A novel type of Sobolev-Poincaré inequality for submanifolds of Euclidean space

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Abstract

For functions on generalised connected surfaces (of any dimensions) with boundary and mean curvature, we establish an oscillation estimate in which the mean curvature enters in a novel way. As application we prove an a priori estimate of the geodesic diameter of compact connected smooth immersions in terms of their boundary data and mean curvature.

These results are developed in the framework of varifolds. For this purpose, we establish that the notion of indecomposability is the appropriate substitute for connectedness and that it has a strong regularising effect; we thus obtain a new natural class of varifolds to study.

Finally, our development leads to a variety of questions that are of substance both in the smooth and the nonsmooth setting.

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Keywords Sobolev-Poincaré inequality · minimal surface · geodesic distance · diameter · varifold · indecomposability · mean curvature · boundary · density.

Contents

1 Introduction

We will firstly discuss our main geometric estimates in the differential geometric context before proceeding to the context of varifold theory. *Suppose throughout this introduction that m and n are positive integers and $2 \leq m < n$.*

1.1 Differential geometric estimates

We begin with an a priori estimate of the geodesic diameter of a compact connected smooth immersion (i.e., immersion of class ∞)[1] in $\mathbf{R}^n$ which may be viewed as global consequence of our Sobolev-Poincaré inequality (Theorem B below). The estimate is new even in the context of *minimal surfaces*, that is, in case $\mathbf{h}(F, \cdot) = 0$.

Theorem A (see 7.10)**.** *Suppose M is a compact connected m dimensional manifold-with-boundary of class 2, $F : M \to \mathbf{R}^n$ is an immersion of class 2, $\mathbf{h}(F, x)$ and $\mathbf{h}(F|\partial M, x)$ denote the mean curvature vectors of F and $F|\partial M$ at x, respectively, g is the Riemannian metric on M induced by F, and σ is the Riemannian distance associated to (M, g).*

Then, for some positive finite number Γ determined by m, there holds

$$\operatorname{diam}_\sigma M \leq \Gamma\big(\textstyle\int_M |\mathbf{h}(F, \cdot)|^{m-1} \, \mathrm{d}\mathscr{H}^m_\sigma + \int_{\partial M} |\mathbf{h}(F|\partial M, \cdot)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1}_\sigma\big);$$

here, by convention, $0^0 = 1$.

The Hausdorff measures $\mathscr{H}^m_\sigma$ and $\mathscr{H}^{m-1}_\sigma$ in this theorem agree with the usual Riemannian measures induced by F on M and ∂M, see 2.9. We notice that, by our convention, if $m = 2$ then $\int_{\partial M} |\mathbf{h}(F|\partial M, \cdot)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1}_\sigma = \mathscr{H}^1_\sigma(\partial M)$. Theorem A generalises results of Topping and Paeng. Namely, Topping treated the case $\partial M = \varnothing$ in [Top08, Theorem 1.1] and Paeng covered the case $m = 2$ under the additional hypothesis of convexity of (M, g) in [Pae14, Theorem 2 (a)]. In those cases, our contribution (see 7.7) lies in identifying Theorem A as consequence of our Sobolev-Poincaré inequality, Theorem B below. The consideration of the mean curvature of $F|\partial M$ to treat the general case is new.

For $m \geq 3$, the preceding theorem naturally leads to the following question which is open even if $\mathbf{h}(F, \cdot) = 0$ and F is an embedding. In case the answer is in the affirmative, Theorem A could be realised as corollary of the resulting statement by applying the latter to both F and $F|\partial M$.

Question 1. May the term $\int_{\partial M} |\mathbf{h}(F, \cdot)|^{m-2} \, \mathrm{d}\mathscr{H}^{m-1}_\sigma$ in Theorem A be replaced by the sum of the diameters of the connected components of ∂M computed with respect to the Riemannian distance on ∂M induced by $F|\partial M$?

To prove Theorem A, one readily reduces the problem to the case that F is an embedding by Whitney-type approximation results (see [Hir94, 2.1.0]). Then, anticipating also the needs of the nonsmooth setting, we observe (see 7.1 and 7.3) a characterisation of the geodesic diameter in terms of the oscillatory behaviour of smooth compactly supported functions: *If M is a closed subset of $\mathbf{R}^n$ and d denotes the diameter of M with respect to the geodesic distance on M, then*

$$d = \sup\{\operatorname{diam} f[M] : 0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}) \text{ and } |\operatorname{D} f(x)| \leq 1 \text{ for } x \in M\},$$

[1] Objects of class k are termed *of class C^k* in [Hir94, p. 9].

where $f[M]$ denotes the image of M under f.

For the oscillatory behaviour of functions on properly embedded submanifolds, we in turn derive the following Sobolev-Poincaré inequality which is of central importance to the developments in this paper. It ultimately rests on the analysis of certain superlevel sets via the isoperimetric inequality.

Theorem B (see 4.5 and 4.6)**.** *Suppose M is a properly embedded, connected m dimensional submanifold-with-boundary of $\mathbf{R}^n$ of class 2, $0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$, $E = \{x : f(x) > 0\}$, and $\kappa = \sup\{|\operatorname{D}(f|M)(x)| : x \in M \sim \partial M\}$. Then, for some positive finite number Γ determined by m, there holds*

$$\begin{aligned}\operatorname{diam} f[M] \leq \Gamma\big(&\mathscr{H}^m(E \cap M)^{1/m} + \textstyle\int_{E\cap M} |\mathbf{h}(M,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m x \\ &+ \mathscr{H}^{m-1}(E \cap \partial M)^{1/(m-1)} + \textstyle\int_{E\cap\partial M} |\mathbf{h}(\partial M,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} x\big)\kappa;\end{aligned}$$

here, by convention, $0^0 = 1$.

The fact that Theorem B is indeed a special case of 4.5 and 4.6 follows from 2.15, 3.13, and [Men16a, 8.7, 9.2]. A version of Theorem B with minimal smoothness requirements on both M and f will be stated in the varifold setting (Theorem B′ below).

In case M is compact, Theorem B takes the flavour of a Poincaré inequality since it then yields (with κ associated to f as in Theorem B) that

$$\begin{aligned}\operatorname{diam} f[M] \leq \Gamma\big(&\mathscr{H}^m(M)^{1/m} + \textstyle\int_{M} |\mathbf{h}(M,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m x \\ &+ \mathscr{H}^{m-1}(\partial M)^{1/(m-1)} + \textstyle\int_{\partial M} |\mathbf{h}(\partial M,x)|^{m-2} \,\mathrm{d}\mathscr{H}^{m-1} x\big)\kappa\end{aligned}$$

whenever $f \in \mathscr{E}(\mathbf{R}^n, \mathbf{R})$.[2] In view of the afore-mentioned observation, Theorem A is now readily deduced from Theorem B and the isoperimetric inequality. In contrast, if M is non-compact (and $0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$), Theorem B takes the flavour of a Sobolev inequality since then $\operatorname{diam} f[M] = \sup\{f(x) : x \in M\}$.

Classically, both a Sobolev inequality (see Allard [All72, 7.1, 7.3] or Michael and Simon [MS73, 2.1]) and a Poincaré inequality (see Hutchinson [Hut90, Theorems 1 and 3]) on a properly embedded submanifold in $\mathbf{R}^n$ require either the addition of a summand involving f and $\mathbf{h}(M, \cdot)$ or the hypothesis that $\int_{E\cap M} |\mathbf{h}(M,\cdot)|^m \,\mathrm{d}\mathscr{H}^m$ does not exceed a small dimensional constant. Under the latter hypothesis, Sobolev inequalities and Poincaré inequalities involving the supremum norm (instead of integral norms) of the function have appeared in [Men16a, 10.1 (2d)] and [Men16a, 10.1 (1d), 10.7 (4), 10.9 (4)], respectively. In view of these inequalities, the following question concerning a possible improvement of Theorem B naturally arises.

Question 2. If, in Theorem B, $\partial M = \varnothing$, $m < q < \infty$, and κ is instead given by $\kappa = \big(\int_M |\operatorname{D}(f|M)(x)|^q \,\mathrm{d}\mathscr{H}^m x\big)^{1/q}$, does there hold, for some positive finite number Γ determined by m and q, the estimate

$$\operatorname{diam} f[M] \leq \Gamma\big(\mathscr{H}^m(E \cap M)^{1/m} + \textstyle\int_{E\cap M} |\mathbf{h}(M,x)|^{m-1} \,\mathrm{d}\mathscr{H}^m x\big)^{1-m/q}\kappa?$$

1.2 Varifold case

In order to establish varifold analogues of Theorems A and B in Theorems A′ and B′ below, we need to formulate appropriate notions of *connectedness*, *mean*

[2] The space $\mathscr{E}(\mathbf{R}^n, \mathbf{R})$ consists of all functions $f : \mathbf{R}^n \to \mathbf{R}$ of class ∞.

curvature, and *boundary*. Our most important conceptual finding in this regard is that *the natural notion of connectedness for varifolds – indecomposability – has a strong regularising effect.* This observation is not only crucial for the present development but also yields a promising new natural class of varifolds to study; some possibilities in this regard are indicated in Question 4 below.

The formulation of each of the three afore-mentioned notions is based on the following basic boundedness condition on the first variation.

Hypothesis 1 (First variation)**.** *Suppose V is an m dimensional varifold in $\mathbf{R}^n$ and $\|\delta V\|$ is a Radon measure.*

Concerning basic properties of the first variation and the regularising effects of the resulting mean curvature alone, [Men17] provides an introductory exposition including 20 examples. We recall (see Allard [All72, 5.5 (1)]) that V is rectifiable if it satisfies Hypothesis 1 and $\boldsymbol{\Theta}^m(\|V\|, x) > 0$ for $\|V\|$ almost all x.

Notions of connectedness

All our notions of indecomposability are based on the question whether certain superlevel sets of functions belonging to a family Ψ, contained in the space $\mathbf{T}(V)$ of generalised weakly differentiable functions on V, split the given varifold V into two nontrivial pieces without introducing additional first variation.

Definition (see 3.2 and 3.3)**.** Suppose that V satisfies Hypothesis 1 and that $\Psi \subset \mathbf{T}(V)$. Then, V is called *indecomposable of type* Ψ if and only if, whenever $f \in \Psi$, the set of $y \in \mathbf{R}$, such that $E(y) = \{x : f(x) > y\}$ satisfies

$$\|V\|(E(y)) > 0, \quad \|V\|(U \sim E(y)) > 0, \quad \delta(V \llcorner E \times \mathbf{G}(n,m)) = (\delta V) \llcorner E,$$

has $\mathscr{L}^1$ measure zero. If $\Psi = \mathbf{T}(V)$, the postfix "of type Ψ" may be omitted.

We are mainly concerned with the three cases $\Psi = \mathbf{T}(V)$, $\Psi = \mathscr{E}(\mathbf{R}^n, \mathbf{R})$, and $\Psi = \mathscr{D}(\mathbf{R}^n, \mathbf{R})$. Each resulting condition has its natural domain of applicability and is weaker – in fact, strictly weaker (see below) – than the preceding one.

Indecomposability of type $\mathbf{T}(V)$ is equivalent to the simpler condition that there is no Borel set E such that

$$\|V\|(E) > 0, \quad \|V\|(U \sim E) > 0, \quad \delta(V \llcorner E \times \mathbf{G}(n,m)) = (\delta V) \llcorner E.$$

This condition was introduced in [Men16a, 6.2, 6.3] and lies at the core of the theory of generalised weakly differential functions on V. Assuming rectifiability, this notion leads to existence of a corresponding decomposition (see [Men16a, 6.12]); without rectifiability, a decomposition may fail to exist (see [MS17, 4.13]).

For a varifold V associated to a properly embedded smooth submanifold-with-boundary M, both types, $\mathbf{T}(V)$ and $\mathscr{E}(\mathbf{R}^n, \mathbf{R})$, of indecomposability are equivalent (see 3.13) to connectedness of M. In contrast, for immersions (see 2.14, 3.9, and 3.11), connectedness of the underlying manifold-with-boundary implies, for the associated varifold V, indecomposability of type $\mathscr{E}(\mathbf{R}^n, \mathbf{R})$, but not of type $\mathbf{T}(V)$. Accordingly, amongst the three notions presently discussed, indecomposability of type $\mathscr{E}(\mathbf{R}^n, \mathbf{R})$ is the natural choice for the study of varifolds associated to immersions.

For a varifold associated to a properly embedded smooth submanifold-with-boundary M, indecomposability of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$ is equivalent to the condition that either M is connected or all connected components of M are non-compact (see 3.13). It turns out that indecomposability of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$ is the natural notion for the study of local properties of varifolds; in fact, all our results below will merely require this weakest type of indecomposability.

Finally, we emphasise two properties of the concept of indecomposability for varifolds that are in sharp contrast to that of topological connectedness for submanifolds. Firstly, decompositions of varifolds may easily be non-unique (see [Men16a, 6.13]). Secondly, all types of indecomposability are extrinsic notions through the influence of the first variation.

Mean curvature

Next, we discuss the degree to which the regularising effect of summability conditions on the mean curvature is strengthened through indecomposability in the absence of boundary (whose influence will summarised thereafter).

Hypothesis 2 (Density and mean curvature)**.** *Suppose V is an m dimensional varifold in $\mathbf{R}^n$, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$, $1 \leq p \leq \infty$, and the generalised mean curvature vector $\mathbf{h}(V, \cdot)$ of V belongs to $\mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$.*

Unlike in the differential geometric case, $\mathbf{h}(V, \cdot)$ may have a nontrivial tangential component. In fact, considering the example of a weighted properly embedded smooth submanifold (see [Men16a, 7.6, 15.2]), the following question seems natural; if V is integral, it has an affirmative answer (see [Men13, 4.8]).

Question 3. Suppose V satisfies Hypothesis 1, $\tau = \operatorname{Tan}^m(\|V\|, \cdot)_\natural$ is the tangent plane[3] function, $\Theta = \boldsymbol{\Theta}^m(\|V\|, \cdot)$, and $\Theta(x) \geq 1$ for $\|V\|$ almost all x. Does it follow that both functions τ and Θ are $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all x, and, if so, does there hold – denoting $(\|V\|, m)$ approximate derivatives by the prefix "ap" – the equation

$$\mathbf{h}(V, x) \bullet u = T(\operatorname{ap} \mathrm{D}\, \tau(x) \circ \tau(x)) \bullet u + (\operatorname{ap} \mathrm{D}(\log \circ \Theta)(x) \circ \tau(x))(u) \quad \text{for } u \in \mathbf{R}^n$$

for $\|V\|$ almost all x, where the trace operator T is as in [Men16a, 15.1]?

If we have $p \geq m$ in Hypothesis 2, then $\operatorname{spt} \|V\|$ is in many ways well-behaved. For instance, there holds $\boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1$ for $x \in \operatorname{spt} \|V\|$ by [Men09, 2.7] – in particular, $\operatorname{spt} \|V\|$ has locally finite $\mathscr{H}^m$ measure –, $\operatorname{spt} \|V\|$ is locally connected (see [Men16a, 6.14 (3)]), decompositions of V are locally finite (see [Men16a, 6.11]) and non-uniquely refine the decomposition of $\operatorname{spt} \|V\|$ into connected components (see [Men16a, 6.13, 6.14 (1)]), connected components of $\operatorname{spt} \|V\|$ are locally connected by paths of finite length (see [Men16a, 14.2]), and the resulting geodesic distance thereon is a continuous Sobolev functions with bounded generalised weak derivative (see [Men16b, 6.8 (1)]).

In contrast, if $p < m$, then $\operatorname{spt} \|V\|$ has substantially less geometric significance; in fact, whenever X is an open subset of $\mathbf{R}^n$, there exists (see [Men16a, 14.1]) a varifold V with $\operatorname{spt} \|V\| = \operatorname{Clos} X$. However, one is at least assured by [Men09, 2.11] that $\mathscr{H}^{m-p}$ almost all x satisfy the dichotomy

$$\text{either } \boldsymbol{\Theta}^m_*(\|V\|, x) \geq 1 \quad \text{or } \boldsymbol{\Theta}^m(\|V\|, x) = 0.$$

[3] For $\|V\|$ almost all x, the closed cone $\operatorname{Tan}^m(\|V\|, x)$ is an m dimensional plane and $\tau(x)$ is the standard orthogonal projection of $\mathbf{R}^n$ onto $\operatorname{Tan}^m(\|V\|, x)$.

Combining Hypothesis 2 with indecomposability, we establish in the present paper that the critical value for the exponent p then drops from m to $m-1$. In particular, we prove the following theorem concerning $p \geq m-1$.

Theorem C (see 3.1, 5.5, and 6.7 (1))**.** *Suppose V and p satisfy Hypothesis 2, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, and $m-1 \leq p < m$. Then, there holds*

$$\boldsymbol{\Theta}_*^m(\|V\|, x) \geq 1 \quad \textit{for } \mathscr{H}^{m-p} \textit{ almost all } x \in \operatorname{spt} \|V\|.$$

Moreover, the dimension $m-p$ in the Hausdorff measure $\mathscr{H}^{m-p}$ may not be replaced by any smaller number determined by m and p.

