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Consensus analysis of systems with time-varying interactions : An event-triggered approach

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Abstract: We present consensus analysis of systems with single integrator dynamics interacting via time-varying graphs under the event-triggered control paradigm. Event-triggered control sparsifies the control applied, thus reducing the control effort expended. Initially, we consider a multi-agent system with persistently exciting interactions and study the behaviour under the application of event-triggered control with two types of trigger functions- *static* and *dynamic* trigger. We show that while in the case of static trigger, the edge-states converge to a ball around the origin, the dynamic trigger function forces the states to reach consensus exponentially. Finally, we extend these results to a more general setting where we consider switching topologies. We show that similar results can be obtained for agents interacting via switching topologies and validate our results by means of simulations.

1 Introduction

A flock of birds, a swarm of bees, a school of fishes, a colony of ants -all display a wonderful coordination and complex patterns which have caught our attention since time immemorial. On closer observation we realize that these seemingly complex patterns emerge out of simple yet powerful, local rules on each agent of the group based on interactions with only the neighbours. When we attempt to algorithmize such a 'decentralized' behaviour, two types of questions can be posed- what would be the result of a particular decentralized control law or what decentralized control law would result in a desired global pattern or formation. Consensus is one of the most common and powerful 'global' behaviours usually studied, primarily because it can be easily extended to other problems like formation, rendezvous, flocking *etc*. Vicsek et al. (1995) proposed an average-based decentralized control law for a multi-agent system based on only local information and observed that the agents attain consensus while Jadbabaie et al. (2003) gave a proof of the same assuming the agents interact via connected graphs that can switch at different time instants. Various results have been proposed by Ren and Beard (2005), Ren and Atkins (2005), Olfati-Saber and Shamma (2005), and others on consensus of a multi-agent system under directed or undirected graphs, with switching topologies and also with time delays.

Studying consensus behaviour for systems with switching or time-varying graphs is naturally of interest as in real-life scenarios it is not possible to assume that each agent has the same set of neighbours at all times. Time-varying graphs are useful when the information from each neighbour is assigned a weight proportional to the reliability of the information or the distance between them. Martin and Girard (2013) proved consensus under the assumptions of *persistent*-connectivity and cut-balance interactions. Chowdhury et al. (2016) obtained bounds on the rate of convergence for single-integrator and double integrator dynamics under persistent interactions.

With control laws being implemented on digital computers, developing discrete-time counterparts to continuous-time control laws is an eventuality. Among discrete control laws event-triggered control is preferred over time-triggered control as it comes into play only when an 'event' is triggered, thus sparsifying control. Event-triggered control of multi-agent systems has been studied by many. Dimarogonas et al. (2012), Seyboth et al. (2011) prove consensus under a connected time-invariant graph for single integrator dynamics, while Yu et al. (2015), Zhu et al. (2014) have extended the results for agents with general linear systems dynamics. Chen and Dai (2016) has studied the consensus of time varying systems with non-linear dynamics under event-triggered control but under a constant interaction topology. Our work considers the broader case of time-varying graphs that can have different spanning-trees and we prove consensus of single-integrator systems under event-triggered control structure.

The paper is organized as follows. We brush up on graph theory and persistent excitation in section 2. The system dynamics for single integrator systems are introduced in section 3. In section 4, we introduce the notion of event-triggered control, define the trigger conditions and evaluate convergence under event-triggered control in section 5. We extend the results to switching graphs in section 5.1 and show simulation results in section 6. The results that we obtained are summarized in section 7.

2 Preliminaries

2.1 Notions of graph theory

In this work we consider agent interactions represented by undirected graphs $\mathcal{G} = (V, E)$, where $V = \{v_1, v_2, \cdots v_n\}$ denotes a non-empty set of nodes and $[V]^2 \supseteq E = \{e_1, e_2, \cdots e_m\}$ is

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the edge set, where $[V]^2$ is the set of all subsets of V containing two elements. Each node represents an agent of the system and each edge (v_i, v_j) signifies that the agents occupying nodes v_i and v_j can exchange information with each other. We define $D(\mathcal{G}) (= [d_{ij}]) \in \mathbb{R}^{n\times m}$ to be the incidence matrix associated with the graph $\mathcal{G}$ by arbitrarily assigning orientation to each edge $e_j \in E$. Then $[d_{ij}] = -1$ if v_i is the tail of e_j ,$[d_{ij}] = 1$ if v_i is the head of e_j, $[d_{ij}] = 0$ otherwise. The Laplacian of $\mathcal{G}$,

$$L(\mathcal{G}) = D(\mathcal{G})\, D(\mathcal{G})^\top \tag{1}$$

is a symmetric square matrix that captures the inter-connections between each pairs of nodes. We model the time-varying interactions between the agents by assuming a constant, underlying graph $\mathcal{G}$ with edge weights that can be time-varying. The Laplacian can be tweaked to reflect the resultant time-varying graph $\tilde{\mathcal{G}}(t)$ as,

