$ by 3.4 and, choosing $1 < \lambda < \rho(A, x)$ so that $0 < t < \lambda$ and employing 4.6 and 3.6(3)(4), we conclude that ξ_A is approximately differentiable at $h_t(x)$, $\text{apD } \xi_A(h_t(x))$ is symmetric and $\text{aplim inf}_{y \rightarrow h_t(x)} \rho(A, y) \geq \lambda/t$, whence, letting $\lambda \rightarrow \rho(A, x)$, we get that $h_t(x)$ is a smooth point of ξ_A and (1) is proved.

We observe that $\text{aplim}_{y \rightarrow x} \rho(A, y) = \rho(A, x)$ for \mathcal{L}^n a.e. $x \in (\text{dmn } \xi_A) \sim A$, by 4.6 and [Fed69, 2.9.13, 2.10.19(4)]; by [Fed69, 3.2.11], the same conclusion holds for \mathcal{H}^{n-1} a.e. $x \in \delta_A^{-1}\{r\} \cap \text{dmn } \xi_A$ and for \mathcal{L}^1 a.e. $r > 0$. Combining 3.5, 3.6(2) and 3.10(3)(4), we infer that ξ_A is approximately differentiable at x and $\text{apD } \xi_A(x)$ is symmetric for \mathcal{H}^{n-1} a.e. $x \in \{y : \rho(A, y) > 1\} \cap \delta_A^{-1}\{r\}$ and for \mathcal{L}^1 a.e. $r > 0$; by [Fed69, 3.2.11], the same conclusion holds at \mathcal{L}^n a.e. $x \in \{y : \rho(A, y) > 1\}$. Therefore (2) is proved and (3) follows from 4.6, 4.3 and 3.10(1).

4.10 Lemma. *Suppose A is a closed subset of \mathbf{R}^n and x is a smooth point of ξ_A . Then the following statements hold.*

(1) If $v, v_1, v_2 \in \mathbf{R}^n$ are such that $\text{ap D } \xi_A(x)(v_1) = \text{ap D } \xi_A(x)(v_2)$, then

$$\begin{aligned} \text{ap D } \xi_A(x)(v) \bullet \text{ap D } \nu_A(x)(v_1) &= \text{ap D } \xi_A(x)(v) \bullet \text{ap D } \nu_A(x)(v_2), \\ \text{ap D } \xi_A(x)(v_1) \bullet \text{ap D } \nu_A(x)(v) &= \text{ap D } \xi_A(x)(v) \bullet \text{ap D } \nu_A(x)(v_1). \end{aligned}$$

(2) If $0 < t < \rho(A, x)$, $y = \xi_A(x) + t(x - \xi_A(x))$ and $v, w, v_1, w_1 \in \mathbf{R}^n$ are such that (see 4.9(1))

$$\text{ap D } \xi_A(y)(w) = \text{ap D } \xi_A(x)(v), \quad \text{ap D } \xi_A(y)(w_1) = \text{ap D } \xi_A(x)(v_1),$$

then

$$\text{ap D } \nu_A(x)(v_1) \bullet \text{ap D } \xi_A(x)(v) = \text{ap D } \nu_A(y)(w_1) \bullet \text{ap D } \xi_A(y)(w).$$

Proof. Let $r = |x - \xi_A(x)|$. In order to prove (1) we use 3.5 and 3.6(2) to compute

$$\begin{aligned} &\text{ap D } \xi_A(x)(v) \bullet \text{ap D } \nu_A(x)(v_1) \\ &= r^{-1}v \bullet [\text{ap D } \xi_A(x)(v_1) - (\text{ap D } \xi_A(x) \circ \text{ap D } \xi_A(x))(v_1)] \\ &= r^{-1}v \bullet [\text{ap D } \xi_A(x)(v_2) - (\text{ap D } \xi_A(x) \circ \text{ap D } \xi_A(x))(v_2)] \\ &= \text{ap D } \xi_A(x)(v) \bullet \text{ap D } \nu_A(x)(v_2), \\ &\text{ap D } \xi_A(x)(v) \bullet \text{ap D } \nu_A(x)(v_1) \\ &= r^{-1}v \bullet [\text{ap D } \xi_A(x)(v_1) - (\text{ap D } \xi_A(x) \circ \text{ap D } \xi_A(x))(v_1)] \\ &= r^{-1} \text{ap D } \xi_A(x)(v_1) \bullet [v - \text{ap D } \xi_A(x)(v)] \\ &= \text{ap D } \xi_A(x)(v_1) \bullet \text{ap D } \nu_A(x)(v). \end{aligned}$$

In order to prove (2) we use 3.5 and 3.6(2)(3)(4) to get

$$\begin{aligned} \text{ap D } \xi_A(y)(w_1) &= \text{ap D } \xi_A(x)(v_1) = \text{ap D } \xi_A(x)(T_{\xi}(v_1)) \\ &= \text{ap D } \xi_A(y)[\text{ap D } \xi_A(x)(v_1) + t(T_{\xi}(v_1) - \text{ap D } \xi_A(x)(v_1))] \\ &= \text{ap D } \xi_A(y)[\text{ap D } \xi_A(y)(w_1) + tr \text{ap D } \nu_A(x)(v_1)], \\ t^{-1}r^{-1}[\text{ap D } \xi_A(y)(w_1) - (\text{ap D } \xi_A(y) \circ \text{ap D } \xi_A(y))(w_1)] \\ &= (\text{ap D } \xi_A(y) \circ \text{ap D } \nu_A(x))(v_1), \\ \text{ap D } \nu_A(x)(v_1) \bullet \text{ap D } \xi_A(x)(v) \\ &= \text{ap D } \nu_A(x)(v_1) \bullet \text{ap D } \xi_A(y)(w) \\ &= (\text{ap D } \xi_A(y) \circ \text{ap D } \nu_A(x))(v_1) \bullet w \\ &= t^{-1}r^{-1}[\text{ap D } \xi_A(y)(w_1) - (\text{ap D } \xi_A(y) \circ \text{ap D } \xi_A(y))(w_1)] \bullet w \\ &= \text{ap D } \nu_A(y)(w_1) \bullet \text{ap D } \xi_A(y)(w). \end{aligned}$$

4.11 Definition. Suppose A is a closed subset of \mathbf{R}^n and (a, u) is a smooth point of $N(A)$. We define

$$T_A(a, u) = \text{im ap D } \xi_A(x) \quad \text{and} \quad Q_A(a, u)(\tau, \tau_1) = \tau \bullet \text{ap D } \nu_A(x)(v_1),$$

whenever x is a smooth point of ξ_A such that $\psi_A(x) = (a, u)$, $\tau \in T_A(a, u)$, $\tau_1 \in T_A(a, u)$ and $v_1 \in \mathbf{R}^n$ such that $\text{ap D } \xi_A(x)(v_1) = \tau_1$.

We call $Q_A(a, u)$ *second fundamental form of A at a in the direction u* .

4.12 Lemma. *Let (a, u) be a smooth point of $N(A)$.*

Then $Q_A(a, u) : T_A(a, u) \times T_A(a, u) \rightarrow \mathbf{R}$ is a symmetric bilinear form and $T_A(a, u) \subseteq \{v : v \bullet u = 0\}$. Moreover if $r > 0$ and $\delta_A(a + ru) = r$, then

$$Q_A(a, u)(\tau, \tau) \geq -r^{-1}|\tau|^2 \quad \text{whenever } \tau \in T_A(a, u).$$

Proof. If x and y are smooth points of ξ_A such that $\psi_A(x) = (a, u) = \psi_A(y)$ then $y = \xi_A(x) + (\delta_A(y)/\delta_A(x))(x - \xi_A(x))$, and the first part of the conclusion follows from 3.7 and 4.10.

If $0 < s < r$ then $a + su$ is a smooth point of ξ_A by 4.9(1), and $\psi_A(a + su) = (a, u)$. If $\tau \in T_A(a, u)$ and $v \in \mathbf{R}^n$ is such that $\text{apD } \xi_A(a + su)(v) = \tau$, then from 3.5 and 3.6(4) we compute

$$\begin{aligned} Q_A(a, u)(\tau, \tau) &= \text{apD } \xi_A(a + su)(v) \bullet \text{apD } \nu_A(a + su)(v) \\ &= s^{-1} \text{apD } \xi_A(a + su)(v) \bullet (T_{\natural}(v) - \text{apD } \xi_A(a + su)(v)) \\ &\geq -s^{-1} |\text{apD } \xi_A(a + su)(v)|^2 = -s^{-1} |\tau|^2. \end{aligned}$$

Letting $s \rightarrow r$ we get the second conclusion.

4.13 Remark. A similar estimate is proved in [San17, 4.12] for different notions of curvature.

4.14 Theorem. *Suppose $A \subseteq \mathbf{R}^n$ is closed, $\theta \in \mathbf{L}_1^{\text{loc}}(\mathcal{H}^{n-1} \llcorner N(A))$, θ is $\mathcal{H}^{n-1} \llcorner N(A)$ almost positive and ψ is the measure over $\mathbf{R}^n \times \mathbf{R}^n$ such that*

$$\psi(S) = \int_{S \cap N(A)}^* \theta d\mathcal{H}^{n-1} \quad \text{whenever } S \subseteq \mathbf{R}^n \times \mathbf{R}^n.$$

For each smooth point (a, u) of $N(A)$ we define $\kappa_1(a, u) \leq \dots \leq \kappa_{n-1}(a, u)$ so that $\kappa_{m+1}(a, u) = \infty$, $\kappa_1(a, u), \dots, \kappa_m(a, u)$ are the eigenvalues of $Q_A(a, u)$ and $m = \dim T_A(a, u)$.

Then the following three statements hold.

