# Stochastic climate models

## Part I. Theory

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### ABSTRACT

A stochastic model of climate variability is considered in which slow changes of climate are explained as the integral response to continuous random excitation by short period "weather" disturbances. The coupled ocean–atmosphere–cryosphere–land system is divided into a rapidly varying "weather" system (essentially the atmosphere) and a slowly responding "climate" system (the ocean, cryosphere, land vegetation, etc.). In the usual Statistical Dynamical Model (SDM) only the average transport effects of the rapidly varying weather components are parameterised in the climate system. The resultant prognostic equations are deterministic, and climate variability can normally arise only through variable external conditions. The essential feature of stochastic climate models is that the non-averaged "weather" components are also retained. They appear formally as random forcing terms. The climate system, acting as an integrator of this short-period excitation, exhibits the same random-walk response characteristics as large particles interacting with an ensemble of much smaller particles in the analogous Brownian motion problem. The model predicts "red" variance spectra, in qualitative agreement with observations. The evolution of the climate probability distribution is described by a Fokker-Planck equation, in which the effect of the random weather excitation is represented by diffusion terms. Without stabilising feedback, the model predicts a continuous increase in climate variability, in analogy with the continuous, unbounded dispersion of particles in Brownian motion (or in a homogeneous turbulent fluid). Stabilising feedback yields a statistically stationary climate probability distribution. Feedback also results in a finite degree of climate predictability, but for a stationary climate the predictability is limited to maximal skill parameters of order 0.5.

## 1. Introduction

A characteristic feature of climatic records is their pronounced variability. The spectral analysis of continuous climatic time series normally reveals a continuous variance distribution encompassing all resolvable frequencies, with higher variance levels at lower frequencies. Combining different data sources of various time scale and resolution (recorded meteorological data, varves, ice and sediment cores, global ice volume) the increase in spectral energy with decreasing frequency can be traced from the high frequency limit of climate variability (approximately 1 cycle per month, following the definitions adopted in GARP Publication 16, 1975) down to frequencies of order 1 cycle per $10^5$ years (cf. GARP-US Committee Report (1975), Appendix A). An understanding of the origin of climatic variability, in the entire spectral range from extreme ice age changes to seasonal anomalies, is a primary goal of climate research. Yet despite the long interest in the ice-age problem and the more recent intensification of climate research there exists today no generally accepted, simple explanation for the observed structure of climate variance spectra.

Various attempts have been made to link climatic changes to variable external factors such as the solar activity, secular changes of the orbital parameters of the earth, or the increased turbidity of the atmosphere following volcanic eruptions (cf. reviews in GARP Publication 16). A persistent difficulty with these investigations is that the postulated input–response relationships, if they exist, are not sufficiently pronounced to be immediately obvious on inspection of the appropriate time series. Thus a detailed statistical analysis is necessary, for which the data base is often only marginally

adequate. Summaries of solar-climate relations extracted by statistical techniques may be found in King (1975) and Wilcox (1975); a critical analysis of the statistical significance of some of the claimed correlations has been given by Monin & Vulis (1971).

Climate variations have also often been discussed in terms of internal atmosphere–ocean–cryosphere–land feed-back mechanisms. Positive feedback amplifies the response of the system to changes in the external parameters and, if sufficiently strong, can produce unstable spontaneous transitions from one climate state to another. Feedback mechanisms have generally been formulated in terms of highly simplified energy-budget models containing only a few "climate" variables, such as the zonally averaged surface temperatures, the area of the ice sheets and the albedo of the earth's surface. A basic difficulty of unstable feedback models (apart from— or possibly because of—their high degree of idealization) is that they tend to predict climatic variations as flip-flop transitions and therefore fail to reproduce the observed continuous spectrum of climatic variability.

In this paper an alternative model of climate variability is investigated which predicts the basic structure of climatic spectra without invoking internal instabilities or variable external boundary conditions. The variability of climate is attributed to internal random forcing by the short time scale "weather" components of the system. Slowly reponding components of the system, such as the ice sheets, oceans, or vegetation of the earth's surface, act as integrators of this random input much in the same way as heavy particles imbedded in an ensemble of much lighter particles integrate the forces exerted on them by the light particles. If feedback effects are ignored, the resultant "Brownian motion" of the slowly responding components yields r.m.s. climate variations—relative to a given initial state—which increase as the square root of time. In the frequency domain, the climate variance spectrum is proportional to the inverse frequency squared. The non-integrable singularity of the spectrum at zero frequency is consistent with the non-stationarity of the process. The spectral analysis for a finite-duration record yields a finite peak at zero frequency proportional in energy to the duration of the record.

In order to obtain a statistically stationary

response, stabilising negative feedback processes must be invoked. Thus from the viewpoint of the present model, the problem of climate variability is not to discover positive feedback mechanisms which enhance the small variations of external inputs or produce instabilities, but rather to identify the negative feedback processes which must be present to balance the continual generation of climatic fluctuations by the random driving forces associated with the internal "weather" interactions.

Following the derivation of the random-walk characteristics of a stochastically driven climate system in Sections 2 and 3, the basic Fokker-Planck equation governing the evolution of such a system is presented in Section 4. Special solutions for a system with linear feedback are given in Section 5, and the results are then applied to the analysis of climate predictability in Section 6.

Some of the concepts underlying the present stochastic model have been expressed previously by Mitchell (1966) in his investigation of sea-surface temperature (SST) anomalies. An application of the present model to SST data and to temperature fluctuations in the seasonal thermocline is given in Part 2 of this paper (Frankignoul & Hasselmann, 1976). In Part 3, the effect of introducing stochastic forcing into simple statistical dynamical models of the Budyko-Sellers type is investigated (Lemke, 1976).