$$L\left(\tilde{\mathcal{G}}(t)\right) = D(\mathcal{G})\, W(t)\, D(\mathcal{G})^\top . \tag{2}$$

where $W(t) \in \mathbb{R}^{m\times m}$ is a diagonal matrix which captures the time-varying nature of each interaction and $w_{ii} \geq 0\ \forall t$. We assume that the underlying graph $\mathcal{G}$ is connected and therefore contains a *spanning-tree*, i.e. the edge set E of $\mathcal{G}$ can be partitioned into two subsets $E = E_\tau \cup E_c$, where E_τ consists of the spanning-tree edges and E_c contains the cycle edges. We also assume that the time-varying edge weight matrix $W(t)$ is piece-wise continuous and satisfies the persistent excitation condition with constants (μ_1, μ_2, T).

2.2 Persistence of excitation

Definition 1. (Sastry and Bodson, 2011, p. 72) The signal $g(\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{n\times m}$ is *Persistently Exciting* (PE) if there exist finite positive constants μ_1, μ_2, T such that,

$$\mu_2 I_n \geq \int_t^{t+T} g(\tau) g(\tau)^T d\tau \geq \mu_1 I_n \qquad \forall t \geq t_0 \tag{3}$$

A function $g(\cdot)$ that satisfies the condition (3) is said to be persistently exciting with constants (μ_1, μ_2, T).
We say that a graph $\tilde{\mathcal{G}}$ is persistently exciting if its associated edge-weight matrix $W(t)$ is persistently exciting.

2.3 Other Conventions

$\|\cdot\|$ denotes the frobenius 2-norm on vectors and the induced 2-norm on matrices. $\lambda_{min}(\cdot)$ and $\lambda_{min}(\cdot)$ operate on square matrices and return the minimum and maximum eigenvalues respectively of the said matrix. Boldfaced 1 and 0 $(\mathbf{1}, \mathbf{0})$ represent vectors with all ones and all zeroes respectively and I is used to denote the identity matrix. Their dimensions can be inferred contextually if not mentioned explicitly.

3 Network Models

In this section we state the single integrator dynamical equations and perform a series of linear transformations, to bring them to a form that we could work with later on. This section has been taken from Chowdhury et al. (2016) and presented here for reference of the readers.

3.1 Single Integrator

Consider a multi-agent system with states $x_i \in \mathbb{R}$ for $i = 1, 2, 3 \cdots n$ with a connected underlying graph $\mathcal{G}$ and the Laplacian of the time-varying graphs $L\left(\tilde{\mathcal{G}}(t)\right) = [l_{ij}(t)]$. The dynamics of each state with control $u_i \in \mathbb{R}$ can be written as,

$$\begin{aligned} \dot{x}_i &= u_i \\ u_i &= -k \sum_{j=1}^{n} l_{ij}(t) x_j \end{aligned} \tag{4}$$

for $t > t_0$ with initial condition $x_i(t_0) \in \mathbb{R}$ for $i = 1, 2, 3.....n$ and positive control gain $k \in \mathbb{R}$. The augmented dynamics of the system can be written in terms of the state vectors $x = [x_1, x_2, \cdots x_n]^\top \in \mathbb{R}^n$ as,

$$\begin{aligned} \dot{x}(t) &= -kL\left(\tilde{\mathcal{G}}(t)\right) x \\ &= -kD(\mathcal{G})\, W(t) D(\mathcal{G})^\top x \end{aligned} \tag{5}$$

Taking cue from Zelazo and Mesbahi (2011), Mesbahi and Egerstedt, 2010, p. 77-81 we transform the consensus problem of equation (5) into a stabilization problem by considering the edge states instead of node states. We effect the conversion through the following transformation,

$$x_e = D(\mathcal{G})^\top x. \tag{6}$$

The dynamics of the edge states, on differentiation of (6) yields,

$$\dot{x}_e = -kL_e(\mathcal{G})\, W(t)\, x_e \tag{7}$$

where $L_e(\mathcal{G}) \in \mathbb{R}^{m\times m}$ represents the edge-Laplacian of the graph $\mathcal{G}$ and can be expressed as, $L_e(\mathcal{G}) = D(\mathcal{G})^\top D(\mathcal{G})$ as shown in Zelazo and Burger (2014). Also, using the fact that the underlying graph contains a spanning-tree $\mathcal{G}_\tau$ we partition the edge states after a suitable permutation as follows,

$$x_e = \begin{bmatrix} x_\tau \\ x_c \end{bmatrix} \tag{8}$$

for $x_\tau \in \mathbb{R}^p$ and $x_c \in \mathbb{R}^{m-p}$ where p is the number of edges in $\mathcal{G}_\tau$. Similarly we can partition $D(\mathcal{G})$, $W(t)$ as $D(\mathcal{G}) = [D(\mathcal{G}_\tau)\ \ D(\mathcal{G}_c)]$, $W(t) = \begin{bmatrix} W_\tau(t) & 0 \\ 0 & W_c(t) \end{bmatrix}$. The edge-Laplacian, in terms of the partitions of $D(\mathcal{G})$ is then,