- (1) *For \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)$, $\text{Tan}^{n-1}(\psi, (a, u))$ is a $(n-1)$ dimensional plane contained in $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner N(A), (a, u))$ and there exist $u_1(a, u), \dots, u_{n-1}(a, u)$ such that $u_1(a, u), \dots, u_{n-1}(a, u), u$ is an orthonormal basis of \mathbf{R}^n and*

$$\left(\frac{1}{(1 + \kappa_i(a, u)^2)} u_i(a, u), \frac{\kappa_i(a, u)}{(1 + \kappa_i(a, u)^2)} u_i(a, u) \right) \quad \text{for } i = 1, \dots, n-1$$

is an orthonormal basis of $\text{Tan}^{n-1}(\psi, (a, u))$. [Here and in the sequel, a function $a : \mathbf{R} \rightarrow \mathbf{R}$ is extended to ∞ by $a(\infty) = \lim_{k \rightarrow \infty} a(k)$.]

- (2) *For \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)$,*

$$T_A(a, u) = \mathbf{p}[\text{Tan}^{n-1}(\psi, (a, u))] \quad \text{and} \quad Q_A(a, u)(\tau, \tau_1) = \tau \bullet \sigma_1$$

whenever $\tau \in T_A(a, u)$, $\tau_1 \in T_A(a, u)$ and $(\tau_1, \sigma_1) \in \text{Tan}^{n-1}(\psi, (a, u))$.

- (3) *For every $(\mathcal{H}^{n-1} \llcorner N(A))$ integrable $\overline{\mathbf{R}}$ valued function f on $N(A)$,*

$$\begin{aligned} &\int_{N(A)} f(a, u) \prod_{i=1}^{n-1} \frac{|\kappa_i(a, u)|}{(1 + \kappa_i(a, u)^2)^{1/2}} d\mathcal{H}^{n-1}(a, u) \\ &= \int_{\mathbf{S}^{n-1}} \int_{\{a : (a, v) \in N(A)\} \times \{v\}} f d\mathcal{H}^0 d\mathcal{H}^{n-1} v. \end{aligned}$$

Proof. We choose $\lambda > 1$. For every $r > 0$ we let N_r to be as in 3.10 and S_r to be the set $x \in N_r$ such that x is a smooth point of ξ_A , $\Theta^n(\mathcal{L}^n \llcorner \mathbf{R}^n \sim A_\lambda, x) = 0$,

$$\begin{aligned} \text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner N_r, x) &= \mathbf{R}^n \cap \{v : v \bullet \nu_A(x) = 0\}, \\ \text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \psi_A[N_r], \psi_A(x)) &\in \mathbf{G}(n, n-1). \end{aligned}$$

By 3.10(1)(3), 4.6, 4.9(2) and [Fed69, 2.10.19(4), 3.2.11, 3.2.19] we infer that $\mathcal{H}^{n-1}(N_r \sim S_r) = 0$ and $\mathcal{H}^{n-1}(\psi_A[N_r] \sim \psi_A[S_r]) = 0$ for \mathcal{L}^1 a.e. $r > 0$. By 3.6(2), 3.10(1), 2.12 and 2.13, we notice that

$$\text{ap D } \psi_A(x) [\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner N_r, x)] = \text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \psi_A[N_r], \psi_A(x)),$$

whenever $x \in S_r$ and for every $r > 0$. Therefore, if $r > 0$, $x \in S_r$,

$$\chi_1 \leq \dots \leq \chi_{n-1} \text{ are the eigenvalues of } \text{ap D } \nu_A(x) | \text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner N_r, x),$$

and v_1, \dots, v_{n-1} is an orthonormal basis of $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner N_r, x)$ such that

$$\text{ap D } \nu_A(x)(v_i) = \chi_i v_i \quad \text{for } i = 1, \dots, n-1,$$

then we apply 3.5 to conclude that

$$\text{ap D } \xi_A(x)(v_i) = (1 - \delta_A(x)\chi_i)v_i \quad \text{for } i = 1, \dots, n-1,$$

$$\chi_i = \delta_A(x)^{-1} \quad \text{for } i > \dim T_A(\psi_A(x)),$$

$$Q_A(\psi_A(x))(v_i, v_j) = \chi_j(1 - \delta_A(x)\chi_j)^{-1} v_i \bullet v_j \quad \text{for } i, j \leq \dim T_A(\psi_A(x)),$$

$$\kappa_i(\psi_A(x)) = \chi_i(1 - \delta_A(x)\chi_i)^{-1} \quad \text{for } i \leq \dim T_A(\psi_A(x)),$$

and an orthonormal basis of $\text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner \psi_A[N_r], \psi_A(x))$ is given by

$$\left(\frac{1}{(1 + \kappa_i(\psi_A(x))^2)^{1/2}} v_i, \frac{\kappa_i(\psi_A(x))}{(1 + \kappa_i(\psi_A(x))^2)^{1/2}} v_i \right) \quad \text{for } i = 1, \dots, n-1.$$

Therefore by 4.3, 3.10(1) and 2.14 we get (1) and (2).

Finally, when f is a nonnegative $(\mathcal{H}^{n-1} \llcorner N(A))$ measurable $\overline{\mathbf{R}}$ valued function, we may apply [Fed69, 3.2.22(3)] with W , Z and f replaced by $\psi_A[N_r]$, \mathbf{S}^{n-1} and $\mathbf{q}|\psi_A[N_r]$ to conclude

$$\begin{aligned} & \int_{\psi_A[N_r]} f(a, u) \prod_{i=1}^{n-1} |\kappa_i(a, u)| (1 + \kappa_i(a, u)^2)^{-1/2} d\mathcal{H}^{n-1}(a, u) \\ &= \int_{\mathbf{S}^{n-1}} \int_{\{a: (a, v) \in \psi_A[N_r]\} \times \{v\}} f d\mathcal{H}^0 d\mathcal{H}^{n-1} v \end{aligned}$$

for \mathcal{L}^1 a.e. $r > 0$ and (3) is a consequence of 4.3 and [Fed69, 2.4.7]. The general case asserted in (3) is then a consequence of [Fed69, 2.4.4].

4.15 Remark. In case $\text{reach}(A) > 0$, it follows from 4.4 and 4.14(2) that Q_A coincides with the second fundamental form of A introduced in [Fu89, 4.5] on \mathcal{H}^{n-1} almost all of $N(A)$.

4.16 Remark. The principal curvatures on $N(A)$ introduced in [HLW04, p. 244], to give an explicit representation of the *support measures* $\mu_0(A; \cdot), \dots, \mu_{n-1}(A; \cdot)$ of A in [HLW04, Corollary 2.5], coincide on \mathcal{H}^{n-1} almost all of $N(A)$ with the functions κ_i introduced in 4.14. This follows from 4.14(1). The support measures arise as coefficient measures of a general Steiner formula for A , see [HLW04, Theorem 2.1].

4.17 Corollary. *If A_1 and A_2 are closed subsets of \mathbf{R}^n , then*

$$Q_{A_1}(a, u) = Q_{A_2}(a, u) \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(A_1) \cap N(A_2).$$

Proof. By 3.10(1) and 4.3 we can choose two Borel functions, θ_1 and θ_2 , such that $\theta_1 = \theta_2$ on \mathcal{H}^{n-1} almost all of $N(A_1) \cap N(A_2)$ and the hypothesis of 4.14 are satisfied with A and θ replaced by A_1 and θ_1 and by A_2 and θ_2 . If we let

$$\psi_1(S) = \int_{S \cap N(A_1)}^* \theta_1 d\mathcal{H}^{n-1} \quad \text{and} \quad \psi_2(S) = \int_{S \cap N(A_2)}^* \theta_2 d\mathcal{H}^{n-1}$$

whenever $S \subseteq \mathbf{R}^n \times \mathbf{R}^n$, we notice that $\psi_1 \llcorner N(A_2) = \psi_2 \llcorner N(A_1)$ and we may apply 2.14 to conclude that

$$\text{Tan}^{n-1}(\psi_2, \zeta) = \text{Tan}^{n-1}(\psi_2 \llcorner N(A_1), \zeta) = \text{Tan}^{n-1}(\psi_1, \zeta) \in \mathbf{G}(n, n-1)$$

for \mathcal{H}^{n-1} a.e. $\zeta \in N(A_1) \cap N(A_2)$. Therefore the conclusion comes from 4.14(2).

The following result is employed in the computation of the formula in 4.19, as well as in the subsequent sections.

4.18 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is a closed set, $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a diffeomorphism of class 2 onto \mathbf{R}^n and $\nu_F : \mathbf{R}^n \times \mathbf{S}^{n-1} \rightarrow \mathbf{R}^n \times \mathbf{S}^{n-1}$ is given by*

$$\nu_F(a, u) = \left(F(a), \frac{(\mathbf{D}F(a)^{-1})^*(u)}{|(\mathbf{D}F(a)^{-1})^*(u)|} \right) \quad \text{whenever } (a, u) \in \mathbf{R}^n \times \mathbf{S}^{n-1}.$$

Then ν_F is a diffeomorphism of class 1 onto $\mathbf{R}^n \times \mathbf{S}^{n-1}$ such that

$$(\nu_F)^{-1} = \nu_{F^{-1}} \quad \text{and} \quad \nu_F[N(A)] = N(F[A]).$$

Proof. A direct computation shows that ν_F is a diffeomorphism of class 1 onto $\mathbf{R}^n \times \mathbf{S}^{n-1}$ with $(\nu_F)^{-1} = \nu_{F^{-1}}$.

If $(a, u) \in N(A)$ and $r > 0$ such that $\mathbf{U}(a + ru, r) \cap A = \emptyset$, we let

$$v = (\mathbf{D}F(a)^{-1})^*(u), \quad W = F[\mathbf{U}(a + ru, r)], \quad S = \text{Bdry } W.$$

Since $S = F[\text{Bdry } \mathbf{U}(a + ru, r)]$, by [Fed69, 3.1.21] we conclude that

$$\mathbf{D}F(a)[\text{Tan}(\text{Bdry } \mathbf{U}(a + ru, r), a)] = \text{Tan}(S, F(a)),$$

and, consequently, $v \in \text{Nor}(S, F(a))$. If $s = \text{reach}(S, F(a))$, then by [Fed59, 4.11, 4.8(12)] we conclude that $s > 0$ and $\mathbf{U}(F(a) + s(v/|v|), s) \cap S = \emptyset$. Therefore,

either $\mathbf{U}(F(a) + s(v/|v|), s) \subseteq W$ or $\mathbf{U}(F(a) + s(v/|v|), s) \subseteq \mathbf{R}^n \sim \text{Clos } W$.