The postscript of Theorem C is based on the construction of a smooth (but not properly embedded) submanifold employing a Cantor-type set. If $p < m-1$, the behaviour completely changes as demonstrated by our next example.

Theorem D (see 3.1 and 5.7)**.** *Whenever $m \geq 3$, there exists a bounded m dimensional submanifold M of class ∞ of $\mathbf{R}^n$ such that the varifold associated to M is indecomposable and satisfies Hypothesis 2 whenever $1 \leq p < m-1$, such that $\mathscr{H}^m((\operatorname{Clos} M) \sim M) = 1$, and such that*

$$\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \textit{for } a \in (\operatorname{Clos} M) \sim M.$$

To construct M, we start with a countably infinite family of small pieces of M ensuring the required size of $(\operatorname{Clos} M) \sim M$ and then connect these pieces by thin cylinders. The threshold $p = m-1$ exactly reflects the behaviour of the mean curvature of cylinders under scaling of their radius. Without indecomposability, the threshold $p = m$ similarly reflects the scaling behaviour of spheres.

Boundary

In the absence of a boundary operator for varifolds (as is available for currents), the distribution $B \in \mathscr{D}'(\mathbf{R}^n, \mathbf{R}^n)$ defined by

$$B(\theta) = (\delta V)(\theta) + \textstyle\int \mathbf{h}(V, x) \bullet \theta(x) \, \mathrm{d}\|V\| \, x \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$$

acts as replacement whenever V satisfies Hypothesis 1; in particular, we have $\|B\| = \|\delta V\|_{\text{sing}}$. One should keep in mind that, even in case V corresponds to a properly embedded submanifold-with-boundary M of class 1 of $\mathbf{R}^n$, the support of B needs not to be contained in ∂M due to possible singular parts of the distributional derivative of the tangent plane function (see [Men17, Example 15]).

Hypothesis 3 (Density and boundary)**.** *Suppose V and W are m and $m-1$ dimensional varifolds in $\mathbf{R}^n$, respectively, $\|\delta V\|$ and $\|\delta W\|$ are Radon measures, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, $\boldsymbol{\Theta}^{m-1}(\|W\|, x) \geq 1$ for $\|W\|$ almost all x,*

$$\|\delta V\| \leq \|V\| \llcorner |\mathbf{h}(V, \cdot)| + \|W\|,$$

and $\|\delta W\|$ is absolutely continuous with respect to $\|W\|$.

The displayed equation is equivalent to requiring $\|B\| \leq \|W\|$. There are good geometric reasons to consider the stronger condition

$$|B(\theta)| \leq \textstyle\int |S_\natural^\perp(\theta(x))| \, \mathrm{d}W \, (x, S) \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n).$$

It may be seen as the boundary part of Almgren's concept (see [Alm66, Subsection 4-3]) of *regular* pair (V, W). The condition is also employed by Ekholm, White, and Wienholtz (see [EWW02, Section 7]). For our present purposes, the slightly weaker condition will be sufficient. Finally, the last condition in Hypothesis 3 excludes the presence of boundary for W.[4]

We may now state the boundary version of Theorem C.

Theorem C′ (see 6.7 (1))**.** *Suppose V and W satisfy Hypothesis 3, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, $m-1 \le p < m$, $\mathbf{h}(V, \cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$, and if $m > 2$ then $\mathbf{h}(W, \cdot) \in \mathbf{L}_{p-1}^{\mathrm{loc}}(\|W\|, \mathbf{R}^n)$. Then, there holds*

$$\textit{either } \boldsymbol{\Theta}_*^m(\|V\|, x) \ge 1 \quad \textit{or } \boldsymbol{\Theta}_*^{m-1}(\|W\|, x) \ge 1$$

for $\mathscr{H}^{m-p}$ almost all $x \in \operatorname{spt}\|V\|$; in particular, $\mathscr{H}^m \llcorner \operatorname{spt}\|V\| \le \|V\|$. Moreover, the dimension $m-p$ in the Hausdorff measure $\mathscr{H}^{m-p}$ may not be replaced by any smaller number determined by m and p.

Varifold-geometric estimates

With the precedingly introduced concepts and hypotheses, the varifold analogue of Theorem A may now be formulated as follows.

Theorem A′ (see 7.4)**.** *Suppose V and W satisfy Hypothesis 3, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, $(\|V\| + \|W\|)(\mathbf{R}^n) < \infty$, and σ denotes the geodesic distance on $\operatorname{spt}\|V\|$. Then, for some positive finite number Γ determined by m, there holds*

$$\operatorname{diam}_\sigma \operatorname{spt}\|V\| \le \Gamma\big(\textstyle\int |\mathbf{h}(V, \cdot)|^{m-1} \,\mathrm{d}\|V\| + \int |\mathbf{h}(W, \cdot)|^{m-2} \,\mathrm{d}\|W\|\big);$$

here, by convention, $0^0 = 1$.

In particular, if the sum on the right hand side of the equation is finite, then $\operatorname{spt}\|V\|$ is a compact subset of $\mathbf{R}^n$ and any two points of $\operatorname{spt}\|V\|$ may be connected by a path of finite length in $\operatorname{spt}\|V\|$. In analogy with the results described for the case $p = m$ of Hypothesis 2, an array of further questions arises. In the absence of boundary, the most immediate ones read as follows. The last item thereof relates to the possible study of *intermediate conditions on the mean curvature*, that is, to $1 < p < m$ in Hypothesis 2 (see [Men16a, p. 990]); a special case of that item was already raised as fifth question in [Sch16, Section A].

Question 4. Suppose V satisfies Hypothesis 3 with $W = 0$, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, and $\mathbf{h}(V, \cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$.

(1) Is $\operatorname{spt}\|V\|$ locally connected?

(2) If so, is $\operatorname{spt}\|V\|$ locally connected by paths of finite length?

(3) If so, is the geodesic distance induced on connected components of $\operatorname{spt}\|V\|$ a Sobolev function with bounded generalised weak derivative and what are the continuity properties of this particular (or, any such) function?

[4]This condition is evidently natural from the differential geometric point of view. However, following Almgren's original approach to compactness (see [Alm65, Theorem 10.8]), one might also study tuples $(V_0, \ldots, V_m)$ consisting of i dimensional varifolds V_i such that V_{i-1} controls the boundary behaviour of V_i for $i > 0$. This would include m dimensional cubes, for instance.

We conclude the exposition of our findings by stating the Sobolev-Poincaré inequality (of which Theorem B is a special case), that is the foundation for all named theorems of this introduction apart of the example, Theorem D. For this purpose, we employ the generalised V weak derivative $V\,\mathbf{D}\,f$ of functions f in $\mathbf{T}(V)$. As the theorem does not contain any indecomposability hypothesis, its formulation has to account for the possibility of a countably infinite decomposition of V such that f is $\|V\| + \|\delta V\|$ almost constant on each component thereof (see [Men09, 1.2] and [Men16a, 8.24, 8.34]). This is accomplished by employing a Borel set Y to track the image of f in its conclusion. For simplicity (see [Men16a, 8.16]), we state the theorem for nonnegative functions.

Theorem B′ (see 4.5 and 4.7)**.** *Suppose V and W satisfy Hypothesis 3,*

$$\mathbf{h}(V,\cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n), \quad \textit{if } m > 2 \textit{ then } \mathbf{h}(W,\cdot) \in \mathbf{L}_{m-2}^{\mathrm{loc}}(\|W\|, \mathbf{R}^n),$$

$0 \leq f \in \mathbf{T}(V) \cap \mathbf{T}(W)$,

$$|V\,\mathbf{D}\,f(x)| \leq 1 \textit{ for } \|V\| \textit{ almost all } x, \quad |W\,\mathbf{D}\,f(x)| \leq 1 \textit{ for } \|W\| \textit{ almost all } x,$$

and $E = \{x : f(x) > 0\}$. Then, there exists a Borel subset Y of $\mathbf{R}$ such that

$$f(x) \in Y \quad \textit{for } \|V\| \textit{ almost all } x$$

and such that, for some positive finite number Γ determined by m,

$$\begin{aligned}\mathscr{L}^1(Y) \leq \Gamma\big(\|V\|(E)^{1/m} &+ \textstyle\int_E |\mathbf{h}(V,\cdot)|^{m-1}\,\mathrm{d}\|V\| \\ &+ \|W\|(E)^{1/(m-1)} + \textstyle\int_E |\mathbf{h}(W,\cdot)|^{m-2}\,\mathrm{d}\|W\|\big);\end{aligned}$$

here, by convention, $0^0 = 1$.

If V is indecomposable of type $\{f\}$, then $\operatorname{spt} f_\# \|V\|$ is an interval and satisfies the bound

$$\operatorname{diam} \operatorname{spt} f_\# \|V\| \leq \mathscr{L}^1(Y)$$

(see 3.14 (1) and 4.6). If f is continuous, then $f[\operatorname{spt} \|V\|] \subset \operatorname{spt} f_\# \|V\|$. Now, the deduction of Theorem A′ from Theorem B′ is based – as in the differential geometric case – on the characterisation of the geodesic diameter of closed subsets of $\mathbf{R}^n$. To infer Theorem C′ (or its special case Theorem C) from Theorem B′, we apply the latter theorem with $f(x) = \sup\{r - |x - a|, 0\}$ to obtain (see 6.3) a conditional lower density ratio bound on $\mathbf{B}(a, r)$ which may then be combined with the afore-mentioned dichotomy from [Men09, 2.11]. Finally, we remark that Theorem B′ and C′ are local in nature and accordingly have appropriate formulations (see 4.5 and 6.7 (1)) for varifolds in open subsets of $\mathbf{R}^n$.

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1.4 Notation

Our notation is that of [Men16a]; the relevant material including a review of less common symbols and terminology appears in Section 1 therein. Thus, we are largely consistent with Federer's terminology in geometric measure theory (see [Fed69, pp. 669–676]) and Allard's notation for varifolds (see [All72]).

Amendments

The terms *immersion* and *embedding* are employed in accordance with [Hir94, p. 21]. Whenever k is a positive integer or $k = \infty$, we mean by a *[sub]manifold-with-boundary of class k* a Hausdorff topological space with a countable base of its topology that is, in the terminology of [Hir94, pp. 29–30], a C^k [sub]manifold. For manifolds-with-boundary M of class k, we similarly adapt the notion of *chart of class k* and *Riemannian metric of class k* $-$ 1 from [Hir94, p. 29, p. 95] and denote by ∂M its *boundary* as in [Hir94, p. 30].

Whenever m is a positive integer, we denote with $\boldsymbol{\gamma}(m)$ the best constant in the general isoperimetric inequality for m dimensional varifolds in Euclidean space (see [MS17, 3.7]). If V is an m dimensional varifold in an open subset U of $\mathbf{R}^n$ and $\|\delta V\|$ is a Radon measure, then $\mathbf{T}(V)$ denotes (see [MS17, 4.2]) the space of generalised V weakly differentiable functions with values in $\mathbf{R}$ and $V\,\mathbf{D}\,f$ denotes the generalised V weak derivative for f in $\mathbf{T}(V)$.

Definitions in the text

The symbols for restriction and push forward, $\phi \llcorner f$ and $f_\# \phi$, appear in 2.1 and 2.4. The terms *Riemannian distance* and *geodesic distance* are fixed in 2.7 and 7.1. For immersions, the notions of *mean curvature* and *associated varifold* are explained in 2.12 and 2.14. The concept of *indecomposability of type* Ψ is introduced 3.2. Finally, the topological space $\mathscr{C}^k(M, Y)$ is defined in 7.8.

2 Preliminaries

The main purpose of this section is to collect basic material on measures (see 2.1–2.6) and some properties of the Riemannian metric, the Riemannian distance, and varifolds associated to immersions (see 2.7–2.15). Moreover, we record a separation lemma (see 2.16). Finally, we provide the link between the coarea formula for varifolds involving Hausdorff measure and that based on the distributional boundary of superlevel sets when both concepts are applicable (see 2.17).

2.1 Definition. Whenever ϕ measures X and f is a $\{y : 0 \leq y \leq \infty\}$ valued function whose domain contains ϕ almost all of X, we define the measure $\phi \llcorner f$ over X by

$$(\phi \llcorner f)(A) = \textstyle\int_A^* f \,\mathrm{d}\phi \quad \text{for } A \subset X.$$

2.2 Remark. Basic properties of this measure are listed in [Fed69, 2.4.10].

2.3 Remark. If X is a locally compact Hausdorff space, ϕ is a Radon measure over X, and $0 \leq f \in \mathbf{L}_1^{\mathrm{loc}}(\phi)$, then $\phi \llcorner f$ is a Radon measure over X, provided X is the union of a countable family of compact subsets of X. The supplementary hypothesis "provided ... of X" may not be omitted; in fact, one may take f to be the characteristic function of the set constructed in [HS75, 9.41 (e)].

2.4 Definition. Whenever ϕ measures X, Y is a topological space, and f is a Y valued function with $\operatorname{dmn} f \subset X$, we define the measure $f_\# \phi$ over Y by

$$f_\# \phi(B) = \phi(f^{-1}[B]) \quad \text{for } B \subset Y.$$

2.5 Remark. This slightly extends [Fed69, 2.1.2], where $\operatorname{dmn} f = X$ is required.

Next, we adapt the regularity of push forwards of measures from the context of proper maps (see [Fed69, 2.2.17]) to that of measurable functions.

2.6 Lemma. *Suppose ϕ is a Radon measure over a locally compact Hausdorff space X, Y is a separable metric space, f is a ϕ measurable Y valued function, and X is ϕ almost equal to the union of a countable family of compact subsets of X.*

Then, $f_\# \phi$ is a Borel regular measure over Y.

Proof. By [Fed69, 2.1.2], all closed subsets of Y are $f_\# \phi$ measurable. To prove the Borel regularity, we employ [Fed69, 2.3.5] to reduce the problem to the case, $C = \operatorname{spt} \phi$ is compact and $f|C$ is continuous. Then, supposing $B \subset Y$ and $\varepsilon > 0$, we employ [Fed69, 2.2.5] to choose an open subset U of X with $f^{-1}[B] \subset U$ and $\phi(U) \leq \varepsilon + f_\# \phi(B)$, define an open subset V of Y by $V = Y \sim f[C \sim U]$, and verify

$$B \subset V, \quad f^{-1}[V] \subset U \cup (X \sim C),$$

whence it follows $f_\# \phi(V) \leq \varepsilon + f_\# \phi(B)$. □

Recall (see [Fed69, 1.10.1]) that $\langle v \odot w, b \rangle$ is an alternative notation for $b(v, w)$ whenever b is a symmetric bilinear form on some vector space V and $v, w \in V$.

2.7 Definition. Suppose the pair (M, g) consists of a connected manifold-with-boundary M of class 1 and a Riemannian metric g on M of class 0.

Then, the *Riemannian distance σ of (M, g)* is the function on $M \times M$ whose value at $(c, z) \in M \times M$ equals the infimum of the set of numbers

$$\textstyle\int_K \langle C'(y) \odot C'(y), g(C(y)) \rangle^{1/2} \, \mathrm{d}\mathscr{L}^1 \, y$$

corresponding to all locally Lipschitzian[5] functions C mapping some compact interval K into M with $C(\inf K) = c$ and $C(\sup K) = z$.

Our setting differs from the familiar setting of a general Riemannian manifold (M, g) together with its induced Riemannian measures on submanifolds of class 1. On the one hand, it is less general since, in the present paper, g is always induced by an immersion into $\mathbf{R}^n$. On the other hand, as induced Riemannian measures would be insufficient (see the proof of 3.9), we make use of the more general concept of Hausdorff measures associated to σ.

2.8 Lemma. *Suppose m and n are positive integers, $m \leq n$, M is a connected m dimensional manifold-with-boundary of class 1, $F : M \to \mathbf{R}^n$ is an immersion of class 1, g is the Riemannian metric on M induced by F, and σ is the Riemannian distance associated to (M, g).*

Then, the function σ is a metric on M inducing the given topology on M and $F_\# \mathscr{H}^k_\sigma = \mathscr{H}^k \llcorner N(F, \cdot)$ whenever $0 \leq k < \infty$.

[5]That is, C is continuous and $\phi \circ C$ is locally Lipschitzian whenever ϕ is a chart of M of class 1. A posteriori, this is equivalent to C being locally Lipschitzian with respect to σ.

