Fine properties of the curvature of arbitrary closed sets

Mario Santilli*
Augsburg Universität

August 14, 2019

Abstract

Given an arbitrary closed set A of $\mathbf{R}^n$, we establish the relation between the eigenvalues of the approximate differential of the spherical image map of A and the principal curvatures of A introduced by Hug-Last-Weil, thus extending a well known relation for sets of positive reach by Federer and Zähle. Then we provide for every $m = 1, \ldots, n-1$ an integral representation for the support measure μ_m of A with respect to the m dimensional Hausdorff measure.

Moreover a notion of second fundamental form Q_A for an arbitrary closed set A is introduced so that the finite principal curvatures of A correspond to the eigenvalues of Q_A. We prove that the approximate differential of order 2, introduced in a previous work of the author, equals in a certain sense the absolutely continuous part of Q_A, thus providing a natural generalization to higher order differentiability of the classical result of Calderon and Zygmund on the approximate differentiability of functions of bounded variation.

MSC-classes 2010. 52A22, 53C65 (Primary); 28A75, 60D05 (Secondary).

Keywords. Parallel sets, nearest point projection, approximate differentiability, second fundamental form, support measures, second order rectifiability.

1 Introduction

Background

The theory of curvature of arbitrary closed subsets of the Euclidean space, which finds its roots in the landmark paper of Federer [Fed59] on sets of positive reach, has been initiated by Stachó in [Sta79] and continued by Hug-Last-Weil in [HLW04]. If $A \subseteq \mathbf{R}^n$ is a closed set and $\boldsymbol{\delta}_A$ is the distance function from A, these authors introduced the *generalized normal bundle of A*,

$$N(A) = (A \times \mathbf{S}^{n-1}) \cap \{(a, u) : \boldsymbol{\delta}_A(a + su) = s \text{ for some } s > 0\}$$

*email: `mario.santilli@math.uni-augsburg.de`

and they observed that there exists a countable collection $A_1, A_2, \dots$ of closed sets of positive reach and compact boundary such that

$$N(A) \subseteq \bigcup_{i=1}^{\infty} N(A_i).$$

On the basis of this fact, it follows that $N(A)$ is a countably $n-1$ rectifiable subset of $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and its $n-1$ dimensional approximate tangent plane coincides with that of one of the sets A_n at $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$. This observation allows to introduce the *principal curvatures of* A,

(i) $$-\infty < \lambda_{A,1}(a,u) \leq \ldots \leq \lambda_{A,n-1}(a,u) \leq \infty,$$

at $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$, using the notion of principal curvature for sets of positive reach introduced by Zähle in [Zäh86]. The support measures $\mu_0, \dots, \mu_{n-1}$ of A are then introduced by

(ii) $$\mu_i(D) = \frac{1}{(n-i)\boldsymbol{\alpha}(n-i)} \int_D H_{n-i-1} d\mathscr{H}^{n-1},$$

whenever $D \subseteq \mathbf{R}^n \times \mathbf{S}^{n-1}$ is an $\mathscr{H}^{n-1}$ measurable set such that the integral on the right side exists (finite or infinite). Here H_j denotes *the j-th symmetric function of the principal curvatures of* A,

(iii) $$H_j = \sum_{\{l_1,\dots,l_j\} \subseteq \{1,\dots,n-1\}} \left(\prod_{i=1}^{n-1} (1+\lambda_{A,i}^2)^{-1/2} \right) \prod_{i=1}^{j} \lambda_{A,l_i}.$$

The main result of the theory, the *Steiner formula* in [HLW04, Theorem 2.1], is phrased in terms of these support measures; see also [KW14, Theorem 1] where a corrected version of this formula is pointed out. Despite this important result, several basic questions in the theory remain undisclosed and it is our aim in this work to investigate some of them.

The theory of curvature for arbitrary closed sets has found applications so far in the study of random closed sets in stochastic geometry (see [HLW04, sections 7-8], [Las06]) and in spatial statistics (see [KW14]). On the other hand, the fact that this is a theory developed with no a priori assumptions on the structure of the sets (e.g. convex, positive reach, etc..), makes it certainly appealing in the study of singular surfaces arising as solutions of variational problems (e.g. varifolds). We will present these applications in subsequent works.

Results of the present paper

Relating the principal curvatures to the eigenvalues of the differential of the spherical image map. If $A \subseteq \mathbf{R}^n$ is a closed set, let $\boldsymbol{\xi}_A$ be the nearest point projection onto A and let $\boldsymbol{\nu}_A$ be the spherical image map, i.e. $\boldsymbol{\nu}_A(x) = \boldsymbol{\delta}_A(x)^{-1}(x - \boldsymbol{\xi}_A(x))$ for $x \in \operatorname{dmn} \boldsymbol{\xi}_A \sim A$. If A is a set of positive reach then it is well known (Federer [Fed59, 4.8] and Zähle [Zäh86]) that $\boldsymbol{\nu}_A$ is differentiable with symmetric differential at $\mathscr{L}^n$ a.e. $x \in \{y : 0 < \boldsymbol{\delta}_A(y) < \operatorname{reach}(A)\}$ and, denoting by $\chi_{A,1}(x) \leq \ldots \leq \chi_{A,n-1}(x)$ the eigenvalues of $\mathrm{D}\,\boldsymbol{\nu}_A(x) | \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$, it follows that the principal curvature $\lambda_{A,i}(a,u)$ of A at $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$ is given by

$$\text{(iv)} \qquad \lambda_{A,i}(a,u) = \frac{\chi_{A,i}(a+ru)}{1 - r\chi_{A,i}(a+ru)} \quad \text{for } 0 < r < \operatorname{reach}(A);$$

in fact the right side does not depend on $r \in (0, \operatorname{reach}(A))$. On the other hand, it has been proved in [Asp73] that if A is an arbitrary closed subset of $\mathbf{R}^n$ then a certain extension of the nearest point projection $\boldsymbol{\xi}_A$ on $\mathbf{R}^n$ is differentiable with symmetric differential at $\mathscr{L}^n$ almost every $x \in \mathbf{R}^n$ (the nearest point projection is not well defined on $\mathbf{R}^n$ unless A is convex). Therefore it is a natural question to understand if the principal curvatures of an arbitrary closed set introduced in [HLW04] can be realized by mean of a suitable extension of (iv). We provide the answer in sections 3 and 4, whose content we now briefly describe. The main purpose of section 3 is to analyse the set of approximate differentiability points of $\boldsymbol{\xi}_A$ for an arbitrary closed set A and to describe the tangential and curvature properties of the level sets $S(A,r)$ of the distance function $\boldsymbol{\delta}_A$ in terms of $\boldsymbol{\nu}_A$ and its approximate differential, see 3.12. This is done introducing a meaningful reach-type function $\rho(A,\cdot)$ in 3.6 and analysing the behaviour of $\boldsymbol{\xi}_A$ on the super-level sets of $\rho(A,\cdot)$, see 3.10. A first consequence of this analysis is contained in 3.13-3.14, where we provide a refined version of the differentiability theorem in [Asp73]. As a second consequence, we obtain in section 4 the answer to the question posed at the beginning of this paragraph, which we summarize in the following theorem.

1.1 Theorem. *If $A \subseteq \mathbf{R}^n$ is a closed set then $\boldsymbol{\nu}_A$ is approximately differentiable with symmetric approximate differential at $\mathscr{L}^n$ a.e. $x \in \mathbf{R}^n \sim A$ and, denoting by $\chi_{A,1}(x) \leq \ldots \leq \chi_{A,n-1}(x)$ the eigenvalues of $\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)|\{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$,*

$$\lambda_{A,i}(a,u) = \frac{\chi_{A,i}(a+ru)}{1 - r\chi_{A,i}(a+ru)}$$

for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$, for every $0 < r < \sup\{s : \boldsymbol{\delta}_A(a+su) = s\}$ and $i = 1, \ldots, n-1$.

The number $\sup\{s : \boldsymbol{\delta}_A(a+su) = s\}$ equals the *reach function of A* at (a,u) introduced in [KW14, p. 292] and it naturally appears in the Steiner formula (see [KW14, Theorem 1]). Moreover we introduce a symmetric bilinear form (which we call *second fundamental form of A at a in the direction u*)

$$\text{(v)} \qquad Q_A(a,u) : T_A(a,u) \times T_A(a,u) \to \mathbf{R},$$

at $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$, whose eigenvalues coincide with the finite principal curvatures of A. Here $T_A(a,u)$ is a linear subspace of $\mathbf{R}^n$ whose dimension can vary from 0 to $n-1$. The second fundamental form will be further investigated in sections 5 and 6.

Integral representation of the support measures. In section 5 we consider the following natural stratification of a closed set A: for each $0 \leq m \leq n$, we define *the m-th stratum of A* as

$$A^{(m)} = A \cap \{a : \dim \boldsymbol{\xi}_A^{-1}\{a\} = n-m\} = A \cap \{a : 0 < \mathscr{H}^{n-m-1}(N(A,a)) < \infty\}$$

(recall that $\boldsymbol{\xi}_A^{-1}\{a\}$ is a convex set for every $a \in A$). The structure of this stratification has been investigated in [MS17], where it is proved (notice 5.2) that $A^{(m)}$ is always countably $(\mathscr{H}^m, m)$ rectifiable of class 2, see [MS17, 4.12]. The main point here is to analyse the behaviour of the principal curvatures of A on each strata, see 5.6 and 5.7(1). Then for each integer $1 \leq m \leq n-1$ we obtain the following integral representation formula of the support measure μ_m with respect to the m dimensional Hausdorff measure $\mathscr{H}^m$. For arbitrary closed sets this result appears to be known only if $m = n-1$, see [HLW04, 4.1] (see also [CH00, 5.5] for the special case of sets of positive reach).

1.2 Theorem. *(see 5.7) If $A \subseteq \mathbf{R}^n$ is a closed set, $\mu_0, \ldots, \mu_{n-1}$ are the support measures of A, $1 \leq m \leq n-1$ is an integer, S is a countable union of Borel subsets with finite $\mathscr{H}^m$ measure and $T \subseteq N(A)|S$ is $\mathscr{H}^{n-1}$ measurable then*

$$\mu_m(T) = \frac{1}{(n-m)\boldsymbol{\alpha}(n-m)} \int_{A^{(m)}} \mathscr{H}^{n-m-1}\{v : (z,v) \in T\} d\mathscr{H}^m z.$$

Second order approximate differentiability. Finally in section 6 we analyse the relation of the present notion of curvature with the notion of approximate curvature for second-order rectifiable sets introduced by the author in [San17]. In the latter, second order rectifiable sets are characterized by the existence of the approximate differential of order 2 at almost every point (we refer to [San17, 1.2] for a precise statement, which actually holds for all possible orders of rectifiability). In this section we complement this characterization with the following result:

1.3 Theorem. *(see 2.7, 2.8 and 6.2) Let $A \subseteq \mathbf{R}^n$ be a closed set, $1 \leq m \leq n-1$ and let $S \subseteq A$ be $\mathscr{H}^m$ measurable and $(\mathscr{H}^m, m)$ rectifiable of class 2. Then there exists $R \subseteq S$ such that $\mathscr{H}^m(S \sim R) = 0$ and*

$$\operatorname{ap}\operatorname{Tan}(S,a) = T_A(a,u) \qquad \operatorname{ap}\operatorname{D}^2 S(a)(\tau,\upsilon) \bullet u = -Q_A(a,u)(\tau,\upsilon)$$

for every $\tau, \upsilon \in T_A(a,u)$ and for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)|R$.

In other words this theorem asserts that "the absolutely continuous part of the second fundamental form Q_A, when restricted on a second order rectifiable subset S of A, coincides with the approximate differential of order 2 of S". This result has an interesting analogy with the classical theorem of Calderon and Zygmund asserting that the absolutely continuous part of the total differential of a function of bounded variation coincides with its approximate gradient. This analogy is further strengthened if we look at the primitive g of the Cantor function f (f is a function of bounded variation whose total differential cannot be fully described by the approximate derivative), see 6.3. The epigraph of g is a closed convex set A of $\mathbf{R}^2$ which admits a subset $T \subseteq \partial A$ such that $\mathscr{H}^1(N(A)|T) > 0$ and

$$T_A(a,u) = \{0\} \quad \text{for } \mathscr{H}^1 \text{ a.e. } (a,u) \in N(A)|T.$$

It follows that the second fundamental form cannot be fully described by the approximate differential of order 2.

Acknowledgements. The author thanks Prof. Ulrich Menne for conversations concerning the content of section 6; moreover the author thanks the referee for his or her careful reading of the manuscript.