$$\begin{aligned} L_e(\mathcal{G}) &= [D(\mathcal{G}_\tau)\ \ D(\mathcal{G}_c)]^\top [D(\mathcal{G}_\tau)\ \ D(\mathcal{G}_c)] \\ &= \begin{bmatrix} L_e(\mathcal{G}_\tau) & D(\mathcal{G}_\tau)^\top D(\mathcal{G}_c) \\ D(\mathcal{G}_c)^\top D(\mathcal{G}_\tau) & L_e(\mathcal{G}_c) \end{bmatrix}. \end{aligned} \tag{9}$$

We know that for connected graphs, x_c can always be written as $x_c = Z^\top x_\tau$ as shown in Sandhu et al. (2005) where $Z = (L_e(\mathcal{G}_\tau))^{-1} D(\mathcal{G}_\tau)^\top D(\mathcal{G}_c)$. This is because the spanning-tree edges essentially capture the behaviour of all the edges. So we can focus our attention only on the spanning-tree edges. Using (7), (8) and (9), we get $\dot{x}_\tau = -kL_e(\mathcal{G}_\tau) RW(t) R^\top x_\tau$ where $R = [I_p\ Z] \in \mathbb{R}^{p\times m}$. The preceding transformations that helped us re-write the system dynamics in terms of x_τ are along the lines of Zelazo and Mesbahi (2011) and Mesbahi and Egerstedt, 2010, p. 77-81. Since $L_e(\mathcal{G}_\tau)$ is symmetric and positive definite, it can be diagonalized as $L_e(\mathcal{G}_\tau) = \Gamma\Lambda\Gamma^\top$ for some orthogonal matrix $\Gamma \in \mathbb{R}^{p\times p}$ and diagonal matrix $\Lambda \in \mathbb{R}^{p\times p}$. Consider a change of variable by the transformation $\Upsilon = \Gamma^\top x_\tau$. The above equation becomes,

$$\dot{\Upsilon} = -k\Lambda M(t)\, \Upsilon \tag{10}$$

where $M(t) = \Gamma^\top RW(t) R^\top \Gamma \in \mathbb{R}^{p\times p}$. As stated earlier, the consensus problem of equation (5) is equivalent to stabilization problem of equation (10). For the sake of further reference, we denote the preceding set of transformations from x to Υ by $\psi := \Gamma^\top [I_p \, 0_{m-p}] \, D(\mathcal{G})^\top$. So we have, $\Upsilon = \psi x$. We state (Chowdhury et al., 2016, Theorem 5) and use the result to obtain the rate of convergence to consensus of a single integrator system defined by equations (5).

Theorem 1. ((Chowdhury et al., 2016, Theorem 5)) Consider the closed-loop consensus dynamics (5). Assume that, the underlying graph $\mathcal{G}$ is connected. The states of the closed-loop dynamics $x(t)$ with time-varying communication topology characterized by $W(t)$, achieve consensus exponentially, if there exists a spanning tree with corresponding edge-weight matrix $W_\tau(t)$ that is persistently exciting. Further the convergence rate α_v to consensus is bounded below by,

$$\alpha_v \geq \frac{1}{2T} \ln \frac{1}{\left[1 - \frac{2k\lambda_{\min}(\Lambda)\mu_1}{\left(1+k\sqrt{p}\|\Lambda\|\mu_2\right)^2}\right]}$$

where, T, μ_1 and μ_2 are the constants appearing in Definition 1 and Λ is a diagonal matrix containing the eigenvalues of the spanning tree edge Laplacian matrix.

Further,

$$\|\Upsilon(t)\| \leq m_v e^{-\alpha_v(t-t_0)} \, \|\Upsilon(t_0)\| \tag{11}$$

where α_v and m_v can be calculated from the underlying graph and control gains. **Note**: The relationship between $\|x_e\|$ and $\|\Upsilon\|$ can be established in the following way.

$$\begin{aligned} \|x_e\| &= \sqrt{\|x_\tau\|^2 + \|x_c\|^2} \\ &\leq \|x_\tau\| \sqrt{1 + \|Z^\top\|^2} \end{aligned}$$

Using the fact that Γ is orthogonal, $x_\tau = \Gamma\Upsilon$,

$$\|x_e\| \leq \rho \, \|\Upsilon\| \tag{12}$$

where $\rho = \|\Gamma\| \sqrt{1 + \|Z^\top\|^2}$. The induced 2-norm of Z can be calculated as $\lambda_{max}\left(Z^\top Z\right)^{\frac{1}{2}}$.

