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Model reduction by iterative error system approximation

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ABSTRACT

The analysis of a posteriori error estimates used in reduced basis methods leads to a model reduction scheme for linear time-invariant systems involving the iterative approximation of the associated error systems. The scheme can be used to improve reduced-order models (ROMs) with initial poor approximation quality at a computational cost proportional to that for computing the original ROM. We also show that the iterative approximation scheme is applicable to parametric systems and demonstrate its performance using illustrative examples.

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1. Introduction

Model order reduction (MOR) of dynamical systems has become a central topic in the computational sciences and engineering. In particular for linear time-invariant (LTI) systems, many MOR methodologies have been proposed in recent years. For a survey we refer to the books [1–3] and the recent survey paper [4]. Here, we will discuss the situation that a reduced-order LTI model has been computed by any projection-based method, and it is desirable to improve its approximation quality. This may be due to the fact that design specifications are not met with the original reduced-order model (ROM), or an a posteriori error analysis shows insufficient accuracy of the approximation. The goal is to employ the already obtained information from the ROM and an a posteriori error analysis in order to improve the approximation quality of the ROM in a systematic manner. Compared with the reduced basis (RB) method, which updates the ROM iteratively, the approach proposed here is more general in that any appropriate model reduction technique can be applied in the iteration steps, and the way the ROM is improved differs from the RB method in several aspects. We are only aware of the approach in [5,6], which yields a procedure similar to the one we suggest in this paper, but is derived from a different perspective. (The relations and differences of both approaches are discussed later in this paper.)

The present paper has been inspired by the *RB approach* to MOR of parametric systems [7]. In particular a careful analysis of the methodology presented in [8] leads to the developments reported later. The essence of our approach is an iterative method which at each step is performing MOR to the *error system*; as a result this error is reduced at each step of the iteration in a targeted fashion, thus yielding good reduced-order systems. Due to the different derivation of the method as compared to [5,6], it also becomes clear quickly that an extension to linear parametric problems is straightforward.

The paper is structured as follows. An analysis of the a posteriori error bounds in [8] leads to Lemma 2.1 which reveals the new structure of the error system. Consequences follow in Section 2.1 and the proposed new procedure is described in Section 3. The paper concludes with numerical

examples illustrating the performance of the method and a generalization to parametric systems.

1.1. Some preliminaries

Consider the single-input single-output (SISO) full-order model (FOM)

$$\mathbf{E} \frac{d}{dt} \mathbf{x} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u}, \mathbf{y} = \mathbf{C} \mathbf{x} \quad (1)$$

of order n (i.e. the state $\mathbf{x}(t)$ is in \mathbb{R}^n at any time t , $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$) with transfer function $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$. We construct the (SISO) ROM

$$\hat{\mathbf{E}} \frac{d}{dt} \hat{\mathbf{x}} = \hat{\mathbf{A}} \hat{\mathbf{x}} + \hat{\mathbf{B}} \mathbf{u}, \hat{\mathbf{y}} = \hat{\mathbf{C}} \hat{\mathbf{x}} \quad (2)$$

of order k by means of a Petrov–Galerkin projection defined by $\mathbf{V}, \mathbf{W} \in \mathbb{C}^{n \times k}$, i.e.

$$\hat{\mathbf{E}} = \mathbf{W}^* \mathbf{E} \mathbf{V}, \hat{\mathbf{A}} = \mathbf{W}^* \mathbf{A} \mathbf{V}, \hat{\mathbf{B}} = \mathbf{W}^* \mathbf{B}, \hat{\mathbf{C}} = \mathbf{C} \mathbf{V}.$$

The transfer function of the ROM, thus, is $\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$.

Assume for the time being that $\mathbf{E} = \mathbf{I}$, and $\mathbf{W}^* \mathbf{V} = \mathbf{I}$ (as in [8]). Given the FOM and ROM defined by Equations (1) and (2), respectively, the *error* and *residual* in *time domain* are defined as follows:

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{V} \hat{\mathbf{x}}(t), \quad (3)$$

$$\mathbf{r}(t) = \mathbf{A} \mathbf{V} \hat{\mathbf{x}}(t) + \mathbf{B} \mathbf{u}(t) - \mathbf{V} \frac{d}{dt} \hat{\mathbf{x}}(t), \quad (4)$$

Hence, the temporal evolution of the error is described by

$$\frac{d}{dt} \mathbf{e}(t) = \mathbf{A} \mathbf{e}(t) + \mathbf{r}(t). \quad (5)$$

Throughout this paper, we assume that \mathbf{A} and all reduced order matrices $\hat{\mathbf{A}}$ (or the associated matrix pairs (\mathbf{A}, \mathbf{E}) , $(\hat{\mathbf{A}}, \hat{\mathbf{E}})$) have all their eigenvalues in the open left half plane. This implies that the associated linear systems are (asymptotically) stable, as well as that the constant γ_1 in the following proposition is finite.

Proposition 1.1 (A posteriori error estimate [8].) *Let $\gamma_1 \geq \sup_{t \geq 0} \|\exp(\mathbf{A}t)\|$. Then the following bound holds:*

$$\|\mathbf{x}(t) - \mathbf{V} \hat{\mathbf{x}}(t)\| \leq \gamma_1 \left[\|\mathbf{e}(0)\| + \int_0^t \|\mathbf{r}(\tau)\| d\tau \right] =: \Delta_{\mathbf{x}}(t). \quad (6)$$

Furthermore: $\|\mathbf{y}(t) - \hat{\mathbf{y}}(t)\| \leq \|\mathbf{C}\| \Delta_{\mathbf{x}}(t)$.

Remark 1.1 Denoting the Laplace transforms of $\hat{\mathbf{x}}, \mathbf{u}$ by $\hat{\mathbf{X}}, \mathbf{U}$, the residual in frequency domain is

$$\mathbf{R}(s) = \mathbf{A} \mathbf{V} \hat{\mathbf{X}}(s) + \mathbf{B} \mathbf{U}(s) - s \mathbf{V} \hat{\mathbf{X}}(s)$$

$$\begin{aligned}
 &= (\mathbf{A} - s\mathbf{I})\mathbf{V}\hat{\mathbf{X}}(s) + \mathbf{B}\mathbf{U}(s) = [(\mathbf{A} - s\mathbf{I})\mathbf{V}[\mathbf{W}^*\Phi(s)\mathbf{V}]^{-1}\mathbf{W}^* + \mathbf{I}]\mathbf{B}\mathbf{U}(s) \\
 &= [\mathbf{I} - \Phi(s)\mathbf{V}(\mathbf{W}^*\Phi(s)\mathbf{V})^{-1}\mathbf{W}^*]\mathbf{B}\mathbf{U}(s),
 \end{aligned}$$

where $\Phi(s) = s\mathbf{I} - \mathbf{A}$. This expression (with $\mathbf{U}(s) = 1$) will appear prominently in the sequel.

1.2. Improved error estimation using the dual system

Following [8], we define the *dual system* on the interval $[0, T]$, with fixed final time T . The dual state will be denoted by \mathbf{x}^{du} :

$$\frac{d}{dt}\mathbf{x}^{\text{du}}(t) = \mathbf{A}^{\text{du}}\mathbf{x}^{\text{du}}(t) \text{ where } \mathbf{A}^{\text{du}} = -\mathbf{A}^*, \mathbf{x}^{\text{du}}(T) = \mathbf{C}^*. \quad (7)$$

Given the *dual projection* defined by $\mathbf{V}^{\text{du}}, \mathbf{W}^{\text{du}} \in \mathbb{C}^{n \times k}$, satisfying $\mathbf{W}^{\text{du}*}\mathbf{V}^{\text{du}} = \mathbf{I}$, the associated ROM is

$$\frac{d}{dt}\hat{\mathbf{x}}^{\text{du}}(t) = \hat{\mathbf{A}}^{\text{du}}\hat{\mathbf{x}}^{\text{du}}(t) \text{ where } \hat{\mathbf{A}}^{\text{du}} = -\mathbf{V}^{\text{du}*}\mathbf{A}^*\mathbf{W}^{\text{du}}, \hat{\mathbf{x}}^{\text{du}}(T) = \mathbf{V}^{\text{du}*}\mathbf{C}^*. \quad (8)$$

In this framework, the projection defined by \mathbf{V}, \mathbf{W} in Section 1.1 is sometimes referred to as the *primal projection*. The resulting error and residual of the dual FOM and ROM are:

$$\mathbf{e}^{\text{du}}(t) = \mathbf{x}^{\text{du}}(t) - \mathbf{W}^{\text{du}}\hat{\mathbf{x}}^{\text{du}} \text{ and} \quad (9)$$

$$\mathbf{r}^{\text{du}}(t) = \mathbf{A}^{\text{du}}\mathbf{W}^{\text{du}}\hat{\mathbf{x}}^{\text{du}}(t) - \mathbf{W}^{\text{du}}\frac{d}{dt}\hat{\mathbf{x}}^{\text{du}}(t) = (\mathbf{A}^{\text{du}}\mathbf{W}^{\text{du}} - \mathbf{W}^{\text{du}}\hat{\mathbf{A}}^{\text{du}})\hat{\mathbf{x}}^{\text{du}}(t). \quad (10)$$