2 Preliminaries

The notation and the terminology agree with [Fed69, pp. 669–676]. Let m be a non negative integer. The symbols $\mathbf{U}(a,r)$ and $\mathbf{B}(a,r)$ denote the open and closed ball with centre a and radius r ([Fed69, 2.8.1]); $\mathbf{S}^m$ is the m dimensional unit sphere in $\mathbf{R}^{m+1}$ ([Fed69, 3.2.13]); $\mathscr{L}^m$ and $\mathscr{H}^m$ are the m dimensional Lebesgue and Hausdorff measure ([Fed69, 2.10.2]); $\boldsymbol{\alpha}(m) = \mathscr{L}^m(\mathbf{U}(0,1))$; given a measure μ, we denote by $\boldsymbol{\Theta}^{*m}(\mu,\cdot)$, $\boldsymbol{\Theta}^m_*(\mu,\cdot)$ and $\boldsymbol{\Theta}^m(\mu,\cdot)$ the m dimensional densities of μ ([Fed69, 2.10.19]); $\mathbf{G}(m,k)$ is the Grassmann manifold of all k dimensional subspaces in $\mathbf{R}^m$ ([Fed69, 1.6.2]). Moreover, given a function f, we denote by $\operatorname{dmn} f$ and $\operatorname{im} f$ the domain and the image of f. The symbol $\bullet$ denotes the standard inner product of $\mathbf{R}^m$. If T is a linear subspace of $\mathbf{R}^m$, then $T_\natural : \mathbf{R}^m \to \mathbf{R}^m$ is the orthogonal projection onto T and

$$T^\perp = \mathbf{R}^m \cap \{v : v \bullet u = 0 \text{ for } u \in T\}.$$

If X and Y are sets, $Z \subseteq X \times Y$ and $S \subseteq X$, then

$$Z|S = Z \cap \{(x,y) : x \in S\}.$$

The maps $\mathbf{p}, \mathbf{q} : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m$ are defined by

$$\mathbf{p}(x,v) = x, \quad \mathbf{q}(x,v) = v.$$

To treat the classical concept of rectifiable sets we adopt the terminology introduced in [Fed69, 3.2.14]. Moreover, if $A \subseteq \mathbf{R}^n$ we say that A *is countably* $(\mathscr{H}^m, m)$ *rectifiable of class* 2 if A can be $\mathscr{H}^m$ almost covered by the union of countably many m dimensional submanifolds of class 2 of $\mathbf{R}^n$; we omit the prefix "countably" when $\mathscr{H}^m(A) < \infty$. We refer to [Fed69, 3.1.21] for the notions of *tangent and normal cone of a set*; moreover, given a measure μ and a positive integer m, the *approximate tangent cone* $\operatorname{Tan}^m(\mu,\cdot)$ is defined as in [Fed69, 3.2.16]. Finally, if X and Y are metric spaces and $f : X \to Y$ is a function such that f and f^{-1} are Lipschitzian functions, then we say that f is a *bi-Lipschitzian homeomorphism.*

Second fundamental form and normal bundle of submanifolds of class 2

2.1 Definition. Suppose $1 \le m \le n$ are integers, M is an m dimensional submanifold of class 2 of $\mathbf{R}^n$ and $a \in M$. Then we call second fundamental form of M at a the unique symmetric 2 linear function

$$\mathbf{b}_M(a) : \operatorname{Tan}(M,a) \times \operatorname{Tan}(M,a) \to \operatorname{Nor}(M,a)$$

such that $\mathbf{b}_M(a)(u,v) \bullet \nu(a) = -\operatorname{D}\nu(a)(u) \bullet v$ for each $u, v \in \operatorname{Tan}(M,a)$, whenever $\nu : M \to \mathbf{R}^n$ is of class 1 relative to M with $\nu(x) \in \operatorname{Nor}(M,x)$ for every $x \in M$.

The following lemma is well known in differential geometry.

2.2 Lemma. *Let* $M \subseteq \mathbf{R}^n$ *be an* m *dimensional submanifold of class* 2 *and let* $N = \operatorname{Nor}(M) \cap (M \times \mathbf{S}^{n-1})$.

Then N is an $n-1$ dimensional submanifold of class 1 of $\mathbf{R}^n \times \mathbf{R}^n$ and, if $(a,u) \in N$ then $\mathrm{Tan}(N,(a,u))$ is the set of $(\tau, v + \mathrm{D}\,\nu(a)(\tau))$ such that $\tau \in \mathrm{Tan}(M,a)$, $v \in \mathrm{Nor}(M,a)$ is orthogonal to u and ν is a unit normal vector field of class 1 on an open neighborhood of a such that $\nu(a) = u$.

Proof. The conclusion is a direct consequence of the fact that, using a normal frame of M in an open neighborhood Z of a, we can locally parametrize N at (a,u) using the product manifold $(M \cap Z) \times \mathbf{S}^{n-m-1}$. $\square$

2.3 Remark. If $(a,u) \in N$, $\tau \in \mathrm{Tan}(M,a)$, $\tau_1 \in \mathrm{Tan}(M,a)$ and $\sigma_1 \in \mathbf{R}^n$ is such that $(\tau_1, \sigma_1) \in \mathrm{Tan}(N,(a,u))$, then

$$\tau \bullet \sigma_1 = -\mathbf{b}_M(a)(\tau, \tau_1) \bullet u.$$

Approximate differentiability for functions and sets

First we recall the following measure-theoretic notions of limit and differentiability for functions, which play a key role in section 3.

2.4 Definition. Let f be a function mapping a subset of $\mathbf{R}^n$ into some set Y and let $a \in \mathbf{R}^n$. If Y is a normed vector space, a point $y \in Y$ is *the approximate limit of f at a* if and only if

$$\mathbf{\Theta}^n(\mathscr{L}^n \llcorner \mathbf{R}^n \sim \{x : |f(x) - y| \le \epsilon\}, a) = 0 \quad \text{for every } \epsilon > 0$$

and we denote it by $\operatorname{ap\,lim}_{x \to a} f(x)$. If $Y = \overline{\mathbf{R}}$, a point $t \in \overline{\mathbf{R}}$ is *the approximate lower limit of f at a* [*the approximate upper limit of f at a*] if and only if

$$t = \sup\{s : \mathbf{\Theta}^n(\mathscr{L}^n \llcorner \{x : f(x) < s\}, a) = 0\}$$

$$\big[t = \inf\{s : \mathbf{\Theta}^n(\mathscr{L}^n \llcorner \{x : f(x) > s\}, a) = 0\}\big]$$

and we denote it by $\operatorname{ap\,lim\,inf}_{x \to a} f(x)$ [$\operatorname{ap\,lim\,sup}_{x \to a} f(x)$].

2.5 Definition. Let $n \ge 1$, $\nu \ge 1$ and $k \ge 0$ be integers, $A \subset \mathbf{R}^n$, $f : A \to \mathbf{R}^\nu$ and $a \in \mathbf{R}^n$.

We say that f is *approximately differentiable of order k at a* if there exists a polynomial function $P : \mathbf{R}^n \to \mathbf{R}^\nu$ of degree at most k such that $P(a) = f(a)$ if $a \in A$, and

$$\operatorname*{ap\,lim}_{x \to a} \frac{|f(x) - P(x)|}{|x - a|^k} = 0.$$

We let $\operatorname{ap} \mathrm{D}^i f(a) = \mathrm{D}^i P(a)$ for $i = 1, \dots, k$.

2.6 Remark. The following statement follows immediately from 2.4 and 2.5. *Suppose n, ν, k, A, f, a are as in 2.5 and $B \subseteq A$. Then $f|B$ is approximately differentiable of order k at a if and only if f is approximately differentiable of order k at a and $\mathbf{\Theta}^n(\mathscr{L}^n \llcorner \mathbf{R}^n \sim B, a) = 0$. In this case $\operatorname{ap} \mathrm{D}^i (f|B)(a) = \operatorname{ap} \mathrm{D}^i f(a)$ for $i = 1, \dots, k$.*

We recall now from [San17, 3.8, 3.19, 3.20] the notion of approximate differentiability for sets.

2.7 Definition. Let $n \geq 1$ and $k \geq 1$ be integers, $A \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$. We say that A is *approximately differentiable of order k at a* if and only if there exist an integer $1 \leq m \leq n$, $T \in \mathbf{G}(n,m)$ and a polynomial function $P : T \to T^\perp$ of degree at most k such that $P(0) = 0$, $\mathrm{D}\,P(0) = 0$ and the following two conditions hold:

(1) for every $\epsilon > 0$ there exists $\eta > 0$ such that

$$\mathscr{H}^m(\mathbf{B}(z, \epsilon r) \cap \{x - a : x \in A\}) \geq \eta r^m$$

for every $z \in T \cap \mathbf{B}(0,r)$ and $0 \leq r \leq \eta$,

(2) for every $\epsilon > 0$,

$$\lim_{r \to 0} \frac{\mathscr{H}^m\left(\{x - a : x \in A\} \cap \mathbf{B}(0,r) \cap \{z : \boldsymbol{\delta}_{\mathrm{gr}(P)}(z) > \epsilon\, r^k\}\right)}{\boldsymbol{\alpha}(m) r^m} = 0,$$

where $\mathrm{gr}\,P = \{\chi + P(\chi) : \chi \in T\}$.

2.8 Definition. Let n, k, A, a, m, T and P as in 2.7. Then we define

$$\mathrm{ap}\,\mathrm{Tan}(A,a) = T, \quad \mathrm{ap}\,\mathrm{Nor}(A,a) = T^\perp,$$

$$\mathrm{ap}\,\mathrm{D}^k\, A(a) = \mathrm{D}^k(P \circ T_\natural)(0).$$

2.9 Remark. One can prove, using a standard density-argument, that if M is an m dimensional submanifold of class 1 [class 2] in $\mathbf{R}^n$ and $A \subseteq M$ is $\mathscr{H}^m$ measurable with $\mathscr{H}^m(A) < \infty$, then

$$\mathrm{Tan}(M,a) = \mathrm{ap}\,\mathrm{Tan}(A,a) \quad \text{for } \mathscr{H}^m \text{ a.e. } a \in A$$

$$[\mathrm{ap}\,\mathrm{D}^2\, A(a) | \mathrm{ap}\,\mathrm{Tan}(A,a) \times \mathrm{ap}\,\mathrm{Tan}(A,a) = \mathbf{b}_M(a) \quad \text{for } \mathscr{H}^m \text{ a.e. } a \in A.]$$

2.10 Remark. For a set $A \subseteq \mathbf{R}^n$ other notions of measure-theoretic tangent planes are well known, see [San17, 1.3, 1.4]. If A is $\mathscr{H}^m$ measurable and $\mathscr{H}^m(A) < \infty$ then the sets of points where these tangent planes exist and belong to $\mathbf{G}(n,m)$ are $\mathscr{H}^m$ almost equal to the set of points where $\mathrm{ap}\,\mathrm{Tan}(A,\cdot)$ exists and belongs to $\mathbf{G}(n,m)$.

2.11 Remark. A characterization of higher order rectifiable sets is obtained in [San17, 3.23, 5.6] in terms of the approximate differentiability given in 2.7.

Level sets of distance function

2.12 Definition. Let $A \subseteq \mathbf{R}^n$ be a closed set. We define

$$\boldsymbol{\delta}_A(x) = \inf\{|x - a| : a \in A\} \quad \text{for } x \in \mathbf{R}^n,$$

$$S(A,r) = \{x : \boldsymbol{\delta}_A(x) = r\} \quad \text{for } r > 0.$$

In this paper we need the following result on the rectifiability properties of the level sets of $\boldsymbol{\delta}_A$.

2.13 Theorem. *Let A be a closed subset of $\mathbf{R}^n$ and $r > 0$.*

(1) If $K \subseteq \mathbf{R}^n$ is compact then $S(A,r) \cap K$ is $n-1$ rectifiable.

(2) $S(A,r)$ is countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable of class 2.

Proof. If A is bounded then the proof of (1) is contained in [RW10, 2.3] (which relies on [Fu85]). If A is unbounded then the proof can be readily reduced to the previous case noting that if $r > 0$ and $K \subseteq \mathbf{R}^n$ is compact then the set

$$C = \bigcup_{x \in S(A,r) \cap K} A \cap \{a : |x - a| = \boldsymbol{\delta}_A(x)\}$$

is compact and $S(A,r) \cap K \subseteq S(C,r)$.

We notice that for each $x \in S(A,r)$ there exists $v \in \mathbf{R}^n \sim \{0\}$ such that $\mathbf{U}(x+v,|v|) = \varnothing$. In fact, we can choose $v = a - x$ for $a \in A$ such that $|x - a| = r$. Therefore (2) comes from [MS17, 4.12]. Notice that [MS17, 4.12] also implies that $S(A,r)$ is countably $n-1$ rectifiable, a piece of information already contained in (1). □

2.14 Remark. The local structure of the level sets of the distance function has been thoroughly studied in the last decades; see [Fer76], [GP72], [Fu85] and [RZ12]. However, here we only use the rectifiability properties in 2.13.

2.15 Definition. If $A \subseteq \mathbf{R}^n$ is a closed set, we define *the positive boundary* $\partial^+ A$ of A as the set of all $x \in A$ such that there exists $v \in \mathbf{R}^n \sim \{0\}$ with $A \cap \mathbf{U}(x+v,|v|) = \varnothing$.

The following result is contained in [RW10, 2.5] when A is a compact set.