If $\gamma(t) = F(a + tu)$ for $t \in \mathbf{R}$, noting that $\dot{\gamma}(0) \bullet v = 1$ and

$$D_t(|\gamma(t) - F(a) - s(v/|v|)|)(0) = -1/|v|,$$

we conclude that $\gamma(t) \in \mathbf{U}(F(a) + s(v/|v|), s)$ for $t > 0$ sufficiently small,

$$\mathbf{U}(F(a) + s(v/|v|), s) \subseteq W \quad \text{and} \quad \nu_F(a, u) \in N(F[A]).$$

Therefore $\nu_F[N(A)] \subseteq N(F[A])$ and replacing F by F^{-1} and A by $F[A]$ we conclude

$$\nu_F[N(A)] = N(F[A]).$$

4.19 Theorem. *Suppose A is a closed subset of \mathbf{R}^n and $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a diffeomorphism of class 2 onto \mathbf{R}^n .*

Then (see 4.18) $DF(a)[T_A(a, u)] = T_{F[A]}(\nu_F(a, u))$ and

$$\begin{aligned} & Q_{F[A]}(\nu_F(a, u)) \circ \odot_2(DF(a)|T_A(a, u)) \\ &= |(DF(a)^{-1})^*(u)|^{-1} Q_A(a, u) \\ & \quad + (D^2 F(a)|\odot_2 T_A(a, u)) \bullet ((DF(a)^{-1})^*(u)/|(DF(a)^{-1})^*(u)|), \end{aligned}$$

for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)$.

Proof. We define, for $(a, u) \in \mathbf{R}^n \times (\mathbf{R}^n \sim \{0\})$,

$$g(a, u) = (DF(a)^{-1})^*(u)/|(DF(a)^{-1})^*(u)|.$$

If $(a, u) \in \mathbf{R}^n \times (\mathbf{R}^n \sim \{0\})$, $(\tau, \sigma) \in \mathbf{R}^n \times \mathbf{R}^n$ and

$$\zeta = (DF(a)^{-1})^*(\sigma) - ((DF(a)^{-1})^* \circ D(DF)(a)(\tau)^* \circ (DF(a)^{-1})^*)(u),$$

then we compute

$$Dg(a, u)(\tau, \sigma) = |(DF(a)^{-1})^*(u)|^{-1}(\zeta - (g(a, u) \bullet \zeta)g(a, u)).$$

If θ and ψ are as in 4.14, noting 4.18, we define the measure μ over $\mathbf{R}^n \times \mathbf{R}^n$ by

$$\mu(S) = \int_{N(F[A]) \cap S}^* \theta \circ \nu_{F^{-1}} d\mathcal{H}^{n-1} \quad \text{whenever } S \subseteq \mathbf{R}^n \times \mathbf{R}^n,$$

and we apply 2.11 with $\gamma = \text{Lip}(\nu_F|_{\nu_{F^{-1}}[K]})$ to conclude that

$$\int_{N(F[A]) \cap K} \theta \circ \nu_{F^{-1}} d\mathcal{H}^{n-1} \leq \gamma^{n-1} \int_{N(A) \cap \nu_{F^{-1}}[K]} \theta d\mathcal{H}^{n-1} < \infty,$$

whenever $K \subseteq \mathbf{R}^n \times \mathbf{S}^{n-1}$ is compact. Combining 4.18, 4.14(1)(2) and 2.12 we conclude

$$\begin{aligned} & \text{Tan}^{n-1}(\psi, (a, u)) \text{ is a } (n-1) \text{ dimensional plane,} \\ & T_A(a, u) = \mathbf{p}[\text{Tan}^{n-1}(\psi, (a, u))], \quad Q_A(a, u)(\tau, \tau_1) = \tau \bullet \sigma_1, \\ & D\nu_F(a, u)[\text{Tan}^{n-1}(\psi, (a, u))] = \text{Tan}^{n-1}(\mu, \nu_F(a, u)), \\ & DF(a)[T_A(a, u)] = T_{F[A]}(\nu_F(a, u)), \end{aligned}$$

$$Q_{F[A]}(\nu_F(a, u))(DF(a)(\tau), DF(a)(\tau_1)) = DF(a)(\tau) \bullet Dg(a, u)(\tau_1, \sigma_1),$$

whenever $\tau \in T_A(a, u)$, $\tau_1 \in T_A(a, u)$, $(\tau_1, \sigma_1) \in \text{Tan}^{n-1}(\psi, (a, u))$ and for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)$. Since

$$DF(a)(\tau) \bullet g(a, u) = |(DF(a)^{-1})^*(u)|^{-1} u \bullet \tau = 0$$

whenever (a, u) is a smooth point of $N(A)$ and $\tau \in T_A(a, u)$ by 4.12, the conclusion follows.

5 Curvature and second-order-rectifiable strata

We introduce a natural stratification in 5.1 and we extend a classical integral-geometric formula to arbitrary closed sets in 5.6. Then we prove in 5.9 the structural theorem on the second fundamental form announced in section 1.

5.1 Definition. Suppose A is a closed subset of \mathbf{R}^n . For each $0 \leq m \leq n$, we define the m -th stratum of A by

$$A^{(m)} = A \cap \{a : \dim \xi_A^{-1}\{a\} = n - m\}.$$

[The dimension of a convex subset K of \mathbf{R}^n is the dimension of the affine hull of K and it is denoted by $\dim K$.]

5.2 Remark. Noting [MS17, 4.4, 4.12] and [Fed69, 3.2.14], we observe that $A^{(m)}$ can be covered, up to a set of \mathcal{H}^m measure zero, by countably many m dimensional submanifolds of class 2 of \mathbf{R}^n and there exists countably many m rectifiable Borel subsets B_i of \mathbf{R}^n (in particular, $\mathcal{H}^m(B_i) < \infty$) such that $B_i \subseteq B_{i+1}$ whenever $i \geq 1$ and

$$A^{(m)} = \bigcup_{i=1}^{\infty} B_i.$$

5.3 Remark. We infer from [MS17, 4.4] that

$$\dim \text{Dis}(A, a) = \dim \xi_A^{-1}\{a\} \quad \text{whenever } a \in A.$$

Moreover, from 4.2, if $m = 0, \dots, n-1$ and $a \in A^{(m)}$ then there exists $P \in \mathbf{G}(n, n-m)$ such that $N(A, a) \subseteq P \cap \mathbf{S}^{n-m-1}$ and

$$0 < \mathcal{H}^{n-m-1}(N(A, a)) < \infty.$$

In particular, $\{a : N(A, a) \neq \emptyset\} = \bigcup_{m=0}^{n-1} A^{(m)}$ and $\{a : N(A, a) = \emptyset\} = A^{(n)}$.

5.4 Remark. Combining 4.18 and 5.3 we conclude that if $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a diffeomorphism of class 2 onto \mathbf{R}^n and $m = 0, \dots, n$, then

$$F[A]^{(m)} = F[A^{(m)}].$$

5.5 Lemma. *If $A \subseteq \mathbf{R}^n$ is closed, $0 \leq m \leq n-1$ is an integer and $x \in \xi_A^{-1}[A^{(m)}]$ satisfies 4.8(1)(2), then $\dim \text{im apD } \xi_A(x) \leq m$.*

In particular, $\dim T_A(a, u) \leq m$ whenever (a, u) is a smooth point of $N(A)$ such that $a \in A^{(m)}$.

Proof. Let $a = \xi_A(x)$, $1 < \lambda < \rho(A, x)$, $B = \{y : \rho(A, y) \geq \lambda\}$ and $C = \xi_A^{-1}[\{a\}] \cap B$. Then C is a convex subset of \mathbf{R}^n and $\dim C = \dim \xi_A^{-1}\{a\}$. In fact, $C = \{y : \delta_A(a + \lambda(y - a)) = \lambda|y - a|\}$ by 4.6 and 3.4 and C is convex by [Fed59, 4.8(2)]. Moreover, if U is the relative interior of $\xi_A^{-1}\{a\}$ (the relative interior of a convex subset K of \mathbf{R}^n is the interior of K relative to the affine hull of K), then $\{y : a + \lambda(y - a) \in U\}$ is contained in C and it is open relative to the affine hull of $\xi_A^{-1}\{a\}$. Therefore $\dim C = \dim \xi_A^{-1}\{a\}$.

By 3.6(2), let $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be an extension of $\xi_A|_B$ that is differentiable at x with $\text{D}F(x) = \text{apD } \xi_A(x)$. Since $F(y) = a$ whenever $y \in C$, we conclude that $\text{D}F(x)(y - x) = 0$ whenever $y \in C$. Therefore $\text{D}F(x)(y - x) = 0$ whenever y belongs to the affine hull of C . Since $\dim C = n - m$, we conclude that $\dim \text{im apD } \xi_A(x) \leq m$. The postscript readily follows.

5.6 Theorem. *If f is a $(\mathcal{H}^{n-1} \llcorner N(A))$ integrable $\overline{\mathbf{R}}$ valued function, then*

$$\begin{aligned} & \int_{N(A)|A^{(m)}} f(a, u) \prod_{i=1}^m \frac{1}{(1 + \kappa_i(a, u)^2)^{1/2}} d\mathcal{H}^{n-1}(a, u) \\ &= \int_{A^{(m)}} \int_{\{z\} \times N(A, z)} f d\mathcal{H}^{n-m-1} d\mathcal{H}^m z, \end{aligned}$$

where $\kappa_1, \dots, \kappa_{n-1}$ are defined as in 4.14.

Proof. We assume $f \geq 0$ on \mathcal{H}^{n-1} almost all of $N(A)$, since, as usual, the general case follows from [Fed69, 2.4.4]. Since $A^{(0)}$ is a countable set by 5.2, the case $m = 0$ is a consequence of [Fed69, 2.4.8]. Therefore we assume $m \geq 1$.