Thus, the dual error satisfies the differential equation:

$$\frac{d}{dt}\mathbf{e}^{\text{du}}(t) = \mathbf{A}^{\text{du}}\mathbf{e}^{\text{du}}(t) + \mathbf{r}^{\text{du}}(t), \text{ where } \mathbf{e}^{\text{du}}(T) = (\mathbf{I} - \mathbf{W}^{\text{du}}\mathbf{V}^{\text{du}*})\mathbf{C}^*. \quad (11)$$

Proposition 1.2 (Dual a posteriori error estimate [8].) *Let $\gamma_1^{\text{du}} \geq \sup_{t \in [0, T]} \|\exp(-\mathbf{A}^{\text{du}}t)\|$. Then the following bound holds:*

$$\|\mathbf{x}^{\text{du}}(t) - \mathbf{W}^{\text{du}}\hat{\mathbf{x}}^{\text{du}}(t)\| \leq \gamma_1^{\text{du}} \left[\|\mathbf{e}^{\text{du}}(T)\| + \int_t^T \|\mathbf{r}^{\text{du}}(\tau)\| d\tau \right] =: \Delta_x^{\text{du}}(t). \quad (12)$$

The next result provides an exact expression for the output error $\mathbf{y} - \hat{\mathbf{y}}$, where the two outputs are defined by Equations (1) and (2), respectively. This expression involves both the primal and the dual reduced systems. In the sequel, the inner product of the complex vectors \mathbf{x}, \mathbf{y} is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^*\mathbf{y}$.

Lemma 1.1 (Output error equality [8].) *The reduced output error at time T satisfies*

$$\mathbf{y}(T) - \hat{\mathbf{y}}(T) = \mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 + \mathcal{T}_4, \quad \text{with}$$

$$\mathcal{T}_1 = \langle \mathbf{e}^{\text{du}}(T), \mathbf{e}(T) \rangle, \text{ where } \mathbf{e}^{\text{du}}(T) = (\mathbf{I} - \mathbf{W}^{\text{du}}\mathbf{V}^{\text{du}*})\mathbf{C}^* \text{ and}$$

$$\mathbf{e}(T) = e^{AT} \mathbf{e}(0) + \int_0^T e^{A(T-\tau)} \mathbf{r}(\tau) d\tau,$$

$$\mathcal{T}_2 = \left\langle \mathbf{W}^{\text{du}} \hat{\mathbf{x}}^{\text{du}}(0), \mathbf{e}(0) \right\rangle, \text{ Where } \hat{\mathbf{x}}^{\text{du}}(t) = e^{\hat{\mathbf{A}}^{\text{du}}(t-T)} \underbrace{\hat{\mathbf{x}}^{\text{du}}(T)}_{\mathbf{V}^{\text{du}*} \mathbf{C}^*} \text{ and}$$

$$\mathbf{e}(0) = (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{x}(0),$$

$$\mathcal{T}_3 = \int_0^T \langle \mathbf{r}^{\text{du}}(t), \mathbf{e}(t) \rangle dt, \quad \text{where } \mathbf{r}^{\text{du}}(t) = (\mathbf{A}^{\text{du}} \mathbf{W}^{\text{du}} - \mathbf{W}^{\text{du}} \hat{\mathbf{A}}^{\text{du}}) \hat{\mathbf{x}}^{\text{du}}(t) \text{ and}$$

$$\mathbf{e}(t) = e^{At} \mathbf{e}(0) + \int_0^t e^{A(t-\tau)} \mathbf{r}(\tau) d\tau,$$

$$\mathcal{T}_4 = \int_0^T \langle \mathbf{W}^{\text{du}} \hat{\mathbf{x}}^{\text{du}}(t), \mathbf{r}(t) \rangle dt, \quad \text{Where } \mathbf{r}(t) = (\mathbf{A}\mathbf{V} - \mathbf{V}\hat{\mathbf{A}}) \hat{\mathbf{x}}(t) + (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{B}\mathbf{u}(t).$$

At this point (following [8]), the primal and dual systems are combined to define an improved reduced system output at time T , namely:

$$\hat{\mathbf{y}}(T) = \hat{\mathbf{y}}(T) + \mathcal{T}_2 + \mathcal{T}_4 \quad (13)$$

Our goal in the sequel is to analyse this expression so as to obtain an **explicit reduced-order** system and hence an explicit error system for this improved reduced system output.

2. Analysis of the improved reduced system

In order to determine the linear system behind Equation (13), we start with some consequences of the above definitions:

$$\hat{\mathbf{y}}(T) = \mathbf{C}\mathbf{V}\mathbf{e}^{\mathbf{W}^* \mathbf{A}\mathbf{V}T} \mathbf{W}^* \mathbf{x}(0) + \int_0^T \mathbf{C}\mathbf{V}\mathbf{e}^{\mathbf{W}^* \mathbf{A}\mathbf{V}(T-\tau)} \mathbf{W}^* \mathbf{B}\mathbf{u}(T-\tau) d\tau,$$

$$\mathcal{T}_2 = \langle \mathbf{W}^{\text{du}} \hat{\mathbf{x}}^{\text{du}}(0), \mathbf{e}(0) \rangle = \mathbf{C}\mathbf{V}^{\text{du}} \mathbf{e}^{(\mathbf{W}^{\text{du}})^* \mathbf{A}\mathbf{V}^{\text{du}} T} (\mathbf{W}^{\text{du}})^* (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{x}(0).$$

In order to investigate the term \mathcal{T}_4 , we note that

$$\mathbf{W}^{\text{du}} \hat{\mathbf{x}}^{\text{du}}(t) = \mathbf{W}^{\text{du}} \mathbf{e}^{-\hat{\mathbf{A}}^{\text{du}}(T-t)} \hat{\mathbf{x}}^{\text{du}}(T) = \mathbf{W}^{\text{du}} \mathbf{e}^{(\mathbf{V}^{\text{du}})^* \mathbf{A}^* \mathbf{W}^{\text{du}}(T-t)} (\mathbf{V}^{\text{du}})^* \mathbf{C}^*$$

Therefore, with $\mathbf{r}(t) = (\mathbf{A}\mathbf{V} - \mathbf{V}\hat{\mathbf{A}}) \hat{\mathbf{x}}(t) + (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{B}\mathbf{u}(t)$, we get

$$\begin{aligned} \mathcal{T}_4 &= \int_0^T \left[\mathbf{W}^{\text{du}} \mathbf{e}^{(\mathbf{V}^{\text{du}})^* \mathbf{A}^* \mathbf{W}^{\text{du}}(T-t)} (\mathbf{V}^{\text{du}})^* \mathbf{C}^* \right]^* [(\mathbf{A}\mathbf{V} - \mathbf{V}\hat{\mathbf{A}}) \hat{\mathbf{x}}(t) + (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{B}\mathbf{u}(t)] dt \\ &= \int_0^T \mathbf{C}\mathbf{V}^{\text{du}} \mathbf{e}^{(\mathbf{W}^{\text{du}})^* \mathbf{A}\mathbf{V}^{\text{du}}(T-t)} (\mathbf{W}^{\text{du}})^* [(\mathbf{A}\mathbf{V} - \mathbf{V}\hat{\mathbf{A}}) \hat{\mathbf{x}}(t) + (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{B}\mathbf{u}(t)] dt \end{aligned}$$

$$\begin{aligned}
 &= \int_0^T \mathbf{C}\mathbf{V}^{\text{du}} \mathbf{e}^{(\mathbf{W}^{\text{du}})^* \mathbf{A}\mathbf{V}^{\text{du}}(T-t)} (\mathbf{W}^{\text{du}})^* (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{B}\mathbf{u}(t) dt \\
 &+ \int_0^T \mathbf{C}\mathbf{V}^{\text{du}} \mathbf{e}^{(\mathbf{W}^{\text{du}})^* \mathbf{A}\mathbf{V}^{\text{du}}(T-t)} (\mathbf{W}^{\text{du}})^* (\mathbf{A}\mathbf{V} - \mathbf{V}\hat{\mathbf{A}}) \hat{\mathbf{x}}(t) dt.
 \end{aligned}$$