2.16 Lemma. *Let $A \subseteq \mathbf{R}^n$ be a closed set and let $P_r = \{x : \boldsymbol{\delta}_A(x) \leq r\}$ for $r > 0$. Then for all $r > 0$ up to a countable set,*

$$\mathscr{H}^{n-1}(S(A,r) \sim \partial^+ P_r) = 0.$$

Proof. If $r > 0$ and $i \geq 1$ is an integer, we define $P_{i,r} = \{x : \boldsymbol{\delta}_{A \cap \mathbf{B}(0,i)}(x) \leq r\}$. We fix two integers $i \geq 1$ and $j \geq 1$ and we prove that for all $0 < r < j$ up to a countable set,

$$\mathscr{H}^{n-1}(S(A,r) \cap \mathbf{U}(0,i) \sim \partial^+ P_r) = 0.$$

Let $0 < r < j$ and $x \in S(A,r) \cap \mathbf{U}(0,i) \cap \partial^+ P_{i+j,r}$. Then there exist $s > 0$ and $v \in \mathbf{R}^n$ with $|v| = 1$ and $\mathbf{U}(x+sv,s) \cap P_{i+j,r} = \varnothing$. Evidently we can choose s small so that $\mathbf{U}(x+sv,s) \subseteq \mathbf{U}(0,i)$. If there was $z \in \mathbf{U}(x+sv,s)$ such that $\boldsymbol{\delta}_A(z) \leq r$ then we could choose $a \in A$ so that $|z-a| = \boldsymbol{\delta}_A(z)$ and infer that

$$a \in A \cap \mathbf{B}(0,i+j), \quad \boldsymbol{\delta}_{A \cap \mathbf{B}(0,i+j)}(z) \leq r,$$

whence we would get a contradiction. Therefore

$$S(A,r) \cap \mathbf{U}(0,i) \cap \partial^+ P_{i+j,r} \subseteq S(A,r) \cap \mathbf{U}(0,i) \cap \partial^+ P_r.$$

Moreover we observe that

$$S(A,r) \cap \mathbf{U}(0,i) \subseteq S(A \cap \mathbf{B}(0,i+j), r) \quad \text{for all } 0 < r < j.$$

Now we employ [RW10, 2.5] to infer

$$\mathscr{H}^{n-1}(S(A,r) \cap \mathbf{U}(0,i) \sim \partial^+ P_r) = 0$$

for all $0 < r < j$, up to a countable set. □

3 Fine properties of the nearest point projection

The main objective of this section is to analyse the fine properties of the nearest point projection $\boldsymbol{\xi}_A$ and relate them to the tangential and curvature properties of the distance sets $S(A, r)$.

We start introducing some basic notation. It will be repeatedly used through the rest of this paper together with the notation already introduced in 2.12.

3.1 Definition (Basic notation)**.** Suppose $A \subseteq \mathbf{R}^n$ is closed and U is the set of all $x \in \mathbf{R}^n$ such that there exists a unique $a \in A$ with $|x - a| = \boldsymbol{\delta}_A(x)$. The *nearest point projection onto* A is the map $\boldsymbol{\xi}_A$ characterised by the requirement

$$|x - \boldsymbol{\xi}_A(x)| = \boldsymbol{\delta}_A(x) \quad \text{for } x \in U.$$

Let $\boldsymbol{\nu}_A$ and $\boldsymbol{\psi}_A$ be the functions on $U \sim A$ such that

$$\boldsymbol{\nu}_A(z) = \boldsymbol{\delta}_A(z)^{-1}(z - \boldsymbol{\xi}_A(z)) \quad \text{and} \quad \boldsymbol{\psi}_A(z) = (\boldsymbol{\xi}_A(z), \boldsymbol{\nu}_A(z)),$$

whenever $z \in U \sim A$. We refer to $\boldsymbol{\nu}_A$ as *the spherical image map of* A. Finally,

$$U(A) = \operatorname{dmn} \boldsymbol{\xi}_A \sim A.$$

3.2 Remark. It is known that $\boldsymbol{\xi}_A$ is continuous by [Fed59, 4.8(4)], $\operatorname{dmn} \boldsymbol{\xi}_A$ is a Borel subset of $\mathbf{R}^n$ by [MS17, 3.5], $\boldsymbol{\xi}_A^{-1}\{a\}$ is a convex subset of $\mathbf{R}^n$ whenever $a \in A$ by [Fed59, 4.8(2)] and

$$\text{(vi)} \qquad \mathscr{L}^n(\mathbf{R}^n \sim \operatorname{dmn} \boldsymbol{\xi}_A) = 0$$

by [Fed59, 4.8(3)] and Rademacher's theorem [Fed69, 3.1.6].

3.3 Remark. Noting 3.2, we readily infer that for every $0 < r < \infty$ the map $\boldsymbol{\psi}_A | U(A) \cap S(A, r)$ is an homeomorphism with

$$(\boldsymbol{\psi}_A | U(A) \cap S(A, r))^{-1}(a, u) = a + ru \quad \text{whenever } (a, u) \in \boldsymbol{\psi}_A[U(A) \cap S(A, r)].$$

3.4 Remark. We notice that if $v \in \mathbf{R}^n \sim \{0\}$, $a \in A$ and $|v| = \boldsymbol{\delta}_A(a + v)$ then

$$a + tv \in U(A) \quad \text{and} \quad \boldsymbol{\xi}_A(a + tv) = a$$

whenever $0 < t < 1$.

3.5 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is closed, $x \in U(A)$, $\boldsymbol{\xi}_A$ is approximately differentiable at x and $T = \mathbf{R}^n \cap \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$.*

Then $\boldsymbol{\delta}_A$ is differentiable at x, $\boldsymbol{\nu}_A$ is approximately differentiable at x,

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x) \bullet \boldsymbol{\nu}_A(x) = 0 \quad \textit{and} \quad \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x) = |x - \boldsymbol{\xi}_A(x)|^{-1}(T_\natural - \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)).$$

In particular $\ker \operatorname{ap} \mathrm{D}\, \boldsymbol{\psi}_A(x) \subseteq T^\perp$.

Proof. Since $\boldsymbol{\delta}_A(y) = |y - \boldsymbol{\xi}_A(y)|$ for $y \in \operatorname{dmn} \boldsymbol{\xi}_A$, we use A.4, A.5 and [Fed59, 4.8(3)] to deduce that $\boldsymbol{\delta}_A$ is differentiable at x and

$$\text{(vii)} \qquad \mathrm{D}\, \boldsymbol{\delta}_A(x)(v) = \boldsymbol{\nu}_A(x) \bullet v \quad \text{for } v \in \mathbf{R}^n.$$

It follows that $\boldsymbol{\nu}_A$ is approximately differentiable at x and employing (vii) one computes

$$\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v) = \frac{T_\natural(v) - \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v)}{\boldsymbol{\delta}_A(x)} \quad \text{for } v \in \mathbf{R}^n.$$

Then we readily infer that $\ker \operatorname{ap}\mathrm{D}\,\boldsymbol{\psi}_A(x) \subseteq T^\perp$.

If $r = |x - \boldsymbol{\xi}_A(x)|$ we use the continuity of $\boldsymbol{\xi}_A$ at x (see 3.2) to select $0 < \delta < r$ such that $|\boldsymbol{\xi}_A(z) - \boldsymbol{\xi}_A(x)| \le r$ and

$$\text{(viii)} \qquad (\boldsymbol{\xi}_A(z) - x) \bullet \boldsymbol{\nu}_A(x) = (\boldsymbol{\xi}_A(z) - \boldsymbol{\xi}_A(x)) \bullet \boldsymbol{\nu}_A(x) - r \le 0$$

whenever $z \in \mathbf{U}(x,\delta) \cap \operatorname{dmn} \boldsymbol{\xi}_A$. Since $|\boldsymbol{\xi}_A - x| \ge r$ and $T_\natural(x - \boldsymbol{\xi}_A(x)) = 0$ we use (viii) to infer

$$\big(r^2 - |T_\natural(\boldsymbol{\xi}_A(z) - \boldsymbol{\xi}_A(x))|^2\big)^{1/2} \le |(\boldsymbol{\xi}_A(z) - x) \bullet \boldsymbol{\nu}_A(x)| = -(\boldsymbol{\xi}_A(z) - x) \bullet \boldsymbol{\nu}_A(x),$$

$$\text{(ix)} \qquad (\boldsymbol{\xi}_A(z) - \boldsymbol{\xi}_A(x)) \bullet \boldsymbol{\nu}_A(x) + (r^2 - |T_\natural(\boldsymbol{\xi}_A(z) - \boldsymbol{\xi}_A(x))|^2)^{1/2} \le r,$$

for $z \in \mathbf{U}(x,\delta) \cap \operatorname{dmn} \boldsymbol{\xi}_A$. Employing A.1 and A.4 we obtain from (ix) that

$$\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x) \bullet \boldsymbol{\nu}_A(x) = 0.$$

□

3.6 Definition. If A is a closed subset of $\mathbf{R}^n$, we define

$$\rho(A,x) = \sup\{t : \boldsymbol{\delta}_A(\boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x))) = t\boldsymbol{\delta}_A(x)\},$$

whenever $x \in U(A)$.

3.7 Remark. We notice that if $x \in U(A)$ then $1 \le \rho(A,x) \le \infty$ and

$$\rho(A,x) \ge \lambda \quad \text{if and only if} \quad \boldsymbol{\delta}_A(\boldsymbol{\xi}_A(x) + \lambda(x - \boldsymbol{\xi}_A(x))) = \lambda\boldsymbol{\delta}_A(x)$$

for $\lambda \ge 1$. It follows from 3.2 that $\rho(A,\cdot) : U(A) \to \mathbf{R} \cup \{+\infty\}$ is an upper-semicontinuous function.

3.8 Definition. If A is a closed subset of $\mathbf{R}^n$ and $\lambda \ge 1$ we define

$$A_\lambda = \{x : \rho(A,x) \ge \lambda\}$$

and $D(A_\lambda)$ to be the set of $x \in A_\lambda$ such that $\boldsymbol{\xi}_A | A_\lambda$ is approximately differentiable at x; see 2.6.

3.9 Remark. If $0 < R = \operatorname{reach}(A)$, $0 < r < R$ and $0 < \boldsymbol{\delta}_A(x) \le r$ it follows from [Fed59, 4.8(6)] that

$$\sup\{t : \boldsymbol{\xi}_A(\boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x))) = \boldsymbol{\xi}_A(x)\} \ge R/r;$$

in particular, $\{x : 0 < \boldsymbol{\delta}_A(x) \le r\} \subseteq A_{R/r}$.

Here we provide a thorough description of the nearest point projection $\boldsymbol{\xi}_A$ on the super level sets A_λ.

3.10 Lemma. *Suppose A is a closed subset of $\mathbf{R}^n$ and define the maps*[1] h_t *on* $U(A)$ *corresponding to* $0 < t < \infty$ *by*

$$h_t(z) = \boldsymbol{\xi}_A(z) + t(z - \boldsymbol{\xi}_A(z)) \quad \text{for } z \in U(A). \tag{x}$$

Then the following statements hold for $1 < \lambda < \infty$ *and* $0 < t < \lambda$.

(1) $\operatorname{Lip}(\boldsymbol{\xi}_A|A_\lambda) \le \lambda(\lambda-1)^{-1}$ *and* $h_t|A_\lambda$ *is a bi-Lipschitzian homeomorphism onto* $A_{\lambda/t}$ *with* $(h_t|A_\lambda)^{-1} = h_{t^{-1}}|A_{\lambda/t}$.

(2) $\mathscr{L}^n(A_\lambda \sim D(A_\lambda)) = 0$.

(3) The map $\boldsymbol{\psi}_A|A_\lambda$ *has an extension* $\Psi : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ *such that* Ψ *is differentiable at every* $x \in D(A_\lambda)$ *with* $\mathrm{D}\,\Psi(x) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\psi}_A(x)$. *Moreover* $\ker \operatorname{ap}\mathrm{D}\,\boldsymbol{\psi}_A(x) = \{s\boldsymbol{\nu}_A(x) : s \in \mathbf{R}\}$ *whenever* $x \in D(A_\lambda)$.

(4) $h_t[D(A_\lambda)] \subseteq D(A_{\lambda/t})$.

(5) If $x \in D(A_\lambda)$ *then* $h_{t^{-1}}$ *is approximately differentiable at* $h_t(x)$ *with*

$$\operatorname{ap}\mathrm{D}\,h_{t^{-1}}(h_t(x)) = \operatorname{ap}\mathrm{D}\,h_t(x)^{-1},$$

$$\operatorname{ap}\mathrm{D}\,\boldsymbol{\psi}_A(x) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\psi}_A(h_t(x)) \circ \operatorname{ap}\mathrm{D}\,h_t(x).$$

(6) If $x \in D(A_\lambda)$ *then the eigenvalues of* $\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)$ *and* $\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)$ *belong to the intervals* $0 \le s \le \lambda(\lambda-1)^{-1}$ *and* $(1-\lambda)^{-1}\boldsymbol{\delta}_A(x)^{-1} \le s \le \boldsymbol{\delta}_A(x)^{-1}$, *respectively. In case* $\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)$ *is a symmetric endomorphism, so are* $\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(h_t(x))$ *and* $\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(h_t(x))$.