Suppose $\lambda > 1$, N_r is as in 3.10 for every $0 < r < \infty$ and B_i is as in 5.2 for every integer $i \geq 1$. We notice that $\kappa_{m+1}(a, u) = \infty$ for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)|A^{(m)}$, by 5.5. Therefore for every $i \geq 1$ we apply [Fed69, 3.2.22(3)] with W, Z and f replaced by $\psi_A[N_{1/i}] \cap \mathbf{p}^{-1}[B_i]$, B_i and \mathbf{p} and, noting 4.14(1), we conclude

$$\begin{aligned} & \int_{\psi_A[N_{1/i}] \cap \mathbf{p}^{-1}[B_i]} f(a, u) \prod_{i=1}^m \frac{1}{(1 + \kappa_i(a, u)^2)^{1/2}} d\mathcal{H}^{n-1}(a, u) \\ &= \int_{B_i} \int_{\mathbf{p}^{-1}[\{z\}] \cap \psi_A[N_{1/i}]} f d\mathcal{H}^{n-m-1} d\mathcal{H}^m z. \end{aligned}$$

Letting $i \rightarrow \infty$, the conclusion comes from [Fed69, 2.4.7], 4.3 and 5.2.

5.7 Remark. We can combine 5.5, 5.6 and 4.16 to infer the following explicit formulas for the support measures of A : for every $m = 0, \dots, n-1$,

$$\mu_m(A; B|A^{(m)}) = \frac{1}{(n-m)\alpha(n-m)} \int_{A^{(m)}} \mathcal{H}^{n-m-1}\{v : (z, v) \in B\} d\mathcal{H}^m z,$$

whenever $B \subseteq N(A)$ is \mathcal{H}^{n-1} measurable. If $m = n-1$ this formula is contained in [HLW04, Proposition 4.1]. If $\text{reach}(A) > 0$, the same formulas are proved in [Hug98, Theorem 3.2].

5.8 Remark. It follows by 5.6 that if $S \subseteq A^{(m)}$ and $\mathcal{H}^m(S) = 0$, then

$$\dim T_A(a, u) \leq m-1 \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(A)|S.$$

5.9 Theorem. *If A is a closed subset of \mathbf{R}^n and $0 \leq m \leq n-1$, then there exists a Borel set $R \subseteq A^{(m)}$ such that*

- (1) $\mathcal{H}^m(A^{(m)} \sim R) = 0$;
- (2) $\mathcal{H}^{n-1}(N(A)|S) = 0$, whenever $S \subseteq R$ such that $\mathcal{H}^m(S) = 0$;
- (3) $Q_A(a, u) = -\mathbf{b}_M(a) \bullet u$ for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)|R \cap M$, whenever M is an m dimensional submanifold of class 2 of \mathbf{R}^n .

Proof. Suppose $\lambda > 1$ and N_r is as in 3.10 for $r > 0$. For every integer $i \geq 1$ we let $C_i = \psi_A[N_{1/i}]$ and B_i to be as in 5.2. By 3.10(1) and [Fed69, 2.2.17] we notice that $\mathcal{H}^m \llcorner B_i$ and $\mathbf{p}_\#(\mathcal{H}^{n-1} \llcorner C_i)$ are Radon measures of \mathbf{R}^n whenever $i \geq 1$. Therefore for every $i \geq 1$ and $j \geq 1$ we can apply [Fed69, 2.9.2] with

ϕ and ψ replaced by $\mathcal{H}^m \llcorner B_i$ and $\mathbf{p}_\#(\mathcal{H}^{n-1} \llcorner C_j)$ to infer the existence of a Borel set $R_{i,j} \subseteq B_i$ such that $\mathcal{H}^m(B_i \sim R_{i,j}) = 0$ and

$$\mathcal{H}^{n-1}(C_j|S) = 0 \quad \text{whenever } S \subseteq R_{i,j} \text{ such that } \mathcal{H}^m(S) = 0.$$

If we let $R = \bigcup_{i=1}^\infty \bigcap_{j=1}^\infty R_{i,j}$, then one may readily check, using 4.3 and 5.2, that R is a Borel set satisfying (1) and (2).

Suppose M is a m dimensional submanifold of class 2 of \mathbf{R}^n , Z is the set of $a \in A^{(m)} \cap M$ such that $N(A, a) \subseteq \text{Nor}(M, a)$ and $N = \text{Nor}(M) \cap (M \times \mathbf{S}^{n-1})$. Since by [Fed69, 2.10.19(4)] we have

$$\text{Tan}(M, a) \subseteq \text{Tan}^m(\mathcal{H}^m \llcorner A^{(m)}, a) \subseteq \text{Tan}(A, a) \quad \text{for } \mathcal{H}^m \text{ a.e. } a \in A^{(m)} \cap M,$$

we conclude by 4.2 that $\mathcal{H}^m(A^{(m)} \cap M \sim Z) = 0$. We choose a measure ψ as in 4.14. Since $N(A)|Z \subseteq N$, we can use 2.14, 4.14(1)(2), 2.15 and 2.16 to get that

$$\text{Tan}^{n-1}(\psi, (a, u)) = \text{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner N(A)|Z, (a, u)) = \text{Tan}(N, (a, u)),$$

$$T_A(a, u) = \text{Tan}(M, a), \quad Q_A(a, u) = -\mathbf{b}_M(a) \bullet u,$$

for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)|Z$. Since $\mathcal{H}^{n-1}(N(A)|R \cap M \sim Z) = 0$ by (2), the conclusion follows.

5.10 Remark. Noting 5.2 and [San17, 3.23], we can describe Q_A on the subsets of R of finite \mathcal{H}^m measure, in terms of the approximate second fundamental form introduced in [San17, p. 3].

5.11 Remark. The conclusion of 5.9(2) may fail to hold if we replace “ $S \subseteq R$ ” with “ $S \subseteq A^{(m)}$ ”. Suppose $C \subseteq \mathbf{R}$ is compact, $0 < s < 1$, $0 < \mathcal{H}^s(C) < \infty$,

$$f(x) = \mathcal{H}^s(C \cap \{z : z \leq x\}) \quad \text{whenever } x \in C,$$

and g is a primitive of f . Then g is a non-decreasing convex function of class 1 on \mathbf{R} . We let $A = \mathbf{R}^2 \cap \{(x, y) : g(x) \leq y\}$ and $S = \{(x, g(x)) : x \in C\}$ and we notice that A is a closed convex set,

$$N(A, (x, g(x))) = \{(1 + f(x)^2)^{-1/2}(f(x), -1)\} \quad \text{whenever } x \in \mathbf{R},$$

$S \subseteq A^{(1)}$, $\mathcal{H}^1(S) = 0$ and $\mathcal{H}^1(\mathbf{q}(N(A))) > 0$. Since $\mathbf{q}(N(A)|A \sim S)$ is a countable subset of \mathbf{S}^1 , we conclude that $\mathcal{H}^1(N(A)|S) > 0$. We notice that $\dim T_A(a, u) = 0$ for \mathcal{H}^1 a.e. $(a, u) \in N(A)|S$ by 5.8. Therefore $\kappa_1(a, u) = \infty$ for \mathcal{H}^1 a.e. $(a, u) \in N(A)|S$; see 4.14.

6 Lusin’s condition (N) for the normal bundle

We establish the area formula for the generalized Gauss map in 6.6 and the general criterion for second-order-differentiability in 6.10 for the special class of closed subsets whose normal bundle satisfies the Lusin condition (N) in 6.1.

6.1 Definition. Suppose $A \subseteq \mathbf{R}^n$ is a closed set, $\Omega \subseteq \mathbf{R}^n$ is an open set and $1 \leq m < n$ is an integer. We say that $N(A)$ satisfies the *m dimensional (Lusin’s) condition (N) in Ω* if and only if

$$\mathcal{H}^{n-1}(N(A)|S) = 0, \quad \text{whenever } S \subseteq A \cap \Omega \text{ such that } \mathcal{H}^m(A^{(m)} \cap S) = 0.$$

In case $\Omega = \mathbf{R}^n$, we say that $N(A)$ satisfies the *m dimensional (Lusin’s) condition (N)*.

6.2 Remark. If A is a closed subset of \mathbf{R}^n , Ω is an open subset of \mathbf{R}^n and $C = \text{Clos}(A \cap \Omega)$, then one may easily check that $N(A)|\Omega = N(C)|\Omega$. Therefore $A^{(m)} \cap \Omega = C^{(m)} \cap \Omega$ for every $m = 0, \dots, n$ by 5.3 and $Q_A(\zeta) = Q_C(\zeta)$ for \mathcal{H}^{n-1} a.e. $\zeta \in N(A)|\Omega$ by 4.17. Consequently, if $N(A)$ satisfies the m dimensional condition (N) , then $N(C)$ satisfies the m dimensional condition (N) in Ω .

6.3 Remark. If $\mathcal{H}^{n-1}(N(A)) > 0$, then there is at most one integer $m = 0, \dots, n-1$ so that $N(A)$ satisfies the m dimensional condition (N) .

6.4 Lemma. *Suppose $U \subseteq \mathbf{R}^n$ is open, $A \subseteq \mathbf{R}^n$ is closed, $N(A)$ satisfies the m dimensional condition (N) in U and $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a diffeomorphism of class 2 onto \mathbf{R}^n .*

Then $N(F[A])$ satisfies the m dimensional condition (N) in $F[U]$.

Proof. Suppose $S \subseteq F[A] \cap F[U]$ such that $\mathcal{H}^m(F[A]^{(m)} \cap S) = 0$. Since $F^{-1}[S] \subseteq A \cap U$ and $0 = \mathcal{H}^m(F^{-1}[S \cap F[A]^{(m)}]) = \mathcal{H}^m(F^{-1}[S] \cap A^{(m)})$ by 5.4, we conclude that

$$\mathcal{H}^{n-1}(N(A)|F^{-1}[S]) = 0.$$

Therefore, by 4.18,

$$\nu_F[N(A)|F^{-1}[S]] = N(F[A])|S, \quad \mathcal{H}^{n-1}(N(F[A])|S) = 0.$$

6.5 Remark. If in 6.1 we replace $\mathcal{H}^{n-1}(N(A)|S) = 0$ by

$$\mathcal{H}^{n-1}(\{v : (a, v) \in N(A)|S\}) = 0,$$

then the resulting property is not preserved under diffeomorphisms of class 2, as the following example shows for $n = 3$ and $m = 2$.