## 2. Relationship between GCM's, SDM's and stochastic forcing models

It is useful to introduce a formal notation which is independent of the individual model structure. Let the instantaneous state of the complete system atmosphere–ocean–cryosphere–land be described by a finite set of discrete variables $\mathbf{z} = (z_1, z_2, ...)$. The state vector $\mathbf{z}$ may be taken to represent the fields of density, velocity, temperature, etc. of the various media, as defined at discrete grid points and levels, or as given by the coefficients of some suitably truncated functional expansion. The evolution of the system will then be described by a series of prognostic equations

$$\frac{dz_i}{dt} = w_i(\mathbf{z}) \tag{2.1}$$

where $w_i$ is a known (in general complicated nonlinear) function of z. For the following we ignore the parameterization problems associated with the projection of the complete system on to a finite set of parameters; we assume that for our purposes the prognostic eqs. (2.1) accurately describe the evolution of the system for all times of interest.

A basic assumption of most models is that the complete system z can be divided into two subsystems, z = (x, y), which are characterised by strongly differing response times $\tau_x$, $\tau_y$. Thus writing eq. (2.1) in terms of the two subsystems,

$$\frac{dx_i}{dt} = u_i(\mathbf{x}, \mathbf{y}) \qquad (2.2)$$

$$\frac{dy_i}{dt} = v_i(\mathbf{x}, \mathbf{y}) \qquad (2.3)$$

it is assumed that

$$O\left(x_i\left(\frac{dx_i}{dt}\right)^{-1}\right) = \tau_x \ll \tau_y = O\left(y_i\left(\frac{dy_i}{dt}\right)^{-1}\right) \qquad (2.4)$$

The fast responding components $x_i$ may be identified with the normal prognostic "weather" variables used in deterministic numerical weather prediction or General Circulation Models (GCM's), whereas the slowly responding "climate" variables $y_i$ may be associated with variables such as the sea surface temperature, ice coverage, land foliage, etc. which are normally set constant in weather prediction models but represent essential prognostic variables on climatic time scales. $\tau_x$ is typically of the order of a few days, whereas most climate variables have response scales $\tau_y$ of the order of several months, years or longer. Thus the inequality (2.4) is generally well satisfied.

With presently available computers it is not possible to integrate the complete coupled system (2.2)–(2.3) over periods of climatic time scale $O(\tau_y)$. High resolution GCM's are normally used to integrate the subset of equations (2.2) over an intermediate period $\tau_i$ in the range $\tau_x < \tau_i < \tau_y$ for which the "climatic" variables can be regarded as constant, but which is still sufficiently long to define the statistics of the weather variables x for a given climatic state y. Thus although GCM's provide important information for climate studies, they are not suitable for the simulation of climate variability as such.

Dynamical investigations of climate variability have been based in the past largely on Statistical Dynamical Models (SDM's), which address the subset of eqs. (2.3). In the usual approach it is argued that for the time scales $\tau_y$ of interest in (2.3), the rapidly fluctuating terms in the prognostic equations can be ignored, so that (2.3) can be averaged over the period $\tau_i$, thereby removing the weather fluctuations while still regarding y in the right hand side of (2.3) as constant,

$$\frac{dy_i}{dt} \approx \langle v_i(\mathbf{x}, \mathbf{y})\rangle \qquad (2.5)$$

Formally, it will be more convenient in the following to regard the average $\langle ...\rangle$ as an ensemble average over a set of realisations x for given y. It is assumed that ergodicity holds, so that ensemble averaging and time averaging are equivalent.

Since $v_i$ is in general a nonlinear function of x, the average rate of change $\langle v_i\rangle$ of $y_i$ will depend on the statistical properties of x as well as on y. To close the problem, the statistics of x must therefore be expressed in terms of y through the introduction of some closure hypothesis. For example, in zonally averaged energy budget models of the Budyko (1969)–Sellers (1969) type the meridional heat fluxes by standing and transient eddies must be parameterised in terms of the mean meriodional temperature distributions.

Although this class of model may be termed statistical in the sense that an averaging operation and a statistical closure hypothesis are involved, the reduced eq. (2.5) is in fact deterministic rather than statistical. It is known that the asymptotic solutions of nonlinear deterministic equations containing a relatively small number of degrees of freedom can already exhibit non-periodic, random-type oscillations similar in character to observed weather or climate fluctuations (cf. Lorenz, 1965). However, simple models with these features appear to have been investigated primarily in relation to weather simulation. Most of the better known simple SDM's predict a unique, time-independent asymptotic state for any given initial state. These models appear inherently incapable of generating internally time variable solutions with continuous variance spectra, as required by observation. In the past climate variability

has therefore been explained in the framework of classical SDM's as the response of the system (2.5) to variations of external boundary conditions, such as the solar radiation and the turbidity of the atmosphere, rather than through internal interactions.

By a natural extension of the SDM, however, one can obtain an alternative climatic model which yields continuous variance spectra with the observed "red" distribution directly through internal interactions. (This, of course, does not exclude the possible significance of additional externally induced climatic changes). Returning to eq. (2.3), let $\delta y = y(t) - y_0$ denote the change of the climate state relative to a given initial state $y(t=0) = y_0$ in a time $t < \tau_y$ sufficiently small that $y$ can still be regarded as constant in the forcing term on the right hand side of the equation. The change may be divided into mean and fluctuating terms, $\delta y = \langle \delta y \rangle + y'$ where the ensemble average is taken here over all $x$ states for fixed $y_0$ (not $y$). The mean change $\langle \delta y \rangle$ follows from (2.5),

$$\langle \delta y_i \rangle = \langle v_i \rangle t \tag{2.6}$$

(for this term it is irrelevant whether the average refers to fixed $y$ or $y_0$). The rate of change of the fluctuating term is given by

$$\frac{dy_i'}{dt} = v_i(\mathbf{x}, \mathbf{y}) - \langle v_i \rangle = v_i' \tag{2.7}$$

where $\langle v_i' \rangle = 0$ and $y_i' = 0$ for $t = 0$.

The statistics of $v_i'(t)$ are defined through the statistics of the weather variables $x(t)$ for given $y_0$. It is assumed that $x(t)$, and therefore $v(t)$, represents a stationary random process.