Proof of (1). If $x \in A_\lambda$ and $y \in A_\lambda$, then we apply [MS17, 4.7(1)] with q, a, b and v replaced by $\lambda|x - \boldsymbol{\xi}_A(x)|$, $\boldsymbol{\xi}_A(x)$, $\boldsymbol{\xi}_A(y)$ and $x - \boldsymbol{\xi}_A(x)$ respectively, to infer that

$$(\boldsymbol{\xi}_A(y) - \boldsymbol{\xi}_A(x)) \bullet (x - \boldsymbol{\xi}_A(x)) \le (2\lambda)^{-1}|\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)|^2,$$

and symmetrically,

$$(\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)) \bullet (y - \boldsymbol{\xi}_A(y)) \le (2\lambda)^{-1}|\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)|^2.$$

Combining the two equations we get

$$|\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)||x-y| \ge (\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)) \bullet (x-y) \ge \lambda^{-1}(\lambda-1)|\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)|^2.$$

By 3.4 one infers $\boldsymbol{\xi}_A(h_t(x)) = \boldsymbol{\xi}_A(x)$ and $h_{t^{-1}}(h_t(x)) = x$ whenever $x \in A_\lambda$, and $h_t[A_\lambda] \subseteq A_{\lambda/t}$. Since $0 < t^{-1} < \lambda/t$, the same conclusions hold with λ and t replaced by λ/t and t^{-1} respectively. Henceforth (1) is proved.

Proof of (2). Since $\boldsymbol{\xi}_A|A_\lambda$ is Lipschitzian then $\mathscr{L}^n(A_\lambda \sim D(A_\lambda)) = 0$ by [San17, 2.11].

Proof of (3). Since $\boldsymbol{\xi}_A|A_\lambda$ is Lipschitzian there exists a Lipschitzian function $F : \mathbf{R}^n \to \mathbf{R}^n$ such that $F|A_\lambda = \boldsymbol{\xi}_A|A_\lambda$ by [Fed69, 2.10.43]. Then, by A.5, the map F is differentiable at every $x \in D(A_\lambda)$ with

$$\mathrm{D}\,F(x) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x).$$

[1] In case A is convex, the map h_t is called "dilation with center A" in [Wal76, §3].

If $x \in D(A_\lambda)$ then $x + s\boldsymbol{\nu}_A(x) \in A_\lambda$ and

$$F(x + s\boldsymbol{\nu}_A(x)) = \boldsymbol{\xi}_A(x + s\boldsymbol{\nu}_A(x)) = \boldsymbol{\xi}_A(x)$$

for $-\boldsymbol{\delta}_A(x) < s < (\lambda - 1)\boldsymbol{\delta}_A(x)$. Differentiating with respect to s we get that

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)(\boldsymbol{\nu}_A(x)) = \mathrm{D}\, F(x)(\boldsymbol{\nu}_A(x)) = 0$$

and $\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)(\boldsymbol{\nu}_A(x)) = 0$ by 3.5. Let $G : \mathbf{R}^n \to \mathbf{R}^n$ be any function such that $G(x) = \boldsymbol{\delta}_A(x)^{-1}(x - F(x))$ for $x \in \mathbf{R}^n \sim A$. Noting 3.5 and [San17, 2.8] we infer that G is differentiable at every $x \in D(A_\lambda)$ with $\mathrm{D}\, G(x) = \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)$. Henceforth $\Psi = (F, G)$ and (3) is proved.

Proof of (4) *and* (5). Let $x \in D(A_\lambda)$ and $y = h_t(x)$. Then h_t is approximately differentiable at x and, noting (1), we can use A.3 and [Buc92, Theorem 1] to infer that $\operatorname{ap} \mathrm{D}\, h_t(x)$ is an isomorphism of $\mathbf{R}^n$ and

$$\boldsymbol{\Theta}^n(\mathscr{L}^n \llcorner \mathbf{R}^n \sim A_{\lambda/t}, y) = 0.$$

For $\epsilon > 0$ we define

$$P_\epsilon = A_\lambda \cap \{w : |h_t(w) - h_t(x) - \operatorname{ap} \mathrm{D}\, h_t(x)(w - x)| \geq \epsilon |w - x|\},$$

$$Q_\epsilon = A_{\lambda/t} \cap \{z : |h_{t^{-1}}(z) - x - \operatorname{ap} \mathrm{D}\, h_t(x)^{-1}(z - y)| \geq \epsilon |z - y|\},$$

we observe that $Q_\epsilon \subseteq h_t(P_{C\epsilon})$ for $C = \| \operatorname{ap} \mathrm{D}\, h_t(x)^{-1} \|^{-1} (\operatorname{Lip}(h_t|A_\lambda)^{-1})^{-1}$ and

$$\mathbf{B}(h_t(x), r) \cap Q_\epsilon \subseteq h_t[P_{C\epsilon} \cap \mathbf{B}(x, (\operatorname{Lip}(h_t|A_\lambda)^{-1})r)] \quad \text{for } r > 0,$$

whence we deduce that

$$\boldsymbol{\Theta}^n(\mathscr{L}^n \llcorner Q_\epsilon, h_t(x)) = 0 \quad \text{for every } \epsilon > 0,$$

the map $h_{t^{-1}}$ is approximately differentiable at y and

$$\operatorname{ap} \mathrm{D}\, h_{t^{-1}}(y) = \operatorname{ap} \mathrm{D}\, h_t(x)^{-1}.$$

Let Ψ be an extension of $\boldsymbol{\psi}_A|A_\lambda$ given by (3). If $z \in A_{\lambda/t}$, being $\lambda > 1$ and noting 3.4, we get that

$$\Psi(h_{t^{-1}}(z)) = \boldsymbol{\psi}_A(h_{t^{-1}}(z)) = \boldsymbol{\psi}_A(z)$$

and we use A.4 to infer that $\boldsymbol{\psi}_A$ is approximately differentiable at y with

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\psi}_A(y) = \operatorname{ap} \mathrm{D}\, \boldsymbol{\psi}_A(x) \circ \operatorname{ap} \mathrm{D}\, h_{t^{-1}}(y).$$

Proof of (6). If $\mu \in \mathbf{R}$, $v \in \mathbf{S}^{n-1}$ and $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)(v) = \mu v$ then, noting that $\operatorname{ap} \mathrm{D}\, h_s(x)$ is injective for $0 < s < \lambda$ by (5), we infer that

$$(1 - s)\mu + s \neq 0 \quad \text{for } 0 < s < \lambda,$$

whence we deduce that

$$0 \leq \mu \leq \lambda(\lambda - 1)^{-1}.$$

If $\mu \neq 0$, $v \in \mathbf{S}^{n-1}$ and $\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)(v) = \mu v$ then

$$v \bullet \boldsymbol{\nu}_A(x) = 0 \quad \text{and} \quad \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)(v) = (1 - \boldsymbol{\delta}_A(x)\mu)v$$

by 3.5, which implies $(1-\lambda)^{-1}\boldsymbol{\delta}_A(x)^{-1} \le \mu \le \boldsymbol{\delta}_A(x)^{-1}$.

If $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)$ is symmetric, then there exists an orthonormal basis $v_1, \dots, v_n$ of $\mathbf{R}^n$ and $0 \le \mu_1 \le \ldots \le \mu_n$ such that $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)(v_i) = \mu_i v_i$ for $i = 1, \dots, n$ and (5) implies that

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(h_t(x))(v_i) = \mu_i((1-t)\mu_i + t)^{-1} v_i \quad \text{whenever } i = 1, \dots, n.$$

Therefore $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(h_t(x))$ is symmetric and so is $\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(h_t(x))$ by 3.5.

3.11 Remark. Combining 3.5 and 3.10(5), if $1 < \lambda < \infty$, $0 < t < \lambda$, $x \in D(A_\lambda)$ and $T = \mathbf{R}^n \cap \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$, then

$$\operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(h_t(x)) = \operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x) \subseteq T,$$

$$\operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(h_t(x)) = \operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x) \subseteq T.$$

Here the tangential and curvature properties of the distance sets $S(A, r)$ are expressed in terms of the spherical image map of A and its approximate differential.

3.12 Lemma. *If A is a closed subset of $\mathbf{R}^n$ then for $\mathscr{L}^1$ a.e. $r > 0$ and for $\mathscr{H}^{n-1}$ a.e. $x \in S(A, r)$ the following three statements hold:*

$$\mathscr{H}^{n-1}\Big(S(A, r) \sim \bigcup_{\lambda > 1} D(A_\lambda)\Big) = 0,$$

$$\operatorname{ap} \operatorname{Tan}(S(A, r), x) = \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\},$$

$$\operatorname{ap} \mathrm{D}^2\, S(A, r)(x)(u, v) \bullet \boldsymbol{\nu}_A(x) = -\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)(u) \bullet v$$

for $u, v \in \operatorname{ap} \operatorname{Tan}(S(A, r), x)$.

Proof. We define $P_r = \{x : \boldsymbol{\delta}_A(x) \le r\}$ for $r > 0$ and $B = \bigcup_{\lambda > 1} A_\lambda$. First we prove that

$$S(A, r) \cap B = \partial^+ P_r \quad \text{for every } r > 0.$$

Let $x \in \partial^+ P_r$. Then $x \in S(A, r)$ and we choose $a \in A$ with $|x - a| = r$, $u \in \mathbf{S}^{n-1}$ and $s > 0$ such that $\mathbf{U}(x + su, s) \cap P_r = \varnothing$. Noting that $\boldsymbol{\delta}_A(x + su) > r$ we apply [Fed59, 4.9] to infer that

$$s = \boldsymbol{\delta}_{P_r}(x + su) = \boldsymbol{\delta}_A(x + su) - r$$

whence we deduce that $r + s \le |x + su - a|$ and $r \le u \bullet (x - a)$. It follows that $x - a$ and u must be linearly dependent and $x - a = ru$. Noting 3.4 we conclude that $\rho(A, x) \ge r^{-1}(r + s)$. We assume now $x \in A_\lambda \cap S(A, r)$ for $\lambda > 1$. Since $\boldsymbol{\delta}_A(\boldsymbol{\xi}_A(x) + \lambda(x - \boldsymbol{\xi}_A(x))) = \lambda r$ it follows from [Fed59, 4.9] that

$$\boldsymbol{\delta}_{P_r}(\boldsymbol{\xi}_A(x) + \lambda(x - \boldsymbol{\xi}_A(x))) = (\lambda - 1)r$$

and, noting that $\boldsymbol{\xi}_A(x) + \lambda(x - \boldsymbol{\xi}_A(x)) = x + (\lambda - 1)r\boldsymbol{\nu}_A(x)$, we conclude that $x \in \partial^+ P_r$.

It follows from 2.16 that $\mathscr{H}^{n-1}(S(A, r) \sim B) = 0$ for all, but countably many $r > 0$, whence we deduce using 3.10(2) and Coarea formula that

$$\text{(xi)} \qquad \mathscr{H}^{n-1}\Big(S(A, r) \sim \bigcup_{\lambda > 1} D(A_\lambda)\Big) = 0 \quad \text{for } \mathscr{L}^1 \text{ a.e. } r > 0.$$

It follows from 3.10(3) and [Fed69, 2.10.19(4), 3.2.16] that for all $r > 0$, $\lambda > 1$ and for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r) \cap D(A_\lambda)$,

$$\text{(xii)} \qquad \boldsymbol{\Theta}^{n-1}(\mathscr{H}^{n-1} \llcorner S(A,r) \sim A_\lambda, x) = 0$$

and $\boldsymbol{\psi}_A$ is $(\mathscr{H}^{n-1} \llcorner S(A,r), n-1)$ approximately differentiable[2] at x with

$$\text{(xiii)} \qquad (\mathscr{H}^{n-1} \llcorner S(A,r), n-1) \operatorname{ap} \mathrm{D}\, \boldsymbol{\psi}_A(x) = \operatorname{ap} \mathrm{D}\, \boldsymbol{\psi}_A(x).$$

Moreover we claim that for $\mathscr{L}^1$ a.e. $r > 0$ and for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$

$$\text{(xiv)} \qquad \operatorname{ap} \operatorname{Tan}(S(A,r), x) = \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}.$$

To prove (xiv), first we notice that it follows from [Fed69, 3.1.6, 3.2.11, 3.1.21], [Fed59, 4.8(3)] and (vi) that $\boldsymbol{\delta}_A$ is differentiable at x with $\operatorname{grad} \boldsymbol{\delta}_A(x) = \boldsymbol{\nu}_A(x)$ and $\operatorname{Tan}(S(A,r), x) \subseteq \{v : v \bullet \operatorname{grad} \boldsymbol{\delta}_A(x) = 0\}$ for $\mathscr{L}^1$ a.e. $r > 0$ and for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$; second we employ 2.13 and [San17, 3.23].

Combining (xi)-(xiv) with 3.10(1) and [San17, 3.25] we conclude that

$$\operatorname{ap} \mathrm{D}^2\, S(A,r)(x)(u,v) \bullet \boldsymbol{\nu}_A(x) = -\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)(u) \bullet v$$

for $u, v \in \operatorname{ap} \operatorname{Tan}(S(A,r), x)$, for $\mathscr{H}^{n-1}$ a.e. $x \in S(A,r)$ and for $\mathscr{L}^1$ a.e. $r > 0$. □

3.13 Definition. If $A \subseteq \mathbf{R}^n$ is a closed set we say that $x \in U(A)$ is a *regular point of* $\boldsymbol{\xi}_A$ if and only if $\operatorname{ap} \lim_{y \to x} \rho(A,y) = \rho(A,x) > 1$ and $\boldsymbol{\xi}_A$ is approximately differentiable at x with symmetric approximate differential.

The set of regular points of $\boldsymbol{\xi}_A$ is denoted by $R(A)$.