Thus, $\mathcal{T}_2 + \mathcal{T}_4$ represents the analytical solution to the system:

$$\frac{d}{dt} \hat{\mathbf{x}}_2(t) = \mathbf{A}_2 \hat{\mathbf{x}}_2(t) + \mathbf{W}^{\text{du}*} \mathbf{r}(t), \quad \hat{\mathbf{y}}_2(t) = \mathbf{C}_2 \hat{\mathbf{x}}_2(t)$$

with initial condition $\hat{\mathbf{x}}_2(0) = (\mathbf{W}^{\text{du}})^* (\mathbf{I} - \mathbf{V}\mathbf{W}^*) \mathbf{x}(0)$. Here, $\mathbf{A}_2 = (\mathbf{W}^{\text{du}})^* \mathbf{A}\mathbf{V}^{\text{du}}$, $\mathbf{B}_2 = (\mathbf{W}^{\text{du}})^*$, $\mathbf{C}_2 = \mathbf{C}\mathbf{V}^{\text{du}}$. Hence putting the equations together yields the following augmented reduced system, where the reduced state of the primal system $\hat{\mathbf{x}}$ is redefined as $\hat{\mathbf{x}}_1$:

$$\begin{aligned}
 \frac{d}{dt} \hat{\mathbf{x}}_1(t) &= \mathbf{A}_1 \hat{\mathbf{x}}_1(t) + \mathbf{B}_1 \mathbf{u}(t), & \hat{\mathbf{y}}_1(t) &= \mathbf{C}_1 \hat{\mathbf{x}}_1(t) \\
 \frac{d}{dt} \hat{\mathbf{x}}_2(t) &= \mathbf{A}_2 \hat{\mathbf{x}}_2(t) + \mathbf{B}_2 [(\mathbf{A}\mathbf{V} - \mathbf{V}\mathbf{A}_1) \hat{\mathbf{x}}_1(t) + \mathbf{\Pi}\mathbf{B}\mathbf{u}(t)], & \hat{\mathbf{y}}_2(t) &= \mathbf{C}_2 \hat{\mathbf{x}}_2(t),
 \end{aligned}$$

where

$$\mathbf{A}_1 = \mathbf{W}^* \mathbf{A}\mathbf{V}, \quad \mathbf{B}_1 = \mathbf{W}^* \mathbf{B}, \quad \mathbf{C}_1 = \mathbf{C}\mathbf{V}, \quad \mathbf{\Pi} = \mathbf{I} - \mathbf{V}\mathbf{W}^*.$$

Hence,

$$\frac{d}{dt} \hat{\mathbf{x}}_2(t) + \mathbf{W}^{\text{du}*} \mathbf{V} \frac{d}{dt} \hat{\mathbf{x}}_1(t) = \mathbf{A}_2 \hat{\mathbf{x}}_2 + \mathbf{W}^{\text{du}*} \mathbf{A}\mathbf{V} \hat{\mathbf{x}}_1(t) + \mathbf{W}^{\text{du}*} \mathbf{B}\mathbf{u}(t).$$

Rewriting these equations with $\hat{\xi} = \begin{bmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix}$ yields

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{W}^{\text{du}*} \mathbf{V} & \mathbf{I} \end{bmatrix} \frac{d}{dt} \hat{\xi} = \begin{bmatrix} \mathbf{W}^* \mathbf{A}\mathbf{V} & \mathbf{0} \\ \mathbf{W}^{\text{du}*} \mathbf{A}\mathbf{V} & \mathbf{W}^{\text{du}*} \mathbf{A}\mathbf{V}^{\text{du}} \end{bmatrix} \hat{\xi} + \begin{bmatrix} \mathbf{W}^* \mathbf{B} \\ \mathbf{W}^{\text{du}*} \mathbf{B} \end{bmatrix} \mathbf{u}, \quad \hat{\mathbf{y}} = [\mathbf{C}\mathbf{V}, \mathbf{C}\mathbf{V}^{\text{du}}] \hat{\xi}.$$

Thus defining

$$\mathbf{V}_1 = \mathbf{V}, \quad \mathbf{V}_2 = \mathbf{V}^{\text{du}}, \quad \mathbf{W}_1 = \mathbf{W}, \quad \mathbf{W}_2 = \mathbf{W}^{\text{du}},$$

and re-inserting $\mathbf{E} \neq \mathbf{I}$, we get the following result.

Lemma 2.1 *The system with output $\tilde{\mathbf{y}}$ as defined in (13) has the following generalized state form:*

$$\mathbf{E}_\xi \frac{d}{dt} \hat{\xi} = \mathbf{A}_\xi \hat{\xi} + \mathbf{B}_\xi \mathbf{u}, \quad \tilde{\mathbf{y}} = \mathbf{C}_\xi \hat{\xi}, \quad \hat{\xi}(0) = \mathbf{W}_\xi^* \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{\Pi}\mathbf{x}(0) \end{bmatrix},$$

where $\mathbf{\Pi} = \mathbf{I} - \mathbf{E}\mathbf{V}(\mathbf{W}^* \mathbf{E}\mathbf{V})^{-1} \mathbf{W}^*$, $\mathbf{V}_\xi = \begin{bmatrix} \mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 \end{bmatrix}$, $\mathbf{W}_\xi = \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_2 \end{bmatrix}$, and

$$\mathbf{E}_\xi = \mathbf{W}_\xi^* \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{E} & \mathbf{E} \end{bmatrix} \mathbf{V}_\xi, \quad \mathbf{A}_\xi = \mathbf{W}_\xi^* \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{A} \end{bmatrix} \mathbf{V}_\xi, \quad \mathbf{B}_\xi = \mathbf{W}_\xi^* \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix}, \quad \mathbf{C}_\xi = [\mathbf{C}, \mathbf{C}] \mathbf{V}_\xi.$$

From the above lemma, it follows for the error of the transfer function $\tilde{\mathbf{H}}$ associated to the improved reduced system with output $\tilde{\mathbf{y}}$:

$$\begin{aligned} \mathbf{H}(s) - \tilde{\mathbf{H}}(s) &= \mathbf{H}(s) - \underbrace{[\mathbf{C}\mathbf{V}_1]}_{\mathbf{C}_1}, \underbrace{[\mathbf{C}\mathbf{V}_2]}_{\mathbf{C}_2} \left[\begin{array}{c} \overbrace{\mathbf{W}_1^*(s\mathbf{E} - \mathbf{A})\mathbf{V}_1}^{\Phi_{11}(s)} \\ \underbrace{\mathbf{W}_2^*(s\mathbf{E} - \mathbf{A})\mathbf{V}_1}_{\Phi_{21}(s)} \\ \mathbf{0} \\ \underbrace{\mathbf{W}_2^*(s\mathbf{E} - \mathbf{A})\mathbf{V}_2}_{\Phi_{22}(s)} \end{array} \right]^{-1} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{W}_1^*\mathbf{B} \\ \mathbf{W}_2^*\mathbf{B} \\ \mathbf{B}_2 \end{bmatrix} \\ &= \mathbf{C}\Phi^{-1}(s)\mathbf{B} - \mathbf{C}_1\Phi_{11}^{-1}(s)\mathbf{B}_1 - \mathbf{C}_2\Phi_{22}^{-1}(s)\mathbf{B}_2 + \mathbf{C}_2\Phi_{22}^{-1}(s)\Phi_{21}(s)\Phi_{11}^{-1}(s)\mathbf{B}_1. \end{aligned}$$

Here, $\Phi(s) = s\mathbf{E} - \mathbf{A}$.

Corollary 2.1 *The transfer function of the error system is*

$$\mathbf{H}_{\text{err}}(s) = \mathbf{H}(s) - \tilde{\mathbf{H}}(s) = \mathbf{C}[\Phi(s)^{-1} - \mathbf{V}_2\Phi_{22}(s)^{-1}\mathbf{W}_2^*]\Phi(s)[\Phi(s)^{-1} - \mathbf{V}_1\Phi_{11}(s)^{-1}\mathbf{W}_1^*]\mathbf{B} \quad (14)$$

Time domain representation of the error system

With the augmented system $\mathbf{E}_a = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{E} & \mathbf{E} \end{bmatrix}$, $\mathbf{A}_a = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{A} \end{bmatrix}$, $\mathbf{B}_a = \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix}$, $\mathbf{C}_a = [\mathbf{C}, \mathbf{C}]$, the augmented state $\mathbf{x}_a(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix}$ satisfies

$$\mathbf{E}_a \frac{d}{dt} \mathbf{x}_a(t) = \mathbf{A}_a \mathbf{x}_a(t) + \mathbf{B}_a \mathbf{u}(t), \mathbf{y}(t) = \mathbf{C}_a \mathbf{x}_a(t) \quad \Rightarrow \quad \mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s),$$

in other words the augmented system is a non-minimal realization of the original system $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$. Consider the augmented error system:

$$\mathbf{E}_a \frac{d}{dt} \mathbf{e}_a(t) = \mathbf{A}_a \mathbf{e}_a(t) + \mathbf{r}_a(t), \mathbf{y}_{\text{err}}(t) = \mathbf{C}_a \mathbf{e}_a(t), \quad (15)$$

where $r_a(t)$ is the associated augmented residual defined by

$$\mathbf{r}_a(t) = \mathbf{A}_a \mathbf{V}_\xi \mathbf{W}_\xi^* \mathbf{x}_a(t) + \mathbf{B}_a \mathbf{u}(t) - \mathbf{E}_a \mathbf{V}_\xi \mathbf{W}_\xi^* \frac{d}{dt} \mathbf{x}_a(t). \quad (16)$$

Corollary 2.2 *Equations (15) and (16) describe the associated error in generalized state space form. That is, there holds: $\mathbf{y}_{\text{err}}(s) = \mathbf{H}_{\text{err}}(s)\mathbf{u}(s)$, where $\mathbf{H}_{\text{err}}(s)$ is defined by (14).*

2.1. Some comments and consequences

(a) The interpretation of the ROM obtained here can be done without reference to the dual. This system namely has triangular structure (the first projection affects the second, but not vice-versa). In addition, while the improved system output in Equation (13) is defined only for time T , here, this restriction is lifted and $\tilde{\mathbf{y}}(t)$ in Lemma 2.1 is defined for all time.