Equation (2.7) is identical to the equations describing the diffusion of a fluid particle in a turbulent fluid, where $y_i'$ represents the coordinate vector of the particle and $v_i'$ the turbulent (Lagrangian) velocity It is well known from this problem (Taylor, 1921, Hinze, 1959) that for statistically stationary $v_i'$, the integration of (2.7) yields a non-stationary process $y_i'$, the covariance matrix $\langle y_i' y_j' \rangle$ growing linearly in time $t$ for $t \gg \tau_x$. Taylor pointed out in his original paper that this result could be interpreted physically as the continuum-mechanical analogy to normal molecular diffusion or to Brownian motion. In fact, for $t \gg \tau_x$ it is immaterial for the (macroscopic) statistical properties of $y_i'$, involving time scales $\gg \tau_x$, whether the forcing is continuous or discontinuous.

The nonstationary response $y_i'$ to stationary random forcing $v_i'$ in the stochastic model implies that climate variations would continue to grow indefinitely if feedback effects were ignored. These, of course, will begin to become effective as soon as the integration is carried into the region $t = O(\tau_y)$. The properties of the random walk model in the ranges $t < \tau_y$ and $t = O(\tau_y)$ will be discussed in more detail in the following sections.

The relationship between GCM's, SDM's and stochastic forcing models may be conveniently summarized in terms of the Brownian motion analogy. The climate variables $y$ and weather variables $x$ may be interpreted in the analogous particle picture as the (position and momentum) coordinates of large and small particles, respectively. The analysis of climate variability in terms of SDM's is then equivalent to determining the large-particle paths by considering only the interactions between the large particles themselves and the *mean* pressure and stress fields set up by the small-particle motions (plus the influence of variable external forces). Numerical experiments with GCM's correspond in this picture to the explicit computation of all paths of the small particles for fixed positions of the large particles. Even if the large particles were allowed to vary during the computation, it would normally not be feasible to carry the integrations sufficiently far to consider appreciable deviations of the large particles from their initial positions. Finally, the approach used in the stochastic forcing model corresponds to the classical statistical treatment of the Brownian motion problem, in which the large-particle dispersion is inferred from the statistics of the small particles with which they interact. In contrast to the Brownian motion problem, the variables $x$ in the real climate–weather system are, of course, not in thermodynamic equilibrium, so that the statistical properties of $x$ cannot be inferred from the statistical thermodynamical theory of energetically closed systems, but must be evaluated from numerical simulations with GCM's (or from real data). A great reduction of computation is nevertheless achieved through a statistical treatment, since relatively little statistical information on $x$ is actually needed, and this can be obtained from GCM experiments of relatively short duration $\tau_i < \tau_y$.

At first sight it may appear surprising that

a statistical reduction of the complete climate–weather system is possible at all without arbitrary closure hypotheses, since one is accustomed to regarding systems involving turbulent geophysical fluid flows as basically irreducible, strongly nonlinear processes. The reduction in this case is a consequence of the time-scale separation (2.4). This property is lacking in the usual turbulent system. However, the condition is familiar from "weak-turbulence" theories for plasmas (cf. Kadomtsev, 1965) or from similar theories of weakly interacting random wave fields in solid state physics, high energy physics and in various geophysical applications (cf. Hasselmann, 1966, 1967). In essence, the property (2.4) enables statistical closure through the application of the Central Limit Theorem, whereby the response of a system is completely determined statistically by the second moments of the input if the forcing consists of a superposition of a large number of small, statistically independent pulses of time scale short compared with the response time of the system.

## 3. The local dispersion rate

For times $t$ in the intermediate range $\tau_z < t < \tau_y$ the integration of (2.7) yields linearly increasing covariances in accordance with Taylor's (1921) relation

$$\langle y_i' y_j' \rangle = 2 D_{ij} t \tag{3.1}$$

where

$$D_{ij} = \tfrac{1}{2} \int_{-\infty}^{\infty} P_{ij}(\tau) \, d\tau \tag{3.2}$$

and $P_{ij}(\tau) = \langle v_i'(t+\tau) v_j'(t) \rangle$ denotes the covariance function.

Physically, the dispersion mechanism may be interpreted as the response to a large number of statistically independent random changes $\Delta y_i = v_i'$. $\Delta t$ induced in $y_i$ at time increments $\Delta t$ of the order of the integral correlation time of $v_i'$.

It is useful to represent the dispersion process also in the Fourier domain. Writing

$$v_i'(t) = \int_{-\infty}^{\infty} V_i(\omega) \, e^{i\omega t} \, d\omega \tag{3.3}$$

the solution of (2.7) may be expressed as the Fourier integral

$$y_i'(t) = \int_{-\infty}^{\infty} Y_i(\omega) \, e^{i\omega t} d\omega - \int_{-\infty}^{\infty} Y_i(\omega) \, d\omega \tag{3.4}$$

where

$$Y_i(\omega) = \frac{V_i(\omega)}{i\omega} \tag{3.5}$$

The second, time independent term on the right hand side of (3.4) arises through the initial condition $y_i' = 0$ for $t = 0$.

For a stationary process, the Fourier components are statistically orthogonal,

$$\langle V_i(\omega) \, V_j^*(\omega') \rangle = \delta(\omega - \omega') \, F_{ij}(\omega)$$

where $F_{ij}(\omega)$ denotes the (two-sided) cross spectrum of $v_i$. The Fourier components $Y_i(\omega)$ are then also statistically orthogonal, and the cross spectrum of $y_i'(t)$ is given by

$$G_{ij}(\omega) = \frac{F_{ij}(\omega)}{\omega^2} \quad (\omega \neq 0) \tag{3.6}$$

The existence of a non-integrable singularity in $G_{ij}$ at $\omega = 0$ is consistent with the non-stationarity of $y_i'$. The fact that the non-stationary contribution to $y_i'$ is concentrated at zero frequency can be confirmed by evaluating the contribution to the covariance from a narrow band of frequencies $-\Delta\omega < \omega < \Delta\omega$ centered at zero frequency. Noting that the second integral in (3.4) represents a zero-frequency contribution, this is given by

$$\langle y_i' y_j' \rangle_{\Delta\omega} = \int_{-\infty}^{\infty} F_{ij}(\omega) \frac{2(1 - \cos \omega t)}{\omega^2} \, d\omega \tag{3.7}$$