3.14 Theorem. *If A is a closed subset of $\mathbf{R}^n$ then $\mathscr{L}^n(\mathbf{R}^n \sim (R(A) \cup A)) = 0$. If $x \in R(A)$ then $\boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x)) \in R(A)$ for every $0 < t < \rho(A,x)$.*

Proof. One infers from 3.12, 3.5 and Coarea formula that $\rho(A,x) > 1$ and $\boldsymbol{\xi}_A$ is approximately differentiable with symmetric approximate differential for $\mathscr{L}^n$ a.e. $x \in \mathbf{R}^n \sim A$. Since $\rho(A,\cdot)$ is a Borel function by 3.7, it follows from [Fed69, 2.9.13] that $\operatorname{ap} \lim_{y \to x} \rho(A,y) = \rho(A,x)$ for $\mathscr{L}^n$ a.e. $x \in U(A)$. Therefore,

$$\mathscr{L}^n(\mathbf{R}^n \sim (R(A) \cup A)) = 0.$$

If $x \in R(A)$ and $0 < t < \rho(A,x)$ we choose λ such that $t < \lambda < \rho(A,x)$ and $\lambda > 1$ and we notice that $x \in D(A_\lambda)$. It follows from 3.10(4)(6) that $\boldsymbol{\xi}_A$ is approximately differentiable at $h_t(x)$ (see (x)) with symmetric approximate differential,

$$\boldsymbol{\Theta}^n(\mathscr{L}^n \llcorner \mathbf{R}^n \sim A_{\lambda/t}, h_t(x)) = 0, \qquad \operatorname{ap} \liminf_{y \to h_t(x)} \rho(A,y) \geq \lambda/t.$$

Since $\rho(A, h_t(x)) = t^{-1}\rho(A,x)$, we conclude that

$$\operatorname{ap} \liminf_{y \to h_t(x)} \rho(A,y) \geq \rho(A, h_t(x)) > 1$$

and it follows from 3.7 that $h_t(x) \in R(A)$. □

[2]Given a measure ϕ on a normed vector space and a positive integer m, we refer to [Fed69, 3.2.16] for the notion of (μ, m) approximate differentiability.

3.15 Remark. The fact that $\boldsymbol{\xi}_A$ is approximately differentiable with symmetric approximate differential at $\mathscr{L}^n$ a.e. $x \in U(A)$ can be alternatively deduced from [Asp73].

3.16 Remark. It follows from Coarea formula and 3.14 that

$$\mathscr{H}^{n-1}(S(A,r) \sim R(A)) = 0 \quad \text{for } \mathscr{L}^1 \text{ a.e. } r > 0.$$

3.17 Definition. If $A \subseteq \mathbf{R}^n$ is a closed set, $1 < \lambda < \infty$ and $0 < r < \infty$ then we define

$$S_\lambda(A,r) = S(A,r) \cap A_\lambda,$$

3.18 Remark. If $r > 0$ we can readily check the following properties.

(1) $\boldsymbol{\psi}_A | S_\lambda(A,r)$ is a bi-Lipschitzian homeomorphism by 3.3 and 3.10(1).

(2) $\boldsymbol{\psi}_A[S_\lambda(A,r)] = (A \times \mathbf{S}^{n-1}) \cap \{(a,u) : \boldsymbol{\delta}_A(a + \lambda r u) = \lambda r\}$ (using 3.4 and 3.7), whence we deduce that $\boldsymbol{\psi}_A[S_\lambda(A,r)]$ is a closed subset of $A \times \mathbf{S}^{n-1}$ and

$$\boldsymbol{\psi}_A[S_\lambda(A,r)] \subseteq \boldsymbol{\psi}_A[S_\lambda(A,s)] \quad \text{if } 0 < s < r < \infty.$$

(3) It follows from 2.13(1) that $\boldsymbol{\psi}_A[S_\lambda(A,r)] | K$ is $n-1$ rectifiable for every $K \subseteq \mathbf{R}^n$ compact.

(4) If $\operatorname{reach}(A) = R > 0$ and $0 < r < R$ it follows from 3.9 that

$$S(A,r) = S_{R/r}(A,r).$$

4 Second fundamental form

In this section we introduce the second fundamental form in (v) and we prove theorem 1.1.

4.1 Definition. Suppose A is a closed subset of $\mathbf{R}^n$. We define

$$N(A) = (A \times \mathbf{S}^{n-1}) \cap \{(a,u) : \boldsymbol{\delta}_A(a + su) = s \text{ for some } s > 0\}.$$

Moreover we let $N(A,a) = \{v : (a,v) \in N(A)\}$ for $a \in A$.

4.2 Remark. We notice that $N(A)$ coincides with the *normal bundle of* A introduced in [HLW04, §2.1] and $N(A) \subseteq \operatorname{Nor}(A)$, see [Fed59, 4.4] or [Fed69, 3.1.21]. If $\operatorname{reach} A > 0$ then $N(A,a) = \operatorname{Nor}(A,a) \cap \mathbf{S}^{n-1}$ for $a \in A$ by [Fed59, 4.8(12)].

4.3 Remark. If $1 < \lambda < \infty$, $(a,u) \in A \times \mathbf{S}^{n-1}$ and $\boldsymbol{\delta}_A(a+su) = s$ for some $s > 0$ it follows from 3.4 that $a + \lambda^{-1} s u \in A_\lambda$ and $\boldsymbol{\psi}_A(a + \lambda^{-1} s u) = (a,u)$. Then we readily infer that

$$N(A) = \boldsymbol{\psi}_A[A_\lambda] = \bigcup_{r>0} \boldsymbol{\psi}_A[S_\lambda(A,r)].$$

It follows from 3.18 that $N(A)$ *is a countably* $n-1$ *rectifiable Borel subset of* $\mathbf{R}^n \times \mathbf{S}^{n-1}$. This fact has been already noticed in [HLW04, p. 243].

4.4 Definition. If $x \in R(A)$ then $\boldsymbol{\psi}_A(x)$ *is a regular point of* $N(A)$. We denote the set of all regular points of $N(A)$ by $R(N(A))$.

4.5 Remark. It follows from 4.3, 3.18 and 3.16 that

$$\mathscr{H}^{n-1}(N(A) \sim R(N(A)) = 0.$$

Moreover it follows from 3.14 that if $(a,u) \in R(N(A))$ then $a + ru \in R(A)$ for $0 < r < \sup\{s : \boldsymbol{\delta}_A(a+su) = s\}$.

The following lemma ensures that the definition in 4.7 is well posed.

4.6 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is a closed set, $x \in R(A)$, $0 < t < \rho(A,x)$ and $y = \boldsymbol{\xi}_A(x) + t(x - \boldsymbol{\xi}_A(x))$, then the following two statements hold.*

(1) If $v, v_1, v_2 \in \mathbf{R}^n$ are such that $\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_2)$, then

$$\begin{aligned}
&\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_2),\\
&\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1).
\end{aligned}$$

(2) If $v, w, v_1, w_1 \in \mathbf{R}^n$ are such that $\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v)$ and $\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w_1) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1)$, then

$$\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(y)(w_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w).$$

Proof. Let $r = |x - \boldsymbol{\xi}_A(x)|$ and we recall that $x \in D(A_\lambda)$ for $1 < \lambda < \rho(A,x)$. To prove (1) we compute, using 3.5 and 3.10(3),

$$\begin{aligned}
&\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1)\\
&\quad = r^{-1} v \bullet [\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) - (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x))(v_1)]\\
&\quad = r^{-1} v \bullet [\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_2) - (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x))(v_2)]\\
&\quad = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_2),\\
&\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1)\\
&\quad = r^{-1} v \bullet [\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) - (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x))(v_1)]\\
&\quad = r^{-1} \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) \bullet [v - \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v)]\\
&\quad = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v);
\end{aligned}$$

to prove (2) we compute, using 3.5 and 3.10(3)(5)(6),

$$\begin{aligned}
&\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w_1) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(T_\natural(v_1))\\
&\quad = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)[\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1) + t(T_\natural(v_1) - \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v_1))]\\
&\quad = \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)[\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w_1) + tr\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1)],\\
&t^{-1}r^{-1}[\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w_1) - (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y))(w_1)]\\
&\quad = (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x))(v_1),\\
&\operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(x)(v)\\
&\quad = \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x)(v_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w)\\
&\quad = (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(x))(v_1) \bullet w\\
&\quad = t^{-1}r^{-1}[\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w_1) - (\operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y) \circ \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y))(w_1)] \bullet w\\
&\quad = \operatorname{ap}\mathrm{D}\,\boldsymbol{\nu}_A(y)(w_1) \bullet \operatorname{ap}\mathrm{D}\,\boldsymbol{\xi}_A(y)(w).
\end{aligned}$$

□

4.7 Definition. Suppose A is a closed subset of $\mathbf{R}^n$ and $(a,u) \in R(N(A))$.
We define

$$T_A(a,u) = \operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x) \quad \text{and} \quad Q_A(a,u)(\tau,\tau_1) = \tau \bullet \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x)(v_1),$$

whenever x is a regular point of $\boldsymbol{\xi}_A$ such that $\boldsymbol{\psi}_A(x) = (a,u)$, $\tau \in T_A(a,u)$, $\tau_1 \in T_A(a,u)$ and $v_1 \in \mathbf{R}^n$ such that $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)(v_1) = \tau_1$.

We call $Q_A(a,u)$ *second fundamental form of A at a in the direction u.*

4.8 Lemma. *If $A \subseteq \mathbf{R}^n$ is a closed set and $(a,u) \in R(N(A))$ then*

$$Q_A(a,u) : T_A(a,u) \times T_A(a,u) \to \mathbf{R}$$

is a symmetric bilinear form and $T_A(a,u) \subseteq \{v : v \bullet u = 0\}$. Moreover if $r > 0$ and $\boldsymbol{\delta}_A(a+ru) = r$, then

$$Q_A(a,u)(\tau,\tau) \geq -r^{-1}|\tau|^2 \quad \textit{whenever } \tau \in T_A(a,u).$$

Proof. If x and y are regular points of $\boldsymbol{\xi}_A$ such that $\boldsymbol{\psi}_A(x) = (a,u) = \boldsymbol{\psi}_A(y)$ then $y = \boldsymbol{\xi}_A(x) + (\boldsymbol{\delta}_A(y)/\boldsymbol{\delta}_A(x))(x - \boldsymbol{\xi}_A(x))$, and the first part of the conclusion follows from 3.11 and 4.6.

If $0 < s < r$ then $a+su$ is a regular point of $\boldsymbol{\xi}_A$ by 4.5 and $\boldsymbol{\psi}_A(a+su) = (a,u)$. If $\tau \in T_A(a,u)$ and $v \in \mathbf{R}^n$ is such that $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v) = \tau$ then, noting that $\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v) \bullet v \geq 0$ by 3.10(6), we use 3.5 to compute

$$\begin{aligned} Q_A(a,u)(\tau,\tau) &= \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v) \bullet \operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(a+su)(v) \\ &= s^{-1} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v) \bullet (T_\natural(v) - \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v)) \\ &= s^{-1} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v) \bullet (v - \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v)) \\ &\geq -s^{-1} |\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+su)(v)|^2 = -s^{-1}|\tau|^2. \end{aligned}$$

Letting $s \to r$ we get the second conclusion. □

4.9 Definition. Let $A \subseteq \mathbf{R}^n$ be closed. For each regular point (a,u) of $N(A)$ we define *the principal curvatures of A at* (a,u),

$$\kappa_{A,1}(a,u) \leq \ldots \leq \kappa_{A,n-1}(a,u),$$

so that $\kappa_{A,m+1}(a,u) = \infty$, $\kappa_{A,1}(a,u), \ldots, \kappa_{A,m}(a,u)$ are the eigenvalues of $Q_A(a,u)$ and $m = \dim T_A(a,u)$. Moreover

$$\chi_{A,1}(x) \leq \ldots \leq \chi_{A,n-1}(x)$$

are the eigenvalues of $\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(x) | \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\}$ for $x \in R(A)$.

Now we clarify the relation between the $\kappa_{A,i}$'s and the $\chi_{A,i}$'s.

4.10 Lemma. *If $A \subseteq \mathbf{R}^n$ is closed and $(a,u) \in R(N(A))$ then*

$$\kappa_{A,i}(a,u) = \frac{\chi_{A,i}(a+ru)}{1 - r\chi_{A,i}(a+ru)}$$

for $0 < r < \sup\{s : \boldsymbol{\delta}_A(a+su) = s\}$ and $i = 1, \ldots, n-1$.

Proof. If $(a,u) \in R(N(A))$ and $0 < r < \sup\{s : \boldsymbol{\delta}_A(a+su) = s\}$ let

$$T = \{v : v \bullet \boldsymbol{\nu}_A(a+ru) = 0\}$$

and let $\{v_1, \ldots, v_{n-1}\}$ be an orthonormal basis of T such that

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\nu}_A(a+ru)(v_i) = \chi_{A,i}(a+ru)v_i \quad \text{for } i = 1, \ldots, n-1.$$

It follows from 3.5 that

$$\operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(a+ru)(v_i) = (1 - r\chi_{A,i}(a+ru))v_i \quad \text{for } i = 1, \ldots, n-1,$$

whence we conclude from the definitions 4.7 and 4.9 that

$$\chi_{A,i}(a+ru) = r^{-1} \quad \text{for } i > \dim T_A(a,u),$$

$$Q_A(a,u)(v_i, v_j) = \chi_{A,j}(a+ru)(1-r\chi_{A,j}(a+ru))^{-1} v_i \bullet v_j \quad \text{for } i,j \le \dim T_A(a,u),$$

$$\kappa_{A,i}(a,u) = \chi_{A,i}(a+ru)(1 - r\chi_{A,i}(a+ru))^{-1} \quad \text{for } 1 \le i \le n-1.$$

□

It is immediate from the following lemma to conclude that the principal curvatures introduced in [HLW04] coincides with those introduced in 4.9, see 4.12.