Proof. We first verify that one may reduce the statement to the case that F is an embedding, hence to case that $M \subset \mathbf{R}^n$ and $F = \mathbf{1}_M$. Clearly, $|z - \zeta| \leq \sigma(z, \zeta)$ for $z, \zeta \in M$, hence $\mathscr{H}^k(S) \leq \mathscr{H}^k_\sigma(S)$ for $S \subset M$. On the other hand, given $1 < \lambda < \infty$ and $c \in M$, there exists $\delta > 0$ such that

$$\sigma(z, \zeta) \leq \lambda |z - \zeta| \quad \text{whenever } z, \zeta \in M \cap \mathbf{B}(c, \delta);$$

in fact, we observe that it is sufficient to note that the chart ψ of $\mathbf{R}^n$ of class 1 occurring in the definition of submanifold-with-boundary of $\mathbf{R}^n$ of class 1 (see [Hir94, p. 30]) may be required to satisfy $\mathrm{D}\,\psi(c) = \mathbf{1}_{\mathbf{R}^n}$. This in particular implies $\mathscr{H}^k_\sigma(S) \leq \lambda^k \mathscr{H}^k(S)$ for $S \subset M \cap \mathbf{B}(c, \delta)$ and the conclusion follows. □

2.9 Remark. Denoting by h the Riemannian metric on ∂M induced by $F|\partial M$ and by ϱ the Riemannian distance associated to $(\partial M, h)$, the preceding lemma yields in particular $\mathscr{H}^{m-1}_\varrho(S) = \mathscr{H}^{m-1}_\sigma(S)$ for $S \subset \partial M$. Moreover, the measures $\mathscr{H}^m_\sigma$ and $\mathscr{H}^{m-1}_\varrho$ agree with the usual Riemannian measures (see [Sak96, Section 2.5]) associated to (M, g) and $(\partial M, h)$, respectively, by [Fed69, 3.2.46].

In our setting, the constancy theorem then takes the following form.

2.10 Theorem. *Suppose m, n, M, F, g, and σ are as in 2.8, $C \subset M$, and $\mathscr{H}^{m-1}_\sigma((\mathrm{Bdry}\, C) \sim \partial M) = 0$.*

Then, either $\mathscr{H}^m_\sigma(C) = 0$ or $\mathscr{H}^m_\sigma(M \sim C) = 0$.

Proof. Noting that $M \sim \partial M$ is connected and $\mathscr{H}^m_\sigma(\partial M) = 0$ by 2.9, we assume $\partial M = \varnothing$. Next, we observe that it is sufficient to prove that there holds either $\mathscr{H}^m_\sigma(C \cap \mathrm{dmn}\, \phi) = 0$ or $\mathscr{H}^m_\sigma((\mathrm{dmn}\, \phi) \sim C) = 0$ whenever ϕ is a chart of M of class 1 satisfying $\mathrm{im}\, \phi = \mathbf{R}^m$, as M is covered by the domains of a countable collection of such charts. To verify this dichotomy, we let $A = \phi[C]$ and infer $\mathscr{H}^{m-1}(\mathrm{Bdry}\, A) = 0$ from [Fed69, 3.2.46], hence $\mathrm{Bdry}\, A = \varnothing$ if $m = 1$ and $\mathscr{I}^{m-1}_1(\mathrm{Bdry}\, A) = 0$ if $m > 1$ by [Fed69, 2.10.15]. This implies A is of locally finite perimeter by [Fed69, 4.5.11] and that $\partial(\mathbf{E}^m \llcorner A) = 0$ by [Fed69, 4.5.6 (1)]. Consequently, there holds $\mathscr{L}^m(A) = 0$ or $\mathscr{L}^m(\mathbf{R}^m \sim A) = 0$ by the constancy theorem [Fed69, 4.1.7], whence we deduce the assertion by [Fed69, 3.2.46]. □

2.11 Remark. Instead of [Fed69, 4.5.6, 11], one could employ an argument based on capacity (see [Men16a, 5.7]) to infer $\partial(\mathbf{E}^m \llcorner A) = 0$ from $\mathscr{H}^{m-1}(\mathrm{Bdry}\, A) = 0$.

Next, we collect properties of immersions and their associated varifolds.

2.12 Definition. Suppose M is a manifold-with-boundary of class 2 and the map $F : M \to \mathbf{R}^n$ is an immersion of class 2. Then, the *mean curvature vector* $\mathbf{h}(F, c)$ along F at $c \in M \sim \partial M$ is characterised by the condition

$$\mathbf{h}(F, c) = \mathbf{h}(F[W], F(c))$$

whenever W is open neighbourhood of c in $M \sim \partial M$ and $F|W$ is an embedding.

2.13 Remark. Whenever ϕ is a chart of M of class 2, $c \in \mathrm{dmn}\, \phi$, $\psi = \phi^{-1}$, and $e_1, \ldots, e_m \in \mathbf{R}^m$ are such that $\langle e_1, \mathrm{D}\,\psi(\phi(c))\rangle, \ldots, \langle e_m, \mathrm{D}\,\psi(\phi(c))\rangle$ form an orthonormal base of $\mathrm{Tan}(F[W], F(c))$, we have (cf. [All72, pp. 423–424])

$$\mathbf{h}(F, c) = \sum_{i=1}^m \left\langle e_i \odot e_i, \mathrm{Nor}(F[W], F(c))_\natural \circ \mathrm{D}^2(F \circ \psi)(\phi(c)) \right\rangle .$$

2.14 Definition. Suppose m and n are positive integers, $m \leq n$, M is an m dimensional manifold-with-boundary of class 1, U is an open subset of $\mathbf{R}^n$, and $F : M \to U$ is a proper immersion of class 1.

Then, we define the *associated varifold* V *to* (F, U) by

$$V(k) = \textstyle\int k(x, \operatorname{Tan}(\operatorname{im} F, x)) N(F, x) \, \mathrm{d}\mathscr{H}^m x \quad \text{for } k \in \mathscr{K}(U \times \mathbf{G}(n, m)).$$

2.15 Remark. We notice that V is rectifiable, $\|V\| = \mathscr{H}^m \llcorner N(F, \cdot)$, and

$$\operatorname{Tan}^m(\|V\|, x) = \operatorname{Tan}(\operatorname{im} F, x), \quad \boldsymbol{\Theta}^m(\|V\|, x) = N(F, x)$$

for $\|V\|$ almost all x. If M and F are of class 2, then we employ [All72, 4.4, 7] to verify firstly that $\|\delta V\|$ is a Radon measure satisfying

$$\|\delta V\| \leq \|V\| \llcorner |\mathbf{h}(V, \cdot)| + \mathscr{H}^{m-1} \llcorner N(F|\partial M, \cdot)$$

with equality in case $F|\partial M$ is an embedding, and secondly (using approximate differentiation) that, for $\|V\|$ almost all x, we have $\mathbf{h}(V, x) = \mathbf{h}(F, a)$ whenever $F(a) = x$. In the terminology of [Men16c, 3.12], we may alternatively express the mean curvature of V through the pointwise differential of second order of the set $\operatorname{im} F$ by

$$\mathbf{h}(V, x) = \operatorname{trace} \operatorname{pt} \mathrm{D}^2(\operatorname{im} F)(x, \operatorname{Tan}(\operatorname{im} F, x)) \quad \text{for } \|V\| \text{ almost all } x;$$

here, the trace of a $\mathbf{R}^n$ valued bilinear form B on $\mathbf{R}^n$ is defined by the requirement $\alpha(\operatorname{trace} B) = \operatorname{trace}(\alpha \circ B)$ for $\alpha \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$.

Anticipating the needs of 3.7 and 7.3, we include here a separation lemma.

2.16 Lemma. *Suppose U is an open subset of $\mathbf{R}^n$.*

Then, the following two statements hold.

(1) *If A is a relatively closed subset of U, then there exists $f \in \mathscr{E}(U, \mathbf{R})$ satisfying $f \geq 0$ and $A = \{x : f(x) = 0\}$.*

(2) *If E_0 and E_1 are disjoint relatively closed subsets of U, then there exists $f \in \mathscr{E}(U, \mathbf{R})$ satisfying $E_i \subset \operatorname{Int}\{x : f(x) = i\}$ for $i \in \{0, 1\}$ and $0 \leq f \leq 1$.*

Proof. To prove (1), we assume $U = \mathbf{R}^n$ and $A \neq \varnothing$. We employ [Zie89, 3.6.1] to construct real numbers $\Delta_0, \Delta_1, \Delta_2, \ldots$ and $g : \mathbf{R}^n \sim A \to \{y : 0 < y < \infty\}$ of class ∞ satisfying, for $i = 0, 1, 2, \ldots$, the estimate

$$\| \mathrm{D}^i g(x) \| \leq \Delta_i \operatorname{dist}(x, A)^{1-i} \quad \text{whenever } x \in \mathbf{R}^n \text{ and } 0 < \operatorname{dist}(x, A) \leq \tfrac{1}{2}.$$

Selecting $h : \mathbf{R} \to \mathbf{R}$ of class ∞ satisfying $h \geq 0$ and $\{y : y \leq 0\} = \{y : h(y) = 0\}$, we verify that

$$\sup \big\{ \| \mathrm{D}^i h(y) \| y^{-j} : i = 0, 1, \ldots, j \text{ and } 0 < y \leq \Delta_0/2 \big\} < \infty,$$
$$\sup \big\{ \| \mathrm{D}^i (h \circ g)(x) \| \operatorname{dist}(x, A)^{-j} : i = 0, 1, \ldots, j \text{ and } 0 < \operatorname{dist}(x, A) \leq \tfrac{1}{2} \big\} < \infty$$

whenever j is a nonnegative integer with the help of the Taylor formula and the general formula for the differentials of a composition, see [Fed69, 3.1.11]. Therefore, we may take f to be the extension of $h \circ g$ to $\mathbf{R}^n$ by 0.

To prove (2), we choose, for $i \in \{0, 1\}$, disjoint relatively closed sets A_i with $E_i \subset \operatorname{Int} A_i$ and, by (1), $g_i \in \mathscr{E}(U, \mathbf{R})$ satisfying $g_i \geq 0$ and $\{x : g_i(x) = 0\} = A_i$, and take $f = g_0/(g_0 + g_1)$. □

Finally, we describe fibres of a real valued Lipschitzian map on a varifold using the notion of distributional boundary of superlevel sets from [Men16a, 5.1].

2.17 Theorem. *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, $f : U \to \mathbf{R}$ is locally Lipschitzian, and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$.*

Then, there exists an $\mathscr{L}^1$ measurable function W with values in $\mathbf{RV}_{m-1}(U)$ endowed with the weak topology such that, for $\mathscr{L}^1$ almost all y,

$$\begin{aligned} \operatorname{Tan}^{m-1}(\|W(y)\|, x) &= \operatorname{Tan}^m(\|V\|, x) \cap \ker V\,\mathbf{D}\,f(x) \in \mathbf{G}(n, m-1), \\ \boldsymbol{\Theta}^{m-1}(\|W(y)\|, x) &= \boldsymbol{\Theta}^m(\|V\|, x) \end{aligned}$$

for $\|W(y)\|$ almost all x and

$$\begin{aligned} \|W(y)\| &= \big(\mathscr{H}^{m-1} \llcorner \{x : f(x) = y\}\big) \llcorner \boldsymbol{\Theta}^m(\|V\|, \cdot), \\ V\,\partial\,E(y)(\theta) &= \textstyle\int \langle \theta, |V\,\mathbf{D}\,f|^{-1} V\,\mathbf{D}\,f\rangle \,\mathrm{d}\|W(y)\| \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n). \end{aligned}$$

Proof. By [Men16a, 8.7], we have $f \in \mathbf{T}(V)$ and $F = V\,\mathbf{D}\,f$ satisfies

$$F(x) = (\|V\|, m)\operatorname{ap}\mathrm{D}\,f(x) \circ \operatorname{Tan}^m(\|V\|, x)_\natural \quad \text{for } \|V\| \text{ almost all } x.$$

Moreover, whenever $y \in \mathbf{R}$, we recall from [Men16a, 8.29] that

$$V\,\partial\,E(y)(\theta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} \textstyle\int_{\{x : y < f(x) \leq y + \varepsilon\}} \langle \theta, F\rangle \,\mathrm{d}\|V\| \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Defining $T \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n)$ as in [Men16a, 8.1], we infer from [Men16a, 8.5] that T is representable by integration and

$$\textstyle\int g \,\mathrm{d}\|T\| = \int g(x, f(x)) |F(x)| \,\mathrm{d}\|V\|\, x, \quad T(\phi) = \int \langle \phi(x, f(x)), F(x)\rangle \,\mathrm{d}\|V\|\, x$$

whenever g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function and $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$. We abbreviate

$$\mu_y = \big(\mathscr{H}^{m-1} \llcorner \{x : f(x) = y\}\big) \llcorner \boldsymbol{\Theta}^m(\|V\|, \cdot)$$

whenever $y \in \mathbf{R}$ and $\mathscr{H}^{m-1}(\{x : f(x) = y\} \sim \operatorname{dmn} \boldsymbol{\Theta}^m(\|V\|, \cdot)) = 0$. Since we have $\boldsymbol{\Theta}^m(\|V\|, x) \in \mathbf{R}$ for $\mathscr{H}^m$ almost all $x \in U$ by [All72, 3.5 (1b)], we observe that [KM17, 3.5 (2)], in conjunction with [Fed69, 2.10.27], yields

$$\begin{aligned} \textstyle\int g \,\mathrm{d}\|T\| &= \textstyle\iint g(x, y) \,\mathrm{d}\mu_y\, x \,\mathrm{d}\mathscr{L}^1\, y, \\ T(\phi) &= \textstyle\iint \langle \phi(x, y), |F(x)|^{-1} F(x)\rangle \,\mathrm{d}\mu_y\, x \,\mathrm{d}\mathscr{L}^1\, y \end{aligned}$$

whenever g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function and $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$; in particular, μ_y is a Radon measure for $\mathscr{L}^1$ almost all y.

Next, we will show: *For $\mathscr{L}^1$ almost all y, there holds*

$$\textstyle\int \langle \theta, |F|^{-1} F\rangle \,\mathrm{d}\mu_y = V\,\partial\,E(y)(\theta) \quad \textit{for } \theta \in \mathscr{D}(U, \mathbf{R}^n);$$

We first notice that, if μ_y is a Radon measure, then, as a function of θ, both terms describe a distribution; hence, by [Men16a, 2.2, 24], it is sufficient to verify that, for each $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, the equation holds for $\mathscr{L}^1$ almost all y. Noting

$$\textstyle\int_y^{y+\varepsilon} \int \langle \theta, |F|^{-1} F\rangle \,\mathrm{d}\mu_\upsilon \,\mathrm{d}\mathscr{L}^1\, \upsilon = \int_{\{x : y < f(x) \leq y + \varepsilon\}} \langle \theta, F\rangle \,\mathrm{d}\|V\|$$

whenever $y \in \mathbf{R}$, $0 < \varepsilon < \infty$, and $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, we then conclude the assertion from [Fed69, 2.8.17, 9.8].

In view of the first paragraph and the assertion of the preceding paragraph, the conclusion now follows from [Men16a, 12.1 (2)]. □

3 Indecomposability

We begin with a basic implication between connectedness and indecomposability (see 3.1) before introducing (see 3.2) the refined concept, indecomposability of type Ψ. In the main part of the present section, we obtain basic properties of the new concept (see 3.3–3.6) and study its relation to topological connectedness (see 3.7–3.13) showing in particular that the three main types considered here are distinct (see 3.11 and 3.13). Finally, we collect (see 3.14) four key implications on f of indecomposability of type $\{f\}$.

3.1 Lemma. *Suppose m and n are positive integers, $m \le n$, U is an open subset of $\mathbf{R}^n$, M is a connected m dimensional submanifold of U of class 2, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and*

$$V(k) = \textstyle\int_M k(x, \operatorname{Tan}(M,x)) \,\mathrm{d}\mathscr{H}^m x \quad \textit{for } k \in \mathscr{K}(U \times \mathbf{G}(n,m)).$$

Then, V is indecomposable.

Proof. Whenever E is $\|V\| + \|\delta V\|$ measurable and $V \,\partial\, E = 0$, we note [Men16a, 5.9 (1)] and employ [All72, 4.6 (3)] to conclude that each $z \in M$ admits a neighbourhood X in M such that $E \cap X$ is $\mathscr{H}^m$ almost equal to $\varnothing$ or $E \cap X$, whence it follows $\|V\|(E) = 0$ or $\|V\|(U \sim E) = 0$ as M is connected. ☐

Next, we introduce the main new concept of the present paper.

3.2 Definition. Suppose m and n are positive integers, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and $\Psi \subset \mathbf{T}(V)$.

Then, V is called *indecomposable of type* Ψ if and only if, whenever $f \in \Psi$, the set of $y \in \mathbf{R}$, such that $E(y) = \{x : f(x) > y\}$ satisfies

$$\|V\|(E(y)) > 0, \quad \|V\|(U \sim E(y)) > 0, \quad V \,\partial\, E(y) = 0,$$

has $\mathscr{L}^1$ measure zero.