(b) The error system (Equation (14)) factors in a product of two residues, the first coming from the original projection and the second from the second (also referred to as dual) projection. Specifically, the transfer function $\mathbf{H}_{\text{err}}(s)$ can be rewritten as

$$\mathbf{H}_{\text{err}}(s) = \underbrace{\mathbf{C}[\mathbf{I} - \mathbf{V}_2\Phi_{22}^{-1}(s)\mathbf{W}_2^*]}_{=:R^{\text{du}}(s)} \Phi^{-1}(s) \underbrace{[\mathbf{I} - \Phi\mathbf{V}_1\Phi_{11}^{-1}(s)\mathbf{W}_1^*]}_{=:R^{\text{pr}}(s)} \mathbf{B} \quad (17)$$

As we have assumed SISO systems, it is straightforward to get

$$\|\mathbf{H}_{\text{err}}(s)\| \leq \|R^{\text{du}}(s)\|_2 \|R^{\text{pr}}(s)\|_2 / \sigma_{\min}(\Phi(s)) =: \Delta(s),$$

where $R^{\text{Pr}}(s)$ is the frequency-domain expression of the residual $\mathbf{r}(t)$ of the ROM (2), and $R^{\text{du}}(s)$ is the frequency-domain expression of the residual $\mathbf{r}^{\text{du}}(t)$ of the reduced dual system (Equation (8)).

Essentially, the error bound $\Delta(s)$ decays quadratically as it is the product of the two residuals $\|R^{\text{du}}(s)\|_2$ and $\|R^{\text{Pr}}(s)\|_2$. $|\mathbf{H}_{\text{err}}(s)|$ is the true error, often decaying faster than quadratic, and sometimes, it is much smaller than the product of $\|R^{\text{du}}(s)\|_2$ and $\|R^{\text{Pr}}(s)\|_2$, see [Figure 7](#) for an experimental illustration, where $\mathbf{H}_{\text{err}}^{(2)}$ is the error system (Equation (17)) of a parametric model.

(c) The primal and dual projections can be of different dimension k , ℓ , respectively; the dimension of the ROM is then $k + \ell$. Notice also that the ROM above *cannot* be obtained by means of an (explicit) Petrov–Galerkin projection applied to $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$. Instead the (block diagonal) projection defined by $\mathbf{V}_\xi, \mathbf{W}_\xi$, must be applied to a *non-minimal realization* of the original system, namely $\mathbf{E}_a = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{E} & \mathbf{E} \end{bmatrix}$, $\mathbf{A}_a = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{A} \end{bmatrix}$, $\mathbf{B}_a = \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix}$, $\mathbf{C}_a = [\mathbf{C}, \mathbf{C}]$.

(d) The error system (14) can be written as

$$\begin{aligned} & \mathbf{C}[\Phi^{-1}(s) - \mathbf{V}_2 \Phi_{22}^{-1}(s) \mathbf{W}_2^*] \Phi(s) [\Phi^{-1}(s) - \mathbf{V}_1 \Phi_{11}^{-1}(s) \mathbf{W}_1^*] \mathbf{B} \\ &= \mathbf{C}[\Phi^{-1}(s) - \mathbf{V}_2 \Phi_{22}^{-1}(s) \mathbf{W}_2^*] \underbrace{[\mathbf{I} - \Phi(s) \mathbf{V}_1 \Phi_{11}^{-1}(s) \mathbf{W}_1^*] \mathbf{B}}_{\mathbf{H}_w^{(1)}} \end{aligned}$$

Thus, one suggestion for making use of the above formula is

Step 1. Choose the first (primal) projection $\mathbf{V}_1, \mathbf{W}_1$.

Step 2. Compute the second (dual) projection $\mathbf{V}_2, \mathbf{W}_2$ by means of *weighted* reduction, where the to-be-reduced system is $\mathbf{C}\Phi^{-1}(s)$ and the (right) weight is $\mathbf{H}_w^{(1)}$.

(e) The output of the error system (Equation (15)) describes the time-domain output error of the improved ROM defined in Lemma 2.1. In [Section 4](#), we plot the time-domain output errors of the successively constructed ROMs for a parametric system in [Figure 6](#), where the monotonic decay of the errors can be observed.

(f) In this framework, it readily follows that three or more stages can be considered. In the case of three stages, we project the system

$$\mathbf{E}_3 = \begin{bmatrix} \mathbf{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{E} & \mathbf{E} & \mathbf{0} \\ \mathbf{E} & \mathbf{E} & \mathbf{E} \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{A} & \mathbf{0} \\ \mathbf{A} & \mathbf{A} & \mathbf{A} \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \\ \mathbf{B} \end{bmatrix}, \quad \mathbf{C}_3 = [\mathbf{C}, \mathbf{C}, \mathbf{C}],$$

by $\mathbf{V} = \text{blkdiag}[\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3]$, and $\mathbf{W} = \text{blkdiag}[\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3]$, to obtain

$$\mathbf{E}_\xi = \mathbf{W}_\xi \mathbf{E}_3 \mathbf{V}_\xi, \quad \mathbf{A}_\xi = \mathbf{W}_\xi \mathbf{A}_3 \mathbf{V}_\xi, \quad \mathbf{B}_\xi = \mathbf{W}_\xi \mathbf{B}_3 \mathbf{V}_\xi, \quad \mathbf{C}_\xi = \mathbf{W}_\xi \mathbf{C}_3 \mathbf{V}_\xi.$$

Let also $\Phi_i(s) = s\mathbf{E}_i - \mathbf{A}_i$, and recall that $\mathbf{C}_3[s\mathbf{E}_3 - \mathbf{A}_3]^{-1} \mathbf{B}_3 = \mathbf{C}[\mathbf{sI} - \mathbf{A}]^{-1} \mathbf{B}$, i.e. $(\mathbf{E}_3, \mathbf{A}_3, \mathbf{B}_3, \mathbf{C}_3)$ is a non-minimal realization of $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$. The transfer function of the associated *error system* is

$$\mathbf{H}_{\text{err}}^{(3)}(s) = \mathbf{C}[\Phi^{-1}(s) - \mathbf{V}_3 \Phi_3^{-1}(s) \mathbf{W}_3^*] \underbrace{\Phi(s) [\Phi^{-1}(s) - \mathbf{V}_2 \Phi_2^{-1}(s) \mathbf{W}_2^*] \Phi(s) [\Phi^{-1}(s) - \mathbf{V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^*] \mathbf{B}}_{\mathbf{H}_w^{(21)}(s)}.$$

Thus, the to-be-approximated system here is still $\mathbf{C}\Phi^{-1}(s)$, while the weighting function is $\mathbf{H}_w^{(21)}$.

Hence, we may conclude that this method leads to model reduction by successive approximation of the ensuing error systems.

(g) Some remarks on the related literature are in order.

- (1) Some of the above considerations have been used in [9], e.g. some of the discussions in the comments (b) and (c).
- (2) In [5,6], Panzer, Wolf, and Lohmann have obtained an expression similar to Equation (14) and its generalization to more than two stages. The motivation that led to these results,

however, is different from the reduced-basis motivation used here. The papers above contain an additional result which is not used here: namely the fact is used that if \mathbf{V} is constructed by means of a rational Krylov method based on \mathbf{E} , \mathbf{A} , \mathbf{B} , then $\mathbf{\Pi AV}$ is rank one, and in particular it can be factored as $\mathbf{\Pi AV} = (\mathbf{\Pi B})(\tilde{\mathbf{C}})$, where $\tilde{\mathbf{C}}$ is a row vector. In these expressions $\mathbf{\Pi} = \mathbf{I} - \mathbf{E V}(\mathbf{W}^* \mathbf{E V})^{-1} \mathbf{W}^*$, and \mathbf{W} is arbitrary.

3. The new procedure: weighted reduction of successive error systems

In the following, we give a detailed description of the iterative procedure to improve a ROM using successive approximation of the error system. We use the following notation throughout:

$$\Phi_i(s) = s \mathbf{W}_i^* \mathbf{E V}_i - \mathbf{W}_i^* \mathbf{A V}_i, \quad i = 1, \dots, q.$$

(A) **Data:** given is the system $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$ of McMillan degree n , with transfer function $\mathbf{H}(s) = \mathbf{C} \Phi^{-1}(s) \mathbf{B}$, where $\Phi(s) = s \mathbf{E} - \mathbf{A}$.