The weighting function $2(1 - \cos \omega t)/\omega^2$ has a maximum value equal to $t^2$ at $\omega = 0$ and a peak width proportional to $1/t$. Thus its integral is proportional to $t$, and in the limit of large $t$, as the peak becomes infinitely sharp, the function can be replaced by the $\delta$-function expression

$$\frac{2(1 - \cos \omega t)}{\omega^2} \approx 2\pi t \delta(\omega) \quad (\omega t \gg 1) \tag{3.8}$$

For large $t$ (3.7) therefore becomes

$$\langle y_i' y_j' \rangle = 2\pi t F_{ij}(0) \tag{3.9}$$
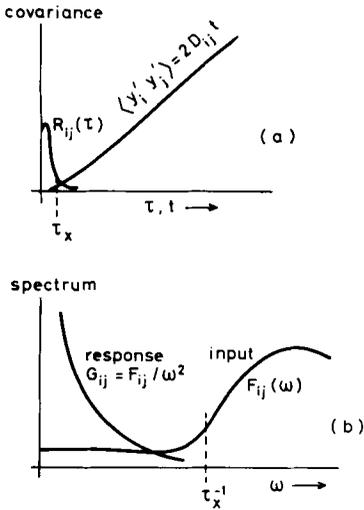
covariance



spectrum



*Fig. 1.* Input and response functions of stochastically forced climate model without feed-back; (a) covariances, (b) spectra.

The subscript $\Delta\omega$ has now been dropped, since the contribution to $\langle y_i' y_j' \rangle$ from frequencies $|\omega| > \Delta\omega$ is constant and therefore becomes negligible compared with the nonstationary contribution for large $t$.

Equation (3.9) represents a special case of the resonant response of an undamped linear system to random external forcing. The general result for such systems states that the energy of the response is concentrated in spectral lines at the eigenfrequencies of the system, and that the energy of each line increases linearly with time at a rate proportional to the spectral density of the input at the eigenfrequency (cf. Hasselmann, 1967). Equation (3.9) corresponds to the case of a system with a single normal mode of frequency $\omega = 0$.

The equivalence of the expressions (3.1), (3.2) and (3.9) can be recognised using the Fourier transform relation

$$F_{ij}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{ij}(\tau) e^{-i\omega\tau} d\tau \qquad (3.10)$$

It follows from (3.10) that normally, for $F_{ij}(0) \neq 0$, the spectrum of any stationary process $v_i$ becomes white (constant) for sufficiently small frequencies (in other words, one need consider only the first term of the Taylor ex-

pansion of the spectrum). Generally, there exists some cut-off time lag $O(\tau_x)$ such that $P_{ij}(\tau) \approx 0$ for $\tau > \tau_x$. For frequencies $\omega < \tau_x^{-1}$, the exponential in (3.10) can then be set equal to one, so that $F_{ij}(\omega) \approx F_{ij}(0)$. In this range equation (3.6) may then be replaced by

$$G_{ij}(\omega) = \frac{F_{ij}(0)}{\omega^2} \quad (\tau_y^{-1} \ll \omega \ll \tau_x^{-1}) \qquad (3.11)$$

The left side of the inequality follows from the restriction to integration times $t < \tau_y$, which limits the definition of the spectrum to frequencies large compared with $\tau_y^{-1}$.

The main features of the random walk response in the time and frequency domain are indicated in Fig. 1.

In most climate applications the reponse will lie in the low frequency range $\omega < \tau_x^{-1}$ where the input spectrum can be regarded as white and equation (3.11) is applicable. For the generalization of the theory in the next section it is important to note that the constant level of the input spectrum at low frequencies can be determined from relatively short time series of the input, the record length required being governed by the time scale of the input, rather than the time scale of the response. The length of the time series need only be long enough to evaluate the covariance function for time lags up to the cut-off time lag of order $\tau_x$. For example, in the problem of the generation of SST anomalies by random fluxes at the sea surface (considered in Part 2 of this paper), the statistical structure of the atmospheric input can normally be adequately determined from time series of a few weeks duration (ignoring the seasonal signal). From this the statistical properties of the random walk response according to (3.1), (3.2), and (3.11) can be evaluated for much longer time periods, of the order of several months. The upper limit $t = O(\tau_y)$ of the response time is determined ultimately by the breakdown of the uncoupled random walk model when internal feedback effects begin to come into play.

The dispersion coefficients $D_{ij}$ can be inferred indirectly, without reference to weather data, from the rate of growth of the covariances $\langle y_i' y_j' \rangle$ as evaluated from climatic time series. Alternatively, if the stochastic forcing is known as a function of the weather variables, the zero frequency level of the spectral input can be determined directly from weather data. By

either method, application of the random walk model, for example, to ice sheet data or SST anomalies indicates that the r.m.s. rate of divergence of climate from its present state by random weather forcing is considerable: without stabilising feedback the random walk model predicts that changes in the extent of the ice cover comparable with ice-age amplitudes would occur within time periods of the order of a century. The inclusion of feedback is thus essential for a realistic climate model. The generalisation to a model including arbitrary internal coupling is carried out in the next section.

## 4. The Fokker-Planck equation for a general stochastic climate model

The inequalities $\tau_x < t < \tau_y$ limiting the range of validity of the random walk model without feedback are characteristic of a two-timing theory. With respect to the rapidly varying components of the system the theory represents an asymptotic infinite-time limit, but at the same time the analysis is valid only for infinitesimal changes of the slowly varying components. The standard way of removing the restriction $t < \tau_y$ is to interpret the infinitesimal changes of the slowly varying components as *rates of change*, thereby obtaining a differential equation which is valid for all times, provided the original conditions on which the local theory was based continue to remain valid.