4 Event-Triggered Control

In this section we outline the event-triggered control strategies and show how it modifies the single-integrator dynamics defined by equation (5). Under the event-triggered control paradigm each agent broadcasts a (piecewise)constant value, $\hat{x}_i$ which is updated to the current value of the state whenever an 'event is triggered'. The control applied by each agent would then be,

$$u_i = -\,k \sum_{j=1}^{n} l_{ij}(t)\hat{x}_j \tag{13}$$

To define an event, we introduce error variables $e_i(t) = \hat{x}_i(t) - x_i(t)$ which denote the difference between the broadcasted value and the current value of the state for each agent. The trigger condition that updates the broadcasted states $\hat{x}$ effectively shapes the behaviour of the system. We define the trigger condition using a trigger function for each state $f_i(t, e_i) : \mathbb{R} \to \mathbb{R}$. An event is said to be 'triggered' when $f_i > 0$. Once an event is 'triggered' say at time t^*, the broadcast value is updated i.e. $\hat{x}_i(t) = x_i(t^*) \implies e_i(t^*) = 0$ for $t^* \leq t < t'$ where t' is the time instant when the next subsequent event is triggered. We define two trigger functions as shown in Seyboth et al. (2011)- the static trigger and the dynamic trigger.

(1) **Static Trigger Function**

$$f_i(e_i(t)) = \|e_i(t)\| - c \tag{14}$$

(2) **Dynamic Trigger Function**

$$f_i(t, e_i(t)) = \|e_i(t)\| - ce^{-\beta(t)} \tag{15}$$

where $c > 0$

The static trigger function can be seen as a special case of the dynamic trigger function with $\beta = 0$. So we will perform our analysis with (15) as the trigger function, and substitute $\beta = 0$ when we want to evaluate the static-trigger case. The dynamics of single integrator systems under event-triggered control can be written as,

$$\dot{x}(t) = -\,kL\left(\tilde{\mathcal{G}}(t)\right)(x+e) \tag{16}$$

where $e = [e_1, e_2 \cdots e_n]^\top \in \mathbb{R}^n$. The equivalent stabilization problem under event triggered control for single integrator will then be,

$$\dot{\Upsilon} = -\,k\Lambda M(t)\left(\Upsilon + \tilde{e}\right) \tag{17}$$

where $\tilde{e} = \psi e$.

4.1 Bounds on the error variables

In this section, we obtain bounds on $\tilde{e}$ in terms of the bounds on e. The trigger function is so designed that each e_i is always upper-bounded. We can see that,

$$\|e_i\| \leq ce^{-\beta t} \tag{18}$$

We can relate the bounds on e to bounds on $\tilde{e}$ in the following way. We know that $\tilde{e} = \psi \, e$. Let us define $\bar{e} := D(\mathcal{G})^\top e$. From the structure of $D(\mathcal{G})$, we get

$$\bar{e}_i = e_j - e_k \tag{19}$$

for $i = 1, 2, \cdots m$ and j, k chosen on the basis of $D(\mathcal{G})$. Using (18), (19) can be rewritten as,

$$\begin{aligned} \|\bar{e}_i\| &= \|e_j - e_k\| \\ &\leq \|e_j\| + \|e_k\| \\ &\leq 2ce^{-\beta t}. \end{aligned} \tag{20}$$

Also,

$$\begin{aligned} \tilde{e} &= \psi e \\ &= \Gamma^\top [I_p \; 0_{m-p}] \, \bar{e} \\ &= \Gamma^\top [\bar{e}_1 \, \bar{e}_2 \cdots \bar{e}_p]^\top . \end{aligned} \tag{21}$$

Using the bound on each $\bar{e}_i$ from (20),

$$\begin{aligned} \|\tilde{e}\| &\leq \|\Gamma\| \, \|[\bar{e}_1 \, \bar{e}_2 \cdots \bar{e}_p]\| \\ &\leq \sqrt{p}\|\Gamma\|\|\bar{e}_i\| \\ &\leq 2c\sqrt{p}\|\Gamma\|e^{-\beta t}. \end{aligned} \tag{22}$$

Defining $C := 2c\sqrt{p}\,\|\Gamma\|$, we can write the above inequality to be,

$$\|\tilde{e}(t)\| \leq Ce^{-\beta t}. \tag{23}$$

5 Consensus Analysis

Theorem 2. Consider a multi-agent system with single integrator dynamics as defined by equation (5). Assume that the underlying graph ($\mathcal{G}$), representing the interaction between the agents be connected, with p edges in the spanning-tree. If the spanning tree edge weight matrix $W_\tau(t)$ is persistently exciting with constants (μ_1,μ_2,T), then

(1) on application of event-triggered control with a static trigger function defined by equation (14), the edge states x_e of the system exponentially converge to a ball around the origin defined by $\|x_e\| \leq \rho\kappa_2^m$.
(2) on application of event-triggered control with a dynamic trigger function defined by equation (15), the edge states x_e of the system exponentially converge to origin. Also the rate of convergence is lower bounded by β as defined in (15).

where, $\kappa_2 = \frac{k\|\Lambda\| m_v C e^{\alpha_v t_0 + 2\alpha_v T}\mu_2}{e^{\alpha_v T}-1}$ and ρ is defined as in (12). Also the closed loop systems in the cases of static and dynamic triggers does not exhibit zeno behaviour when β is chosen to be greater than α_v.