3.3 Remark. If V is indecomposable, then V is indecomposable of type Ψ whenever $\Psi \subset \mathbf{T}(V)$. For $\Psi = \mathbf{T}(V)$, the converse implication holds; in fact, [MS17, 4.14] readily yields that, whenever E is a $\|V\| + \|\delta V\|$ measurable set satisfying $V \,\partial\, E = 0$, its characteristic function belongs to $\mathbf{T}(V)$.

3.4 Remark. If $\operatorname{spt} \|V\|$ is compact, then indecomposability of types $\mathscr{E}(U, \mathbf{R})$ and $\mathscr{D}(U, \mathbf{R})$ agree. In general, these concepts differ as will be shown in 3.13.

3.5 Remark. If V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$, $a \in U$, $0 < r < \infty$, $\mathbf{B}(a, r) \subset U$, and $f : U \to \mathbf{R}$ satisfies $f(x) = \sup\{r - |x - a|, 0\}$ for $x \in U$, then V is indecomposable of type $\{f\}$, as may be verified by approximation.

3.6 Remark. If V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$ and A is a relatively closed subset of U, then $V | \mathbf{2}^{(U \sim A) \times \mathbf{G}(n,m)}$ is indecomposable of type $\mathscr{D}(U \sim A, \mathbf{R})$, as may be verified using the canonical extension map of $\mathscr{D}(U \sim A, \mathbf{R})$ into $\mathscr{D}(U, \mathbf{R})$.

The following two theorems (one basic, and one more subtle) give relations between indecomposability of type $\mathscr{E}(U, \mathbf{R})$ and connectedness of related objects.

3.7 Theorem. *Suppose m and n are positive integers, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and V is indecomposable of type $\mathscr{E}(U, \mathbf{R})$.*

Then, $\operatorname{spt} \|V\|$ *is connected.*

Proof. If spt $\|V\|$ were not connected, there would exist nonempty disjoint relatively closed subsets E_0 and E_1 of U with spt $\|V\| = E_0 \cup E_1$, and 2.16 (2) would yield f satisfying spt $\|V\| \cap \{x : f(x) > y\} = E_1$ for $0 \leq y < 1$, in contradiction to $\|V\|(E_1) > 0$, $\|V\|(U \sim E_1) > 0$, and $V \partial E_1 = 0$ by [Men16a, 6.5]. □

3.8 Remark. In view of 3.3, the preceding theorem extends [Men16a, 6.5].

3.9 Theorem. *Suppose m and n are positive integers, $m \leq n$, M is an m dimensional manifold-with-boundary of class 2, U is an open subset of $\mathbf{R}^n$, $F : M \to U$ is a proper immersion of class 2, and V is the associated varifold to (F, U).*

Then, connectedness of M implies indecomposability of type $\mathscr{E}(U, \mathbf{R})$ of V.

Proof. Clearly, V is rectifiable and $\|\delta V\|$ is a Radon measure by 2.15. Suppose M is connected and $f : U \to \mathbf{R}$ is locally Lipschitzian. We will show that V is indecomposable of type $\{f\}$. For this purpose, we let

$$B(y) = \{x : f(x) = y\}, \quad E(y) = \{x : f(x) > y\}$$

for $y \in \mathbf{R}$, note $\operatorname{Bdry} F^{-1}[E(y)] \subset F^{-1}[B(y)]$, and define σ as in 2.8. Since $\boldsymbol{\Theta}^m(\|V\|, x) = N(F, x)$ for $\mathscr{H}^m$ almost all $x \in U$ by [All72, 3.5 (1b)] and 2.15, we employ 2.17, [Fed69, 2.10.27], and 2.8 to infer

$$\begin{aligned} \|V \partial E(y)\|(U) &= \textstyle\int_{B(y)} \boldsymbol{\Theta}^m(\|V\|, x) \,\mathrm{d}\mathscr{H}^{m-1} x \\ &= \textstyle\int_{B(y)} N(F, x) \,\mathrm{d}\mathscr{H}^{m-1} x = (F_\# \mathscr{H}^{m-1}_\sigma)(B(y)) \end{aligned}$$

for $\mathscr{L}^1$ almost all y. Whenever $V \partial E(y) = 0$ for such y, we apply 2.10 with $C = F^{-1}[E(y)]$ to conclude that

$$\text{either } F_\# \mathscr{H}^m_\sigma(E(y)) = 0 \quad \text{or } F_\# \mathscr{H}^m_\sigma(U \sim E(y)) = 0.$$

Since $F_\# \mathscr{H}^m_\sigma = \|V\|$ by 2.8 and 2.15, the conclusion follows. □

3.10 Remark. Clearly, we have in fact established indecomposability of V of type $\mathbf{R}^U \cap \{f : f \text{ is locally Lipschitzian}\}$; however, the possible differences of that concept to indecomposability of type $\mathscr{E}(U, \mathbf{R})$ will not be studied in this paper.

3.11 Remark. The preceding theorem shows in particular that the concepts of indecomposability of types $\mathbf{T}(V)$ and $\mathscr{E}(U, \mathbf{R})$ differ; in fact, one may consider $V \in \mathbf{V}_1(\mathbf{R}^2)$ associated to the union of two distinct touching circles in $\mathbf{R}^2$. This answers the second question posed in [Sch16, Section A].

3.12 Remark. Simple examples with $N(F, x) = 2$ for $x \in \operatorname{im} F$, yield that conversely even indecomposability of V need not imply connectedness of M.

With the preceding theory at hand, one may now readily explore the case that the varifold is associated to a properly embedded submanifold-with-boundary.

3.13 Example. Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, M is a m dimensional submanifold-with-boundary in U of class 2, the inclusion map $F : M \to U$ is proper, V is associated to (F, U), and Φ is the family of connected components of M. Then, $\|\delta V\|$ is a Radon measure by 2.15 and one verifies the equivalence of the following four conditions using 3.1 and 3.7:

(1) The submanifold-with-boundary M is connected.

(2) The submanifold $M \sim \partial M$ is connected.

(3) The varifold V is indecomposable.

(4) The varifold V is indecomposable of type $\mathscr{E}(U, \mathbf{R})$.

Hence, $V \llcorner C \times \mathbf{G}(n,m)$ *is indecomposable and* $V \partial C = 0$ *for* $C \in \Phi$, as C is relatively open in M. Next, we notice that [Men16a, 5.2] may be used to obtain

$$V \, \partial \, E(\theta) = \sum_{C \in \Phi} (V \llcorner C \times \mathbf{G}(n,m)) \, \partial \, E(\theta),$$
$$\|V \, \partial \, E\|(k) = \sum_{C \in \Phi} \|(V \llcorner C \times \mathbf{G}(n,m)) \, \partial \, E\|(k)$$

whenever E *is* $\|V\| + \|\delta V\|$ *measurable,* $\theta \in \mathscr{D}(U, \mathbf{R}^n)$*, and* $k \in \mathscr{K}(U \times \mathbf{G}(n,m))$. These equations are readily used to verify that $\{V \llcorner C \times \mathbf{G}(n,m) : C \in \Phi\}$ *is the unique decomposition of* V and that the following two conditions are equivalent:

(5) If C is compact for some $C \in \Phi$, then M is connected.

(6) The varifold V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$.

We conclude this section with a key lemma on indecomposability of type $\{f\}$. It acts as a tool to fully exploit our main oscillation estimate (see 4.5–4.6).

3.14 Lemma. *Suppose* m *and* n *are positive integers,* $m \leq n$*,* U *is an open subset of* $\mathbf{R}^n$*,* $V \in \mathbf{V}_m(U)$*,* $\|\delta V\|$ *is a Radon measure,* $f \in \mathbf{T}(V)$*,* V *is indecomposable of type* $\{f\}$*, and* $\mu = f_\# \|V\|$.

Then, the following four statements hold.

(1) The set $\operatorname{spt} \mu$ *is an interval.*

(2) For $\mathscr{L}^1$ *almost all* $y \in \operatorname{spt} \mu$*, we have* $V \partial \{x : f(x) > y\} \neq 0$.

(3) If $Y \subset \mathbf{R}$ *and* $V \mathbf{D} f(x) = 0$ *for* $\|V\|$ *almost all* $x \in f^{-1}[Y]$*, then we have* $\mathscr{L}^1(Y \cap \operatorname{spt} \mu) = 0$.

(4) If $V \mathbf{D} f(x) = 0$ *for* $\|V\|$ *almost all* x*, then* f *is* $\|V\| + \|\delta V\|$ *almost constant.*

Proof. Let $I = \mathbf{R} \cap \{y : \inf \operatorname{spt} \mu \leq y \leq \sup \operatorname{spt} \mu\}$, choose compact sets K_i with $K_i \subset \operatorname{Int} K_{i+1}$ and $\bigcup_{i=1}^\infty K_i = U$, and pick $\varepsilon_i > 0$ such that (see 2.3 and 2.6)

$$\nu = \sum_{i=1}^\infty \varepsilon_i f_\# \big((\|V\| \llcorner K_i \cap \{x : |f(x)| \leq i\}) \llcorner |V \mathbf{D} f| \big)$$

satisfies $\nu(\mathbf{R}) < \infty$. Define $B = \mathbf{R} \cap \{y : V \partial \{x : f(x) > y\} = 0\}$. Clearly, $\mathscr{L}^1(B \cap I) = 0$; in particular (2) holds. [MS17, 4.11] and [Men16a, 8.29] yield

$$\{y : \mathbf{\Theta}^1(\nu, y) = 0\} \subset B.$$

If Y satisfies the hypotheses of (3), then we have $\nu(Y) = 0$ which entails firstly $\mathscr{L}^1(Y \sim \{y : \mathbf{\Theta}^1(\nu, y) = 0\}) = 0$ by [Fed69, 2.10.19 (4)], and then $\mathscr{L}^1(Y \cap I) = 0$. Now, (1) and (3) follow; in particular $I = \operatorname{spt} \mu$. Under the hypothesis of (4), we conclude $\nu = 0$ which implies $B = \mathbf{R}$, $\mathscr{L}^1(I) = 0$, and that f is $\|V\|$ almost constant, whence we deduce that f is also $\|V\| + \|\delta V\|$ almost constant by [MS17, 4.11] and [Men16a, 8.12, 13 (2) (3), 33]. □

4 Sobolev-Poincaré inequality

The main aim of this section is to establish (see 4.5) our new Sobolev-Poincaré inequality, Theorem B′ in the introduction. The key ingredient for this purpose is a monotonicity lemma (see 4.1) based on the isoperimetric inequality. We also include a simpler version of our Sobolev-Poincaré inequality for varifolds that are suitably indecomposable and have no boundary (see 4.9).

The next lemma will presently be applied only with $q = \infty$. We still include the case $q < \infty$ since, firstly, it yields a new proof of previous Sobolev inequalities (see 4.3) and, secondly, it relates to Question 2 in the introduction. We recall that the space $\mathbf{T}_{\operatorname{Bdry} U}(V)$ introduced in [Men16a, 9.1] is the subspace of *nonnegative* functions of $\mathbf{T}(V)$ realising the concept of *zero boundary values* on $\operatorname{Bdry} U$.

4.1 Lemma. *Suppose m and n are positive integers, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x, $f \in \mathbf{T}_{\operatorname{Bdry} U}(V)$, $0 \le s \le r \le \|V\|_{(\infty)}(f)$,*

$$\text{either,}\quad m = q = 1 \text{ and } \lambda = 1,$$
$$\text{or,}\quad m < q \le \infty \text{ and } \lambda = \big((1/m - 1/q)/(1 - 1/q)\big)^{1-1/q},$$

$V \mathbf{D} f \in \mathbf{L}_q(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}))$, $0 < \varepsilon < \boldsymbol{\gamma}(m)^{-1}$,

$$\|V\| \{x : f(x) \ge y\} < \infty, \quad \|\delta V\| \{x : f(x) \ge y\} \le \varepsilon \, \|V\|(\{x : f(x) \ge y\})^{1-1/m}$$

for $s < y < r$, and $\delta = \boldsymbol{\gamma}(m)^{-1} - \varepsilon$.

Then, the quantities

$$\|V\|(\{x : f(x) \ge y\})^{1/m-1/q} (\|V\| \llcorner \{x : f(x) \ge y\})_{(q)}(V \mathbf{D} f) + \delta\lambda y, \quad \text{if } q < \infty,$$
$$\|V\|(\{x : f(x) \ge y\})^{1/m} \|V\|_{(\infty)}(V \mathbf{D} f) + \delta\lambda y, \quad \text{if } q = \infty,$$

are nonincreasing in y, for $s < y < r$.

Proof. We treat the case $m < q < \infty$. The cases $m = q = 1$ and $q = \infty$ follow by a similar but simpler argument. Abbreviate $\alpha = 1 - 1/q$ and $\beta = 1 - 1/m$. Let $i : U \to \mathbf{R}^n$ denote the inclusion, define

$$E(\upsilon) = \{x : f(x) \ge -\upsilon\}, \quad g(\upsilon) = \|V\|(E(\upsilon)), \quad G(\upsilon) = g(\upsilon)^{1-\beta/\alpha},$$
$$h(\upsilon) = \textstyle\int_{E(\upsilon)} |V \mathbf{D} f|^q \, \mathrm{d}\|V\|, \quad W_\upsilon = i_\#(V \llcorner E(\upsilon) \times \mathbf{G}(n, m)) \in \mathbf{V}_m(\mathbf{R}^n)$$

for $-r < \upsilon < -s$, and notice that

$$\boldsymbol{\Theta}^m(\|W_\upsilon\|, x) \ge 1 \quad \text{for } \|W_\upsilon\| \text{ almost all } x$$

by [Fed69, 2.8.9, 18, 9.11]. Furthermore, as $(\|V\| + \|\delta V\|) \{x : f(x) = \upsilon\} = 0$ for all but countably many υ, we have (see [Men16a, 9.1])

$$\|\delta W_\upsilon\| \le i_\#\big(\|\delta V\| \llcorner E(\upsilon) + \|V \, \partial \, E(\upsilon)\|\big) \quad \text{for } \mathscr{L}^1 \text{ almost all } -r < \upsilon < -s.$$

For such υ, the isoperimetric inequality (see [MS17, 3.5, 7]) yields

$$\boldsymbol{\gamma}(m)^{-1} g(\upsilon)^{1-1/m} \le \|\delta V\|(E(\upsilon)) + \|V \, \partial \, E(\upsilon)\|(U).$$

In view of [Men16a, 8.29] and [Fed69, 2.9.19], we deduce the inequalities

$$0 < (\boldsymbol{\gamma}(m)^{-1} - \varepsilon) g(\upsilon)^{1-1/m} \le \|V \,\partial\, E(\upsilon)\|(U) \le g'(\upsilon)^{1-1/q} h'(\upsilon)^{1/q},$$
$$\delta\lambda = (\boldsymbol{\gamma}(m)^{-1} - \varepsilon)(1 - \beta/\alpha)^{\alpha} \le (g^{1-\beta/\alpha})'(\upsilon)^{\alpha} h'(\upsilon)^{1-\alpha} = G'(\upsilon)^{\alpha} h'(\upsilon)^{1-\alpha}$$

for $\mathscr{L}^1$ almost all $-r < \upsilon < -s$. Therefore, noting $0 < \int_{-r}^{\upsilon} h' \,\mathrm{d}\mathscr{L}^1 \le h(\upsilon)$ for $-r < \upsilon < -s$ by [Fed69, 2.9.19], we obtain (using the inequality relating geometric and arithmetic means)

$$\delta\lambda \le \alpha G'(\upsilon) G(\upsilon)^{\alpha-1} h(\upsilon)^{1-\alpha} + (1-\alpha) h'(\upsilon) h(\upsilon)^{-\alpha} G(\upsilon)^{\alpha} = (G^{\alpha} h^{1-\alpha})'(\upsilon)$$

for $\mathscr{L}^1$ almost all $-r < \upsilon < -s$, whence the conclusion follows by integration with respect to $\mathscr{L}^1$ using [Fed69, 2.9.19]. □

4.2 Remark. For $q = \infty$, the pattern of the preceding proof is that of Allard [All72, 8.3].

4.3 Remark. The preceding lemma in particular entails the estimates [Men16a, 10.1 (2b) (2d)] with a different, somewhat more explicit constant.

We next gather the set of conditions on density and first variation that we assume for both varifolds occurring in the Sobolev-Poincaré estimate in 4.5.

4.4. Suppose U is an open subset of $\mathbf{R}^n$, V is a varifold in U, $m = \dim V$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x, and, either $m = 1$ and $\phi = \|V\|$, $m = 2$ and $\phi = \|\delta V\|$, or $m > 2$, $\mathbf{h}(V, \cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$, and $\phi = \|V\| \llcorner |\mathbf{h}(V, \cdot)|^{m-1}$.