If $A = \mathbf{R}^3 \cap \{(x, y, z) : z = |x|\}$ and $F : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is defined by $F(x, y, z) = (x, y, z + 1 - x^2 - y^2)$ for $(x, y, z) \in \mathbf{R}^3$, then

$$A^{(0)} = \emptyset, \quad A^{(1)} = \mathbf{R}^3 \cap \{(x, y, z) : x = z = 0\}, \quad A^{(2)} = A \sim A^{(1)},$$

$$\mathcal{H}^2(\{v : (a, v) \in N(A)|S\}) = 0 \quad \text{whenever } S \subseteq A \text{ and } \mathcal{H}^2(S \cap A^{(2)}) = 0,$$

$$\{v : (a, v) \in N(F[A])|F[A]^{(1)}\} \quad \text{has not empty relatively interior in } \mathbf{S}^2.$$

The area formula for the generalized Gauss map in the next result is a consequence of 4.14(3) and 5.6.

6.6 Theorem. *Suppose $1 \leq m < n$ is an integer, $\Omega \subseteq \mathbf{R}^n$ is open, $A \subseteq \mathbf{R}^n$ is closed and $N(A)$ satisfies the m dimensional condition (N) in Ω .*

Then for every \mathcal{H}^{n-1} measurable set $B \subseteq N(A)|\Omega$,

$$\begin{aligned} & \int_{\mathbf{S}^{n-1}} \mathcal{H}^0\{a : (a, u) \in B\} d\mathcal{H}^{n-1}u \\ &= \int_A \int_{\{z\} \times \{v : (z, v) \in B\}} |\text{discr } Q_A| d\mathcal{H}^{n-m-1} d\mathcal{H}^m z. \end{aligned}$$

Proof. If $\kappa_1, \dots, \kappa_{n-1}$ are as in 4.14, we notice, by 5.2 and 5.9, that

$$\kappa_{m+1}(a, u) = \infty \quad \text{and} \quad \text{discr } Q_A(a, u) = \prod_{i=1}^m \kappa_i(a, u),$$

for \mathcal{H}^{n-1} a.e. $(a, u) \in N(A)|\Omega$. Therefore, using 4.14(3) and 5.6, we compute

$$\begin{aligned}
& \int_{\mathbf{S}^{n-1}} \mathcal{H}^0\{a : (a, v) \in B\} d\mathcal{H}^{n-1}v \\
&= \int_B \prod_{i=1}^{n-1} |\kappa_i(a, u)|(1 + \kappa_i(a, u)^2)^{-1/2} d\mathcal{H}^{n-1}(a, u) \\
&= \int_{B|A^{(m)}} |\text{discr } Q_A(a, u)| \prod_{i=1}^m (1 + \kappa_i(a, u)^2)^{-1/2} d\mathcal{H}^{n-1}(a, u) \\
&= \int_{A^{(m)}} \int_{\{z\} \times \{v : (z, v) \in B\}} |\text{discr } Q_A| d\mathcal{H}^{n-m-1} d\mathcal{H}^m z,
\end{aligned}$$

whenever $B \subseteq N(A)|\Omega$ is \mathcal{H}^{n-1} measurable.

We need the following elementary estimate in the proof of 6.9.

6.7 Lemma. *Suppose V and W are finite dimensional vector spaces with inner products such that $\dim V = m$ and $\dim W = n$, $f \in \text{Hom}(V, W)$, $0 < t < \infty$ and $b \in \odot^2 W$ such that $b(w, w) \leq t|w|^2$ whenever $w \in W$.*

Then

$$\|f\|^2 \text{trace}(b) + (1 - n)t\|f\|^2 \leq \text{trace}(b \circ \odot_2 f) \leq mt\|f\|^2.$$

Proof. By [Fed69, 1.7.3] we can choose an orthonormal basis v_1, \dots, v_m of V and an orthonormal basis w_1, \dots, w_n of W such that

$$(f^* \circ f)(v_i) \bullet v_j = 0 \quad \text{and} \quad b(w_i, w_j) = 0,$$

whenever $i \neq j$. If we define $c(w, z) = t(w \bullet z) - b(w, z)$ whenever $w, z \in W$, noting $\|f\| = \|f^*\|$ by [Fed69, 1.7.6], we compute

$$\begin{aligned}
\text{trace}(c \circ \odot_2 f) &= \sum_{i=1}^m \sum_{j=1}^n (f(v_i) \bullet w_j)^2 c(w_j, w_j) \\
&= \sum_{j=1}^n |f^*(w_j)|^2 c(w_j, w_j) \leq \|f\|^2 (nt - \text{trace } b), \\
\text{trace}(c \circ \odot_2 f) &= t \sum_{i=1}^m |f(v_i)|^2 - \text{trace}(b \circ \odot_2 f) \geq t\|f\|^2 - \text{trace}(b \circ \odot_2 f).
\end{aligned}$$

Combining the two equations we get the left side. The right side is trivial.

6.8 Definition. If $0 < t < \infty$, $a \in \mathbf{R}^n$ and $T \in \mathbf{G}(n, n-1)$, we define

$$C_t(T, a) = \mathbf{R}^n \cap \{x : |T_{\frac{1}{2}}(x - a)| < t, |T_{\frac{1}{4}}(x - a)| < t\}.$$

The criterion for second-order-differentiability in 6.10, that is the central result of this section, can be deduced by standard arguments from the somewhat more subtle result in 6.9.

6.9 Theorem. *If $1 \leq m < n$ are integers, then there exist $0 < \delta < \infty$ and $0 < \sigma < \infty$ such that the following statement holds.*

If $A \subseteq \mathbf{R}^n$ is a closed set, $N(A)$ satisfies the m dimensional condition (N) , $a \in A$, $0 < r < \infty$, $T \in \mathbf{G}(n, n-1)$ and the following two hypothesis hold,

(1) *there exists $v \in \mathbf{S}^{n-1}$ such that $T_{\frac{1}{2}}(v) = 0$ and*

$$\sup\{v \bullet (x - a) : x \in \text{Clos}(A \cap C_{4r}(T, a))\} \leq r/16;$$

(2) there exists a nonnegative \mathcal{H}^{n-1} measurable function f on $N(A)$ such that

$$\text{trace } Q_A(x, u) \leq f(x, u) \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (x, u) \in N(A) \setminus C_{2r}(T, a),$$

$$\int_{C_{2r}(T, a) \cap A} \int_{\{z\} \times N(A, z)} f^m d\mathcal{H}^{n-m-1} d\mathcal{H}^m z \leq \delta;$$

then there exists a Borel set $M \subseteq N(A) \setminus \text{Clos } C_{(11/8)r}(T, a)$ such that

$$\delta_A(x + (r/2)u) = r/2 \quad \text{whenever } (x, u) \in M,$$

$$\mathcal{H}^{n-m-1}\{v : (x, v) \in M\} > 0 \quad \text{whenever } x \in \mathbf{p}[M],$$

and $\mathcal{H}^m(\mathbf{p}[M]) \geq \sigma r^m$.

Proof. We assume $a = 0$ and we let $C_t = C_t(T, 0)$ whenever $0 < t < \infty$.

By 6.2 we notice that $N(\text{Clos}(A \cap C_{4r}))$ satisfies the m dimensional condition (N) in C_{4r} and we replace A with $\text{Clos}(A \cap C_{4r})$. We consider the diffeomorphism $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ given by

$$F(x) = x + (r/8)v - (4r)^{-1}|T_{\frac{1}{2}}(x)|^2 v \quad \text{whenever } x \in \mathbf{R}^n$$

and we compute

$$F^{-1}(x) = x - (r/8)v + (4r)^{-1}|T_{\frac{1}{2}}(x)|^2 v,$$

$$D F(x)(u) = u - (2r)^{-1}(T_{\frac{1}{2}}(x) \bullet T_{\frac{1}{2}}(u))v,$$

$$D F^{-1}(x)(u) = u + (2r)^{-1}(T_{\frac{1}{2}}(x) \bullet T_{\frac{1}{2}}(u))v,$$

$$D^2 F^{-1}(x)(u_1, u_2) = (2r)^{-1}(T_{\frac{1}{2}}(u_1) \bullet T_{\frac{1}{2}}(u_2))v,$$

whenever $x, u, u_1, u_2 \in \mathbf{R}^n$. Moreover $F^{-1}[\text{Clos } C_r] \subseteq \text{Clos } C_{(11/8)r}$,

$$\sup_{x \in A} F(x) \bullet v \leq 3r/16, \quad \sup_{x \in A, |T_{\frac{1}{2}}(x)| \geq r} F(x) \bullet v \leq -r/16.$$

Suppose L is the set of $(z, \eta) \in (F[A] \cap \text{Clos } C_r) \times \mathbf{S}^{n-1}$ such that $(w - z) \bullet \eta \leq 0$ whenever $w \in F[A]$, and we observe that L is compact and

$$L \subseteq N(F[A]) \setminus F[\text{Clos } C_{(11/8)r}].$$

We prove (see 4.18) that if $(z, \eta) \in L$ then

$$(**) \quad \delta_A(F^{-1}(z) + (r/2)(\mathbf{q} \circ \nu_{F^{-1}})(z, \eta)) = r/2.$$