Since $y$ represents a random variable, the appropriate differential equation should be formulated for the probability density distribution $p(y, t)$ of climatic states in the climatic phase space $y$. For a system in which the mean value and covariance tensor of the infinitesimal changes $\delta y_i = y_i(t) - y_{i,0}$ in an infinitesimal time interval $\delta t < \tau_y$ are both proportional to $\delta t$ (the effects of the higher moments can be shown to be small on account of the two-timing condition (2.4)) the evolution of the probability distribution $p(y, t)$ is governed by a Fokker-Planck equation (cf. Wang and Uhlenbek, 1945)

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial y_i}(\hat{v}_i p) - \frac{\partial}{\partial y_i}\left(D_{ij}\frac{\partial p}{\partial y_j}\right) = 0 \qquad (4.1)$$

where

$$D_{ij} = \frac{\langle y_i' y_j' \rangle}{2\delta t} = \pi F_{ij}(0) \qquad (4.2)$$

with $y_i' = \delta y_i - \langle \delta y_i \rangle$ as before,

and $\hat{v}_i = \langle \delta y_i \rangle / \delta t - \partial D_{ij} / \partial y_j$ or, from (2.6) and (3.1), (3.9)

$$\hat{v}_i = \langle v_i \rangle - \pi \frac{\partial}{\partial y_j} F_{ij}(0) \qquad (4.3)$$

Provided the two-scale approximation remains valid, eq. (4.1) describes the evolution of an ensemble of climatic states with an arbitrary initial distribution for arbitrary large times. The propagation and diffusion coefficients $\hat{v}_i$, $D_{ij}$ will generally be functions of $y$, both directly and through their dependence on the statistical properties of the weather variables $x$. The equation includes both direct internal coupling through the propagation term $\hat{v}_i$ and indirect feedback through the dependence of the diffusion coefficients on the climatic state.

In practice, the expectation values and spectra in (4.2) and (4.3), defined as averages over an $x$-ensemble for fixed $y$, will normally be determined from time averages, rather than through ensemble averaging. In order that the average values can be regarded as local with respect to the climatic time scale $\tau_y$ but still remain adequately defined statistically with respect to the weather variability of time scale $\tau_x$, the averaging time $T$ must satisfy the two-sided inequality $\tau_x < T < \tau_y$. The inequalities imply that the spectral density $F_{ij}(0)$ at "zero frequency" in eqs. (4.2), (4.3) must be interpreted more accurately as the level of the spectrum in the frequency range $\tau_y^{-1} < \omega < \tau_x^{-1}$—as was already pointed out in connection with eq. (3.11). The variance spectra of $v_i$ for lower frequencies $\omega = O(\tau_y^{-1})$ must be attributed, within the framework of the two-timing theory, to the slow variations of the *mean* variables $\langle v_i \rangle$ on the climatic time scale. Since $\langle v_i \rangle$ depends on the local climatic state, the increase of the variance spectra of the climatic variables $y_i$ towards lower frequencies will normally be associated with a corresponding increase of the variance spectra of $v_i$ (and the "weather" variables $x_i$) in this range. This is not in conflict with the basic premise of a white input spectrum at "low" frequencies. Essential for the application of the two-timing concept is that there exists a spectral gap between the "weather" and "climate" frequency ranges in which the input spectra are flat (cf. Fig. 2).
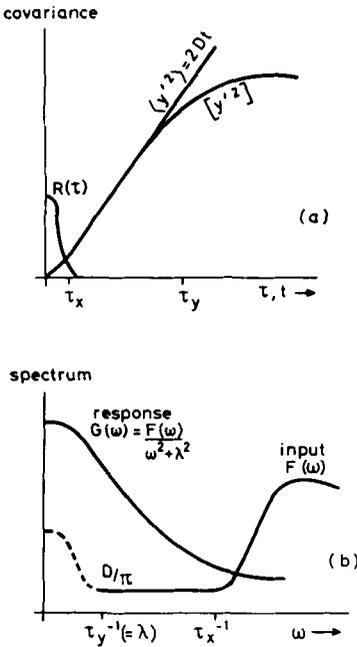
*Fig. 2.* Input and response of stochastically forced (single component) climate model with linear feed-back; (a) covariances, (b) spectra. In the ranges $\tau_x \ll \tau \ll \tau_y$ and $\tau_y^{-1} \ll \omega \ll \tau_x^{-1}$ the models with and without feedback are identical. In the range $\omega \lesssim \tau_y^{-1}$ the spectrum $F(\omega)$ cannot be regarded as part of the "weather input", but is coupled to the climate response.

The presence of the diffusion terms in (4.1) implies that climate evolution is necessarily a statistical rather than a deterministic phenomenon. Even if a well defined climate state is prescribed initially in the form of a $\delta$-function distribution for $p$, the diffusion term immediately leads to a finite spread of the probability distribution $p$ at later times. Without the diffusion term, an initial $\delta$-function distribution would retain its $\delta$-function character and simply propagate along the characteristics $dy_i/dt = \hat{v}_i$ in the climatic phase space.

The analytical integration of eq. (4.1) for an arbitrary nonlinear climate model with several degrees of freedom will normally not be possible. However, solutions can be constructed, for example, by the Monte Carlo method, in which eq. (2.5) and (2.7) are integrated numerically (without the restriction $t < \tau_y$) for an ensemble of realisations using an appropriate statistical simulation of $v_i'$. Within the approximations of

the two-timing theory, $v_i'$ can be represented very simply as a zero'th order Markov process.

For the special case of linear feedback and constant diffusion coefficients, equation (4.1) can be solved explicitly. These solutions are appropriate for climatic systems with small excursions. However, several properties of the linear case discussed in the following two sections may also be expected to apply qualitatively to more general climate models.

Although eq. (4.1) describes the evolution of $p(\mathbf{y}, t)$ in closed form (given the x-statistics for given $\mathbf{y}$), the probability distribution $p(\mathbf{y}, t)$ provides only a partial statistical description of the random process $y(t)$. A complete statistical description would require, for example, the set of joint probability distributions $p(\mathbf{y}_1, ..., \mathbf{y}_p)$ of the climate states for any set of times $t_1, ..., t_p$, or the set of all moments $\langle y_1 ... y_p \rangle$ for all $p > 0$. Generalised Fokker-Planck equations similar to (4.1) can be derived also for multi-time probability distributions, but these will normally be of less immediate interest. In practice, Monte Carlo methods of solving (4.1) actually generate the complete statistics of the process $\mathbf{y}$, as well as yielding $p(\mathbf{y}, t)$, so that the generalised Fokker-Planck equations need not be considered explicitly.