4.5 Theorem. *Suppose m and n are positive integers, $2 \le m \le n$, U is an open subset of $\mathbf{R}^n$, $V_1 \in \mathbf{V}_m(U)$ and $V_2 \in \mathbf{V}_{m-1}(U)$ satisfy the conditions of 4.4,*

$$V_2 = 0 \quad \textit{if } m = 2, \qquad \|\delta V_1\| \le \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\| \quad \textit{if } m > 2,$$
$$\|\delta V_2\| \textit{ is absolutely continuous with respect to } \|V_2\| \quad \textit{if } m > 3,$$
$$f \in \mathbf{T}_{\mathrm{Bdry}\, U}(V_i), \qquad |V_i \,\mathbf{D}\, f(x)| \le 1 \quad \textit{for } \|V_i\| \textit{ almost all } x,$$
$$\phi_i \textit{ are associated to } V_i \textit{ as in 4.4},$$

for $i \in \{1, 2\}$, and $E = \{x : f(x) > 0\}$.

Then, there exists a Borel subset Y of $\mathbf{R}$ such that

$$f(x) \in Y \quad \textit{for } \|V_1\| \textit{ almost all } x,$$
$$\mathscr{L}^1(Y) \le \Gamma\big(\|V_1\|(E)^{1/m} + \phi_1(E) + \|V_2\|(E)^{1/(m-1)} + \phi_2(E)\big),$$

where Γ is a positive finite number determined by m.

Proof. We assume $(\|V_i\| + \phi_i)(E) < \infty$ for $i \in \{1, 2\}$; in particular, we have $\|\delta V_i\|(E) < \infty$. Define $I = \mathbf{R} \cap \{y : y > 0\}$,

$$\mu_i = f_{\#}\|V_i\|, \quad \nu_i = f_{\#}\|\delta V_i\|, \quad \text{and} \quad \omega_i = f_{\#}\phi_i$$

for $i \in \{1, 2\}$. Let $\alpha = \omega_1$ if $m = 2$ and $\alpha = f_{\#}(\|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)|)$ if $m > 2$. With

$$\Delta_1 = 2m\boldsymbol{\gamma}(m), \quad \Delta_2 = \sup\big\{2(m-1)\boldsymbol{\gamma}(m-1), 2\Delta_1(2\boldsymbol{\gamma}(m))^{1/(m-1)}\big\},$$

we define

$$\lambda_i = \Delta_i \, \mu_i(I)^{1/\dim V_i} \quad \text{for } i \in \{1,2\}$$

and functions $r_i : \mathbf{R} \to \mathbf{R}$, for $i \in \{1,2\}$, by

$$r_i(b) = \sup \big\{ s : 0 \le s < b \text{ and } \Delta_i \, \mu_i(\mathbf{B}(b,s))^{1/\dim V_i} \ge s \big\} \quad \text{whenever } b \in \mathbf{R}.$$

Since the sets $(\mathbf{R} \times \mathbf{R}) \cap \big\{(b,s) : 0 \le s < b \text{ and } \Delta_i \, \mu_i(\mathbf{B}(b,s))^{1/\dim V_i} \ge s\big\}$ are relatively closed in $(\mathbf{R} \times \mathbf{R}) \cap \{(b,s) : s < b\}$, we may deduce that r_i are Borel functions for $i \in \{1,2\}$. We also note $r_i(b) \le \lambda_i$ for $b \in \mathbf{R}$ and $i \in \{1,2\}$. Let $C = \{b : \mu_1 \{b\} > 0\}$ and notice that C is countable. Moreover, we define

$$Q_i = \mathbf{R} \cap \{b : r_i(b) > 0\} \quad \text{for } i \in \{1,2\}, \qquad B = \{b : r_1(b) > r_2(b)\}.$$

Our two estimates below rest on the basic fact that

$$\nu_i \, \mathbf{B}(b, r_i(b)) \ge (2\boldsymbol{\gamma}(\dim V_i))^{-1} \mu_i(\mathbf{B}(b, r_i(b)))^{1-1/\dim V_i}$$

whenever $\lambda_i < b \in Q_i$ and $i \in \{1,2\}$; in fact, we note [Men16a, 8.12, 13, 9.9] and apply, for small $s > 0$, 4.1 with m, V, s, r, q, $f(x)$, and ε replaced by $\dim V_i$, V_i, 0, s, ∞, $\sup\{r_i(b) + s - |f(x) - b|, 0\}$, and $(2\boldsymbol{\gamma}(\dim V_i))^{-1}$.

Next, the following two estimates will be proven

$$\mathscr{L}^1(Q_2 \cap \{b : b > \lambda_2\}) \le \Delta_3 \, \omega_2(I), \quad \mathscr{L}^1(B \cap \{b : b > \lambda_1\}) \le \Delta_4 \, \omega_1(I),$$

where $\Delta_3 = 2^{m+1} \Delta_2 \boldsymbol{\gamma}(m-1)^{m-2}$ and $\Delta_4 = 2^{m(m-1)+3} \boldsymbol{\gamma}(m)^{m-1} \Delta_1$. Whenever $\lambda_2 < b \in Q_2$, the basic fact and Hölder's inequality yield

$$\Delta_2^{-1} r_2(b) \le \mu_2(\mathbf{B}(b, r_2(b)))^{1/(m-1)} \le (2\boldsymbol{\gamma}(m-1))^{m-2} \omega_2 \, \mathbf{B}(b, r_2(b)),$$

whence we deduce the first estimate by Vitali's covering theorem (see [Fed69, 2.8.5, 8] with $\delta = \operatorname{diam}$ and $\tau = 3/2$). To similarly prove the second estimate suppose $\lambda_1 < b \in B$. We first notice that $b > r_1(b) > r_2(b)$ yields

$$\begin{aligned} \mu_2(\mathbf{B}(b, r_1(b)))^{1/(m-1)} &\le \Delta_2^{-1} r_1(b) \\ &\le 2^{-1} (2\boldsymbol{\gamma}(m))^{1/(1-m)} \mu_1(\mathbf{B}(b, r_1(b))^{1/m} < \infty. \end{aligned}$$

Combining this estimate with the following consequence of the basic fact,

$$\begin{aligned} &\mu_1(\mathbf{B}(b, r_1(b)))^{1/m} \le (2\boldsymbol{\gamma}(m))^{1/(m-1)} \nu_1(\mathbf{B}(b, r_1(b)))^{1/(m-1)} \\ &\qquad \le (2\boldsymbol{\gamma}(m))^{1/(m-1)} \big(\alpha(\mathbf{B}(b, r_1(b)))^{1/(m-1)} + \mu_2(\mathbf{B}(b, r_1(b)))^{1/(m-1)} \big), \end{aligned}$$

we first obtain

$$2^{-1} \mu_1(\mathbf{B}(b, r_1(b)))^{1/m} \le (2\boldsymbol{\gamma}(m))^{1/(m-1)} \alpha(\mathbf{B}(b, r_1(b)))^{1/(m-1)},$$

and then, using Hölder's inequality,

$$\Delta_1^{-1} r_1(b) \le \mu_1(\mathbf{B}(b, r_1(b)))^{1/m} \le 2^{m(m-1)} \boldsymbol{\gamma}(m)^{m-1} \, \omega_1 \, \mathbf{B}(b, r_1(b)).$$

Vitali's covering theorem now yields the second estimate.

We now define $Y = C \cup Q_1$. As $Q_1 \subset B \cup Q_2 \subset I$, the preceding two estimates imply that the asserted property of Y may be established by proving

$$\mu_1(\mathbf{R} \sim (C \cup Q_1)) = 0.$$

For this purpose, we define

$$\Upsilon = (\operatorname{spt}\mu_1) \cap \left\{ y : y > 0 \text{ and } \limsup_{s \to 0+} \frac{\nu_1 \mathbf{B}(y,s)}{\mu_1(\mathbf{B}(y,s))^{1-1/m}} \geq (2\boldsymbol{\gamma}(m))^{-1} \right\}$$

and notice that $\mu_1(\Upsilon \sim C) = 0$ by [Fed69, 2.8.9, 18, 9.5]. The assertion then follows verifying $(\operatorname{spt}\mu_1) \sim \Upsilon \subset Q_1 \cup \{0\}$; in fact, for $0 < b \in (\operatorname{spt}\mu_1) \sim \Upsilon$ and small $s > 0$, we note [Men16a, 8.12, 13, 9.9] and apply 4.1 with V, s, r, q, $f(x)$, and ε replaced by V_1, 0, s, ∞, $\sup\{s - |f(x) - b|, 0\}$, and $(2\boldsymbol{\gamma}(m))^{-1}$, to infer $r_1(b) > 0$. □

4.6 Remark. In the situation of the preceding theorem, we notice that, if V_1 is indecomposable of type $\{f\}$, then $\operatorname{diam}\operatorname{spt} f_\# \|V_1\| \leq \mathscr{L}^1(Y)$; in fact, applying 3.14 (3) with V and Y replaced by V_1 and $\mathbf{R} \sim Y$, this follows from 3.14 (1). If additionally $\|V_1\| \{x : f(x) \leq y\} > 0$ for $0 < y < \infty$, then

$$\operatorname{diam}\operatorname{spt} f_\# \|V_1\| = \|V_1\|_{(\infty)}(f).$$

4.7 Remark. We notice that $\mathbf{T}_{\operatorname{Bdry} U}(V_i) = \mathbf{T}(V_i) \cap \{f : f \geq 0\}$ in case $U = \mathbf{R}^n$ by [Men16a, 9.2].

Similarly, to prepare for the case without boundary, we collect a set of hypotheses on density and first variation that is assumed to hold in 4.9.

4.8. Suppose U is an open subset of $\mathbf{R}^n$, V is a varifold in U, $2 \leq m = \dim V$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, and, either $m = 2$ and $\psi = \|\delta V\|$, or $m > 2$, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, $\mathbf{h}(V, \cdot) \in \mathbf{L}_{m-1}^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$, and $\psi = \|V\| \llcorner |\mathbf{h}(V, \cdot)|^{m-1}$.

4.9 Corollary. *Suppose U, V, and ψ are as in 4.8, $\Gamma = 2^{m+3} m \boldsymbol{\gamma}(m)$, $f \in \mathbf{T}_{\operatorname{Bdry} U}(V)$, V is indecomposable of type $\{f\}$, and*

$$|V \mathbf{D} f(x)| \leq 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

Then, there holds

$$\operatorname{diam}\operatorname{spt} f_\# \|V\| \leq \Gamma\big(\|V\|(\{x : f(x) > 0\})^{1/m} + \boldsymbol{\gamma}(m)^{m-1} \psi \{x : f(x) > 0\}\big).$$

Proof. With a possibly larger number Γ, this follows from 4.5 and 4.6 with $V_1 = V$ and $V_2 = 0$. We verify the eligibility of the present number Γ by noting that, for $V_2 = 0$, we can take $\Delta_4 = 2^{m+2} \Delta_1 \boldsymbol{\gamma}(m)^{m-1}$ in the proof of 4.5; in fact,

$$\Delta_1^{-1} r_1(b) \leq \mu_1(\mathbf{B}(b, r_1(b)))^{1/m} \leq (2\boldsymbol{\gamma}(m))^{m-1} \omega_1 \mathbf{B}(b, r_1(b))$$

whenever $\lambda_1 < b \in B$ by the basic fact and Hölder's inequality. □

4.10 Remark. As in 4.6, we note that in case $\|V\| \{x : f(x) \leq y\} > 0$ for $0 < y < \infty$ we have $\operatorname{diam}\operatorname{spt} f_\# \|V\| = \|V\|_{(\infty)}(f)$.

5 Examples

The purpose of the present section is to construct submanifolds (see 5.5 and 5.7) to provide the sharpness part of Theorem C and Theorem D in the introduction. As preparations, we firstly list an arithmetic formula and terminology for cylinders (see 5.1–5.2) and then indicate a procedure to smooth out corners (see 5.3–5.4).

5.1. If $0 \leq x < 1$ and i is a nonnegative integer, then

$$\sum_{j=i}^{\infty} (j+1)x^j = (1-x)^{-2}\big((i+1)x^i - ix^{i+1}\big).$$

5.2. Whenever $u \in \mathbf{S}^{n-1}$, $a \in \mathbf{R}^n$, $0 < r < \infty$, and $0 \leq h \leq \infty$, we define

$$Z(a,r,u,h) = \mathbf{R}^n \cap \big\{x : |x-a|^2 = ((x-a) \bullet u)^2 + r^2, 0 \leq (x-a) \bullet u \leq h\big\}.$$

5.3 Lemma. *Suppose n is an integer, $n \geq 2$, Y is an $n-1$ dimensional submanifold-with-boundary of class ∞ of $\mathbf{R}^{n-1}$, ∂Y is connected and compact, $\varepsilon > 0$, and, identifying $\mathbf{R}^n \simeq \mathbf{R}^{n-1} \times \mathbf{R}$, the subsets Q and U of $\mathbf{R}^n$ satisfy*

$$Q \simeq Y \times \{t : 0 \leq t < \infty\}, \quad U \simeq (Y \cap \{y : \operatorname{dist}(y, \partial Y) < \varepsilon\}) \times \{z : 0 \leq z < \varepsilon\}.$$

Then, there exists a properly embedded n dimensional submanifold-with-boundary M of class ∞ of $\mathbf{R}^n$ such that ∂M is connected and $M \sim U = Q \sim U$.

Proof. As ∂Y is compact, we adapt [GT01, Lemma 14.16] to construct $\delta > 0$ such that the function $f : G \to \mathbf{R}$, with $G = \mathbf{R}^{n-1} \cap \{y : \operatorname{dist}(y, \partial Y) < \delta\}$ and

$$f(y) = \operatorname{dist}(y, \partial Y) \quad \text{if } y \in Y, \qquad f(y) = -\operatorname{dist}(y, \partial Y) \quad \text{else},$$

for $y \in G$, is of class ∞ and satisfies $|\operatorname{D} f(y)| = 1$ for $y \in G$. Then, defining $g : G \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ by $g(y,z) = (f(y), z)$ for $y \in G$ and $z \in \mathbf{R}$ and noting $\operatorname{im} \operatorname{D} g(y,z) = \mathbf{R} \times \mathbf{R}$ for $y \in G$ and $z \in \mathbf{R}$, the assertion reduces (e.g., by [Fed69, 3.1.18]) to the case $n = 2$ and $Y = \{y : 0 \leq y < \infty\}$ which is elementary. ☐

5.4 Remark. By induction on n, the preceding lemma implies the following proposition: *If $2 \leq n \in \mathbf{Z}$, $-\infty < a_k < b_k < \infty$ for $k = 1, \ldots, n$, $\varepsilon > 0$, and*

$$Q = \mathbf{R}^n \cap \{x : a_k \leq x \bullet e_k \leq b_k \text{ for } k = 1, \ldots, n\},$$
$$U = Q \cap \big\{x : \operatorname{dist}(x, Q \cap \{\chi : \operatorname{card}\{k : \chi \bullet e_k = a_k \text{ or } \chi \bullet e_k = b_k\} \geq 2\}) < \varepsilon\big\},$$

where $e_1, \ldots, e_n$ form the standard base of $\mathbf{R}^n$, then there exists a properly embedded n dimensional submanifold-with-boundary M of $\mathbf{R}^n$ of class ∞ such that ∂M is connected and $M \sim U = Q \sim U$.

In the next example related to the sharpness of Theorem C, we are in fact able to control the second fundamental form $\mathbf{b}(M, \cdot)$ instead of the mean curvature.