(B) **1st step:** find $\mathbf{V}_1, \mathbf{W}_1 \in \mathbb{R}^{n \times k_1}$, and construct the corresponding ROM:

$$\mathbf{H}_{\text{red}}^{(1)}(s) = \mathbf{C}_1 \Phi_1^{-1}(s) \mathbf{B}_1, \quad \text{where}$$

$$\mathbf{C}_1 = \mathbf{C V}_1, \quad \mathbf{E}_1 = \mathbf{W}_1^* \mathbf{E V}_1, \quad \mathbf{A}_1 = \mathbf{W}_1^* \mathbf{A V}_1, \quad \mathbf{B}_1 = \mathbf{W}_1^* \mathbf{B}.$$

It follows that the error system can be written as

$$\mathbf{H}_{\text{err}}^{(1)}(s) = \mathbf{H}(s) - \mathbf{H}_{\text{red}}^{(1)}(s) = \mathbf{C} \Phi^{-1} \mathbf{B} - \mathbf{C}_1 \Phi_1^{-1} \mathbf{B}_1 = \mathbf{C} \Phi^{-1}(s) [\mathbf{I} - \Phi(s) \mathbf{V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^*] \mathbf{B}.$$

Introducing the projection $\mathbf{\Pi}_1 = \mathbf{I} - \mathbf{E V}_1 (\mathbf{W}_1^* \mathbf{E V}_1)^{-1} \mathbf{W}_1^*$, we can eliminate $\Phi(s)$ as follows:

$$\mathbf{I} - \Phi(s) \mathbf{V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^* = \mathbf{\Pi}_1 + \mathbf{\Pi}_1 \mathbf{A V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^*. \quad (18)$$

Hence the error system above can be expressed as follows, where $\mathbf{H}_w^{(1)}(s)$ is the weight:

$$\mathbf{H}_{\text{err}}^{(1)}(s) = \mathbf{C} \Phi^{-1}(s) \underbrace{[\mathbf{\Pi}_1 + \mathbf{\Pi}_1 \mathbf{A V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^*]}_{\mathbf{H}_w^{(1)}(s)} \mathbf{B}. \quad (19)$$

(C) **2nd step:** we now seek $\mathbf{V}_2, \mathbf{W}_2 \in \mathbb{R}^{n \times k_2}$, where the resulting reduced system with

$$\mathbf{V}_{12} = \text{blkdiag}[\mathbf{V}_1, \mathbf{V}_2], \quad \mathbf{W}_{12} = \text{blkdiag}[\mathbf{W}_1, \mathbf{W}_2], \quad \text{is}$$

$$\mathbf{H}_{\text{red}}^{(2)}(s) = [\mathbf{C}, \mathbf{C}] \mathbf{V}_{12} \left[\mathbf{W}_{12}^* \begin{bmatrix} \Phi & \mathbf{0} \\ \Phi & \Phi \end{bmatrix} \mathbf{V}_{12} \right]^{-1} \mathbf{W}_{12}^* \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix}.$$

It readily follows that the resulting error system is

$$\mathbf{H}_{\text{err}}^{(2)}(s) = \mathbf{H}(s) - \mathbf{H}_{\text{red}}^{(2)}(s) = [\mathbf{C} \Phi^{-1}(s) - \mathbf{C V}_2 \Phi_2^{-1}(s) \mathbf{W}_2^*] \underbrace{[\mathbf{I} - \Phi(s) \mathbf{V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^*]}_{\mathbf{H}_w^{(1)}(s)} \mathbf{B}.$$

Consequently, the second step amounts to the approximation of $\mathbf{C} \Phi^{-1}(s)$, subject to the weight

$$\mathbf{H}_w^{(1)}(s) = [\mathbf{\Pi}_1 + \mathbf{\Pi}_1 \mathbf{A V}_1 \Phi_1^{-1}(s) \mathbf{W}_1^*] \mathbf{B}. \quad (20)$$

(Note that $\mathbf{C} \Phi^{-1} \mathbf{H}_w^{(1)}$ is nothing but the first *error system* $\mathbf{H}_{\text{err}}^{(1)}(s)$ (Equation 19). Therefore, approximation of $\mathbf{C} \Phi^{-1}(s)$, subject to the weight $\mathbf{H}_w^{(1)}(s)$, implicates that $\mathbf{V}_2, \mathbf{W}_2$ are obtained from applying MOR to the *error system* $\mathbf{H}_{\text{err}}^{(1)}(s)$.)

(D) **q -th step:** we seek $\mathbf{V}_q, \mathbf{W}_q \in \mathbb{R}^{n \times k_q}$, where the resulting reduced system with

$\mathbf{V}_{1q} = \text{blkdiag}[\mathbf{V}_1, \dots, \mathbf{V}_2, \mathbf{V}_q]$, $\mathbf{W}_{1q} = \text{blkdiag}[\mathbf{W}_1, \dots, \mathbf{W}_2, \mathbf{W}_q]$, is

$$\mathbf{H}_{\text{red}}^{(q)}(s) = [\mathbf{C}, \dots, \mathbf{C}, \mathbf{C}] \mathbf{V}_{1q} \begin{bmatrix} \mathbf{W}_{1q}^* \begin{bmatrix} \Phi & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \Phi & & \Phi & \mathbf{0} \\ \Phi & \dots & \Phi & \Phi \end{bmatrix} \mathbf{V}_{1q} \end{bmatrix}^{-1} \mathbf{W}_{1q}^* \begin{bmatrix} \mathbf{B} \\ \vdots \\ \mathbf{B} \\ \mathbf{B} \end{bmatrix}.$$

It readily follows that the resulting error system is

$$\begin{aligned} \mathbf{H}_{\text{err}}^{(q)}(s) &= \mathbf{H}(s) - \mathbf{H}_{\text{red}}^{(q)}(s) \\ &= \left[\mathbf{C}\Phi^{-1}(s) - \mathbf{C}\mathbf{V}_q\Phi_q^{-1}(s)\mathbf{W}_q^* \right] \underbrace{\left[\mathbf{I} - \Phi(s)\mathbf{V}_{q-1}\Phi_{q-1}^{-1}(s)\mathbf{W}_{q-1}^* \right]}_{\mathbf{H}_w^{(q-1)}(s)} \cdots \underbrace{\left[\mathbf{I} - \Phi(s)\mathbf{V}_1\Phi_1^{-1}(s)\mathbf{W}_1^* \right]}_{\mathbf{H}_w^{(1)}(s)} \mathbf{B} \end{aligned}$$

Consequently the q -th step amounts to the approximation of $\mathbf{C}\Phi^{-1}(s)$, subject to the weight $\mathbf{H}_w^{(q-1)} \cdots \mathbf{B}_w^{(1)}$, the newest component of the weight being

$$\mathbf{H}_w^{(q-1)}(s) = \left[\mathbf{\Pi}_{q-1} + \mathbf{\Pi}_{q-1}\mathbf{A}\mathbf{V}_{q-1}\Phi_{q-1}^{-1}(s)\mathbf{W}_{q-1}^* \right], \quad (21)$$

where $\mathbf{\Pi}_{q-1} = \mathbf{I} - \mathbf{E}\mathbf{V}_{q-1} \left(\mathbf{W}_{q-1}^* \mathbf{E}\mathbf{V}_{q-1} \right)^{-1} \mathbf{W}_{q-1}^*$.

Remark 3.1 (Computation of $\mathbf{V}_i, \mathbf{W}_i$, $i > 1$) From the above procedure, we know that at each iteration step, a pair of matrices $\mathbf{V}_i, \mathbf{W}_i$, $i = 1, \dots, q$, is computed, such that an updated ROM (with error $\mathbf{H}_{\text{err}}^{(i)}(s)$) of the original system is obtained. We observe that

$$\mathbf{H}_{\text{err}}^{(i)}(s) = \mathbf{H}_{\text{err}}^{(i-1)}(s) - \hat{\mathbf{H}}_{\text{err}}^{(i-1)}(s), \quad i = 2, \dots, \quad (22)$$

where $\hat{\mathbf{H}}_{\text{err}}^{(i-1)}(s) = \mathbf{C}\mathbf{V}_i[\Phi_i^{-1}(s)]\mathbf{W}_i^*\mathbf{B}_w^{(i)}$, and $\mathbf{B}_w^{(i)} = \mathbf{H}_w^{(i-1)}(s) \cdots \mathbf{H}_w^{(1)}(s)\mathbf{B}$.

Comparing $\hat{\mathbf{H}}_{\text{err}}^{(i-1)}(s)$ with

$$\mathbf{H}_{\text{err}}^{(i-1)}(s) = \mathbf{C}[\Phi^{-1}(s)]\mathbf{B}_w^{(i)}, \quad (23)$$

it is easy to see that the matrices $\mathbf{V}_i, \mathbf{W}_i$ are obtained by performing MOR to the *error system* $\mathbf{H}_{\text{err}}^{(i-1)}(s)$, where $\hat{\mathbf{H}}_{\text{err}}^{(i-1)}(s)$ is the resulting *reduced error system*. After $\mathbf{V}_i, \mathbf{W}_i$ are computed, they are combined with $\mathbf{V}_j, \mathbf{W}_j$, $j < i$, to form $\mathbf{V}_{1i}, \mathbf{W}_{1i}$, so as to construct the i th updated ROM $\mathbf{H}_{\text{red}}^{(i)}(s)$ of the original system.

For $i > 1$, if the balanced truncation (BT) method is used to compute $\mathbf{V}_i, \mathbf{W}_i$, $i = 1, \dots, q$, it consists of applying BT to the *error system* $\mathbf{H}_{\text{err}}^{(i-1)}(s)$ (23), i.e. the weighted system $\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B}_w^{(i)}$.