## 5. Linear feed-back models

### (a) *Solution of the Fokker-Planck equation*

For small excursions of the climatic states about an equilibrium state $\mathbf{y} = 0$, say, the diffusion and velocity coefficients in (4.1) can be expanded with respect to $\mathbf{y}$. Since the feedback terms must vanish for the equilibrium state, the coefficients are given to lowest order by

$$D_{ij} = \text{const} \tag{5.1}$$

$$\hat{v}_i = V_{ij} y_j, \quad V_{ij} = \text{const} \tag{5.2}$$

For a stable equilibrium state, the matrix $V_{ij}$ must be negative definite.

The general solution of (4.1) for an arbitrarily prescribed initial distribution $p(\mathbf{y}, t = 0) = p_0(\mathbf{y})$ may be constructed by superposition from the Green-function solution for an initial $\delta$-function distribution $p_0(\mathbf{y}) = \delta(y_1 - y_{10}) ... \delta(y_n - y_{n0})$ at an arbitrary point $\mathbf{y}_0$. This is given by the normal distribution

$$p(\mathbf{y}, t) = (2\pi)^{-n/2} |R|^{-1/2}$$

$$\times \exp\left( -\frac{R_{ij}^{-1}}{2} (y_i - [y_i])(y_j - [y_j]) \right) \quad (5.3)$$

where the mean $[y_i]$ and covariance tensor $R_{ij} = [(y_i - [y]) \cdot (y_j - [y_j])]$ are time dependent functions satisfying the differential equations and initial conditions

$$\frac{d[y_i]}{dt} = V_{ik}[y_k], \quad [y_i] = y_{i0} \text{ for } t = 0 \quad (5.4)$$

$$\frac{dR_{ij}}{dt} = 2D_{ij} + R_{ik} V_{jk} + R_{kj} V_{ik}, \quad R_{ij} = 0 \text{ for } t = 0$$

$$(5.5)$$

The square parentheses [ ] denote averages over the ensemble of climatic states $\mathbf{y}$. Equations (5.4), (5.5) can be verified by substitution of (5.3) in (4.1) or can be derived directly from (2.5), (4.1), (4.2) and (4.3). In matrix notation, the solutions may be written

$$[\mathbf{y}] = e^{Vt}\mathbf{y}_0 \quad (5.6)$$

$$R = R_\infty - e^{Vt} R_\infty e^{Vt^+} \quad (5.7)$$

where $V^+$ denotes the transpose of $V$ and $R_\infty$ is the asymptotic stationary solution of (5.5),

$$2D_{ij} + (R_\infty)_{ik} V_{jk} + (R_\infty)_{kj} V_{ik} = 0 \quad (5.8)$$

$R_\infty$ and the corresponding asymptotic equilibrium distribution $p_\infty$ (with $[\mathbf{y}]_\infty = 0$) are independent of the initial state $\mathbf{y}_0$.

The expressions become particularly simple if the matrix $V$ is diagonal, $V_{ij} = \delta_{ij}\lambda_{(i)}$ (parentheses around the index indicate that the index is excluded from the summation convention). Normally, this can be achieved by a suitable linear transformation of $\mathbf{y}$ to new coordinates. Equations (5.6), (5.7) then become

$$[y_i] = y_{i0} \exp(\lambda_{(i)} t) \quad (5.9)$$

$$R_{ij} = (R_\infty)_{ij}[1 - \exp(\lambda_{(i)} + \lambda_{(j)}) t] \quad (5.10)$$

$$(R_\infty)_{ij} = -\frac{2D_{ij}}{\lambda_{(i)} + \lambda_{(j)}} \quad (5.11)$$

(b) *Spectral decomposition of the variance*

The Gaussian form (5.3) of the probability distribution $p(\mathbf{y}, t)$ could have been inferred

directly from the Central Limit Theorem, without invoking the Fokker-Planck equation. The theorem states that, under very general conditions, the response of a linear system driven by a statistically stationary input consisting of a continuous sequence of infinitely short, statistically independent pulses is Gaussian, independent of the detailed statistical structure of the input. This property holds not only for the probability distribution $p$, but generally for the multi-time joint probability distribution. Thus the statistical structure of the process $\mathbf{y}$ is completely specified if the first moments (given by (5.6)) and the second moments

$$\hat{S}_{ij}(t + \tau) = [(y_i(t + \tau) - [y_i(t + \tau)]) \cdot (y_j(t) - [y_j(t)])] \quad (5.12)$$

are known.

The latter are given by the solution

$$\hat{S}(t, \tau) = e^{V\tau} R(t) \quad (\tau > 0) \quad (5.13)$$

of the differential equation

$$\frac{\partial \hat{S}_{ij}(t, \tau)}{\partial \tau} = V_{ik} \hat{S}_{kj} \quad (\tau > 0) \quad (5.14)$$

under the initial condition $\hat{S}_{ij}(t, \tau = 0) = R_{ij}(t)$, with $R_{ij}(t)$ given by (5.7). Equation (5.14) follows from (2.5), (2.7) and (5.2), noting that in the two-timing limit $v_1'(r + \tau) = (v_i(t + \tau) - \langle v_i(t + \tau)\rangle)$ and $y_j(t)$ are statistically uncorrelated for $\tau > 0$, since the correlation time scale of the random forcing is regarded as infinitely short compared with the correlation time scale of the response. This argument does not hold for $\tau < 0$, since $y_j(t)$ in this case includes the response to $v_i'$ at the earlier time $t + \tau$. However, the solution for $\tau < 0$ can be obtained from (5.13) by interchanging the indices and redefining the time variables.