5.5 Theorem. *Whenever m and n are positive integers, $2 \leq m < n$, and $m - 1 < q < m$, there exists a bounded, connected m dimensional submanifold M of class ∞ of $\mathbf{R}^n$ such that*

$$\textstyle\int_M \|\mathbf{b}(M,x)\|^p \, \mathrm{d}\mathscr{H}^m x < \infty \quad \textit{whenever } 1 \leq p < q,$$
$$\textstyle\int_M \operatorname{Tan}(M,x)_\natural \bullet \operatorname{D}\theta(x) \, \mathrm{d}\mathscr{H}^m x = -\int_M \mathbf{h}(M,x) \bullet \theta(x) \, \mathrm{d}\mathscr{H}^m x$$

for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$, and such that $A = (\operatorname{Clos} M) \sim M$ satisfies

$$\mathscr{H}^{m-q}(A) = \boldsymbol{\alpha}(m-q)2^{1-m+q}, \qquad \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \textit{for } a \in A.$$

Proof. We assume $n = m + 1$ and let $d = m - q$. Whenever J is a compact subinterval of $\mathbf{R}$, we denote by $\Phi(J)$ the family consisting of the two disjoint subintervals

$$\big\{t : \inf J \le t \le \inf J + 2^{-1/d} \operatorname{diam} J\big\}, \quad \big\{t : \sup J - 2^{-1/d} \operatorname{diam} J \le t \le \sup J\big\}$$

of J. Letting $G_0 = \{\{t : -1/2 \le t \le 1/2\}\}$, we define $G_i = \bigcup\{\Phi(S) : S \in G_{i-1}\}$ for every positive integer i and $C = \bigcap_{i=0}^{\infty} \bigcup G_i$. By [Fed69, 2.10.28], there holds

$$\mathscr{H}^d(C) = \boldsymbol{\alpha}(d) 2^{-d}.$$

For every nonnegative integer i, we let $r_i = 2^{-i/d}$ and $s_i = \sum_{j=i}^{\infty} (j+1) r_j$, hence $\operatorname{diam} J = r_i$ whenever $J \in G_i$ and, using 5.1, we compute

$$s_i = (1 - 2^{-1/d})^{-1} \big(i + (1 - 2^{-1/d})^{-1}\big) r_i.$$

Suppose $e_1, \ldots, e_n$ form the standard base of $\mathbf{R}^n$. We observe that 5.3 may be employed to construct a subset N of the cube

$$\mathbf{R}^n \cap \big\{x : 0 < x \bullet e_n < 1 \text{ and } |x \bullet e_k| < \tfrac{1}{2} \text{ for } k = 1, \ldots, n-1\big\}$$

such that its union with (see 5.2)

$$Z\big(\tfrac{r_1 - 1}{2} e_1 + e_n, \tfrac{r_1}{4}, e_n, \infty\big) \cup Z\big(\tfrac{1 - r_1}{2} e_1 + e_n, \tfrac{r_1}{4}, e_n, \infty\big) \cup Z\big(0, \tfrac{1}{4}, -e_n, \infty\big)$$

is a properly embedded, connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$. In particular, there exists $0 \le \kappa < \infty$ satisfying

$$\mathscr{H}^m(N) \le \kappa, \quad \sup \operatorname{im} \|\mathbf{b}(N, \cdot)\| \le \kappa.$$

With $N(a, r) = \mathbf{R}^n \cap \{x : r^{-1}(x - a) \in N\}$ for $a \in \mathbf{R}^n$ and $0 < r < \infty$, we use

$$X_i = \big\{Z\big(\tfrac{\sup J + \inf J}{2} e_1 + (s_0 - s_i) e_n, \tfrac{r_i}{4}, e_n, i r_i\big) : J \in G_i\big\},$$
$$\Psi_i = \big\{N\big(\tfrac{\sup J + \inf J}{2} e_1 + (s_0 - s_i + i r_i) e_n, r_i\big) : J \in G_i\big\},$$

$H_i = \bigcup_{j=0}^{i} \bigcup (X_j \cup \Psi_j)$, and $M_i = \{x : x \in H_i \text{ or } x - 2(x \bullet e_n) e_n \in H_i\}$, to define

$$M = \bigcup \{M_i : i \text{ is a nonnegative integer}\}.$$

Clearly, M is a connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$ and

$$A = \mathbf{R}^n \cap \{x : x \bullet e_1 \in C, \ |x \bullet e_n| = s_0, \text{ and } x \bullet e_k = 0 \text{ for } k = 2, \ldots, n-1\},$$

where $A = (\operatorname{Clos} M) \sim M$.

Next, we will show the following assertion. *There holds*

$$\mathscr{H}^m(M \cap \mathbf{B}(a, r)) \le 2^{2 + m/d} (1 - 2^{1 - m/d})^{-2} (m \boldsymbol{\alpha}(m) + \kappa)(i+1)^{1 + d - m} r^m$$

whenever $a \in A$, i is a nonnegative integer, and $s_{i+1} \le r \le s_i$. For this purpose, we let $I = \{t : |t - a \bullet e_1| \le r\}$ and firstly estimate

$$\lambda_i = \operatorname{card}\{J : I \cap J \ne \varnothing, J \in G_i\} \le 4 (1 - 2^{-1/d})^{-2} (i+1)^d;$$

in fact, G_i is special for $\{t : -1/2 \le t \le 1/2\}$ by [Fed69, 2.10.28 (2) (4)], hence

$$\operatorname{card}\{J : I \supset J \in G_i\} r_i^d \le (2r)^d \le 2(1 - 2^{-1/d})^{-2}(i+1)^d r_i^d.$$

Since $2^{-i} \sum_{j=i}^{\infty} 2^j (j+1) r_j^m \le (1 - 2^{1-m/d})^{-2}(i+1) r_i^m$ by 5.1, we estimate

$$\mathscr{H}^m(M \cap \mathbf{B}(a,r)) \le \lambda_i (m\boldsymbol{\alpha}(m) + \kappa)(1 - 2^{1-m/d})^{-2}(i+1) r_i^m.$$

Hence, together with the first estimate and $r_i \le 2^{1/d}(1 - 2^{-1/d})(i+1)^{-1} s_{i+1}$, the assertion follows.

The assertion of the preceding paragraph implies

$$\boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \text{whenever } a \in A$$

and, as $\operatorname{Clos} M$ is compact, also $\mathscr{H}^m(M) < \infty$. Noting

$$\textstyle\int_{Z(a,r,u,h)} \|\mathbf{b}(Z(a,r,u,h),x)\|^p \, \mathrm{d}\mathscr{H}^m x \le m\boldsymbol{\alpha}(m) r^{m-p}(h/r),$$
$$\textstyle\int_{N(a,r)} \|\mathbf{b}(N(a,r),x)\|^p \, \mathrm{d}\mathscr{H}^m x \le \kappa^{1+p} r^{m-p}$$

for $a \in \mathbf{R}^n$, $0 < r < \infty$, $u \in \mathbf{S}^{n-1}$, and $0 \le h \le \infty$, we estimate

$$\textstyle\int_M \|\mathbf{b}(M,x)\|^p \, \mathrm{d}\mathscr{H}^m x \le 2(\kappa^{1+p} + 4m\boldsymbol{\alpha}(m)) \sum_{i=0}^{\infty} (i+1) 2^{i(1+(p-m)/d)} < \infty$$

whenever $1 \le p < q$. Since $\mathscr{H}^{m-1}(\partial M_i) \le m\boldsymbol{\alpha}(m) 2^{i(1+(1-m)/d)}$ for every positive integer i, the conclusion now readily follows. ☐

5.6 Remark. The preceding theorem answers the third question posed in [Sch16, Section A].

Similarly, in the next example for Theorem D, we in fact control the summability of the second fundamental form of the submanifold.

5.7 Theorem. *Whenever m and n are positive integers and $3 \le m < n$, there exists a bounded, connected m dimensional submanifold M of class ∞ of $\mathbf{R}^n$ such that*

$$\textstyle\int_M \|\mathbf{b}(M,x)\|^p \, \mathrm{d}\mathscr{H}^m x < \infty \quad \textit{whenever } 1 \le p < m-1,$$
$$\textstyle\int_M \operatorname{Tan}(M,x)_\natural \bullet \mathrm{D}\,\theta(x) \, \mathrm{d}\mathscr{H}^m x = -\int_M \mathbf{h}(M,x) \bullet \theta(x) \, \mathrm{d}\mathscr{H}^m x$$

for $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$, and such that

$$\mathscr{H}^m((\operatorname{Clos} M) \sim M) = 1, \qquad \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M, a) = 0 \quad \textit{for } a \in (\operatorname{Clos} M) \sim M.$$

Proof. We assume $n = m+1$. With each $A \subset \mathbf{R}^n$, we associate sets

$$A(a,r) = \mathbf{R}^n \cap \big\{x : r^{-1}(x-a) \in A\big\} \quad \text{for } a \in \mathbf{R}^n \text{ and } 0 < r < \infty.$$

Let $e_1, \ldots, e_n$ denote the standard base vectors of $\mathbf{R}^n$. We define

$$D = \mathbf{R}^n \cap \{x : x \bullet e_n = 0 \text{ and } |x| < 1\},$$
$$H = \mathbf{R}^n \cap \{u : \text{for some } k \in \{1, \ldots, m\},\ u = e_k \text{ or } u = -e_k\}.$$

We define $\gamma : \mathbf{R}^n \to \mathbf{2}^H$ by

$$\gamma(x) = H \sim \{u : \text{for some } k,\ x \bullet e_k = 1 \text{ and } u = e_k \text{ or } x \bullet e_k = 0 \text{ and } u = -e_k\}$$

for $x \in \mathbf{R}^n$. Notice that 5.3 may be used to construct a subset R of the cylinder

$$\mathbf{R}^n \cap \big\{x : 0 \leq x \bullet e_n < \tfrac{1}{4} \text{ and } |x - (x \bullet e_n)e_n| < \tfrac{1}{4}\big\}$$

such that $R \cup \big(\mathbf{R}^n \cap \{x : x \bullet e_n = 0\} \sim D(0, \frac{1}{4})\big) \cup Z(\frac{e_n}{4}, \frac{1}{8}, e_n, \infty)$, see 5.2, is a properly embedded, connected m dimensional submanifold of $\mathbf{R}^n$ of class ∞. Let S denote the reflection $\{x : x - 2(x \bullet e_n)e_n \in R\}$ of R along $\mathbf{R}^n \cap \{x : x \bullet e_n = 0\}$. Considering the submanifold furnished by 5.4 applied with $\varepsilon = \frac{1}{16}$ and (a_k, b_k) replaced by $(-\frac{3}{8}, \frac{3}{8})$ if $k < n$ and $(-\frac{1}{4}, \frac{1}{4})$ if $k = n$, we observe that 5.3 may also be employed to construct, for $G \subset H$, a subset N_G of the cuboid

$$\mathbf{R}^n \cap \big\{x : |x \bullet e_n| \leq \tfrac{1}{4} \text{ and } |x \bullet e_k| < \tfrac{1}{2} \text{ for } k = 1, \ldots, m\big\}$$

such that N_G contains $D(\frac{e_n}{4}, \frac{1}{4}) \cup D(-\frac{e_n}{4}, \frac{1}{4})$ and such that

$$N_G \cup \bigcup \big\{Z(\tfrac{u}{2}, \tfrac{1}{8}, u, \infty) : u \in G\big\}$$

is a properly embedded, connected m dimensional submanifold of $\mathbf{R}^n$ of class ∞. Clearly, there exists $0 \leq \kappa < \infty$ satisfying

$$\mathscr{H}^m(R) \leq \kappa, \quad \sup \operatorname{im} \|\mathbf{b}(R, \cdot)\| \leq \kappa, \quad \mathscr{H}^m(N_G) \leq \kappa, \quad \sup \operatorname{im} \|\mathbf{b}(N_G, \cdot)\| \leq \kappa$$

whenever $G \subset H$.

In this paragraph, we define various objects for each positive integer i. Let $r_i = 2^{-i(i+1)}$ and define C_i to consist of those $x \in \mathbf{R}^n$ such that

$$x \bullet e_n = 2^{-i}, \quad 0 \leq x \bullet e_k \leq 1, \quad \text{and} \quad 2^{i-1} x \bullet e_k \in \mathbf{Z}$$

for $k = 1, \ldots, m$. We have $\operatorname{card} C_i = (2^{i-1} + 1)^m$. Then, noting $\frac{r_i}{2} < 2^{-i}$, we define $X_i(u)$, for $u \in H$, to be the family consisting of the sets

$$Z\big(x + \tfrac{r_i}{2} u, \tfrac{r_i}{8}, u, 2^{-i} - \tfrac{r_i}{2}\big)$$

corresponding to $x \in C_i$ with $u \in \gamma(x)$. We have $\operatorname{card} X_i(u) = (2^{i-1}+1)^{m-1} 2^{i-1}$. With $C_0 = \varnothing$, we define Ψ_i to be the family consisting of the sets

$$N_{\gamma(x)}(x, r_i) \sim D(x - \tfrac{r_i}{4} e_n, \tfrac{r_{i+1}}{4})$$

corresponding to $x \in C_i$ with $x + 2^{-i} e_n \notin C_{i-1}$ as well as the sets

$$N_{\gamma(x)}(x, r_i) \sim \big(D(x + \tfrac{r_i}{4} e_n, \tfrac{r_i}{4}) \cup D(x - \tfrac{r_i}{4} e_n, \tfrac{r_{i+1}}{4})\big)$$

corresponding to $x \in C_i$ with $x + 2^{-i} e_n \in C_{i-1}$. Noting $\frac{r_i}{4} + \frac{3 r_{i+1}}{4} < r_i \leq 2^{-i-1}$, we also define Ω_i to be the family consisting of the sets

$$S\big(x - \tfrac{r_i}{4} e_n, r_{i+1}\big) \cup Z\big(x - (\tfrac{r_i}{4} + \tfrac{r_{i+1}}{4}) e_n, \tfrac{r_{i+1}}{8}, -e_n, 2^{-i-1} - \tfrac{r_i}{4} - \tfrac{3 r_{i+1}}{4}\big) \\ \cup R\big(x - (2^{-i-1} - \tfrac{r_{i+1}}{4}) e_n, r_{i+1}\big)$$

corresponding to $x \in C_i$. Clearly, we have $\operatorname{card} \Psi_i = \operatorname{card} C_i = \operatorname{card} \Omega_i$. Finally, we let $M_i = \bigcup_{j=1}^{i} \bigcup_{u \in H} \bigcup (X_j(u) \cup \Psi_j \cup \Omega_j)$.

Now, we define $M = \bigcup_{i=1}^{\infty} M_i$ and notice that M is a bounded, connected m dimensional submanifold of class ∞ of $\mathbf{R}^n$ such that

$$(\operatorname{Clos} M) \sim M = \mathbf{R}^n \cap \{x : x \bullet e_n = 0 \text{ and } 0 \leq x_k \leq 1 \text{ for } k = 1, \ldots, m\}.$$

Since we have $\mathscr{H}^{m-1}(\partial D(a,r)) = m\boldsymbol{\alpha}(m)r^{m-1}$ and

$$\begin{gathered}\mathscr{H}^m(Z(a,r,u,h)) = m\boldsymbol{\alpha}(m)r^{m-1}h, \quad \sup \operatorname{im} \|\mathbf{b}(Z(a,r,u,h),\cdot)\| = r^{-1}, \\ \mathscr{H}^m(R(a,r)) \le \kappa r^m, \quad \sup \operatorname{im} \|\mathbf{b}(R(a,r),\cdot)\| \le \kappa r^{-1}, \\ \mathscr{H}^m(N_G(a,r)) \le \kappa r^m, \quad \sup \operatorname{im} \|\mathbf{b}(N_G(a,r),\cdot)\| \le \kappa r^{-1}\end{gathered}$$

whenever $a \in \mathbf{R}^n$, $0 < r < \infty$, $u \in \mathbf{S}^{n-1}$, $0 < h < \infty$, and $G \subset H$, one may use the fact that $\sum_{i=1}^\infty 2^{i\lambda} r_i^\varepsilon < \infty$ whenever $\lambda \in \mathbf{R}$ and $\varepsilon > 0$ to conclude

$$\begin{gathered}\lim_{i\to\infty} 2^{im}\mathscr{H}^m(M \sim M_{i-1}) = 0, \quad \lim_{i\to\infty} \mathscr{H}^{m-1}(\partial M_i) = 0, \\ \textstyle\int_M \|\mathbf{b}(M,x)\|^p \,\mathrm{d}\mathscr{H}^m\, x < \infty \quad \text{for } 1 \le p < m-1,\end{gathered}$$

whence we readily deduce the asserted conclusion. □

5.8 Remark. The construction bears some similarities with [Men09, 1.2].

6 Lower density bounds

The purpose of this section is to establish (see 6.7) the main part of Theorem C′ in the introduction. The key to this result are conditional lower density ratio bounds (see 6.3–6.6) based on the Sobolev-Poincaré inequality (see 4.5). Moreover, to treat small positive density ratios, a compactness lemma (see 6.1) is employed.

6.1 Lemma. *Suppose $1 \le M < \infty$.*

Then, there exists a positive, finite number Γ with the following property.

If m and n are positive integers, $m \le n \le M$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{V}_m(\mathbf{U}(a,r))$, $\|\delta V\|\,\mathbf{U}(a,r) \le \Gamma^{-1} r^{m-1}$,

$$\|V\|\,\mathbf{B}(a,s) \ge M^{-1}\boldsymbol{\alpha}(m)s^m \quad \textit{whenever } 0 < s < r,$$

and $\boldsymbol{\Theta}^m(\|V\|,x) \ge 1$ for $\|V\|$ almost all x, then

$$\|V\|\,\mathbf{U}(a,r) \ge (1-M^{-1})\boldsymbol{\alpha}(m)r^m.$$

Proof. If the lemma were false for some M, there would exist sequences Γ_i with $\Gamma_i \to \infty$ as $i \to \infty$ and sequences m_i, n_i, a_i, r_i, and V_i showing that $\Gamma = \Gamma_i$ does not have the asserted property.