If an interpolatory method is used, such as moment-matching (MM), selection of good interpolation points is important. At each step $i > 1$ of updating the ROM, the interpolation point can be selected as the one which corresponds to the largest magnitude of $\mathbf{H}_{\text{err}}^{(i-1)}(s)$. The idea of selecting the interpolation point is analogous to the greedy algorithm of the RB method [7], where at each iteration step, the parameter corresponding to the biggest error is selected to compute a new basis vector, and usually a single vector is added to enrich the RB. For MM, several vectors could be selected at once.

Remark 3.2 (Decay of the ROM error) The behaviour of $\mathbf{H}_{\text{err}}^{(i)}(s)$, for $i \geq 1$, depends on the construction of the matrix pairs \mathbf{W}_i and \mathbf{V}_i used for the projection. From the analysis in Remark 3.1, we see that $\mathbf{V}_i, \mathbf{W}_i$ are computed such that

$$\mathbf{H}_{\text{err}}^{(i-1)} \approx \hat{\mathbf{H}}_{\text{err}}^{(i-1)}(s) = \mathbf{C}\mathbf{V}_i[\Phi_i^{-1}(s)]\mathbf{W}_i^*\mathbf{B}_w^{(i)}.$$

Usually, it is required that $\mathbf{V}_i, \mathbf{W}_i$ are computed so that the relative error of $\hat{\mathbf{H}}_{err}^{(i-1)}$ is below a certain tolerance $\epsilon < 1$, i.e.

$$|\mathbf{H}_{err}^{(i-1)} - \hat{\mathbf{H}}_{err}^{(i-1)}| \leq \epsilon |\mathbf{H}_{err}^{(i-1)}|$$

From the relation in Equation (22), we get

$$|\mathbf{H}_{err}^{(i)}(s)| \leq \epsilon |\mathbf{H}_{err}^{(i-1)}|$$

Therefore, if \mathbf{W}_i and \mathbf{V}_i are computed to obtain a ROM $\hat{\mathbf{H}}_{err}^{(i-1)}$ of the error system meeting the above accuracy requirement, the magnitude of the error system $\mathbf{H}_{err}^{(i)}(s), i > 1$ should monotonically decrease.

Remark 3.3 (Generalization to linear parametric systems.) Consider either the first-order parametric system

$$\begin{aligned} \mathbf{E}(\mathbf{p}) \frac{dx}{dt} &= \mathbf{A}(\mathbf{p})\mathbf{x} + \mathbf{B}(\mathbf{p})\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p})\mathbf{x}, \end{aligned} \quad (24)$$

or the second-order parametric system

$$\begin{aligned} \mathbf{M}(\mathbf{p}) \frac{d^2x}{dt^2} + \mathbf{K}(\mathbf{p}) \frac{dx}{dt} + \mathbf{A}(\mathbf{p})\mathbf{x} &= \mathbf{B}(\mathbf{p})\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p})\mathbf{x}, \end{aligned} \quad (25)$$

where \mathbf{p} is a vector of parameters. $\mathbf{E}, \mathbf{A}, \mathbf{M}, \mathbf{K} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times n_i}$, and $\mathbf{C} \in \mathbb{R}^{n_o \times n}$. Using the Laplace transformation, we get the parametric system in the frequency domain,

$$\begin{aligned} s\mathbf{E}(\mathbf{p})\mathbf{x} &= \mathbf{A}(\mathbf{p})\mathbf{x} + \mathbf{B}(\mathbf{p})\mathbf{U}(s), \\ \mathbf{y}(\mu) &= \mathbf{C}(\mathbf{p})\mathbf{x}, \end{aligned}$$

or

$$\begin{aligned} s^2\mathbf{M}(\mathbf{p})\mathbf{x} + s\mathbf{K}(\mathbf{p})\mathbf{x} + \mathbf{A}(\mathbf{p})\mathbf{x} &= \mathbf{B}(\mathbf{p})\mathbf{U}(s), \\ \mathbf{y}(\mu) &= \mathbf{C}(\mathbf{p})\mathbf{x}. \end{aligned}$$

Either of the above equations can be generally written as

$$\begin{aligned} \Phi(\mu)\mathbf{x} &= \mathbf{B}(\mu)\mathbf{U}(\mu), \\ \mathbf{y}(\mu) &= \mathbf{C}(\mu)\mathbf{x}, \end{aligned} \quad (26)$$

where $\mu = (\mathbf{p}, s)^T$, so that the transfer function of Equation (24) or (25) is $\mathbf{H}(\mu) = \mathbf{C}(\mu)[\Phi(\mu)]^{-1}\mathbf{B}(\mu)$.

The new procedure explored in the current section can, more or less straightforwardly, be applied to these parametric systems by replacing $\Phi(s)$ with $\Phi(\mu)$, \mathbf{C} with $\mathbf{C}(\mu)$, and \mathbf{B} with $\mathbf{B}(\mu)$. For the second-order system in Equation (25), in addition to the block matrices of \mathbf{A} introduced in Section 2.1, the block matrices of \mathbf{M}, \mathbf{K} should also be introduced to compute the updated ROMs as explained in the comment (c) in Section 2.1. The block structures of $\mathbf{M}, \mathbf{K}, \mathbf{A}$ are the same as the block structures of \mathbf{E}, \mathbf{A} for the first-order system. In the next section, we also demonstrate the new procedure applied to an example of a second-order parametric system.

4. Examples

We will now illustrate the above considerations by means of three examples. For the weighted model reduction needed in the successive approximation procedure, we can use weighted

balanced truncation (W-BT). For completeness, we sketch this method here, more details can be found in [1].

Let $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{D})$ be a realization of the original system and $(\mathbf{C}_w, \mathbf{E}_w, \mathbf{A}_w, \mathbf{B}_w, \mathbf{D}_w)$ a realization of the weighting function.¹ We define the composite system with transfer function $\hat{\mathbf{H}}(s) = \mathbf{H}(s)\mathbf{H}_w(s)$:

$$\hat{\mathbf{E}} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_w \end{bmatrix}, \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_w \\ \mathbf{0} & \mathbf{A}_w \end{bmatrix}, \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}\mathbf{D}_w \\ \mathbf{B}_w \end{bmatrix}, \hat{\mathbf{C}} = [\mathbf{C}, \mathbf{D}\mathbf{C}_w], \hat{\mathbf{D}} = \mathbf{D}\mathbf{D}_w.$$

Let \mathcal{P} be the controllability Gramian of $(\hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}})$, and \mathcal{Q} the observability Gramian of $(\hat{\mathbf{C}}, \hat{\mathbf{A}})$. Finally, let \mathcal{P}_n be the leading $n \times n$ minor of \mathcal{P} . With $\mathcal{P}_n = \mathbf{U}^*\mathbf{U}$ and $\mathcal{Q} = \mathbf{L}^*\mathbf{L}$, we compute the SVD $\mathbf{U}\mathbf{L}^* = \mathbf{W}\Sigma\mathbf{V}^*$. The projection matrices are then

$$\mathbf{W}^* = \Sigma_1^{-\frac{1}{2}}\mathbf{V}_1^*\mathbf{L} \text{ and } \mathbf{V} = \mathbf{U}^*\mathbf{W}_1\Sigma_1^{-\frac{1}{2}}$$

Example 1. The first system has order 16 and is defined as follows:

$$\mathbf{A} = \text{blkdiag} \left[\begin{bmatrix} -.1 & 40 \\ -40 & -.1 \end{bmatrix}, \begin{bmatrix} -.01 & 25 \\ -25 & -.01 \end{bmatrix}, \begin{bmatrix} -.02 & 10 \\ -10 & -.02 \end{bmatrix}, -\text{diag}(1 : 1 : 10) \right];$$

$$\mathbf{B} = \text{ones}(16, 1); \mathbf{C} = [2, 1, -1, 3, 1, -1, -1, -2, -2, 5, 3, 1, -1, -2, -4, 1];$$

The W-BT procedure described above is applied in three steps. The reduction order at each step is 2, i.e. the three reduced systems have order 2, 4, 6. In Table 1, the \mathcal{H}_2 - and the \mathcal{H}_∞ -norms of the three error systems are shown, together with the norms of the sixth order BT (last row of the table). Here, Σ is the original system, $\Sigma_2, \Sigma_4, \Sigma_6$ are the systems obtained by successive W-BT, and Σ_6^{bal} is the sixth order reduced system obtained by BT (listed for comparison purposes). Recall that BT of order 6 involves a full, as opposed to block triangular, projection and hence the reduced system has a better fit than the one obtained by a three step-reduction where each step is of order 2.

In this particular case, the iterative procedure is very close to BT, and furthermore the resulting poles of the ROM,

$$\lambda(\mathbf{A}_6) = \begin{bmatrix} -9.9364e-02 + 3.9999e+01i \\ -9.9364e-02 - 3.9999e+01i \\ -1.9821e-02 + 1.0000e+01i \\ -1.9821e-02 - 1.0000e+01i \\ -1.0021e-02 + 2.5000e+01i \\ -1.0021e-02 - 2.5000e+01i \end{bmatrix}$$

are very close to the three complex poles $-.1 \pm 40i, -.01 \pm 25i, -.02 \pm 10i$ of the original system. At each iteration, one of the peaks of the Bode plot is captured together with the corresponding pole. Notice also the monotonicity of the decrease of the error norms as the order of the approximants increases. The corresponding plots are shown in Figure 1.