In fact, if $x = F^{-1}(z)$, $\zeta = (\mathbf{q} \circ \nu_{F^{-1}})(z, \eta)$ and $y \in A$, we compute

$$\begin{aligned} |D F(x)^*(\eta)|^{-1} \eta &= (D F(x)^*)^{-1}(\zeta) = \zeta + (2r)^{-1}(v \bullet \zeta)T_{\frac{1}{2}}(x), \\ &0 \geq (D F(x)^*)^{-1}(\zeta) \bullet (F(y) - F(x)) \\ &= \zeta \bullet (y - x) + (4r)^{-1}(|T_{\frac{1}{2}}(x)|^2 - |T_{\frac{1}{2}}(y)|^2)(v \bullet \zeta) \\ &\quad + (2r)^{-1}(T_{\frac{1}{2}}(x) \bullet (y - x))(v \bullet \zeta) \\ &= \zeta \bullet (y - x) - (4r)^{-1}|T_{\frac{1}{2}}(y - x)|^2(v \bullet \zeta), \\ |y - x - (r/2)\zeta|^2 &= |y - x|^2 + (r^2/4) - r(y - x) \bullet \zeta \\ &\geq |y - x|^2 + (r^2/4) - (1/4)|T_{\frac{1}{2}}(y - x)|^2(v \bullet \zeta) \geq r^2/4. \end{aligned}$$

If $C = \{(1-t)F(0) + tx : x \in T \cap \mathbf{B}(0, 4r), 0 \leq t < \infty\}$ then C is a closed convex set and $\text{DualNor}(C, F(0)) = \text{Tan}(C, F(0)) = \{z - F(0) : z \in C\}$. Observe $\{z : z \bullet v \leq 0, |T_{\mathfrak{h}}(z)| \leq 4r\} \subseteq C$. Then the following assertions will be proved,

$$\text{Nor}(C, F(0)) \cap \mathbf{S}^{n-1} \subseteq \mathbf{q}[L],$$

$$\mathcal{H}^{n-1}(\text{Nor}(C, F(0)) \cap \mathbf{S}^{n-1}) \geq \mathcal{L}^{n-1}(\mathbf{U}(0, 1/(1+32^2)^{1/2})).$$

Let $\eta \in \text{Nor}(C, F(0)) \cap \mathbf{S}^{n-1}$. If $z \in F[A]$ and $(z - F(0)) \bullet \eta > 0$, then $z \notin C$ and $|T_{\mathfrak{h}}(z)| \leq 4r$, whence we deduce that $z \bullet v > 0$, $|T_{\mathfrak{h}}(z)| < r$, $|T_{\mathfrak{h}}^{\perp}(z)| < 3r/16$ and $z \in C_r$. Therefore if $s = \sup\{(z - F(0)) \bullet \eta : z \in F[A]\} > 0$, then we select $z_0 \in F[A]$ such that $(z_0 - F(0)) \bullet \eta = s$, we observe that $(w - z_0) \bullet \eta \leq 0$ for every $w \in F[A]$ and we conclude that $(z_0, \eta) \in L$. Moreover a direct computation shows that

$$\mathbf{S}^{n-1} \cap \text{Nor}(C, F(0)) = \mathbf{S}^{n-1} \cap \{\eta : 32/(1+32^2)^{1/2} \leq \eta \bullet v \leq 1\}$$

and the desired lower bound for the \mathcal{H}^{n-1} measure of $\mathbf{S}^{n-1} \cap \text{Nor}(C, F(0))$ readily follows.

We notice that $N(F[A])$ satisfies the m dimensional condition (N) in $F[C_{4r}]$ by 6.4. By 5.9, 4.19, 4.12 and (**) we infer that

$$\dim T_{F[A]}(z, \eta) = m, \quad \text{D} F^{-1}(z)[T_{F[A]}(z, \eta)] = T_A(\nu_{F^{-1}}(z, \eta)),$$

$$Q_{F[A]}(z, \eta) \geq 0, \quad Q_A(\nu_{F^{-1}}(z, \eta))(\tau, \tau) \geq -(r/2)^{-1} |\tau|^2$$

$$Q_{F[A]}(z, \eta) = |(\text{D} F(F^{-1}(z)))^*(\eta)| Q_A(\nu_{F^{-1}}(z, \eta)) \circ \odot_2 (\text{D} F^{-1}(z)[T_{F[A]}(z, \eta)]) \\ - (\text{D}^2 F^{-1}(z)| \odot_2 T_{F[A]}(z, \eta)) \bullet \text{D} F(F^{-1}(z))^*(\eta)$$

for \mathcal{H}^{n-1} a.e. $(z, \eta) \in L$ and for every $\tau \in T_A(\nu_{F^{-1}}(z, \eta))$. In particular, by [Fed69, 2.10.25], the same conclusion holds for \mathcal{H}^m a.e. $z \in \mathbf{p}[L]$ and for \mathcal{H}^{n-m-1} a.e. $\eta \in \{\zeta : (z, \zeta) \in L\}$. We combine 6.6 and the classical inequality relating the arithmetic and the geometric means of a family of non negative numbers (see [Roc70, pp. 29]) to estimate

$$\begin{aligned} \mathcal{H}^{n-1}(\mathbf{q}[L]) &\leq \int_{\mathbf{S}^{n-1}} \mathcal{H}^0\{z : (z, \eta) \in L\} d\mathcal{H}^{n-1}\eta \\ &= \int_{F[A]} \int_{\{z\} \times \{\eta : (z, \eta) \in L\}} \text{discr } Q_{F[A]} d\mathcal{H}^{n-m-1} d\mathcal{H}^m z \\ &\leq m^{-m} \int_{F[A]} \int_{\{z\} \times \{\eta : (z, \eta) \in L\}} (\text{trace } Q_{F[A]})^m d\mathcal{H}^{n-m-1} d\mathcal{H}^m z. \end{aligned}$$

We observe that if $z \in \text{Clos } C_r$, $\eta \in \mathbf{S}^{n-1}$ and $S \in \mathbf{G}(n, m)$, then

$$\|\text{D} F(F^{-1}(z))\| \leq 3/2, \quad \|\text{D} F^{-1}(z)\| \leq 3/2,$$

$$|\text{trace}((\text{D}^2 F^{-1}(z)| \odot_2 S) \bullet \text{D} F(F^{-1}(z))^*(\eta))| \leq (3/4)mr^{-1}.$$

Therefore, using 6.7, we infer there exists $c_1 > 0$ depending only on m such that

$$(\text{trace } Q_{F[A]}(z, \eta))^m \leq c_1 (f(\nu_{F^{-1}}(z, \eta))^m + r^{-m}),$$

for \mathcal{H}^{n-1} a.e. $(z, \eta) \in L$.

By 3.10(1) and 4.3 we infer that L is a countable union of compact sets with finite \mathcal{H}^{n-1} measure. Therefore $D = \{z : \mathcal{H}^{n-m-1}\{\zeta : (z, \zeta) \in L\} > 0\}$ is a Borel subset of \mathbf{R}^n by [Fed69, 2.10.26, p. 190]. We let

$$M = \nu_{F^{-1}}[L|D],$$

and we notice that M is a Borel subset of $N(A)$ by 4.18, $\mathbf{p}[M] \subseteq \text{Clos } C_{(11/8)r}$, $\delta_A(x + (r/2)u) = r/2$ whenever $(x, u) \in M$ and $\mathcal{H}^{n-m-1}\{v : (x, v) \in M\} > 0$ whenever $x \in \mathbf{p}[M]$. By [Fed69, 2.2.7], 5.3 and 2.11, we infer the following estimates,

$$\begin{aligned} \text{Lip}(F| \text{Clos } C_{(11/8)r}) &\leq \sup_{z \in \text{Clos } C_{(11/8)r}} \|D F(z)\| \leq 27/16, \\ \int_{F[A]} \mathcal{H}^{n-m-1}(L_z) d\mathcal{H}^m z &\leq (27/16)^m \mathcal{H}^{n-m-1}(\mathbf{S}^{n-m-1}) \mathcal{H}^m(\mathbf{p}[M]), \\ \int_{F[A]} \int_{\{\zeta : (z, \zeta) \in L\}} f(\nu_{F^{-1}}(z, \eta))^m d\mathcal{H}^{n-m-1} \eta d\mathcal{H}^m z \\ &\leq c_2 \int_A \int_{\{\zeta : (z, \zeta) \in M\}} f(z, \eta)^m d\mathcal{H}^{n-m-1} \eta d\mathcal{H}^m z, \end{aligned}$$

where c_2 is a constant depending on m and n . Therefore,

$$\begin{aligned} \mathcal{H}^{n-1}(\mathbf{q}[L]) &\leq m^{-m} c_1 c_2 \int_A \int_{\{\zeta : (z, \zeta) \in M\}} f(z, \eta)^m d\mathcal{H}^{n-m-1} \eta d\mathcal{H}^m z \\ &\quad + m^{-m} c_1 (27/16)^m \mathcal{H}^{n-m-1}(\mathbf{S}^{n-m-1}) \mathcal{H}^m(\mathbf{p}[M]) r^{-m}, \end{aligned}$$

whence we get the conclusion.

6.10 Corollary. *Suppose $1 \leq m < n$ are integers, $A \subseteq \mathbf{R}^n$ is a closed set with locally finite \mathcal{H}^m measure, $N(A)$ satisfies the m dimensional condition (N), for \mathcal{H}^m a.e. $a \in A$ there exists $v \in \mathbf{S}^{n-1}$ such that*

$$\lim_{r \rightarrow 0} r^{-1} \sup\{v \bullet (x - a) : x \in \mathbf{B}(a, r) \cap A\} = 0,$$

and there exists a nonnegative \mathcal{H}^{n-1} measurable function f on $N(A)$ such that

$$\text{trace } Q_A(a, u) \leq f(a, u) \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(A),$$

$$\int_{K \cap A} \int_{\{z\} \times N(A, z)} f^m d\mathcal{H}^{n-m-1} d\mathcal{H}^m z < \infty,$$

whenever $K \subseteq \mathbf{R}^n \times \mathbf{S}^{n-1}$ is compact.

Then $\mathcal{H}^m(A \sim A^{(m)}) = 0$. In particular, A is countably (\mathcal{H}^m, m) rectifiable of class 2.