Note: When a connected graph has multiple spanning-trees, κ_2^M and κ_2^m are the largest and smallest values of κ_2 that can be calculated considering each different spanning-tree.

Proof. Let $\phi(t, t_0)$ be the state transition matrix corresponding to system defined by equation (10) . The solution of the system can be written using $\phi(t, t_0)$ as,

$$\Upsilon(t) = \phi(t, t_0)\Upsilon(t_0)$$

We obtain a bound on $\|\phi(t, t_0)\|$ by the following steps.

$$\begin{aligned}\|\Upsilon(t)\| &= \|\phi(t, t_0)\Upsilon(t_0)\| \\ &= \frac{\|\phi(t, t_0)\Upsilon(t_0)\|}{\|\Upsilon(t_0)\|}\|\Upsilon(t_0)\| \end{aligned} \tag{24}$$

Comparing (24) and (11) we can conclude that,

$$\frac{\|\phi(t, t_0)\Upsilon(t_0)\|}{\|\Upsilon(t_0)\|} \leq m_v e^{-\alpha_v(t-t_0)}$$

$$\underset{\Upsilon(t_0)}{sup}\frac{\|\phi(t, t_0)\Upsilon(t_0)\|}{\|\Upsilon(t_0)\|} = \|\phi(t, t_0)\| \leq m_v e^{-\alpha_v(t-t_0)}. \tag{25}$$

The last statement holds true because $\Upsilon(t_0)$ can be arbitrary. Consider the single integrator multi-agent system with event-triggered control defined by the equation (17). The solution of this system can be expressed as ,

$$\Upsilon(t) = \phi(t, t_0)\Upsilon(t_0) - k\int_{t_0}^{t}\phi(t, \tau)\Lambda M(\tau)\tilde{e}(\tau)d\tau \tag{26}$$

We get the following inequality from the above equation,

$$\begin{aligned}\|\Upsilon(t)\| \leq& \|\phi(t, t_0)\|\,\|\Upsilon(t_0)\| + \\ & k\|\Lambda\|\int_{t_0}^{t}\|\phi(t, \tau)\|\,\|M(\tau)\|\,\|\tilde{e}(\tau)\|\,d\tau\end{aligned}$$

Using the bounds on ϕ from equation (25) and $\tilde{e}(\tau)$ from equation (23) we get,

$$\begin{aligned}\|\Upsilon(t)\| \leq& m_v e^{-\alpha_v(t-t_0)}\|\Upsilon(t_0)\| + \\ & k\|\Lambda\| m_v C e^{-\alpha_v t}\int_{t_0}^{t} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|\,d\tau \end{aligned} \tag{27}$$

To obtain a bound on the integral in the above inequality, we divide the interval $[t_0, t]$ into partitions of size T i.e. $[t_0, t_0+T]$, $[t_0+T, t_0+2T]$ and so on . The number of such partitions of length T possible will be given by $\theta = \lfloor\frac{t-t_0}{T}\rfloor$, where $\lfloor\cdot\rfloor$ is the floor function . The last partition can then be written as $[t_0+\theta T, t]$. The integral in (27) can then be written as, $\int_{t_0}^{t_0+T} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|\,d\tau + \int_{t_0+T}^{t_0+2T} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|\,d\tau \cdots + \int_{t_0+\theta T}^{t} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|\,d\tau$. Applying Hölder inequality with $p=\infty$ and $q=1$ and using the fact that $M(\tau)$ is persistently exciting and $\beta < \alpha_v$ we can obtain the following bound on each interval,

$$\begin{aligned}\int_{t'}^{t'+T} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|\,d\tau \leq& \mu_2 sup_{\tau\in[t', t'+T]}e^{-(\beta-\alpha_v)\tau} \\ \leq& \mu_2 e^{-(\beta-\alpha_v)(t'+T)}\end{aligned}$$

Using this upper bound on each integral we get

$$\begin{aligned}\int_{t_0}^{t} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|\,d\tau \leq \mu_2 e^{-(\beta-\alpha_v)T}\Big(e^{-(\beta-\alpha_v)t_0} + \\ e^{-(\beta-\alpha_v)(t_0+T)} + \cdots e^{-(\beta-\alpha_v)(t_0+\theta T)}\Big)\end{aligned}$$