We could assume for some positive integers m and n that $m \le n \le M$, $m = m_i$, $n = n_i$, $a_i = 0$, and $r_i = 1$ whenever i is a positive integer. Defining $V \in \mathbf{V}_m(\mathbf{R}^n \cap \mathbf{U}(0,1))$ to be the limit of some subsequence of V_i, we would obtain

$$\|V\|\,\mathbf{U}(0,1) \le (1-M^{-1})\boldsymbol{\alpha}(m), \quad 0 \in \operatorname{spt}\|V\|, \quad \delta V = 0.$$

Finally, using Allard [All72, 5.6, 8.6, 5.1 (2)], we would then conclude that

$$\begin{gathered}\boldsymbol{\Theta}^m(\|V\|,x) \ge 1 \quad \text{for } \|V\| \text{ almost all } x, \\ \boldsymbol{\Theta}^m(\|V\|,0) \ge 1, \quad \|V\|\,\mathbf{U}(0,1) \ge \boldsymbol{\alpha}(m),\end{gathered}$$

a contradiction. □

6.2 Remark. The pattern of the preceding proof is that of [Men16a, 7.3].

The conditional lower density bounds follow rather immediately from the Sobolev-Poincaré inequality (see 4.5) and its corollary (see 4.9), respectively.

6.3 Lemma. *Suppose m and n are positive inters, $2 \leq m \leq n$, U is an open subset of $\mathbf{R}^n$, $V_1 \in \mathbf{V}_m(U)$ and $V_2 \in \mathbf{V}_{m-1}(U)$ satisfy the conditions of 4.4, V_1 is indecomposable of type $\mathscr{D}(U, \mathbf{R})$,*

$$V_2 = 0 \quad \textit{if } m = 2, \qquad \|\delta V_1\| \leq \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\| \quad \textit{if } m > 2,$$
$$\|\delta V_2\| \textit{ is absolutely continuous with respect to } \|V_2\| \quad \textit{if } m > 3,$$
$$\phi_i \textit{ are associated to } V_1 \textit{ as in 4.4 for } i \in \{1, 2\},$$

$a \in \operatorname{spt} \|V_1\|$, $0 < r < \infty$, $\mathbf{B}(a, r) \subset U$, and $(\operatorname{spt} \|V_1\|) \sim \mathbf{U}(a, r) \neq \varnothing$.
Then,

$$\Gamma_{4.5}(m)^{-1} r \leq \|V_1\|(\mathbf{U}(a, r))^{1/m} + \phi_1 \, \mathbf{U}(a, r) + \|V_2\|(\mathbf{U}(a, r))^{1/(m-1)} + \phi_2 \, \mathbf{U}(a, r).$$

Proof. In view of [MS17, 4.6 (1)], [Men16a, 9.2, 4], and 3.5, one may apply 4.5 and 4.6 with $f(x)$ replaced by $\sup\{r - |x - a|, 0\}$. □

We also include a version without boundary with more explicit constants.

6.4 Lemma. *Suppose U, V, and ψ are as in 4.8, $a \in \operatorname{spt} \|V\|$, $0 < r < \infty$, $\mathbf{B}(a, r) \subset U$, $(\operatorname{spt} \|V\|) \sim \mathbf{U}(a, r) \neq \varnothing$, and V is indecomposable of type $\mathscr{D}(U, \mathbf{R})$.*
Then,

$$2^{-m-3} m^{-1} \boldsymbol{\gamma}(m)^{-1} r \leq \|V\|(\mathbf{U}(a, r))^{1/m} + \boldsymbol{\gamma}(m)^{m-1} \psi \, \mathbf{U}(a, r).$$

Proof. In view of [MS17, 4.6 (1)], [Men16a, 9.2, 4], and 3.5, one may apply 4.9 and 4.10 with $f(x)$ replaced by $\sup\{r - |x - a|, 0\}$. □

6.5 Remark. If either $m < n$ or $m = n = 2$, considering small spheres or small disks, respectively, shows that neither the nonemptyness hypothesis nor the indecomposability hypothesis may be omitted.

6.6 Remark. If $m = 1$ and V otherwise is as in 4.8, then $r \leq \|V\| \, \mathbf{U}(a, r)$; in fact, the indecomposability hypothesis implies $\{x : |x - a| = s\} \cap \operatorname{spt} \|V\| \neq \varnothing$ for $0 < s < r$, whence the inequality follows, since $\mathscr{H}^1 \llcorner \operatorname{spt} \|V\| \leq \|V\|$ by [All72, 3.5 (1b)] and [Men16a, 4.8 (4)].

Apart of the postscript, Theorem C′ is the content of the first item of the next theorem. The remaining items discuss, for special dimensions, a slightly more general boundary condition than Hypothesis 3 by omitting the differential geometric condition that the boundary should not have boundary itself.

6.7 Theorem. *Suppose m and n are positive inters, $2 \leq m \leq n$, U is an open subset of $\mathbf{R}^n$, $V_1 \in \mathbf{V}_m(U)$, $V_2 \in \mathbf{V}_{m-1}(U)$,*

$$\boldsymbol{\Theta}^{\dim V_i}(\|V_i\|, x) \geq 1 \quad \textit{for } \|V_i\| \textit{ almost all } x \textit{ and } i \in \{1, 2\},$$
$$\|\delta V_1\| \textit{ is a Radon measure}, \quad \|\delta V_2\| \textit{ is a Radon measure},$$

V_1 is indecomposable of type $\mathscr{D}(U, \mathbf{R})$, and $\lambda = 2^{-4} \boldsymbol{\alpha}(2)^{-1} \boldsymbol{\gamma}(2)^{-2}$.
Then, the following three statements hold.

(1) *If* $m-1 \le p < m$, $\|\delta V_1\| \le \|V_1\| \llcorner |\mathbf{h}(V_1,\cdot)| + \|V_2\|$, $\|\delta V_2\|$ *is absolutely continuous with respect to* $\|V_2\|$, $\mathbf{h}(V_1,\cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V_1\|, \mathbf{R}^n)$, *and, in case* $m > 2$, *additionally* $\mathbf{h}(V_2,\cdot) \in \mathbf{L}_{p-1}^{\mathrm{loc}}(\|V_2\|, \mathbf{R}^n)$, *then*

$$\mathscr{H}^{m-p}\big(\operatorname{spt}\|V_1\| \cap \big\{x : \mathbf{\Theta}_*^m(\|V_1\|, x) < 1 \text{ and } \mathbf{\Theta}_*^{m-1}(\|V_2\|, x) < 1\big\}\big) = 0.$$

(2) *If* $m = 3$, $\|\delta V_1\| \le \|V_1\| \llcorner |\mathbf{h}(V_1,\cdot)| + \|V_2\|$, $\mathbf{h}(V_1,\cdot) \in \mathbf{L}_2^{\mathrm{loc}}(\|V_1\|, \mathbf{R}^n)$, *and*

$$A = \operatorname{spt}\|V_1\| \cap \big\{x : \mathbf{\Theta}_*^3(\|V_1\|, x) < 1 \text{ and } \mathbf{\Theta}_*^2(\|V_2\|, x) < \lambda\big\},$$

then

$$\mathscr{H}^1 \llcorner A \le \sup\big\{2^7 \boldsymbol{\gamma}(2)^2, 2^2 \Gamma_{4.5}(3)\big\} \|\delta V_2\| \llcorner A.$$

(3) *If* $m = 2$, $V_2 = 0$, *and* $A = \operatorname{spt}\|V_1\| \cap \big\{x : \mathbf{\Theta}_*^2(\|V_1\|, x) < \lambda\big\}$, *then*

$$\mathscr{H}^1 \llcorner A \le 2^8 \boldsymbol{\gamma}(2)^2 \|\delta V_1\| \llcorner A.$$

In particular, in all cases, $\mathscr{H}^m \llcorner \operatorname{spt}\|V_1\| \le \|V_1\|$.

Proof. Firstly, we notice that in case of (1) we may assume $V_2 = 0$ if $m = 2$; in fact, since $\mathbf{\Theta}^1(\|V_2\|, x) \ge 1$ for $x \in \operatorname{spt}\|V_2\|$ by [Men16a, 4.8 (4)], we may otherwise replace U by $U \sim \operatorname{spt}\|V_2\|$ by 3.6.

Secondly, we notice for $i \in \{1,2\}$ that $V_i \in \mathbf{RV}_{\dim V_i}(U)$ by Allard [All72, 5.5 (1)] and that

$$\|V_i\| = \mathscr{H}^{\dim V_i} \llcorner \mathbf{\Theta}^{\dim V_i}(\|V_i\|, \cdot)$$

by [All72, 3.5 (1b)]. Taking $p = m-1$ in case of (2) or (3), we define

$$\psi_1 = \|V_1\| \llcorner |\mathbf{h}(V_1,\cdot)|^p \text{ in case of (1) or (2)}, \quad \psi_1 = \|\delta V_1\| \text{ in case of (3)},$$
$$\psi_2 = \|V_2\| \llcorner |\mathbf{h}(V_2,\cdot)|^{p-1} \text{ in case of (1)}, \quad \psi_2 = \|\delta V_2\| \text{ in case of (2) or (3)}.$$

Taking $\varepsilon = \inf\big\{2^{-7}\boldsymbol{\gamma}(2)^{-2}, 2^{-2}\Gamma_{4.5}(3)^{-1}\big\}$ and

$$\lambda_1 = 1 \text{ in case of (1) or (2)}, \quad \lambda_1 = \lambda \text{ in case of (3)},$$
$$\lambda_2 = 1 \text{ in case of (1) or (3)}, \quad \lambda_2 = \lambda \text{ in case of (2)},$$
$$\delta = 2^{-m}\boldsymbol{\alpha}(m)^{-1}\Gamma_{4.5}(m)^{-m} \text{ in case of (1) or (2)}, \quad \delta = 0 \text{ in case of (3)},$$
$$\varepsilon_1 = 0 \text{ in case of (1) or (2)}, \quad \varepsilon_1 = 2^{-8}\boldsymbol{\gamma}(2)^{-2} \text{ in case of (3)},$$
$$\varepsilon_2 = 0 \text{ in case of (1) or (3)}, \quad \varepsilon_2 = \varepsilon \text{ in case of (2)},$$

we furthermore define

$$A_1 = \operatorname{spt}\|V_1\| \cap \big\{x : \mathbf{\Theta}_*^m(\|V_1\|, x) < \lambda_1\big\}, \quad A_2 = \big\{x : \mathbf{\Theta}_*^{m-1}(\|V_2\|, x) < \lambda_2\big\},$$
$$Q_1 = \big\{x : \mathbf{\Theta}_*^m(\|V_1\|, x) > \delta\big\}, \quad Q_2 = \big\{x : \mathbf{\Theta}^{*m-1}(\|V_2\|, x) > 0\big\},$$
$$X_i = \big\{x : \mathbf{\Theta}^{*m-p}(\psi_i, x) > \varepsilon_i\big\} \quad \text{for } i \in \{1,2\}.$$

Clearly, we have $X_2 = \varnothing$ in case of (3). Moreover, we observe that [Fed69, 2.10.19 (3)] may be employed (cf. [FZ73, p. 152, l. 9–16]) to conclude

$$2^{-8}\boldsymbol{\gamma}(2)^{-2}\,\mathscr{H}^1 \llcorner X_1 \le \psi_1 \text{ in case of (3)}, \quad \varepsilon\,\mathscr{H}^1 \llcorner X_2 \le \psi_2 \text{ in case of (2)},$$
$$\mathscr{H}^{m-p}(X_1) = 0 \text{ in case of (1) or (2)}, \quad \mathscr{H}^{m-p}(X_2) = 0 \text{ in case of (1)}.$$

Clearly, we have $Q_2 = \varnothing$ in case of (3). Applying [Men09, 2.10] with ε, Γ, and s replaced by $(2\boldsymbol{\gamma}(2))^{-1}$, $2^4\boldsymbol{\gamma}(2)$, and 1, we obtain

$$\mathscr{H}^1(A_1 \cap Q_1 \sim X_1) = 0 \text{ in case of (3)}, \quad \mathscr{H}^1(A_2 \cap Q_2 \sim X_2) = 0 \text{ in case of (2)}.$$

According to [Men09, 2.11], there holds

$$\mathscr{H}^{m-p}(A_2 \cap Q_2) = 0 \text{ in case of (1)}.$$

Moreover, we obtain

$$A_1 \cap Q_1 \subset X_1 \cup Q_2 \text{ in case of (1) and (2);}$$

in fact, whenever $\sup\{n, 1/\delta\} \leq M < \infty$ and $a \in Q_1 \sim (X_1 \cup Q_2)$, all sufficiently small $r > 0$ satisfy

$$\mathbf{B}(a,r) \subset U, \qquad \|V_1\| \, \mathbf{B}(a,s) \geq M^{-1}\boldsymbol{\alpha}(m)s^m \quad \text{for } 0 < s < r,$$
$$(\boldsymbol{\alpha}(m)r^m)^{1-1/p}\psi_1(\mathbf{U}(a,r))^{1/p} + \|V_2\| \, \mathbf{U}(a,r) \leq \Gamma_{6.1}(M)^{-1}r^{m-1},$$

whence we infer $\|V_1\| \, \mathbf{U}(a,r) \geq (1-M^{-1})\boldsymbol{\alpha}(m)r^m$ by 6.1 and Hölder's inequality. Next, we verify

$$\operatorname{spt} \|V_1\| \subset Q_1 \cup X_1 \cup Q_2 \cup X_2;$$

in fact, this follows from 6.3 and Hölder's inequality in case of (1), from 6.3 alone in case of (2), and from 6.4 in case of (3). Therefore, we obtain

$$A_1 \cap A_2 \subset (A_2 \cap Q_2) \cup X_1 \cup X_2 \text{ in case of (1) or (2),}$$
$$A_1 \subset (A_1 \cap Q_1) \cup X_1 \text{ in case of (3),}$$

whence the main conclusion follows.

Since $\lambda_2 \, \mathscr{H}^{m-1} \llcorner U \sim A_2 \leq \|V_2\|$ by [Fed69, 2.10.19 (3)] and

$$\mathscr{H}^m \{x : 0 < \boldsymbol{\Theta}^m(\|V_1\|, x) < 1\} = 0,$$

the main conclusion yields $\boldsymbol{\Theta}^m(\|V_1\|, x) \geq 1$ for $\mathscr{H}^m$ almost all $x \in \operatorname{spt} \|V_1\|$ and the postscript follows. □

6.8 Remark. The exponent of the Hausdorff measure $\mathscr{H}^{m-p}$ in (1) may not be replaced by any smaller number determined by m and p even if $V_2 = 0$ by 3.1 and 5.5. Similarly, if $m \geq 3$, the hypothesis $m-1 \leq p$ in (1) may not be replaced by $m-1-\varepsilon \leq p$ for any $0 < \varepsilon \leq 1$ determined by m and p even if $V_2 = 0$ by 3.1 and 5.7.

6.9 Remark. The case $m = 1$, not treated here, was studied in [Men16a, 4.8]; similarly, results on the case $p = m$ and $V_2 = 0$ are summarised in [Men16a, 7.6].

7 Geodesic diameter

In this section, we establish (see 7.4 and 7.10) the Theorems A′ and A in the introduction. For this purpose, we firstly study (and characterise in 7.3) the geodesic diameter of closed subsets of Euclidean space (see 7.1–7.3). Then, we deduce (see 7.4–7.7) the bounds on the geodesic diameter in the varifold setting. As a corollary, we finally transfer the estimate to immersions (see 7.8–7.10).

7.1 Definition (see [Men16b, 6.6]). Whenever X is a boundedly compact metric space, the *geodesic distance* on X is the pseudometric on X whose value at $(a, x) \in X \times X$ equals the infimum of the set of numbers

$$\mathbf{V}_{\inf I}^{\sup I} C$$

corresponding to all continuous maps $C : \mathbf{R} \to X$ such that $C(\inf I) = a$ and $C(\sup I) = x$ for some compact non-empty subinterval I of $\mathbf{R}$.

7.2 Remark (see [Men16b, 6.3]). The same definition results if one considers maps $C : \{y : 0 \leq y \leq b\} \to X$ with $\operatorname{Lip} C \leq 1$ and $b = \mathbf{V}_0^b C$ corresponding to $0 \leq b < \infty$. (In fact, if it is finite, the infimum is attained by some such C.)

7.3 Lemma. *Suppose X is a closed subset of $\mathbf{R}^n$ and d denotes the diameter of X with respect to the geodesic distance (see 7.1) on X.*

Then, there holds

$$d = \sup\{\operatorname{diam} f[X] : 0 \leq f \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}) \text{ and } |\operatorname{D} f(x)| \leq 1 \text{ for } x \in X\}.$$

Proof. In view of [Fed69, 2.9.20] and 7.2, the supremum does not exceed d.