Proof. Combining [Fed69, 2.10.19(5)] and 6.9 and [Fed69, 2.4.11], we infer that

$$\liminf_{r \rightarrow 0} \frac{\mathcal{H}^m(A^{(m)} \cap \mathbf{B}(a, r))}{\mathcal{H}^m(A \cap \mathbf{B}(a, r))} > 0 \quad \text{for } \mathcal{H}^m \text{ a.e. } a \in A.$$

If $V = \{(a, \mathbf{B}(a, r)) : a \in \mathbf{R}^n, 0 < r < \infty\}$ we apply [Fed69, 2.8.18, 2.9.12] with ϕ replaced by $\mathcal{H}^m \llcorner A$ to deduce that V is a $\mathcal{H}^m \llcorner A$ Vitali relation, the $(\mathcal{H}^m \llcorner A, V)$ density function of $A^{(m)}$ is positive on \mathcal{H}^m almost all of A and it is $\mathcal{H}^m \llcorner A$ almost equal to the characteristic function of $A^{(m)}$. Therefore $\mathcal{H}^m(A \sim A^{(m)}) = 0$ and the postscript follows from 5.2.

6.11 Remark. As already mentioned in section 1, this theorem can be compared to the classical result in [Tru89, Theorem 1] asserting the twice superdifferentiability of viscosity subsolutions of certain second order elliptic operators. The central idea of [Tru89, Theorem 1] is to use the Alexandroff maximum principle in [GT01, 9.1], which in turn is based on the area formula for functions, to get a lower bound for the measure of the set of superdifferentiability points of subsolutions; whence the conclusion follows by classical density arguments. The same strategy has been adapted to our case, using the area formula in 6.6.

7 Generalized minimal submanifolds

Following [Sav17] we introduce the class of viscosity minimal sets in 7.1 and we establish the Lusin condition (N) for their normal bundle in 7.5. Second order differentiability almost everywhere is deduced in 7.8 as an application of the results in section 6. Finally, a regularity result for stationary varifolds of arbitrary codimension is given in 7.11.

7.1 Definition. Suppose $1 \leq m < n$ are integers and $A \subseteq \mathbf{R}^n$ is closed. We say that A is a *m dimensional viscosity minimal set of \mathbf{R}^n* if and only if whenever $x \in A$ and ψ is a \mathbf{R} valued function of class 2 on a neighbourhood of x such that $\psi|_A$ has a local maximum at x and $\text{grad } \psi(x) \neq 0$, then there exists $L \in \mathbf{G}(n, m)$ such that

$$\text{grad } \psi(x) \in L^\perp \quad \text{and} \quad \Delta_{x+L} \psi(x) \leq 0,$$

where Δ_{x+L} denotes the Laplace operator on the affine subspace $\{x+u : u \in L\}$.

7.2 Remark. The notion of viscosity minimal set has been recently introduced in [Sav17], where an Allard-type local regularity result is proved for graphical viscosity minimal sets ([Sav17, Theorem 1.5]) by means of nonvariational methods and weak Harnack inequality.

Every m dimensional minimal submanifolds of class 2 is an m dimensional viscosity minimal set, see [Sav17, p. 2].

7.3 Lemma. Suppose $1 \leq m < n$ are integers, $T \in \mathbf{G}(n, n-1)$, $\eta \in T^\perp$, $|\eta| = 1$, $f : T \rightarrow T^\perp$ is pointwise differentiable of order 2 at 0, $f(0) = 0$, $Df(0) = 0$, $\chi_1 \geq \dots \geq \chi_{n-1}$ are the eigenvalues of $\text{pt } D^2 f(0) \bullet \eta$ and A is a m dimensional viscosity minimal set of \mathbf{R}^n such that $0 \in A$ and

$$A \cap V \subseteq \{z : z \bullet \eta \leq f(T_{\mathfrak{h}}(z)) \bullet \eta\},$$

for some open neighbourhood V of 0.

Then

$$\sum_{i=1}^m \chi_i \geq 0.$$

Proof. For $\epsilon > 0$ we let

$$P_\epsilon(\zeta) = (1/2)(\text{pt } D^2 f(0)(\zeta^2) \bullet \eta + \epsilon|\zeta|^2)\eta \quad \text{whenever } \zeta \in T,$$

$$M_\epsilon = \{\zeta + P_\epsilon(\zeta) : \zeta \in T\},$$

$$\psi_\epsilon(z) = \begin{cases} \delta_{M_\epsilon}(z) & \text{if } z \bullet \eta > P_\epsilon(T_{\frac{1}{2}}(z)) \bullet \eta \\ -\delta_{M_\epsilon}(z) & \text{if } z \bullet \eta \leq P_\epsilon(T_{\frac{1}{2}}(z)) \bullet \eta \end{cases}$$

and we observe that ψ_ϵ is of class ∞ on a neighbourhood of 0 by [GT01, 14.16], $\psi_\epsilon|_A$ has a local maximum at 0 and $\psi_\epsilon(\zeta + P_\epsilon(\zeta)) = 0$ whenever $\zeta \in T$. Differentiating the last equation we get

$$\text{grad } \psi_\epsilon(0) = \eta \quad \text{and} \quad \text{grad } \psi_\epsilon(0) \bullet D^2 P_\epsilon(0) = -D^2 \psi_\epsilon(0)|_{T \times T}.$$

Therefore there exists $L \in \mathbf{G}(n, m)$ such that

$$\eta \in L^\perp \quad \text{and} \quad \Delta_L \psi_\epsilon(0) \leq 0,$$

and, noting that $-\chi_1 - \epsilon, \dots, -\chi_{n-1} - \epsilon$ are the eigenvalues of $D^2 \psi_\epsilon(0)|_{T \times T}$, we employ [JT03, Lemma 2.3] to conclude

$$\sum_{i=1}^m (\chi_i + \epsilon) \geq 0.$$

Letting $\epsilon \rightarrow 0$, we get the conclusion.

We need the following elementary consequence of coarea formula in 7.5.

7.4 Lemma. *Suppose $0 \leq \mu \leq m$ are integers, W is a (\mathcal{H}^m, m) rectifiable and \mathcal{H}^m measurable subset of \mathbf{R}^n , $S \subseteq \mathbf{R}^\nu$ is a countable union of sets with finite \mathcal{H}^μ measure and $f : W \rightarrow \mathbf{R}^\nu$ is a Lipschitzian map such that*

$$\mathcal{H}^m(W \cap \{w : \|\bigwedge_\mu((\mathcal{H}^m \llcorner W, m) \text{ ap } D f(w))\| = 0\}) = 0,$$

$$\mathcal{H}^\mu(S \cap \{z : \mathcal{H}^{m-\mu}(f^{-1}\{z\}) > 0\}) = 0.$$

$$\text{Then } \mathcal{H}^m(f^{-1}[S]) = 0.$$

Proof. Firstly we reduce the problem to the case $\mathcal{H}^\mu(S) < \infty$; then, by [Fed69, 2.1.4, 2.10.26], to the case of a Borel subset S of \mathbf{R}^ν . Now the conclusion comes from the coarea formula in [Fed78, p. 300].

7.5 Theorem. *If A is a m dimensional viscosity minimal set and A is a countable union of sets with finite \mathcal{H}^m measure, then $N(A)$ satisfies the m dimensional condition (N) and*

$$\text{trace } Q_A(a, u) \leq 0 \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(A).$$

Proof. For $\lambda > m$ we let N_r to be as in 3.10 whenever $r > 0$.

We select $r > 0$ such that 3.10(3)(4) holds at \mathcal{H}^{n-1} a.e. $x \in N_r$; then we fix $x \in N_r$ to be one of these points. We let $T = \{v : v \bullet \nu_A(x) = 0\}$, we assume $\xi_A(x) = 0$ and we notice that $T_{\frac{1}{2}}(x) = 0$. Suppose $f : T \rightarrow T^\perp$ is a Lipschitzian function pointwise differentiable of order 2 at 0 such that $D f(0) = 0$,

$$\text{pt } D^2 f(0)(u, v) \bullet \nu_A(x) = -\text{ap } D \nu_A(x)(u) \bullet v \quad \text{whenever } u, v \in T,$$

$$\mathbf{U}(x, s) \cap \delta_A^{-1}\{r\} = \mathbf{U}(x, s) \cap \{\chi + f(\chi) : \chi \in T\} \quad \text{for some } 0 < s < r/2.$$

We let $U = T_{\mathfrak{h}}[\mathbf{U}(x, s) \cap \{\chi + f(\chi) : \chi \in T\}]$ and $g(\zeta) = f(\zeta) - x$ whenever $\zeta \in T$. Since U is open relative to T , the set $V = \{y - x : y \in T_{\mathfrak{h}}^{-1}[U] \cap \mathbf{U}(x, s)\}$ is open in \mathbf{R}^n and we claim that

$$V \cap A \subseteq \{z : z \bullet \nu_A(x) \leq g(T_{\mathfrak{h}}(z)) \bullet \nu_A(x)\}.$$

In fact, if $y \in \mathbf{U}(x, s) \cap T_{\mathfrak{h}}^{-1}[U]$ and $y - x \in A$, then we notice that

$$T_{\mathfrak{h}}(y) + f(T_{\mathfrak{h}}(y)) \in \mathbf{U}(x, s) \cap \delta_A^{-1}\{r\}, \quad |T_{\mathfrak{h}}(y) + f(T_{\mathfrak{h}}(y)) - y| < r,$$

and we compute

$$r \leq |T_{\mathfrak{h}}(y) + f(T_{\mathfrak{h}}(y)) - (y - x)| = r - (y - f(T_{\mathfrak{h}}(y))) \bullet \nu_A(x);$$

whence the claim readily follows. If $\chi_1 \leq \dots \leq \chi_{n-1}$ are the eigenvalues of $\text{ap D } \nu_A(x)|T$, then $1 - \chi_1 r, \dots, 1 - \chi_{n-1} r$ are the eigenvalues of $\text{ap D } \xi_A(x)|T$ by 3.5 and $-\chi_1, \dots, -\chi_{n-1}$ are the eigenvalues of $\text{pt D}^2 g(0) \bullet \nu_A(x)$. Therefore we apply 7.3 with f and η replaced by g and $\nu_A(x)$ to conclude that

$$\sum_{i=1}^m \chi_i \leq 0.$$

Since $\chi_j \geq -(\lambda - 1)^{-1} r^{-1}$ whenever $j = 1, \dots, n - 1$ by 3.10(2), we conclude

$$\chi_j - (m - 1)(\lambda - 1)^{-1} r^{-1} \leq \sum_{i=1}^m \chi_i \leq 0 \quad \text{and} \quad \chi_j < r^{-1},$$

whenever $j = 1, \dots, m$; therefore,

$$\| \bigwedge_m ((\mathcal{H}^{n-1} \llcorner N_r, n - 1) \text{ap D } \xi_A(x)) \| > 0.$$

In particular all the assertions of this paragraph hold for \mathcal{H}^{n-1} a.e. $x \in N_r$ and for \mathcal{L}^1 a.e. $r > 0$.