Using the property of the sum of terms in a geometric progression, we get

$$\begin{aligned}&\int_{t_0}^{t} e^{-(\beta-\alpha_v)\tau}\|M(\tau)\|d\tau \leq \\ &\qquad \mu_2 e^{-(\beta-\alpha_v)(t_0+T)}\left(\frac{1-e^{-(\beta-\alpha_v)\theta T}}{1-e^{-(\beta-\alpha_v)T}}\right)\end{aligned} \tag{28}$$

Using inequality (28) in inequality (27),

$$\begin{aligned}&\|\Upsilon(t)\| \leq m_v e^{-\alpha_v(t-t_0)}\|\Upsilon(t_0)\| + \\ &\quad k\|\Lambda\| m_v C e^{-(\beta-\alpha_v)(t_0+T)}\mu_2 e^{-\alpha_v t}\left(\frac{1-e^{-(\beta-\alpha_v)\theta T}}{1-e^{-(\beta-\alpha_v)T}}\right)\end{aligned} \tag{29}$$

We define $\kappa_1 = e^{\alpha_v t_0}m_v\|\Upsilon(t_0)\| - \kappa_3$, $\kappa_2 = e^{(\alpha_v+\beta)T}\kappa_3$,

$$\kappa_3 = \frac{k\|\Lambda\| m_v C e^{-(\beta-\alpha_v)(t_0+T)}\mu_2}{e^{-(\beta-\alpha_v)T}-1}$$

to be able to express (29) as,

$$\|\Upsilon(t)\| \leq \kappa_1 e^{-\alpha_v t} + \kappa_3 e^{-\beta\theta T - \alpha_v(\theta T - t)} \tag{30}$$

Using the properties of floor function, we can simplify the above inequality further,

$$\begin{aligned}\|\Upsilon(t)\| \leq& \kappa_1 e^{-\alpha_v t} + \kappa_3 e^{-\beta(\theta T - t) - \alpha_v(\theta T - t) - \beta t} \\ \leq& \kappa_1 e^{-\alpha_v t} + \kappa_2 e^{-\beta t}\end{aligned} \tag{31}$$

$$\implies \|x_e(t)\| \leq \rho\left(\kappa_1 e^{-\alpha_v t} + \kappa_2 e^{-\beta t}\right). \tag{32}$$

From the above expression it can clearly be seen that for the dynamic trigger case, $lim_{t\to\infty}\|x_e(t)\| = 0$. This guarantees consensus of the states x. Also as $\beta < \alpha_v$ by choice, the rate of decay of the RHS of (32) will be dominated by β. For the static trigger case, $\beta = 0$

$$\begin{aligned}\|x_e(t)\| \leq& \rho\left(\kappa_1 e^{-\alpha_v t} + \kappa_2\right) \\ lim_{t\to\infty}\|x_e(t)\| \leq& lim_{t\to\infty}\rho\kappa_1 e^{-\alpha_v t} + lim_{t\to\infty}\rho\kappa_2 \\ \leq& \rho\kappa_2\end{aligned}$$

It can be seen that the edge-states converge to a ball around the origin $\|x_e\| \leq \rho\kappa_2$ exponentially with a rate of convergence bounded below by α_v.

Ruling out zeno behaviour in closed loop systems

The system states x_i and error states e_i together form a hybrid system, which makes it necessary for us to ensure that zeno behaviour does not occur. Say that for the i^{th} agent, an event is triggered at a time instant t_1 and another consecutive event is triggered at time t_2 for some $t_0 \leq t_1 < t_2$. To rule out the occurrence of zeno behaviour, it is sufficient to show that there exists a positive, non-zero lower bound on $\gamma := t_2 - t_1$. We have $\dot{e}_i = -\dot{x}_i$. Consider the following set of inequalities for time $t_1 \leq t < t_2$.

$$\|\dot{e}_i\| = \|\dot{x}_i\| \le \|\dot{x}\| = \|kD(\mathcal{G})\, W(t) D(\mathcal{G})^\top x(t_1)\|$$
$$\le k\|D(\mathcal{G})\|\, \|W(t)\|\, \|x_e(t_1)\|$$

Using (32) we can rewrite the above inequality as,

$$\|\dot{e}_i\| \le k\rho\|D(\mathcal{G})\|\, \|W(t)\|\, \left(\kappa_1 e^{-\alpha_v t_1} + \kappa_2 e^{-\beta t_1}\right). \quad (33)$$

Also,

$$\int_{t_1}^{t_2} \|\dot{e}_i\| dt \ge \left\| \int_{t_1}^{t_2} \dot{e}_i\, dt \right\| = \|e_i(t_2)\|. \quad (34)$$