Of particular interest is the asymptotic stationary solution

$$S(\tau) = \lim_{t \to \infty} S(t, \tau) = e^{V\tau} R_\infty \quad (5.15)$$

which can be compared with the statistical properties of observed, quasi-stationary climatic time series. If the second moments of the input (i.e. $D_{ij}$) are specified, it is known from linear systems analysis that $S(\tau)$ completely

determines the linear response characteristics (transfer functions) of the system.

The relation corresponding to (5.15) for the climate cross spectrum $G_{ij}$ can best be derived by direct substitution of the Fourier integral representation (3.3) in the basic climate equation

$$\frac{dy_i}{dt} = V_{ij}y_j + v_i'$$

One obtains

$$G_{ij}(\omega) = T_{ik}T_{jl}^* F_{kl}(0) \qquad (5.16)$$

where $T = (i\omega I - V)^{-1}$ ($I$ = unit matrix). For diagonal $V$, eq. (5.16) becomes

$$G_{ij}(\omega) = \frac{F_{ij}(0)}{(\omega - i\lambda_{(i)})(\omega + i\lambda_{(j)})} \qquad (5.17)$$

Equations (5.15), (5.16) may be compared with the corresponding relations (3.1), (3.11) for a system without feedback. The deviation covariance $\langle y_i' y_j' \rangle$ considered in section 3 should be compared in the case of a stationary y-process with the expression $[y_i' y_j'] = [(y_i - y_{i,0})(y_j - y_{j,0})]$ (also known as the "structure function", cf. Tatarski (1961)). This can be expressed in terms of the covariance function as

$$[y_i' y_j'] = (S_{ij}(0) - S_{ij}(\tau)) + (S_{ji}(0) - S_{ji}(\tau)) \qquad (5.18)$$

The general form of the functions $G_{ij}$, $\langle y_i' y_j' \rangle$ and $[y_i' y_j']$ for a system with and without linear feed-back is shown in Fig. 2. For $\tau_x \ll \tau \ll \tau_y$ and $\tau_y^{-1} \ll \omega \ll \tau_x^{-1}$ the behaviour of both systems is identical, but for $\tau = O(\tau_y)$ and $\omega = O(\tau_y^{-1})$ the unbounded response of the system without feedback begins to diverge from the bounded response functions of the linearly stabilised system.

# 6. Climate predictability

The evolution of the probability distribution $p(\mathbf{y}, t)$ as governed by the Fokker-Planck equation (4.1) determines the degree of climate predictability. If the climate state $\mathbf{y}_0$ at time $t = 0$ is known, the initial probability distribution $p_0$ is a $\delta$-function. For a fully predictable system, $p(\mathbf{y}, t)$ remains a $\delta$-function for all times $t > 0$. As pointed out in Section 4, however, the dif-

fusive term in (4.1) results in a broadening of the probability distribution for $t > 0$, and climate prediction therefore always entails some degree of statistical uncertainty.

A simple quantitative measure of the predictive skill can be defined in terms of the mean climatic state $[y_i]$ and the covariance matrix $R_{ij} = [(y_i - [y_i]) \cdot (y_j - [y_j])]$. The mean may be regarded as the climate "prediction". (In the case of a linear system, this is identical with the most probable climatic state, but in general the most probable state and the mean state will differ.) In order to introduce a measure of skill as a simple number, the distance $\delta_1$ of the predicted climate state from the initial state and the r.m.s. deviation $\varepsilon$ from the mean must be defined in terms of some suitable positive definite matrix $M_{ij}$,

$$\delta_1 = \{M_{ij}([y_i] - y_{i,0})([y]_j - y_{j,0})\}^{\frac{1}{2}} \qquad (6.1)$$

$$\varepsilon = \{M_{ij}R_{ij}\}^{\frac{1}{2}} \qquad (6.2)$$

The usual definition of the skill parameter is then given by the ratio "signal to signal-plus-noise",

$$s_1 = \frac{\delta_1}{(\varepsilon_1^2 + \delta_1^2)^{1/2}} \qquad (6.3)$$

For small times $t \ll \tau_y$, the predicted change $\delta_1$ increases linearly with time

$$\delta_1 \approx (M_{ij}v_{i,0}v_{j,0})^{\frac{1}{2}}t \qquad (6.4)$$

whereas the r.m.s. error grows as $t^{\frac{1}{2}}$,

$$\varepsilon \approx (2D_{ij}M_{ij})^{\frac{1}{2}}t^{\frac{1}{2}} \qquad (6.5)$$

Thus initially the skill parameter $s_1 \sim t^{\frac{1}{2}}$; the random deviations from the initial state induced by the stochastic forcing dominate over the deterministic changes produced by the internal coupling within the climatic system, and the predictive skill is small.

For very large $t$, $\delta_1$ and $\varepsilon$ will normally approach the limiting values

$$\delta_{1\infty} = \{M_{ij}([y_i]_\infty - y_{i,0})([y_j]_\infty - y_{j,0})\}^{\frac{1}{2}}$$

$$\varepsilon_\infty = (M_{ij}(R_\infty)_{ij})^{\frac{1}{2}}$$

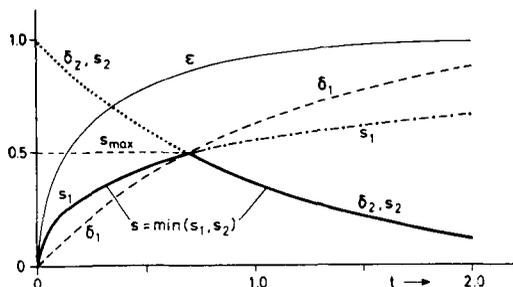appropriate to the stationary equilibrium distribution $p_\infty(\mathbf{y})$—assuming such a distribution

*Fig. 3.* Predicted climate changes $\delta_1$ relative to initial state and $\delta_2$ relative to asymptotic state, statistical error $\varepsilon$, and skill parameters $s_1$, $s_2$ and $s = \min(s_1, s_2)$, for a linear (single component) climate system. The initial value is chosen as $y_0 = (R_\infty)^{\frac{1}{2}} = 1$ (in this case $s_2$ and $\delta_2$ happen to coincide).

exists—and the skill parameter $s_1$ will become constant.