If $x \in A$, $r > 0$ and $\mathcal{H}^{n-m-1}(\xi_A^{-1}[\{x\}] \cap N_r) > 0$, then by 3.10(1), 4.3 and 5.3 we conclude that

$$\mathcal{H}^{n-m-1}(N(A, x)) > 0 \quad \text{and} \quad x \in \bigcup_{i=0}^m A^{(i)}.$$

Suppose now $S \subseteq A$ such that $\mathcal{H}^m(S \cap A^{(m)}) = 0$. Since $\mathcal{H}^m(A^{(i)}) = 0$ whenever $0 \leq i \leq m - 1$ by 5.2, we get

$$\mathcal{H}^m(S \cap \{x : \mathcal{H}^{n-m-1}(\xi_A^{-1}[\{x\}] \cap N_r) > 0\}) = 0 \quad \text{whenever } r > 0,$$

and, employing 3.10(1)(2) and 7.4, we infer that

$$\mathcal{H}^{n-1}(\xi_A^{-1}[S] \cap N_r) = 0 \quad \text{for } \mathcal{L}^1 \text{ a.e. } r > 0.$$

We use 3.10(1) to get $\mathcal{H}^{n-1}(\psi_A[N_r]|S) = 0$ for \mathcal{L}^1 a.e. $r > 0$ and 4.3 to conclude $\mathcal{H}^{n-1}(N(A)|S) = 0$. Therefore $N(A)$ satisfies the m dimensional condition (N).

Employing 5.9, 3.10(1) and 4.3 we get that $\dim T_A(\psi_A(x)) = m$ for \mathcal{H}^{n-1} a.e. $x \in N_r$ and for every $r > 0$. Therefore, letting $\kappa_1, \dots, \kappa_{n-1}$ to be as in 4.14, the proof of 4.14 allows to conclude that

$$\sum_{j=1}^m \frac{\kappa_j(\psi_A(x))}{1 + r\kappa_j(\psi_A(x))} \leq 0,$$

for \mathcal{H}^{n-1} a.e. $x \in N_r$ and for \mathcal{L}^1 a.e. $r > 0$. Choosing a sequence $r_i > 0$ converging to 0 such that for the subset M_i of N_{r_i} whose points satisfy the inequality above, the following condition holds

$$\mathcal{H}^{n-1}(N_{r_i} \sim M_i) = 0 \quad \text{whenever } i \geq 1,$$

we can easily verify that $\text{trace } Q_A(a, u) \leq 0$ for every $(a, u) \in \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \psi_A[M_j]$. Moreover, employing 3.10(1) and 4.3,

$$\mathcal{H}^{n-1}(N(A) \sim \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \psi_A[M_j]) = 0.$$

7.6 Remark. A somehow similar approach is adopted in [Alm86, §5, §6] to prove, among the other things, an equation relating the perpendicular part of the variational mean curvature of certain varifolds with the principal curvatures of the level sets of the distance function to the convex hull of their supports (see [Alm86, §6(2)]).

The idea to deduce the Lusin condition (N) from the fact that the approximate Jacobian of the nearest point projection ξ_A is positive on \mathcal{H}^{n-1} almost all of N_r for \mathcal{L}^1 a.e. $r > 0$, originates from unpublished lecture notes of Ulrich Menne, where the aforementioned approach of [Alm86] is employed to study a kind of weaker Lusin condition (N) in the case of certain varifolds of bounded mean curvature.

7.7 Remark. The second conclusion of 7.5 is the natural extension of [CCKS96, 3.4] to viscosity minimal sets.

7.8 Corollary. *If A is an m dimensional viscosity minimal set of \mathbf{R}^n with locally finite \mathcal{H}^m measure and for \mathcal{H}^m a.e. $a \in A$ there exists $v \in \mathbf{S}^{n-1}$ such that*

$$\lim_{r \rightarrow 0} r^{-1} \sup\{v \bullet (x - a) : x \in \mathbf{B}(a, r) \cap A\} = 0,$$

then $\mathcal{H}^m(A \sim A^{(m)}) = 0$; in particular A is countably (\mathcal{H}^m, m) rectifiable of class 2.

Proof. Combine 6.10 and 7.5.

7.9 Remark. Noting 6.11, this result is the natural extension of [Tru89, Theorem 1] and [CCKS96, 3.5] to viscosity minimal sets.

The remaining part of this section is devoted to state and comment our regularity result for stationary varifolds. We adopt the notation of [All72].

7.10. Three facts on stationary varifolds, that are immediate consequences of well known general results in varifold theory, are provided here for reader's convenience. Suppose $1 \leq m \leq n - 1$ are integers.

- (1) If $V \in \mathbf{V}_m(\mathbf{R}^n)$ and $\delta V = 0$ (i.e. V is an arbitrary m dimensional stationary varifold) then $\text{spt } \|V\|$ is an m dimensional viscosity minimal set. This follows from [Whi10, Theorem 1].

- (2) If $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\delta V = 0$, $0 < d < \infty$ and $\Theta^m(\|V\|, x) \geq d$ for $\|V\|$ a.e. $x \in \mathbf{R}^n$, then

$$\mathcal{H}^m(A \cap \text{spt } \|V\|) \leq d^{-1} \|V\|(A) \quad \text{whenever } A \subseteq \mathbf{R}^n;$$

in particular, $\text{spt } \|V\|$ has locally finite \mathcal{H}^m measure. This follows from [All72, 5.5(1), 3.5(1), 8.6].

- (3) If $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\delta V = 0$, $0 < d < \infty$ and $\Theta^m(\|V\|, x) \geq d$ for $\|V\|$ a.e. $x \in \mathbf{R}^n$, then for $\|V\|$ a.e. $x \in \mathbf{R}^n$ there exists $T \in \mathbf{G}(n, m)$ such that

$$\lim_{r \rightarrow 0} r^{-1} \sup\{\delta_T(y - x) : y \in \mathbf{B}(x, r) \cap \text{spt } \|V\|\} = 0.$$

This follows from [Sim83, 17.11].

7.11 Corollary. *If $1 \leq m \leq n - 1$ are integers, $V \in \mathbf{V}_m(\mathbf{R}^n)$, $\delta V = 0$, $0 < d < \infty$ and $\Theta^m(\|V\|, x) \geq d$ for $\|V\|$ a.e. $x \in \mathbf{R}^n$, then the following three statements hold.*

- (1) $N(\text{spt } \|V\|)$ satisfies the m dimensional condition (N) and

$$\begin{aligned} & \int_{\mathbf{S}^{n-1}} \mathcal{H}^0\{a : (a, u) \in B\} d\mathcal{H}^{n-1}u \\ &= \int_{\text{spt } \|V\|} \int_{\{z\} \times \{v : (z, v) \in B\}} |\text{discr } Q_{\text{spt } \|V\|}| d\mathcal{H}^{n-m-1}z d\mathcal{H}^m z, \end{aligned}$$

for every \mathcal{H}^{n-1} measurable set $B \subseteq N(\text{spt } \|V\|)$;

- (2) $\text{trace } Q_{\text{spt } \|V\|}(a, u) \leq 0$ for \mathcal{H}^{n-1} a.e. $(a, u) \in N(\text{spt } \|V\|)$;

- (3) $\mathcal{H}^m((\text{spt } \|V\|) \sim (\text{spt } \|V\|)^{(m)}) = 0$; in particular $\text{spt } \|V\|$ is countably (\mathcal{H}^m, m) rectifiable of class 2.

Proof. Combine 7.10, 7.5, 7.8 and 6.6.

7.12 Remark. In case V is integral, then

$$\text{trace } Q_{\text{spt } \|V\|}(a, u) = 0 \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } (a, u) \in N(\text{spt } \|V\|);$$

this can be deduced combining 7.11(1), 5.9, 5.2 and [Sch09, Corollary 4.2].

However, it is not known if the hypothesis of integrality is essential here.

7.13 Remark. In [Men12] a result similar to 7.11(1) has been announced for the case of m dimensional *integral* varifolds V of \mathbf{R}^{m+1} with mean curvature $\mathbf{h}(V; \cdot) \in \mathbf{L}_m^{\text{loc}}(\|V\|, \mathbf{R}^n)$, $\|\delta V\|_{\text{sing}} = 0$ and $m \geq 2$.

7.14 Remark. The main result of [Men13] proves that the support of every m dimensional *integral* varifold V of \mathbf{R}^n ($1 \leq m \leq n - 1$) with mean curvature $\mathbf{h}(V; \cdot) \in \mathbf{L}_1^{\text{loc}}(\|V\|, \mathbf{R}^n)$ can be $\|V\|$ almost covered by the union of countably m dimensional submanifolds of class 2. In this regard, the main novelty in 7.11(3) lies in the fact that the stationary varifolds are required to be only rectifiable. Moreover our approach completely differs from [Men13].

7.15 Remark. The regularity result in 7.11 is a substantial consequence of 6.9, whose proof is based on techniques adapted from the theory of viscosity solutions of elliptic PDE's. Such a theory has been previously applied to prove regularity results for *integral* varifolds of codimension 1 in [Sch04]. In this regard, the conclusions in 7.11(2)(3) are conceptually similar to [Sch04, Theorem 6.1] (notice that, by [San17, 3.23], the conclusion in 7.11(3) implies that $\text{spt } \|V\|$ is approximately differentiable of order 2 at \mathcal{H}^m almost all points of $\text{spt } \|V\|$).

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