To prove the converse inequality, we define pseudometrics $\sigma_\delta : X \times X \to \overline{\mathbf{R}}$ by letting $\sigma_\delta(a, x)$, for $(a, x) \in X \times X$ and $0 < \delta \leq 1$, denote the infimum of the set of numbers

$$\sum_{i=1}^{j} |x_i - x_{i-1}|$$

corresponding to all finite sequences $x_0, x_1, \ldots, x_j \in X$ with $x_0 = a$, $x_j = x$, and $|x_i - x_{i-1}| \leq \delta$ for $i = 1, \ldots, j$. One readily verifies that

$$\sigma_\delta(\chi, a) \leq \sigma_\delta(x, a) + |x - \chi| \quad \text{whenever } a, x, \chi \in X \text{ and } |x - \chi| \leq \delta;$$

in particular, $\operatorname{Lip}(\sigma_\delta(\cdot, a)|\mathbf{B}(x, \delta)) \leq 1$ in case $\sigma_\delta(a, x) < \infty$. Denoting by ϱ the geodesic distance on X, we have

$$|a - x| \leq \sigma_\delta(a, x) \leq \varrho(a, x) \quad \text{and} \quad \lim_{\delta \to 0+} \sigma_\delta(a, x) = \varrho(a, x) \quad \text{for } a, x \in X$$

by [Men16b, 6.3]. Consequently, one readily verifies[6]

$$d \leq \sup\{\operatorname{diam} \operatorname{im} \sup\{s - \sigma_\delta(\cdot, a), 0\} : 0 \leq s < \infty,\ 0 < \delta \leq 1, \text{ and } a \in X\}.$$

This estimate implies that the conclusion is a consequence of the following assertion: if $\varepsilon > 0$, $0 \leq s < \infty$, $0 < \delta \leq 1$, $a \in X$, and $\zeta = \sup\{s - \sigma_\delta(\cdot, a), 0\}$, then there exists a nonnegative function $Z \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ such that

$$|Z(x) - \zeta(x)| \leq \varepsilon \quad \text{and} \quad |\operatorname{D} Z(x)| \leq 1 \quad \text{whenever } x \in X.$$

To prove this assertion, we first observe that ζ is a real valued function with $\operatorname{Lip}(\zeta|\mathbf{B}(x, \delta)) \leq 1$ for $x \in X$. Moreover, since $\sup \operatorname{im} \zeta < \infty$, it is sufficient to prove the assertion with $|\operatorname{D} Z(x)| \leq 1$ replaced by $|\operatorname{D} Z(x)| \leq 1 + \varepsilon$. For

[6] In fact, as σ_δ is real valued in case $d < \infty$, we have

$$d = \sup\{\operatorname{diam} \operatorname{im} \sup\{s - \sigma_\delta(\cdot, a), 0\} : 0 \leq s < \infty,\ 0 < \delta \leq 1, \text{ and } a \in X\}.$$

this purpose, we will employ the partition of unity given in [Fed69, 3.1.13]; in particular, let V_1 be the number constructed there, $\kappa = \sup\{1, V_1\}$, and define

$$\Phi = \{\mathbf{U}(\chi,\delta) : \chi \in X \cap \mathbf{U}(a, s+\delta)\} \cup \{\mathbf{R}^n \sim \mathbf{B}(a,s)\}, \quad U = \bigcup \Phi,$$
$$h(x) = \tfrac{1}{20} \sup \big\{ \inf\{\operatorname{dist}(x, \mathbf{R}^n \sim T), 1\} : T \in \Phi \big\} \quad \text{for } x \in U.$$

Employing a Lipschitzian extension (see [Fed69, 2.10.44]) and convolution, we construct, for each $T \in \Phi$, a nonnegative function $g_T \in \mathscr{E}(\mathbf{R}^n, \mathbf{R})$ satisfying

$$|g_T(x) - \zeta(x)| \le (129)^{-n} (20\kappa)^{-1} \delta\varepsilon \quad \text{for } x \in T, \qquad |\operatorname{D} g_T(x)| \le 1 \quad \text{for } x \in \mathbf{R}^n,$$

where we may assume that $g_T = 0$ if $T = \mathbf{R}^n \sim \mathbf{B}(a,s)$, since $\operatorname{spt} \zeta \subset \mathbf{B}(a,s)$. Taking S, S_x, and v_s, for $s \in S$, as in [Fed69, 3.1.13] and choosing $\tau : S \to \Phi$ such that $\operatorname{spt} v_s \subset \tau(s)$ for $s \in S$, we define $G = \sum_{s\in S} v_s g_{\tau(s)}$. Clearly, we have $|G(x) - \zeta(x)| \le \varepsilon$ for $x \in X$ and $\operatorname{spt} G \subset \mathbf{U}(a, s+2\delta)$. Noting $h(x) \ge \frac{\delta}{20}$ for $x \in X$, we furthermore estimate

$$\Big| \sum_{s\in S} \operatorname{D} v_s(x) g_{\tau(s)}(x) \Big| \le \sum_{s \in S_x} |\operatorname{D} v_s(x)| |g_{\tau(s)}(x) - \zeta(x)| \le \varepsilon, \quad |\operatorname{D} G(x)| \le 1 + \varepsilon$$

for $x \in X$. Applying 2.16 (2) with U, E_0, and E_1 replaced by $\mathbf{R}^n$, $\mathbf{R}^n \sim U$, and X to obtain f with the properties listed there, we may now take $Z \in \mathscr{D}(\mathbf{R}^n, \mathbf{R})$ defined by $Z(x) = f(x)G(x)$ for $x \in U$ and $Z(x) = 0$ for $x \in \mathbf{R}^n \sim U$. □

Next, we turn to Theorem A′, the general a priori estimate of the geodesic diameter in the varifold setting with boundary.

7.4 Theorem. *Suppose m and n are positive integers, $2 \le m \le n$, $V_1 \in \mathbf{V}_m(\mathbf{R}^n)$ and $V_2 \in \mathbf{V}_{m-1}(\mathbf{R}^n)$ satisfy the conditions of 4.4 with $U = \mathbf{R}^n$, V_1 is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, $(\|V_1\| + \|V_2\|)(\mathbf{R}^n) < \infty$,*

$$V_2 = 0 \quad \textit{if } m = 2, \qquad \|\delta V_1\| \le \|V_1\| \llcorner |\mathbf{h}(V_1, \cdot)| + \|V_2\| \quad \textit{if } m > 2,$$
$$\|\delta V_2\| \textit{ is absolutely continuous with respect to } \|V_2\| \quad \textit{if } m > 3,$$

ϕ_i are associated to V_i as in 4.4, for $i \in \{1,2\}$, and d denotes the diameter of $\operatorname{spt} \|V\|$ with respect to its geodesic distance (see 7.1).

Then, there holds, for some positive finite number Γ determined by m,

$$d \le \Gamma (\phi_1 + \phi_2)(\mathbf{R}^n).$$

Proof. The isoperimetric inequality (see [MS17, 3.5, 7]) and Hölder's inequality yield

$$\|V_2\|(\mathbf{R}^n)^{1/(m-1)} \le \boldsymbol{\gamma}(m-1)^{m-2}\, \phi_2(\mathbf{R}^n).$$

We will show

$$\|V_1\|(\mathbf{R}^n)^{1/m} \le (2\boldsymbol{\gamma}(m))^{m-1}\, \phi_1(\mathbf{R}^n) + (2\boldsymbol{\gamma}(m))^{1/(m-1)} \boldsymbol{\gamma}(m-1)^{m-2}\, \phi_2(\mathbf{R}^n);$$

in fact, if $m = 2$, then $\|V_1\|(\mathbf{R}^n)^{1/2} \le \boldsymbol{\gamma}(2)\, \phi_1(\mathbf{R}^n)$ by the isoperimetric inequality, and, if $m > 2$, then we may assume $\|V_1\|(\mathbf{R}^n)^{1-1/m} > 2\boldsymbol{\gamma}(m)\, \|V_2\|(\mathbf{R}^n)$, in which case the isoperimetric inequality may be used to obtain

$$\|V_1\|(\mathbf{R}^n)^{1-1/m} \le 2\boldsymbol{\gamma}(m) \textstyle\int |\mathbf{h}(V_1, x)| \,\mathrm{d}\|V_1\|\, x,$$

whence the asserted inequality follows by Hölder's inequality.

Next, suppose $X = \operatorname{spt} \|V_1\|$ and f satisfies the conditions of 7.3. Then, $f \in \mathbf{T}_\varnothing(V_i)$ and $\|V_i\|_{(\infty)}(V_i \, \mathbf{D} f) \leq 1$ for $i \in \{1, 2\}$ by [MS17, 4.6 (1)] and [Men16a, 9.2]. Hence, 4.5 and 4.6 yield

$$\operatorname{diam} \operatorname{spt} f_\# \|V_1\| \leq \Delta(\phi_1 + \phi_2)(\mathbf{R}^n),$$

where $\Delta = \Gamma_{4.5}(m)\big(1 + \boldsymbol{\gamma}(m-1)^{m-2}(1 + (2\boldsymbol{\gamma}(m))^{1/(m-1)}) + (2\boldsymbol{\gamma}(m))^{m-1}\big)$. Finally, we notice $f[X] \subset \operatorname{spt} f_\# \|V_1\|$ as f is continuous.[7] □

7.5 Remark. The preceding theorem answers the fourth question posed in [Sch16, Section A].

In the case without boundary, somewhat more explicit constants may be obtained by using 4.9 instead of 4.5.

7.6 Corollary. *Suppose V and ψ are as in 4.8 with $U = \mathbf{R}^n$, $\|V\|(\mathbf{R}^n) < \infty$, V is indecomposable of type $\mathscr{D}(\mathbf{R}^n, \mathbf{R})$, d denotes the diameter of* $\operatorname{spt} \|V\|$ *with respect to its geodesic distance (see 7.1), and $\Gamma = 2^{m+4} m \boldsymbol{\gamma}(m)^m$.*

Then, there holds

$$d \leq \Gamma \, \psi(\mathbf{R}^n).$$

Proof. With a possibly larger number Γ, this follows from 7.4 with $V_1 = V$ and $V_2 = 0$. We verify the eligibility of the present number Γ by noting that

$$\operatorname{diam} \operatorname{spt} f_\# \|V\| \leq 2^{m+3} m \boldsymbol{\gamma}(m)\big(\|V\|(\mathbf{R}^n)^{1/m} + \boldsymbol{\gamma}(m)^{m-1} \, \psi(\mathbf{R}^n)\big) \leq \Gamma \, \psi(\mathbf{R}^n)$$

by 4.9 in conjunction with the isoperimetric inequality and Hölder's inequality, whenever f satisfies the conditions of 7.3 with $X = \operatorname{spt} \|V\|$. □

7.7 Remark. Here, we compare our proof with Topping's proof of the analogous result for immersions (see [Top08, Theorem 1.1]). The principal geometric idea – suitable smallness of mean curvature implies lower density ratio bounds in balls – is the same. Our formulation (see 4.1 and 6.4) may be traced back to Allard [All72, 8.3], whereas Topping's formulation (see [Top08, Lemma 1.2]) was inspired by a local non-collapsing result for Ricci flow (see Topping [Top05, Theorem 4.2]). However, to implement this geometric idea for varifolds, one faces the difficulty that one cannot, a priori, assume the existence of either geodesics or lower density bounds; the latter are employed to obtain [Top08, Lemma 1.2]. Instead, our proof avoids these tools (though, lower density bounds are independently proven in 6.7) and proceeds through the characterisation of geodesic diameter (see 7.3) and the Sobolev-Poincaré inequality (see 4.5) in conjunction with basic properties from the study of indecomposability (see 3.14).

To prepare for the use of the Whitney-type approximation results in 7.10, we firstly define the appropriate topological function space.

7.8 Definition. Suppose k is a positive integer, M is a compact manifold-with-boundary of class k, and Y is a Banach space.

Then, $\mathscr{C}^k(M, Y)$ is defined (see [Men16b, 2.4]) to be the locally convex space of all maps from M into Y of class k topologised by the family of all seminorms,

[7] In fact, as f is closed, we have $f[X] = \operatorname{spt} f_\# \|V_1\|$.

that correspond to charts ϕ of M of class k and compact subsets K of $\operatorname{dmn}\phi$, and have value

$$\sup\bigl(\{0\}\cup\bigl\{\|\operatorname{D}^l(F\circ\phi^{-1})(x)\| : x\in K\sim\phi[\partial M], l=0,\ldots,k\bigr\}\bigr)$$

at $F\in\mathscr{C}^k(M,Y)$.

7.9 Remark. Choosing a positive integer ι and charts ϕ_i of M of class k with compact subsets K_i of $\operatorname{dmn}\phi_i$, for $i=1,\ldots,\iota$, satisfying $M=\bigcup_{i=1}^{\iota}\operatorname{Int}K_i$, the topology of the locally convex space $\mathscr{C}^k(M,Y)$ is induced by the norm ν on $\mathscr{C}^k(M,Y)$ whose value at $F\in\mathscr{C}^k(M,Y)$ equals

$$\sup\bigl(\{0\}\cup\bigl\{\|\operatorname{D}^j(F\circ\phi_i^{-1})(x)\| : x\in K_i\sim\phi_i[\partial M],\ i=0,\ldots,\iota,\text{ and } j=0,\ldots,k\bigr\}\bigr);$$

in fact, each seminorm occurring in 7.8 is bounded by a finite multiple of ν by the general formula for the differentials of a composition, see [Fed69, 3.1.11]. Similarly, we see that the topology of $\mathscr{C}^k(M,\mathbf{R}^n)$ agrees with that of the space named "$C^k_W(M,\mathbf{R}^n)$" in [Hir94, p. 35]. Consequently, if $n>2\dim M$, the set of embeddings of M into $\mathbf{R}^n$ of class 2 is dense in $\mathscr{C}^2(M,\mathbf{R}^n)$ by [Hir94, 2.1.0].

To conclude our paper, we present Theorem A, the a priori estimate of the geodesic diameter of immersions of compact manifolds-with-boundary.

7.10 Corollary. *Suppose m and n are positive integers, $2\le m\le n$, M is a compact connected m dimensional manifold-with-boundary of class 2, the map $F:M\to\mathbf{R}^n$ is an immersion of class 2, g is the Riemannian metric on M induced by F, and σ is the Riemannian distance associated to (M,g).*

Then, there holds

$$\operatorname{diam}_\sigma M\le\Gamma_{7.4}(m)\bigl(\textstyle\int_M|\mathbf{h}(F,\cdot)|^{m-1}\,\mathrm{d}\mathscr{H}^m_\sigma+\int_{\partial M}|\mathbf{h}(F|\partial M,\cdot)|^{m-2}\,\mathrm{d}\mathscr{H}^{m-1}_\sigma\bigr);$$

here $0^0=1$.

Proof. First, the *special case*, that F is an embedding, will be treated. We define $V_1\in\mathbf{V}_m(\mathbf{R}^n)$ to be associated to $(F,\mathbf{R}^n)$ and $V_2\in\mathbf{V}_{m-1}(\mathbf{R}^n)$ to be 0 if $m=2$ and to be associated to $(F|\partial M,\mathbf{R}^n)$ if $m>2$. Hence, 2.8 and 2.15 yield

$$\mathbf{\Theta}^{\dim V_i}(\|V_i\|,x)\ge 1\quad\text{for } \|V_i\| \text{ almost all } x \text{ and } i\in\{1,2\},$$
$$\|V_1\|=F_\#\mathscr{H}^m_\sigma,\quad \|\delta V_1\|=\|V_1\|\llcorner|\mathbf{h}(F[M\sim\partial M],\cdot)|+F_\#(\mathscr{H}^{m-1}_\sigma\llcorner\partial M),$$
$$\|V_2\|=F_\#(\mathscr{H}^{m-1}_\sigma\llcorner\partial M)\text{ if } m>2,\quad \|\delta V_2\|=\|V_2\|\llcorner|\mathbf{h}(F[\partial M],\cdot)|\text{ if } m>2.$$

As V_1 is indecomposable by 3.13 (1) (3) and F induces an isometry between σ and the geodesic distance on $F[M]$, the special case now follows from 7.4.

In the general case, we assume $n>2m$ and obtain from 7.9 a sequence of embeddings $F_i:M\to\mathbf{R}^n$ of class 2 converging to F in $\mathscr{C}^2(M,\mathbf{R}^n)$ as $i\to\infty$; in particular, $\mathbf{h}(F_i,x)\to\mathbf{h}(F,x)$, uniformly for $x\in M$, as $i\to\infty$ by 2.13 and 7.9. Moreover, denoting by g_i the Riemannian metrics on M induced by F_i and by σ_i the Riemannian distance of (M,g_i), we observe that, given $1<\lambda<\infty$, all sufficiently large i satisfy

$$\lambda^{-2}\langle w\odot w,g(z)\rangle\le\langle w\odot w,g_i(z)\rangle\le\lambda^2\langle w\odot w,g(z)\rangle$$

whenever $z\in M$ and w belongs to the tangent space of M at z, whence we infer $\lambda^{-1}\sigma\le\sigma_i\le\lambda\sigma$ and $\lambda^{-k}\mathscr{H}^k_\sigma\le\mathscr{H}^k_{\sigma_i}\le\lambda^k\mathscr{H}^k_\sigma$ for $0\le k<\infty$. Therefore, the conclusion follows from the special case applied with F replaced by F_i. □

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