Table 1. Example 1, error norms for reduction of 16th order system by W-BT.

	\mathcal{H}_2	\mathcal{H}_∞
$\ \Sigma\ $	1.5349e+01	4.9852e+01
$\ \Sigma - \Sigma_2\ $	1.3058e+01	4.9890e+01
$\ \Sigma - \Sigma_4\ $	4.2125e+00	7.4014e+00
$\ \Sigma - \Sigma_6\ $	9.5388e-01	1.3787e+00
$\ \Sigma - \Sigma_6^{\text{bal}}\ $	9.5371e-01	1.3790e+00

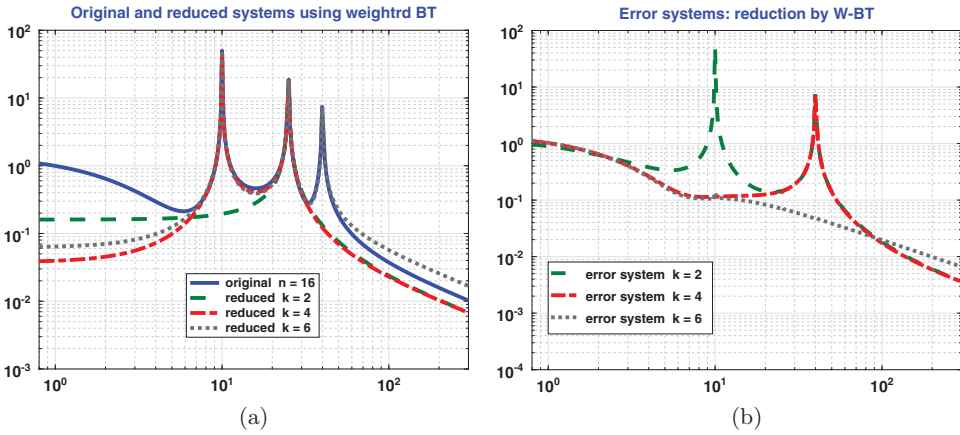


Figure 1. Example 1, reduction of a 16th order system by Weighted-BT – Bode plots of FOM and ROMs (left) and the error systems (right).

Example 2. The second system is the well-known CD player model with McMillan degree 120 [10]. The reduction is also performed in three steps (orders of the resulting ROMs: 6, 12, 18). We apply two reduction methods for each successive step. The first is W-BT as described earlier and the second is Rational Krylov, where the interpolation points are chosen from the peaks of each error system. The necessity for the second method arose from the fact that W-BT missed the second peak of the amplitude Bode plot (between 10⁴ and 10⁵ Hz); this is due to the low magnitude of this part of the frequency response compared with the magnitude of the dominant peak. Capturing the secondary peak is thus achieved at the expense of obtaining higher error norms as shown in Table 2. There, Σ is the original system; $\Sigma_6, \Sigma_{12}, \Sigma_{18}$ are the systems obtained by successive W-BT or rational Krylov; and Σ_{18}^{bal} is the 18th-order reduced system obtained by BT. Again, the monotonicity of the decay of these norms holds. Finally, for comparison purposes, we add the norm of the 18th-order reduced system obtained by BT.

The corresponding Bode plots of the FOM and ROM transfer functions as well as the resulting error systems are shown in Figures 2 and 3.

Example 3. We study a parametric model of a plate (floor inside a building near a highway). The parametric model is of second-order form,

$$\mathbf{M} \frac{d^2 \mathbf{x}}{dt^2} + p\mathbf{E} \frac{d\mathbf{x}}{dt} + \mathbf{A}\mathbf{x} = \mathbf{B}, \quad \mathbf{y} = \mathbf{C}\mathbf{x},$$

The parameter p is the damping coefficient. The size of the original system is $n = 22$.

We use the parametric MOR algorithm proposed in [11] to compute the ROM. From the algorithm, a projection matrix \mathbf{V} is computed, and the ROM is obtained by Galerkin projection as below,

Table 2. Example 2, error norms.

Norms	Reduction of CD player by W-BT			Reduction of CD player by R-Krylov		
	$\ \Sigma\ $	\mathcal{H}_2	\mathcal{H}_∞	$\ \Sigma\ $	\mathcal{H}_2	\mathcal{H}_∞
$\ \Sigma - \Sigma_6\ $	2.6367e + 02	2.7375e + 00	6.8492e + 01	$\ \Sigma - \Sigma_6\ $	2.6367e + 02	6.8492e + 01
$\ \Sigma - \Sigma_{12}\ $	1.3057e + 00	1.3057e + 00	8.4375e - 01	$\ \Sigma - \Sigma_{12}\ $	2.6464e + 02	6.7444e + 01
$\ \Sigma - \Sigma_{18}\ $	1.0567e + 00	1.0567e + 00	1.4160e - 01	$\ \Sigma - \Sigma_{18}\ $	8.4270e + 00	1.9912e + 00
$\ \Sigma - \Sigma_{18}^{bal}\ $	5.0463e - 01	5.0463e - 01	7.0622e - 02	$\ \Sigma - \Sigma_{18}^{bal}\ $	8.3346e + 00	1.9807e + 00
			2.2918e - 02	$\ \Sigma - \Sigma_{18}^{bal}\ $	5.0463e - 01	2.2918e - 02

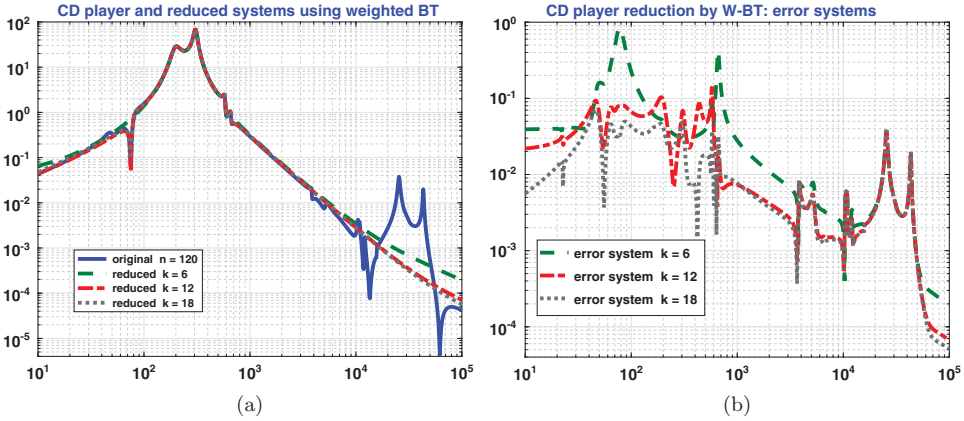


Figure 2. Example 2, reduction of a CD player model by weighted-BT – Bode plots of FOM and ROMs (left) and the error systems (right).

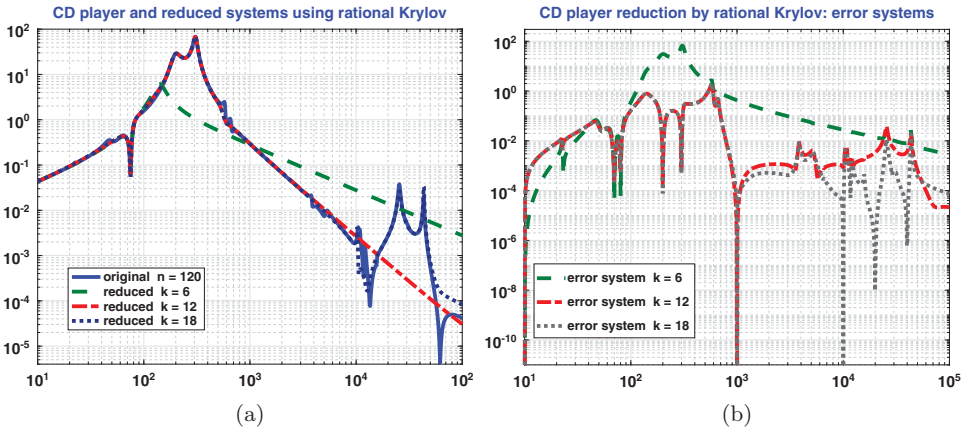


Figure 3. Example 2, reduction of a CD player model by rational Krylov–Bode plots of FOM and ROMs (left) and the error systems (right).

$$\hat{\mathbf{M}} \frac{d^2 \hat{\mathbf{x}}}{dt^2} + p \hat{\mathbf{E}} \frac{d \hat{\mathbf{x}}}{dt} + \hat{\mathbf{A}} \hat{\mathbf{x}} = \hat{\mathbf{B}}, \quad \hat{\mathbf{y}} = \mathbf{C} \mathbf{V} \hat{\mathbf{x}},$$

where $\hat{\mathbf{M}} = \mathbf{V}^T \mathbf{M} \mathbf{V}$, $\hat{\mathbf{E}} = \mathbf{V}^T \mathbf{E} \mathbf{V}$, $\hat{\mathbf{A}} = \mathbf{V}^T \mathbf{A} \mathbf{V} \in \mathbb{R}^{r \times r}$, $\hat{\mathbf{B}} = \mathbf{V}^T \mathbf{B} \in \mathbb{R}^{r \times n_i}$, $\hat{\mathbf{C}} = \mathbf{C} \mathbf{V} \in \mathbb{R}^{n_o \times r}$, and $\hat{\mathbf{x}} \in \mathbb{R}^r$, $\hat{\mathbf{y}} \in \mathbb{R}^{n_o}$.