4.11 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is closed and θ is $\mathscr{H}^{n-1} \llcorner N(A)$ measurable and $\mathscr{H}^{n-1} \llcorner N(A)$ almost positive function such that $\theta \mathscr{H}^{n-1} \llcorner N(A)$ is a Radon measure over $\mathbf{R}^n \times \mathbf{S}^{n-1}$. Let $\psi = \theta \mathscr{H}^{n-1} \llcorner N(A)$.*

Then the following three statements hold.

(1) For $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$, $\operatorname{Tan}^{n-1}(\psi, (a,u))$ is a $(n-1)$ dimensional plane contained in $\operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner N(A), (a,u))$ and there exist $u_1, \ldots, u_{n-1} \in \mathbf{R}^n$ such that $\{u_1, \ldots, u_{n-1}, u\}$ is an orthonormal basis of $\mathbf{R}^n$ and

$$\left\{ \left(\frac{1}{(1+\kappa_{A,i}(a,u)^2)^{1/2}} u_i, \frac{\kappa_{A,i}(a,u)}{(1+\kappa_{A,i}(a,u)^2)^{1/2}} u_i \right) : 1 \le i \le n-1 \right\}$$

is an orthonormal basis of $\operatorname{Tan}^{n-1}(\psi, (a,u))$[3].

(2) For $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)$,

$$T_A(a,u) = \mathbf{p}[\operatorname{Tan}^{n-1}(\psi,(a,u))] \quad \textit{and} \quad Q_A(a,u)(\tau, \tau_1) = \tau \bullet \sigma_1$$

whenever $\tau \in T_A(a,u)$, $\tau_1 \in T_A(a,u)$ and $(\tau_1, \sigma_1) \in \operatorname{Tan}^{n-1}(\psi, (a,u))$.

(3) For every $(\mathscr{H}^{n-1} \llcorner N(A))$ integrable $\overline{\mathbf{R}}$ valued function f on $N(A)$,

$$\int_{N(A)} f(a,u) \prod_{i=1}^{n-1} \frac{|\kappa_{A,i}(a,u)|}{(1+\kappa_{A,i}(a,u)^2)^{1/2}} \, d\mathscr{H}^{n-1}(a,u)$$
$$= \int_{\mathbf{S}^{n-1}} \int_{\{a:(a,v)\in N(A)\}\times\{v\}} f \, d\mathscr{H}^0 \, d\mathscr{H}^{n-1} v.$$

[3] If $\kappa_{A,i}(a,u) = \infty$ the corresponding vector equals $(0, u_i)$.

Proof. The first part of (1) directly follows from B.4 and 4.3. We fix now $\lambda > 1$. For $r > 0$ let P_r be the set of $x \in R(A) \cap D(A_\lambda) \cap S(A,r)$ such that the following two conditions are satisfied:

$$\operatorname{ap}\operatorname{Tan}(S_\lambda(A,r),x) = \mathbf{R}^n \cap \{v : v \bullet \boldsymbol{\nu}_A(x) = 0\},$$

$$\operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner \boldsymbol{\psi}_A[S_\lambda(A,r)], \boldsymbol{\psi}_A(x)) = \operatorname{Tan}^{n-1}(\psi, \boldsymbol{\psi}_A(x)) \text{ is an } n-1 \text{ dimensional plane.}$$

If $r > 0$ and $x \in P_r$ it follows from 3.10(3), 3.18, B.2 and B.3 that

$$\operatorname{ap}\operatorname{D}\boldsymbol{\psi}_A(x)[\operatorname{ap}\operatorname{Tan}(S_\lambda(A,r),x)] = \operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner \boldsymbol{\psi}_A[S_\lambda(A,r)], \boldsymbol{\psi}_A(x)),$$

$$\mathbf{p}[\operatorname{Tan}^{n-1}(\psi, \boldsymbol{\psi}_A(x))] = \operatorname{im}\operatorname{ap}\operatorname{D}\boldsymbol{\xi}_A(x),$$

$$Q_A(\boldsymbol{\psi}_A(x))(\tau,\tau_1) = \tau \bullet \sigma_1$$

for $\tau, \tau_1 \in T_A(\boldsymbol{\psi}_A(x))$ and $(\tau_1, \sigma_1) \in \operatorname{Tan}^{n-1}(\psi, \boldsymbol{\psi}_A(x))$ and if $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal basis of $\operatorname{ap}\operatorname{Tan}(S_\lambda(A,r),x)$ such that $\operatorname{ap}\operatorname{D}\boldsymbol{\nu}_A(x)(v_i) = \chi_{A,i}(x)v_i$ for $i = 1, \ldots, n-1$, then we can easily check using 4.10 that

$$\left\{\left(\frac{1}{(1+\kappa_{A,i}(\boldsymbol{\psi}_A(x))^2)^{1/2}}v_i, \frac{\kappa_{A,i}(\boldsymbol{\psi}_A(x))}{(1+\kappa_{A,i}(\boldsymbol{\psi}_A(x))^2)^{1/2}}v_i\right) : \ 1 \le i \le n-1\right\}$$

is an orthonormal basis of $\operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner \boldsymbol{\psi}_A[S_\lambda(A,r)], \boldsymbol{\psi}_A(x))$. Noting that

$$\mathscr{H}^{n-1}(S_\lambda(A,r) \sim P_r) = 0 \quad \text{and} \quad \mathscr{H}^{n-1}(\boldsymbol{\psi}_A[S_\lambda(A,r)] \sim \boldsymbol{\psi}_A[P_r]) = 0$$

for $\mathscr{L}^1$ a.e. $r > 0$ and 4.3, the second part of (1) and (2) follow.

Finally, when f is a nonnegative $(\mathscr{H}^{n-1} \llcorner N(A))$ measurable $\overline{\mathbf{R}}$ valued function, we may apply [Fed69, 3.2.22(3)] with W, Z and f replaced by $\boldsymbol{\psi}_A[S_\lambda(A,r)]$, $\mathbf{S}^{n-1}$ and $\mathbf{q}|\boldsymbol{\psi}_A[S_\lambda(A,r)]$ to conclude

$$\begin{aligned}\int_{\boldsymbol{\psi}_A[S_\lambda(A,r)]} & f(a,u) \prod_{i=1}^{n-1} |\kappa_{A,i}(a,u)|(1+\kappa_{A,i}(a,u)^2)^{-1/2}\, d\mathscr{H}^{n-1}(a,u) \\ &= \int_{\mathbf{S}^{n-1}} \int_{\{a:(a,v)\in\boldsymbol{\psi}_A[S_\lambda(A,r)]\}\times\{v\}} f\, d\mathscr{H}^0\, d\mathscr{H}^{n-1}v\end{aligned}$$

for $\mathscr{L}^1$ a.e. $r > 0$ and (3) is a consequence of 4.3 and [Fed69, 2.4.7]. The general case asserted in (3) is then a consequence of [Fed69, 2.4.4]. □

4.12 Remark. It follows from 4.11(1) that the principal curvatures on $N(A)$ introduced in [HLW04, p. 244] coincide on $\mathscr{H}^{n-1}$ almost all of $N(A)$ with the principal curvatures introduced in 4.9.

4.13 Remark. In case $\operatorname{reach}(A) > 0$, it follows from 4.11(2) that Q_A coincides with the second fundamental form of A introduced in [Fu89, 4.5] on $\mathscr{H}^{n-1}$ almost all of $N(A)$.

4.14 Remark. It is not difficult to check using 4.11(2) that *if A and B are closed subsets of $\mathbf{R}^n$ then*

$$Q_A(a,u) = Q_B(a,u) \quad \textit{for } \mathscr{H}^{n-1} \textit{ a.e. } (a,u) \in N(A) \cap N(B).$$

5 Stratification and support measures

Recalling that $\boldsymbol{\xi}_A^{-1}\{a\}$ is a convex subset for every $a \in A$, see 3.2, we introduce the following stratification.

5.1 Definition. Suppose A is a closed subset of $\mathbf{R}^n$. For each $0 \le m \le n$, we define *the m-th stratum of A* by[4]

$$A^{(m)} = A \cap \{a : \dim \boldsymbol{\xi}_A^{-1}\{a\} = n - m\}.$$

5.2 Remark. In [MS17, 4.1] the *distance bundle of A* is defined as

$$\operatorname{Dis}(A) = (\mathbf{R}^n \times \mathbf{R}^n) \cap \{(a, v) : a \in A,\ |v| = \boldsymbol{\delta}_A(a + v)\}$$

and $\operatorname{Dis}(A, a) = \{v : (a, v) \in A\}$ is a closed convex subset of $\operatorname{Nor}(A, a)$ with $0 \in \operatorname{Dis}(A, a)$ for every $a \in A$, see [MS17, 4.2]. One readily sees that

$$N(A) = \{(a, |v|^{-1}v) : 0 \ne v \in \operatorname{Dis}(A, a)\}$$

and it follows from [MS17, 4.4] that

$$\dim \operatorname{Dis}(A, a) = \dim \boldsymbol{\xi}_A^{-1}\{a\} \quad \text{whenever } a \in A,$$

$$A^{(m)} = A \cap \{a : \dim \operatorname{Dis}(A, a) = n - m\}.$$

It is proved in [MS17, 4.12] that $A^{(m)}$ is a countably m rectifiable Borel set which can be $\mathscr{H}^m$ almost covered by the union of a countable family of m dimensional submanifolds of $\mathbf{R}^n$ of class 2. Finally one may use Coarea formula to infer that

$$A^{(m)} = A \cap \{a : 0 < \mathscr{H}^{n-m-1}(N(A, a)) < \infty\} \quad \text{for } m = 0, \dots, n - 1.$$

5.3 Lemma. *Suppose $A \subseteq \mathbf{R}^n$ is closed, $0 \le m \le n - 1$ is an integer and $x \in \boldsymbol{\xi}_A^{-1}[A^{(m)}]$ such that $\operatorname{ap}\liminf_{y \to x} \rho(A, y) \ge \rho(A, x) > 1$ and $\boldsymbol{\xi}_A$ is approximately differentiable at x.*

Then $\dim \operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x) \le m$. In particular, $\dim T_A(a, u) \le m$ if (a, u) is a regular point of $N(A)$ such that $a \in A^{(m)}$.

Proof. Let $a = \boldsymbol{\xi}_A(x)$, $1 < \lambda < \rho(A, x)$ and $C = \boldsymbol{\xi}_A^{-1}[\{a\}] \cap A_\lambda$. First we prove that C is a convex subset of $\mathbf{R}^n$ and

$$\dim C = \dim \boldsymbol{\xi}_A^{-1}\{a\} = n - m.$$

In fact, $C = \{y : \boldsymbol{\delta}_A(a + \lambda(y - a)) = \lambda|y - a|\}$ by 3.7 and 3.4 and C is convex by [Fed59, 4.8(2)]. Moreover, if U is the relative interior[5] of $\boldsymbol{\xi}_A^{-1}\{a\}$, then $\{y : a + \lambda(y - a) \in U\}$ is contained in C and it is open relative to the affine hull of $\boldsymbol{\xi}_A^{-1}\{a\}$. Therefore $\dim C = \dim \boldsymbol{\xi}_A^{-1}\{a\}$.

By 3.10(3), let $F : \mathbf{R}^n \to \mathbf{R}^n$ be an extension of $\boldsymbol{\xi}_A | A_\lambda$ that is differentiable at x with $\mathrm{D}\, F(x) = \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x)$. Since $F(y) = a$ whenever $y \in C$, we conclude that $\mathrm{D}\, F(x)(y - x) = 0$ whenever $y \in C$. Therefore $\mathrm{D}\, F(x)(y - x) = 0$ whenever y belongs to the affine hull of C. Since $\dim C = n - m$, we conclude

$$\dim \operatorname{im} \operatorname{ap} \mathrm{D}\, \boldsymbol{\xi}_A(x) \le m.$$

□

[4] The dimension of a convex subset K of $\mathbf{R}^n$ is the dimension of the affine hull of K and it is denoted by $\dim K$.

[5] The relative interior of a convex subset K of $\mathbf{R}^n$ is the interior of K relative to the affine hull of K.

We point out a Coarea-type formula for the generalized normal bundle.