We substitute $\|e_i(t_2)\| = ce^{-\beta t_2}$ as an event is triggered at t_2. Integrating the inequality (33) with limits t_1 and t_2 and using the fact that the edge weights are bounded such that $\|W(t)\| \le \omega$ and (34) we get,

$$\|e_i(t_2)\| \le \int_{t_1}^{t_2} k\rho\omega\|D(\mathcal{G})\|\, \left(\kappa_1 e^{-\alpha_v t_1} + \kappa_2 e^{-\beta t_1}\right) dt$$
$$ce^{-\beta t_2} \le k\rho\omega\|D(\mathcal{G})\|\, \left(\kappa_1 e^{-\alpha_v t_1} + \kappa_2 e^{-\beta t_1}\right)\gamma$$
$$ce^{-\beta\gamma} \le k\rho\omega\|D(\mathcal{G})\|\, \left(\kappa_1 e^{-(\alpha_v - \beta)t_1} + \kappa_2\right)\gamma$$

Rearranging the above inequality we get,

$$\gamma\, e^{\beta\gamma} \ge \frac{c}{k\rho\omega\|D(\mathcal{G})\|\, \left(\kappa_1 e^{-(\alpha_v-\beta)t_1} + \kappa_2\right)} \quad (35)$$

It is evident that $\gamma > 0$ as the RHS of (35) is strictly positive. The minimum value of γ is a solution of the following equation,

$$\gamma\, e^{\beta\gamma} = \frac{c}{k\rho\omega\|D(\mathcal{G})\|\, (\kappa_1 + \kappa_2)} \quad (36)$$

This proves that zeno behaviour does not occur in the dynanic trigger case. When $\beta = 0$, the lower bound on gamma is the RHS of equation (36), which rules out zeno behaviour in static trigger case as well. The preceding arguments on ruling out zeno behaviour are similar to those provided in Seyboth et al. (2011) for time-invariant graphs.

5.1 Consensus in Switching Topologies

Theorem 2 is not valid for switching topologies because our spanning tree and thereby the edge states x_e itself can be changing. The following corollary extends the aforementioned theorem to agents interacting via switching topologies.

Corollary 1. Let $t_1, t_2, \cdots$ be the infinite time sequence of graph switching instants with, $t_{i+1} - t_i \ge t_L$ for some positive t_L, and $i = 0, 1, \cdots$. Consider the agent dynamics (16) corresponding to a single integrator system with event-triggered control. If there exists an infinite sequence of contiguous, non-empty and uniformly bounded time intervals $\tau(j,1)$ $\left(Let\ \tau(j,l) = \left[t_{i_j}, t_{i_{j+l}}\right)\right)$; $j = 1, 2, \cdots$ starting at $t_{i_1} = t_0$, with the property that the union of the undirected graphs across each such interval has a spanning tree, then

(1) with static-trigger, the edge-state x_e converges to a ball of radius $\|x_e\| \le \rho\kappa_2^M$.
(2) with dynamic-trigger, the edge-state x_e converges exponentially to origin at rate greater than β.

Proof. The case of switching topologies differs from the setting of theorem 2 by the fact that the union of graphs in each time interval $\tau(j,1)$ contains a different spanning-tree compared to the occurrence of the same spanning tree in theorem 2. We show that Corollary 1 can be treated as an extension of Theorem 2 using the results of *Van der Waerden's theorem* ((Graham et al., 1980, p. 29)). The collection of possible spanning-tress forms a finite, non-empty set. Taking each possible spanning-tree to be a different colour, by Theorem (Graham et al., 1980, p. 29) it is possible to find N such that in the interval (t_{i_1}, t_{i_N}), the same spanning-tree occurs in each $\tau(1 + (j-1)d, 1)$ for $j = 1, 2, \cdots k$ and some $d > 0$. This allows us to select a persistence window $T > (d+1)t_{max}$ where $t_{max} = max_j\ \tau(j,1)$ for $j = 1, 2, \cdots N$. With the aforementioned selection of T, Theorem 2 can be invoked to prove the convergence of the edge states x_e to a ball around origin in the case of static trigger and to origin in the case of dynamic trigger. As we cannot predict which particular spanning-tree repeats in each interval $\tau(1 + (i-1)d, 1)$, the least conservative bounds are chosen.

6 Simulations

A multi-agent system with four agents under switching graphs was simulated using Matlab© for the static and dynamic trigger cases. The underlying communication topology considered is shown in figure 1 and the different spanning-trees that were switched between are shown in figure 2.