We have computed $\mathbf{H}_{\text{err}}^{(1)}(s, p)$, $\mathbf{H}_{\text{err}}^{(2)}(s, p)$, and $\mathbf{H}_{\text{err}}^{(3)}(s, p)$, which are the transfer functions of the error systems of the first, second and the third ROMs of sizes $r = 4, 7, 10$, respectively. The i th ROM is obtained by interpolating the $i - 1$ st error system $\mathbf{H}_{\text{err}}^{(i-1)}$, where the interpolation point of (s, p) corresponds to the peak of $\mathbf{H}_{\text{err}}^{(i-1)}$, for $i = 2, 3$. In **Figure 4** on the left, we compare $\mathbf{H}_{\text{err}}^{(1)}(s, p)$ with $\mathbf{H}_{\text{err}}^{(2)}(s, p)$, and in **Figure 4** on the right, we compare $\mathbf{H}_{\text{err}}^{(2)}(s, p)$ with $\mathbf{H}_{\text{err}}^{(3)}(s, p)$, where the error systems are plotted along the axes of frequency f and the parameter p . The figures are plotted by randomly choosing 50 samples of $f \in [0, 200]$ Hz and 10 samples of $p \in [95300, 95310]$.

From **Figure 4(a)**, we see that the peak of $\mathbf{H}_{\text{err}}^{(1)}$ is close to $f = 140$ Hz. After interpolation of $\mathbf{H}_{\text{err}}^{(1)}$ at the point close to $f = 140$ Hz, the error of the second ROM, i.e. the error system $\mathbf{H}_{\text{err}}^{(2)}$, has the lowest magnitude at that interpolation point. Similarly in the right plot of **Figure 4**, the peak of $\mathbf{H}_{\text{err}}^{(2)}$ is at a point close to $f = 40$ Hz. After interpolating $\mathbf{H}_{\text{err}}^{(2)}$ at that point, the error of the third

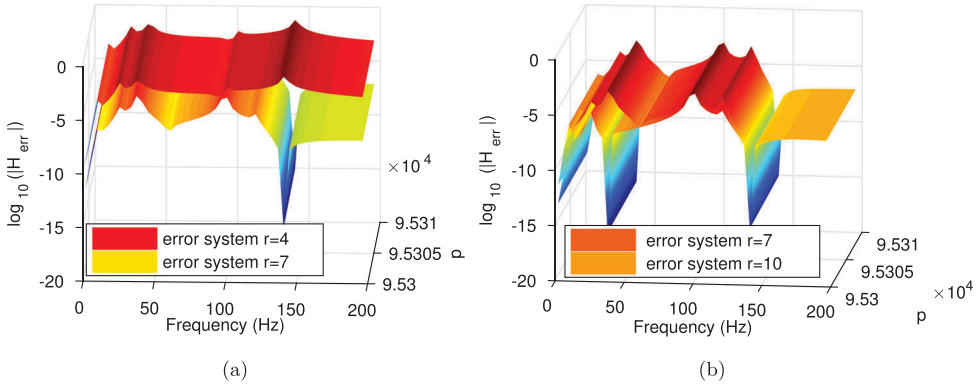


Figure 4. Example 3, the error systems $\mathbf{H}_{\text{err}}^{(1)}$, $\mathbf{H}_{\text{err}}^{(2)}$ and $\mathbf{H}_{\text{err}}^{(3)}$.

Table 3. Example 3, \mathcal{H}_∞ -norm of the error systems.

	$\ \Sigma - \Sigma_{r=4}\ $	$\ \Sigma - \Sigma_{r=7}\ $	$\ \Sigma - \Sigma_{r=10}\ $
$\max_p \mathcal{H}_\infty$	0.1027	0.0013	0.0012

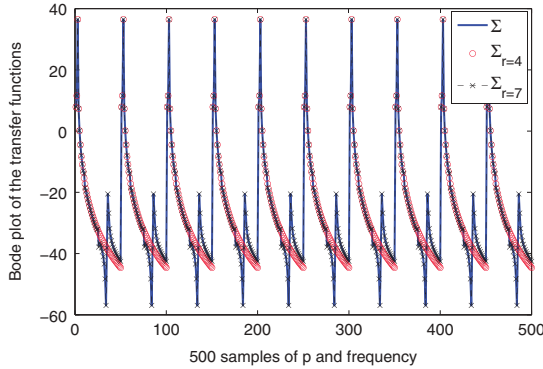


Figure 5. Example 3, the transfer functions of Σ , $\Sigma_{r=4}$, $\Sigma_{r=7}$.

ROM, i.e. the error system $\mathbf{H}_{\text{err}}^{(3)}$, has the lowest magnitude at the same point. It is entirely in agreement with the theoretical analysis in Remark 3.1. The maximal \mathcal{H}_∞ -norm of the error systems over the parameter sample space p is given in Table 3.

In Figure 5, we plot the magnitudes of the transfer functions of the original system Σ , and those of the first ROM $\Sigma_{r=4}$, and the second ROM $\Sigma_{r=7}$ at 500 random samples of frequency and the parameter p . The second ROM is already indistinguishable from the original model.

In addition, we plot the error behaviour of the ROMs in the time domain in Figure 6. The plots clearly describe the behaviour of the output error according to the time evolution as well as the parameter variation.

In Figure 7, the norms of the two residuals $\|R^{\text{du}}\|_2$ and $\|R^{\text{pr}}\|_2$, as well as the magnitude of $\mathbf{H}_{\text{err}}^{(2)}$ are plotted, where $|\mathbf{H}_{\text{err}}^{(2)}|$ is indeed smaller than the product of $\|R^{\text{du}}\|_2$ and $\|R^{\text{pr}}\|_2$, as implied by the error bound discussed in the comment (b) in Section 2.1.

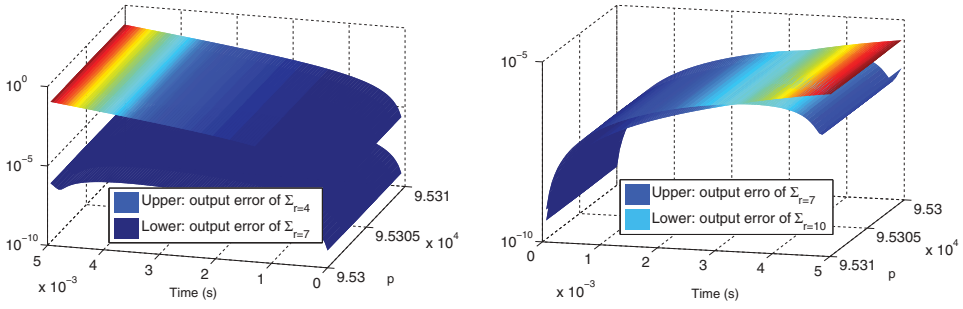


Figure 6. Example 3, behaviour of the output error in the time domain.

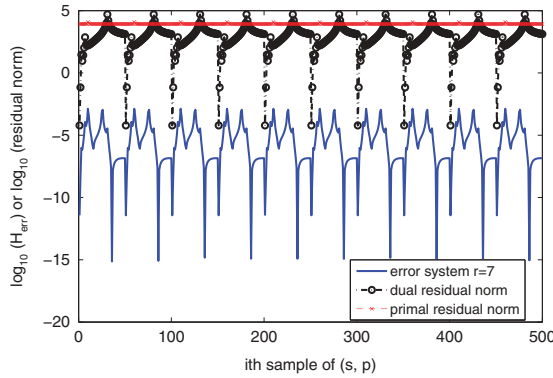


Figure 7. Example 3, magnitude of the error system $H_{\text{err}}^{(2)}$ and the norms of the two associated residuals $\|R_2^{du}\|_2$ and $\|R_2^{pr}\|_2$.

5. Conclusions

We have discussed an iterative procedure to improve a ROM of a LTI or parametric system computed by an arbitrary method. The procedure is based on successively reducing the error systems associated with the current reduced systems. The formulas used are derived by analysing known a posteriori error estimates used in RB methods. The iterative process uses either weighted BT or rational Krylov methods to reduce the error systems, represented as weighted transfer functions. Numerical examples illustrate the performance of the suggested procedure.

Note

1. If the weight is the product of two transfer functions defined by means of $(C^i; E^i; A^i; B^i; D^i)$; $i = 1; 2$, then

$$E_w = \begin{bmatrix} E^1 & \mathbf{0} \\ \mathbf{0} & E^2 \end{bmatrix}, \quad A_w = \begin{bmatrix} A^1 & B^1 C^2 \\ \mathbf{0} & A^2 \end{bmatrix}, \quad B_w = \begin{bmatrix} B^1 D^2 \\ B^2 \end{bmatrix}, \quad C_w = [C^1, D^1 C^2], \quad D_w = D^1 D^2.$$

Disclosure statement

No potential conflict of interest was reported by the authors.

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