5.4 Lemma. *If $A \subseteq \mathbf{R}^n$ is closed set, f is a $(\mathscr{H}^{n-1} \llcorner N(A))$ integrable $\overline{\mathbf{R}}$ valued function and $0 \le m \le n-1$ then*

$$\int_{N(A)|A^{(m)}} f(a,u) \prod_{i=1}^{m} \frac{1}{(1+\kappa_{A,i}(a,u)^2)^{1/2}}\, d\mathscr{H}^{n-1}(a,u)$$
$$= \int_{A^{(m)}} \int_{\{z\}\times N(A,z)} f\, d\mathscr{H}^{n-m-1}\, d\mathscr{H}^m z.$$

Proof. We assume $f \ge 0$ on $\mathscr{H}^{n-1}$ almost all of $N(A)$, since, as usual, the general case follows from [Fed69, 2.4.4]. Since $A^{(0)}$ is a countable set by 5.2, the case $m = 0$ is clear. Therefore we assume $m \ge 1$, we let $\lambda > 1$ and we define $C_i = \boldsymbol{\psi}_A[S_\lambda(A, 1/i)]$ for every integer $i \ge 1$. Since $\kappa_{A,m+1}(a,u) = \infty$ for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)|A^{(m)}$ by 5.3, noting 3.18, the conclusion can be easily derived in two simple steps: first we apply Coarea formula [Fed78, p. 300] with W, f and S replaced by C_i, $\mathbf{p}|C_i$ and $A^{(m)}$ respectively, second we let $i \to \infty$ and we recall 4.3. □

5.5 Remark. If $\operatorname{reach}(A) > 0$ and f is the characteristic funtion of a Borel subset of $N(A)$ then the conclusion of 5.4 is essentially contained in [Hug98, 3.2].

5.6 Remark. The following corollary can be deduced from 5.4. *If $S \subseteq A$ and $1 \le m \le n-1$ then $\mathscr{H}^m(S \cap A^{(m)}) = 0$ if and only if*

$$\kappa_{A,m}(a,u) = \infty \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (a,u) \in N(A)|S \cap A^{(m)}.$$

We obtain here an integral representation for the support measures.

5.7 Theorem. *Suppose $A \subseteq \mathbf{R}^n$ is a closed set, $\mu_0, \dots, \mu_{n-1}$ are the support measures of A, $1 \le m \le n-1$ is an integer and S is a countable union of Borel subsets with finite $\mathscr{H}^m$ measure.*

Then the following two statements hold.

(1) If $j > m$ then $\kappa_{A,m}(x,u) = \infty$ for $\mathscr{H}^{n-1}$ a.e. $(x,u) \in N(A)|S \cap A^{(j)}$;

(2) if $T \subseteq N(A)|S$ is $\mathscr{H}^{n-1}$ measurable then

$$\mu_m(T) = \frac{1}{(n-m)\boldsymbol{\alpha}(n-m)} \int \mathscr{H}^{n-m-1}\{v : (z,v) \in T\} d\mathscr{H}^m z.$$

Proof. Suppose $S_1, S_2, \dots$ is a sequence of Borel subsets with finite $\mathscr{H}^m$ measure whose union equals S and $S_i \subseteq S_{i+1}$ for $i \ge 1$. Let $\lambda > 1$ and $C_i = \boldsymbol{\psi}_A[S_\lambda(A, 1/i)]$. We apply the co-area formula in [Fed78, p. 300] with W, f and S replaced by C_i, $\mathbf{p}|C_i$ and $S_i \cap A^{(j)}$ to infer that

$$\int_{C_i|S_i \cap A^{(j)}} \| \textstyle\bigwedge_m [\mathbf{p}| \operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner C_i, (x,u))] \| \, d\mathscr{H}^{n-1}(x,u) = 0$$

whenever $j > m$. It follows that

$$\dim \mathbf{p}[\operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner C_i, (x,u))] < m,$$

whence we deduce that $\kappa_{A,m}(x,u)=\infty$ for $\mathscr{H}^{n-1}$ a.e. $(x,u)\in C_i|S_i\cap A^{(j)}$ and for $j>m$ by 4.11(2). Then we obtain (1) letting $i\to\infty$ and noting 4.3.

Since $\kappa_{A,m}(x,u)=\infty$ for $\mathscr{H}^{n-1}$ a.e. $(x,u)\in N(A)|A^{(j)}$ if $j<m$ by 5.3, we conclude from (iii) that

$$H_{n-m-1}(x,u)=0 \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (x,u)\in N(A)|S\cap A^{(j)},$$

if $j\neq m$. Since $\kappa_{A,m+1}(x,u)=\infty$ for $\mathscr{H}^{n-1}$ a.e. $(x,u)\in N(A)|A^{(m)}$ by 5.3, it follows that

$$H_{n-m-1}(x,u)=\prod_{i=1}^{m}\frac{1}{(1+\kappa_{A,i}(x,u)^2)^{1/2}} \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (x,u)\in N(A)|A^{(m)}.$$

Then (2) follows from 5.4. □

5.8 Remark. The integral representation in 5.7(2) has been proved in [CH00, 5.5] for sets of positive reach.

5.9 Remark. Since $A^{(n-1)}$ is countably $(n-1)$ rectifiable and $\mathscr{H}^{n-1}(A^{(i)})=0$ for $i<n-1$ (see 5.2) it follows from 5.7 that *if $T\subseteq N(A)$ is $\mathscr{H}^{n-1}$ measurable then*

$$\mu_{n-1}(T)=\frac{1}{2}\int \mathscr{H}^0\{v:(z,v)\in T\}\,d\mathscr{H}^{n-1}z.$$

This formula is equivalent to [HLW04, 4.1].

6 Relation with second order rectifiability

In this final section we prove that, in a certain sense, the "absolutely continuous part" of the second fundamental form introduced in section 4 can be described by the approximate differential of order 2 introduced by the author in [San17], see 6.2.

6.1 Lemma. *Suppose $A\subseteq\mathbf{R}^n$ is closed, $1\le m\le n-1$ and let M be an m dimensional submanifold of class 2.*

Then there exists $R\subseteq A\cap M$ such that $\mathscr{H}^m((A\cap M)\sim R)=0$ and

$$Q_A(a,u)=-\mathbf{b}_M(a)\bullet u \quad \textit{for } \mathscr{H}^{n-1} \textit{ a.e. } (a,u)\in N(A)|R.$$

Proof. We define

$$R=A\cap M\cap\{a:N(A,a)\subseteq N(M,a)\}.$$

Since $\mathbf{\Theta}^m(\mathscr{H}^m \llcorner M\sim A,a)=0$ for $\mathscr{H}^m$ a.e. $a\in A\cap M$ by [Fed69, 2.10.19(4)], it follows from [Fed69, 3.2.16] that

$$\operatorname{Tan}(M,a)=\operatorname{Tan}^m(\mathscr{H}^m\llcorner A\cap M,a)\subseteq\operatorname{Tan}(A,a)$$

for $\mathscr{H}^m$ a.e. $a\in A\cap M$. Henceforth, we deduce from 4.2 that $N(A,a)\subseteq N(M,a)$ for $\mathscr{H}^m$ a.e. $a\in A\cap M$ and

$$\mathscr{H}^m((A\cap M)\sim R)=0.$$

We recall that $N(M)$ is an $n-1$ dimensional submanifold of class 1 in $\mathbf{R}^n \times \mathbf{S}^{n-1}$, see 2.2. Since $N(A)|R \subseteq N(M)$, we may use [Fed69, 2.10.19(4), 3.2.16] to get $\mathbf{\Theta}^{n-1}(\mathscr{H}^{n-1} \llcorner N(M) \sim N(A)|R, (a,u)) = 0$ and

$$\operatorname{Tan}(N(M),(a,u)) = \operatorname{Tan}^{n-1}(\mathscr{H}^{n-1} \llcorner N(A)|R,(a,u))$$

for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)|R$. Finally, if ψ is a Radon measure as in 4.11, we combine 4.11(1) and B.4 to deduce that

$$\operatorname{Tan}(N(M),(a,u)) = \operatorname{Tan}^{n-1}(\psi,(a,u)) \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (a,u) \in N(A)|R.$$

Now we conclude employing 4.11(2) and 2.3. □

6.2 Theorem. *Let $A \subseteq \mathbf{R}^n$ be a closed set, $1 \leq m \leq n-1$ and let $S \subseteq A$ be $\mathscr{H}^m$ measurable and $(\mathscr{H}^m, m)$ rectifiable of class 2. Then there exists $R \subseteq S$ such that $\mathscr{H}^m(S \sim R) = 0$ and*[6]

$$\operatorname{ap}\operatorname{Tan}(S,a) = T_A(a,u) \qquad \operatorname{ap}\mathrm{D}^2 S(a) \bullet u = -Q_A(a,u) \circ \bigodot\nolimits_2 T_A(a,u)_\natural$$

for $\mathscr{H}^{n-1}$ a.e. $(a,u) \in N(A)|R$.

Proof. If $\{M_i : i \geq 1\}$ is a sequence of m dimensional submanifolds of class 2 covering $\mathscr{H}^m$ almost all of S, we apply [Fed69, 2.10.19(4)] and [San17, 3.22] to obtain

$$\operatorname{ap}\operatorname{Tan}(S,a) = \operatorname{Tan}(M_i,a), \quad \operatorname{ap}\mathrm{D}^2 S(a) = \mathbf{b}_{M_i}(a) \circ \bigodot\nolimits_2 \operatorname{Tan}(M_i,a)_\natural,$$

for $\mathscr{H}^m$ a.e. $a \in M_i \cap S$ and for every $i \geq 1$. Now the conclusion easily follows applying 6.1. □

The following lemma shows that the approximate differential of order 2 of a second order rectifiable closed set $S \subseteq \mathbf{R}^n$ does not always fully describe its second fundamental form Q_S. The same phenomenon arises in the theory of functions of bounded variation: the total differential is not always fully described by the approximate gradient. It seems to be not a coincidence that the following example considers exactly the primitive of a function of bounded variation whose total differential cannot be fully described by the approximate derivative. Recall that the boundary of a convex set of $\mathbf{R}^n$ is always countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable of class 2.

6.3 Lemma. *There exists a closed convex set $A \subseteq \mathbf{R}^2$ and a subset T of the topological boundary of A such that $\mathscr{H}^1(T) = 0$, $\mathscr{H}^1(N(A)|T) > 0$ and*

$$T_A(a,u) = \{0\} \quad \textit{for } \mathscr{H}^1 \textit{ a.e. } (a,u) \in N(A)|T.$$

Proof. Let $0 < s < 1$ and let $C \subseteq \mathbf{R}$ be a compact set with $0 < \mathscr{H}^s(C) < \infty$. Define

$$f(x) = \mathscr{H}^s(C \cap \{z : z \leq x\}) \quad \text{for } x \in \mathbf{R},$$

and let g be a primitive of f. Then g is a non-decreasing convex function of class 1 on $\mathbf{R}$ and we define

$$A = \mathbf{R}^2 \cap \{(x,y) : g(x) \leq y\}, \quad T = \{(x, g(x)) : x \in C\}.$$

[6] If $f : V \to W$ is a linear map between vector spaces then $\bigodot_2 f : V \times V \to W \times W$ is defined by $\bigodot_2 f(u,v) = (f(u), f(v))$ for $(u,v) \in V \times V$. Note that this notation does not agree with [Fed69, 1.9.1]

We notice that A is a closed convex set, $T \subseteq A^{(1)}$, $\mathscr{H}^1(T) = 0$ and

$$N(A, (x, g(x))) = \{(1 + f(x)^2)^{-1/2}(f(x), -1)\} \quad \text{whenever } x \in \mathbf{R}.$$

It follows that $\mathscr{H}^1(\mathbf{q}(N(A))) > 0$. Moreover, since f is constant on each connected component of $\mathbf{R} \sim C$, it follows that $\mathbf{q}(N(A)|A \sim T)$ is a countable subset of $\mathbf{S}^1$; in particular $\mathscr{H}^1(\mathbf{q}(N(A)|A \sim T)) = 0$. Therefore one easily infers that

$$\mathscr{H}^1(N(A)|T) > 0.$$

Finally we notice that $T_A(a, u) = \{0\}$ for $\mathscr{H}^1$ a.e. $(a, u) \in N(A)|T$ by 5.6. □

6.4 Remark. If M is an m dimensional submanifold of class 1 in $\mathbf{R}^n$ that meets every m dimensional submanifold of class 2 in a set of $\mathscr{H}^m$ measure zero then it follows from [MS17, 4.12] that $\mathscr{H}^m(M^{(m)}) = 0$. Since $M^{(i)} = \varnothing$ if $i < m$ by 4.2, it follows from 5.6 and 5.7(1) that

$$\dim T_M(a, u) \leq m - 1 \quad \text{for } \mathscr{H}^{n-1} \text{ a.e. } (a, u) \in N(M).$$

The existence of such M can be inferred from [Koh77].

Appendix

In this appendix we collect for the reader's convenience some remarks that are simple consequences of known facts.

A On approximate differentiability

Basic facts on approximate differentiability for functions are collected in [San17, §2]. Here we point out some additional remarks.

A.1 Lemma. *Suppose $n \geq 1$ is an integer, $B \subseteq A \subseteq \mathbf{R}^n$, $a \in A$ and $f : A \to \mathbf{R}$ are such that f is approximately differentiable at a, $\mathbf{\Theta}^{*n}(\mathscr{L}^n \llcorner B, a) = 1$ and $f(x) \leq f(a)$ for every $x \in B$.*

Then $\operatorname{ap} \mathrm{D} f(a) = 0$.