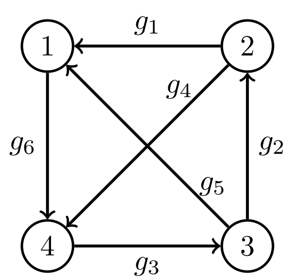

Fig. 1. Underlying graph of arbitrary orientation

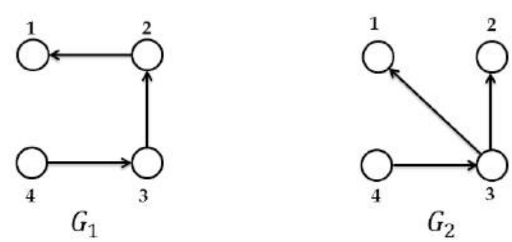

Fig. 2. Spanning Trees Considered

The edge-weights were chosen as g_i = square$(4*t, 20-(i-1)0.1\pi)+1).*sin(5*t)$ for $i = 1, 2\cdots 4$ and $g_6 = 0$, where the function square(at, b) for $a, b \in \mathbb{R}$ generates a square wave of unit amplitude, period $\frac{2\pi}{a}$ and duty-cycle $\left(\frac{T_{\text{on}}}{T_{\text{off}}+T_{\text{on}}}\right)$ b. The aforementioned g_i are defined to emulate real scenarios where there might be instances when no edges are active. The initial value of the states was taken to be $x_0 = [1\ 2\ 0.3\ 0.4]^\top$ Plots 3 and 4 show the evolution of the system states x and the norm of the edge states $\|x_e\|$ under the static trigger with $c = 0.5$ (refer equation (14)). The evolution of the states with dynamic trigger function with $\beta = 0.06$ and $c = 0.5$ (refer equation (15)) is plotted in figures (5) and (6). The bounds were calculated using inequalities presented in Corollary 1 and plotted along with the norm of the edge states. These plots show the convergence of the edge states x_e to a ball around the origin in case of static trigger and consensus of states x in the case of dynamic trigger.

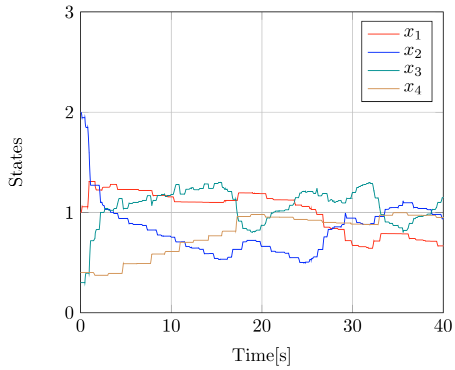

Fig. 3. Evolution of states under static trigger with the graph switching between spanning trees $\mathcal{G}_1$ and $\mathcal{G}_2$ in contiguous intervals

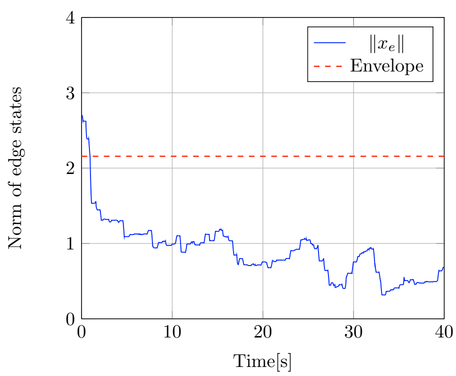

Fig. 4. Evolution of norm of edge-states under static trigger with the graph switching between spanning trees $\mathcal{G}_1$ and $\mathcal{G}_2$ in contiguous intervals

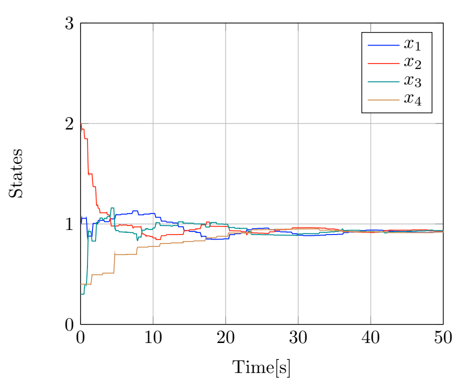

Fig. 5. Evolution of states under dynamic trigger with the graph switching between spanning trees $\mathcal{G}_1$ and $\mathcal{G}_2$ in contiguous intervals

7 Conclusions

The application of event-triggered control to classical consensus algorithms with time-varying, persistently exciting topologies guarantees consensus with dynamic trigger function. Under the more practically implementable static trigger function, the edge-states converge to a ball around the origin. For switching topologies, we utilize the work of Chowdhury et al. (2016) to show that we can extend the results of the persistent, continuously varying graphs to the case of switching topologies. The convergence bounds thus obtained depend on the 'slowest' spanning tree.

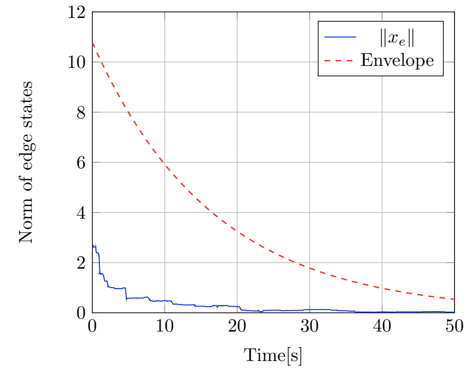

Fig. 6. Evolution of norm of edge-states under dynamic trigger with the graph switching between spanning trees $\mathcal{G}_1$ and $\mathcal{G}_2$ in contiguous intervals

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