The predicted climatic state for large $t$ is simply the stationary climatic mean state $[y]_\infty$. This prediction may be regarded as trivial in the same way as the prediction through persistence for small $t$ is trivial. Since the contribution from straight persistence was subtracted in the definition of $s_1$, it apears more appropriate to introduce an alternative skill parameter

$$s_2 = \delta_2/(\varepsilon^2 + \delta_2^2)^{1/2} \qquad (6.6)$$

for large $t$, where

$$\delta_2 = \{M_{ij}([y_i] - [y_i]_\infty)([y_j] - [y_j]_\infty)\}^{\frac{1}{2}} \qquad (6.7)$$

is the deviation of the predicted climatic state from the stationary climatic mean. The net skill parameter may then be defined as $s = \min(s_1, s_2)$.

The behaviour of $s(t)$ in the intermediate range $t = O(\tau_y)$ between the limiting regions in which either $s_1$ or $s_2$ is very small depends in detail on the structure of the climate model. The general properties of $s(t)$ to be expected in this range may be inferred, however, from the solution for a linear system, cf. Fig. 3. Provided the initial deviation from the stationary climatic mean is of the same order as the variability of the stationary asymptotic distribution (for each degree of freedom separately), the maximal value of the net skill parameter generally lies in the neighbourhood of 0.5. This is due to the fact that the relaxation times for $\delta_1$ and $\varepsilon$ are of the same magnitude, since both are governed

by the same internal feedback processes. Thus both $\delta_1$ and $\varepsilon$ increase at approximately the same rate (after the initial period $t < \tau_y$), and the non-trivial (i.e. non-persistent) component of the prediction and the statistical error always remain of comparable magnitude.

These results may be expected to carry over, at least qualitatively, to nonlinear systems, provided there exists a unique stationary equilibrium distribution—i.e. provided the system is transitive in Lorenz' (1968) sense. In fact, the basic properties of the skill parameters $s_1$, $s_2$ outlined above are largely independent of the detailed dynamics of the climate system and follow simply from the fact that the evolution of the system corresponds to a first-order Markov process. The prediction problem becomes more complex in the case of intransitive systems, in which more than one stationary distribution may exist (for example, for dynamically disconnected regions of the climate phase space) or for nearly intransitive systems, characterised by two or more quasi-stationary, weakly interacting distributions. However, the discussion of these more complex cases must necessarily remain rather academic without reference to a specific climate model and will not be pursued further here.

## 7. Conclusions

The principal features of the stochastic climate model discussed in this paper may be summarised as follows:

(1) The time scales of the "weather system" and "climate system" are well separated.

(2) As a consequence of the time-scale separation, the response of the climate system to the random forcing by the weather components can be described as a continuous random walk or diffusion process (first-order Markov process). The response can be completely characterised by a diffusion tensor, which is proportional to the constant spectral density of the random forcing at low frequencies.

(3) The evolution of the climate system is described by a Fokker-Planck equation for the climate probability distribution; the propagation and diffusion coefficients of the equation depend on the instantaneous climate state, both directly and via the weather statistics.

(4) Without stabilising internal feedback

mechanisms, climate variability would grow indefinitely.

(5) Despite the stochastic nature of climate variability, the internal feedback terms in climate models imply a finite degree of predictability. However, the maximal predicitive skill for a statistically stationary climate system is generally no larger than 0.5 and is always significantly less than unity.

The discussion in this part of the paper has been restricted to the general structure of stochastic models, without reference to a specific model. It should be pointed out, however, that the extension of a typical SDM of, say, the Budyko-Sellers type to a stochastic model requires no basic modification of the internal structure of the model, but simply the addition of random driving terms. The relevant statistical properties of the stochastic forcing functions can be obtained directly from numerical experiments with GCM's or from meteorological data. Thus some of the general properties of stochastic climate models described in this paper can be tested rather easily by comparing observed climatic variability with theoretical predictions obtained with existing SDM's after incorporation of appropriate stochastic forcing terms (Lemke, 1976).

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## СТОХАСТИЧЕСКИЕ МОДЕЛИ КЛИМАТА

Рассматривается стохастическая модель изменчивости климата, в которой медленные изменения климата объясняются как интегральная реакция на непрерывное случайное возбуждение короткопериодными «погодными» возмущениями. Взаимодействующая система океан–атмосфера–криосфера–суша разделяется на быстро изменяющуюся «погодную» систему (атмосфера) и на медленно откликающуюся «климатическую» систему (океан, криосфера, растительность суши и т. д.). В обычной статистически-динамической модели (СДМ) только средние эффекты переноса быстро меняющихся погодных компонент параметризуются в климатической системе. Результирующие прогностические уравнения детерминистичны и климатические вариации обычно могут возникать только при изменении внешних условий. Существенной особенностью стохастических климатических моделей является то, что неосредненные «погодные» компоненты также сохраняются. Формально они появляются как случайные вынуждающие силы. Климатическая система, действующая как интегратор этого короткопериодного возбуждения, про-

являет те же самые характеристики реакции случайного блуждания, как крупные частицы, взаимодействующие с ансамблем гораздо более мелких частиц в аналогичной задаче броуновского движения. Модель предсказывает «красные» спектры изменений параметров в качественном согласии с наблюдениями. Эволюция распределения вероятностей климата описывается уравнением Фоккера–Планка, в котором эффект случайного погодного возбуждения описывается диффузионными членами. Без стабилизирую-

щей обратной связи модель предсказывает непрерывное увеличение изменчивости климата по аналогии с непрерывной неограниченной дисперсией частиц при броуновском движении (или в однородном турбулентном потоке). Стабилизирующая обратная связь дает статистически стационарное распределение вероятностей климата. Обратная связь проявляется также в конечной степени предсказуемости климата, но предсказуемость ограничивается максимальной величиной параметра умения предсказывать порядка 0,5.