Proof. Assume $a = 0$ and $f(0) = 0$. If $\operatorname{ap} \mathrm{D} f(0) \neq 0$ then there would be $\epsilon > 0$ and a non empty open cone C such that $\operatorname{ap} \mathrm{D} f(0)(x) \geq 2\epsilon|x|$ for every $x \in C$. Therefore $f(x) - \operatorname{ap} \mathrm{D} f(0)(x) \leq -2\epsilon|x|$ for every $x \in C \cap B$ and

$$\mathbf{\Theta}^{*n}(\mathscr{L}^n \llcorner B \sim C, 0) < 1, \quad \mathbf{\Theta}^{*n}(\mathscr{L}^n \llcorner B \cap C, 0) > 0,$$
$$\mathbf{\Theta}^{*n}(\mathscr{L}^n \llcorner \mathbf{R}^n \sim \{x : |f(x) - \operatorname{ap} \mathrm{D} f(0)(x)| \leq \epsilon|x|\}, 0) > 0.$$

This would be a contradiction. □

A.2 Remark. We observe that a similar argument proves that *if f is approximately differentiable of order 2 at a then* $\operatorname{ap} \mathrm{D}^2 f(a) \leq 0$.

A.3 Lemma. *Suppose $n \geq 1$ and $\nu \geq 1$ are integers, $B \subseteq A \subseteq \mathbf{R}^n$, $a \in B$ and $f : A \to \mathbf{R}^\nu$ are such that f is approximately differentiable at a, $f|B$ is a bi-Lipschitzian homeomorphism and $\mathbf{\Theta}^n(\mathscr{L}^n \llcorner \mathbf{R}^n \sim B, a) = 0$.*

Then $\ker \operatorname{ap} \mathrm{D} f(a) = \{0\}$.

Proof. If $\Gamma = (1/2)(\operatorname{Lip}(f|B)^{-1})^{-1}$ then $|f(y) - f(x)| \geq 2\Gamma|y - x|$ whenever $y, x \in B$. If there was $v \in \mathbf{R}^n \sim \{0\}$ such that $\operatorname{ap} \mathrm{D} f(a)(v) = 0$, then there would exist a non empty open cone C such that

$$|\operatorname{ap} \mathrm{D} f(a)(u)| \leq \Gamma|u| \quad \text{whenever } u \in C.$$

Choosing $0 < \epsilon < \Gamma$ and letting $D = \{u + a : u \in C\}$ and

$$E = A \cap \{x : |f(x) - f(a) - \operatorname{ap} \mathrm{D} f(a)(x - a)| \leq \epsilon|x - a|\},$$

we would notice that $\boldsymbol{\Theta}^n(\mathscr{L}^n \llcorner \mathbf{R}^n \sim E, a) = 0$ and $B \cap D \cap E = \varnothing$ and we would get a contradiction. □

A.4 Lemma. *If m, n, ν are positive integers, $D \subseteq \mathbf{R}^m$, $U \subseteq \mathbf{R}^n$ is open, $f : D \to \mathbf{R}^n$, $g : U \to \mathbf{R}^\nu$, $x \in D$, $f(x) \in U$, f is approximately differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is approximately differentiable at x with*

$$\operatorname{ap} \mathrm{D}(g \circ f)(x) = \mathrm{D}\, g(f(x)) \circ \operatorname{ap} \mathrm{D} f(x).$$

Proof. Combine [San17, 2.8] and [Fed69, 3.1.1(2)]. □

A.5 Lemma. *If $n, \nu \geq 1$ are integers, $D \subseteq \mathbf{R}^n$, $z \in D$ and $g : \mathbf{R}^n \to \mathbf{R}^\nu$ is a Lipschitzian function such that $g|D$ is approximately differentiable at z, then g is differentiable at z with $\operatorname{ap} \mathrm{D}(g|D)(z) = \mathrm{D}\, g(z)$.*

Proof. This is proved in [Fed69, 3.1.5]. □

B On the tangent cone of a measure

The concept of approximate tangent vector to a measure is introduced in [Fed69, 3.2.16]. Besides the fundamental results given in [Fed69, 3.2.16–3.2.22, 3.3.18], we point out here some useful consequences.

First, the following elementary inequality is useful here and elsewhere.

B.1 Lemma. *If X and Y are metric spaces, $m \geq 1$ is an integer, $\theta(x) \geq 0$ for $\mathscr{H}^m$ a.e. $x \in X$, $0 \leq \gamma < \infty$ and $f : X \to Y$ is an univalent Lipschitzian map onto Y such that γ is a Lipschitz constant for f^{-1}, then*

$$\int_X^* \theta \, d\mathscr{H}^m \leq \gamma^m \int_Y^* \theta \circ f^{-1} d\mathscr{H}^m.$$

Proof. We assume $\int_Y^* \theta \circ f^{-1} d\mathscr{H}^m < \infty$. Then the conclusion easily follows from the definition of upper integral in [Fed69, 2.4.2], using approximation by upper functions. □

B.2 Lemma. *Suppose X and Y are normed vector spaces, $P \subseteq X$, $m \geq 1$ is an integer, $\theta(x) \geq 0$ for $\mathscr{H}^m$ a.e. $x \in P$, $a \in P$ and $f : X \to Y$ is a function differentiable at a such that $f|P$ is a bi-Lipschitzian homeomorphism. Additionally, we define the measures*

$$\psi = \theta \mathscr{H}^m \llcorner P, \quad \mu = (\theta \circ (f|P)^{-1}) \mathscr{H}^m \llcorner f(P).$$

Then $\mathrm{D} f(a)[\operatorname{Tan}^m(\psi, a)] \subseteq \operatorname{Tan}^m(\mu, f(a))$.

Proof. Firstly we prove that $\mathbf{\Theta}^m(\psi \llcorner X \sim f^{-1}[T], a) = 0$, whenever $T \subseteq Y$ such that $\mathbf{\Theta}^m(\mu \llcorner Y \sim T, f(a)) = 0$. In fact, for such a subset T, if $S = f^{-1}[T]$, γ is a Lipschitz constant for $f|P$ and$(f|P)^{-1}$ and $r > 0$, we observe that

$$f[(P \sim S) \cap \mathbf{B}(a, r)] \subseteq (f[P] \sim T) \cap \mathbf{B}(f(a), \gamma r),$$

and we employ B.1 to get that $\psi(\mathbf{B}(a, r) \sim S) \leq \gamma^m \mu(\mathbf{B}(f(a), \gamma r) \sim T)$. Therefore $\mathrm{D}\, f(a)[\mathrm{Tan}^m(\psi, a)] \subseteq \mathrm{Tan}^m(\mu, f(a))$ by [Fed69, 3.1.21, p. 234] and [Fed69, 3.2.16, p. 252]. □

B.3 Remark. If θ is the characteristic function of P then, by [Fed69, 2.4.5], we have that $\psi = \mathscr{H}^m \llcorner P$ and $\mu = \mathscr{H}^m \llcorner f[P]$.

B.4 Lemma. *Suppose $1 \leq k \leq \nu$ are integers, $E \subseteq \mathbf{R}^\nu$ is countably $(\mathscr{H}^k, k)$ rectifiable and $\mathscr{H}^k$ measurable and θ is a $\mathscr{H}^k \llcorner E$ measurable $\mathscr{H}^k \llcorner E$ almost positive function such that*

$$\psi = \theta \mathscr{H}^k \llcorner E$$

is a Radon measure over $\mathbf{R}^\nu$.

Then $\mathrm{Tan}^k(\psi, z)$ is a k dimensional plane contained in $\mathrm{Tan}^k(\mathscr{H}^k \llcorner E, z)$ for $\mathscr{H}^k$ a.e. $z \in E$ and

$$\mathrm{Tan}^k(\mathscr{H}^k \llcorner F, z) \subseteq \mathrm{Tan}^k(\psi, z) \quad \text{for } \mathscr{H}^k \text{ a.e. } z \in F,$$

whenever $F \subseteq E$ is $\mathscr{H}^k$ measurable such that $\mathscr{H}^k(F) < \infty$.

Proof. Firstly we observe that $\psi(S) = 0$ if and only if $\mathscr{H}^k(S) = 0$. Therefore $\mathbf{R}^\nu$ is (ψ, k) rectifiable and, employing [Fed69, 2.4.10, 2.10.19(3)],

$$\mathbf{\Theta}^{*k}(\psi, z) < \infty \quad \text{for } \psi \text{ a.e. } z \in \mathbf{R}^\nu.$$

We apply [Fed69, 3.3.18] to conclude that $\mathrm{Tan}^k(\psi, z) \in \mathbf{G}(n, k)$ for $\mathscr{H}^k$ a.e. $z \in E$. If $F \subseteq E$ is $\mathscr{H}^k$ measurable and $\mathscr{H}^k(F) < \infty$, we define

$$F_i = F \cap \{z : \theta(z) \geq i^{-1}\} \quad \text{for every integer } i \geq 1,$$

we observe that $\mathrm{Tan}^k(\mathscr{H}^k \llcorner F, z) = \mathrm{Tan}^k(\mathscr{H}^k \llcorner F_i, z)$ for $\mathscr{H}^k$ a.e. $z \in F_i$ by [Fed69, 2.10.19(4)], and we use [Fed69, 3.2.16] to conclude

$$\mathrm{Tan}^k(\mathscr{H}^k \llcorner F, z) \subseteq \mathrm{Tan}^k(\psi, z) \quad \text{for } \mathscr{H}^k \text{ a.e. } z \in F.$$

Since by [Fed69, 3.2.14] the set E can be $\mathscr{H}^k$ almost covered by countably many $\mathscr{H}^k$ measurable k rectifiable subsets of $\mathbf{R}^\nu$, we may apply [Fed69, 3.2.19] to conclude that $\mathrm{Tan}^k(\psi, z) \subseteq \mathrm{Tan}^k(\mathscr{H}^k \llcorner E, z)$ for $\mathscr{H}^k$ a.e. $z \in E$. □

References

[Asp73] Edgar Asplund. Differentiability of the metric projection in finite-dimensional Euclidean space. *Proc. Amer. Math. Soc.*, 38:218–219, 1973.

[Buc92] Zoltán Buczolich. Density points and bi-Lipschitz functions in $\mathbf{R}^m$. *Proc. Amer. Math. Soc.*, 116(1):53–59, 1992.

[CH00] Andrea Colesanti and Daniel Hug. Hessian measures of semi-convex functions and applications to support measures of convex bodies. *Manuscripta Math.*, 101(2):209–238, 2000.

[Fed59] Herbert Federer. Curvature measures. *Trans. Amer. Math. Soc.*, 93:418–491, 1959.

[Fed69] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[Fed78] Herbert Federer. Colloquium lectures on geometric measure theory. *Bull. Amer. Math. Soc.*, 84(3):291–338, 1978.

[Fer76] Steve Ferry. When ϵ-boundaries are manifolds. *Fund. Math.*, 90(3):199–210, 1975/76.

[Fu85] Joseph Howland Guthrie Fu. Tubular neighborhoods in Euclidean spaces. *Duke Math. J.*, 52(4):1025–1046, 1985.

[Fu89] Joseph H. G. Fu. Curvature measures and generalized Morse theory. *J. Differential Geom.*, 30(3):619–642, 1989.

[GP72] Ronald Gariepy and W. D. Pepe. On the level sets of a distance function in a Minkowski space. *Proc. Amer. Math. Soc.*, 31:255–259, 1972.

[HLW04] Daniel Hug, Günter Last, and Wolfgang Weil. A local Steiner-type formula for general closed sets and applications. *Math. Z.*, 246(1-2):237–272, 2004.

[Hug98] Daniel Hug. Generalized curvature measures and singularities of sets with positive reach. *Forum Math.*, 10(6):699–728, 1998.

[Koh77] Robert V. Kohn. An example concerning approximate differentiation. *Indiana Univ. Math. J.*, 26(2):393–397, 1977.

[KW14] E. V. Khmaladze and W. Weil. Differentiation of sets—the general case. *J. Math. Anal. Appl.*, 413(1):291–310, 2014.

[Las06] Günter Last. On mean curvature functions of Brownian paths. *Stochastic Process. Appl.*, 116(12):1876–1891, 2006.

[MS17] U. Menne and M. Santilli. A geometric second-order-rectifiable stratification for closed subsets of Euclidean space. *ArXiv e-prints*, March 2017. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*.

[RW10] Jan Rataj and Steffen Winter. On volume and surface area of parallel sets. *Indiana Univ. Math. J.*, 59(5):1661–1685, 2010.

[RZ12] Jan Rataj and Luděk Zajíček. Critical values and level sets of distance functions in Riemannian, Alexandrov and Minkowski spaces. *Houston J. Math.*, 38(2):445–467, 2012.

[San17] M. Santilli. Rectifiability and approximate differentiability of higher order for sets. *ArXiv e-prints*, January 2017. To appear in *Indiana Univ. Math. J.*

[Sta79] L. L. Stachó. On curvature measures. *Acta Sci. Math. (Szeged)*, 41(1-2):191–207, 1979.

[Wal76] Rolf Walter. Some analytical properties of geodesically convex sets. *Abh. Math. Sem. Univ. Hamburg*, 45:263–282, 1976.

[Zäh86] M. Zähle. Integral and current representation of Federer's curvature measures. *Arch. Math. (Basel)*, 46(6):557–567, 1986.