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A PERTURBATION APPROACH TO COHERENT PROPAGATION OF ENERGETIC CHARGED PARTICLES IN RANDOM MAGNETIC FIELDS

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ABSTRACT

The coherent propagation of energetic charged particles in random magnetic fields first discussed by Earl is extended to more general magnetic field models by using a perturbation approach. Furthermore, it is shown that the parameter range for the slab model, for which coherent propagation holds, is larger than found by Earl. In the overlapping parameter region the transport coefficients given by Earl differ from the ones calculated in this paper, although they are of the same order of magnitude if the spectral index q is not too close to $q = -3$. For isotropic magnetic field fluctuations no pure diffusive solutions of the Fokker-Planck equation exist. However, coherent propagation is possible for spectral indices q between -3 and -1.

Subject headings: cosmic rays: general — hydromagnetics — interplanetary medium

I. INTRODUCTION

The propagation of charged particles in fluctuating magnetic fields has been treated in the weak interaction approximation, e.g., by Jokipii (1966) or Hasselmann and Wibberenz (1968). The random fluctuations lead to pitch-angle scattering of particles travelling along the guiding mean magnetic field. In general the propagation is described by a Fokker-Planck equation (Jokipii 1971).

Under the assumption of weak spatial gradients of the density the Fokker-Planck equation can be integrated to yield a diffusion equation which depends on space coordinates only. A spatial diffusion coefficient $K_{\parallel}$ parallel to the mean field can be calculated. This, however, is not possible if the magnetic field spectra are too steep, in which case $K_{\parallel}$ diverges. For the slab model, for instance, for which the magnetic fluctuations depend on the parallel coordinate only, $K_{\parallel}$ is finite for fluctuation spectra of the power-law type with spectral indices q in the range $-2 < q \leq -1$, but $K_{\parallel}$ diverges for $q \to -2$ (Jokipii 1966; Hasselmann and Wibberenz 1968).

Earl (1973*b*) first treated the problem of charged particles propagating in random magnetic fields with spectra too steep for a pure diffusive mode of transport. For the slab model Earl found solutions of the Fokker-Planck equation which represent a transport mode different from pure diffusion and which holds for spectral indices q between -3 and -2. He calls this particle transport "coherent propagation." His notation is adopted in the present paper.

$K_{\parallel}$ diverges when the velocity diffusion coefficient approaches zero too rapidly for $v_{\parallel} \to 0$. This occurs when there is not enough power in the high-frequency range of the magnetic field spectrum. Then there is no diffusive interconnection between the $v_{\parallel} > 0$ and the $v_{\parallel} < 0$ hemispheres, and an approximately isotropic particle distribution over the total $v_{\parallel}$-range from $-U$ to $+U$ cannot be obtained ($v_{\parallel}$ is the particle velocity parallel to the mean magnetic field and U the constant total velocity). This, however, is a necessary requirement for the particle propagation to be governed by diffusive rather than streaming processes.

When $K_{\parallel}$ diverges, it becomes reasonable to consider positive and negative velocity hemispheres separately. In this case mean densities are introduced for positive or negative velocities only and not, as usual, for both velocity hemispheres together. Then there is no averaging across the critical point $v_{\parallel} = 0$.

Particles described by a density distribution averaged in this way undergo coherent propagation instead of purely spatial diffusion. For this type of propagation, the magnitude of the convective velocity is one-half the total particle velocity. The direction of this propagation is positive or negative, depending on the hemisphere considered. There is no diffusive connection between the $v_{\parallel} > 0$ and the $v_{\parallel} < 0$ regions.

The ranges of validity for the purely diffusive and the coherent transport mode are discussed. Results are presented for three magnetic field models, which are often discussed in the literature.

II. ANALYSIS

We consider a density distribution of charged particles in magnetic fields which consist of a dominating constant guiding field and superposed random fluctuations. The particle density $n(x_{\parallel}, v_{\parallel}, t)$ is assumed to depend on the time t and on only one space coordinate $x_{\parallel}$ and on one velocity coordinate $v_{\parallel}$. Both $x_{\parallel}$ and $v_{\parallel}$ have the same direction as the mean magnetic field.

The time evolution of n is governed by the Fokker-Planck equation

$$\frac{\partial n}{\partial t} + v_{\|} \frac{\partial n}{\partial x_{\|}} - \frac{\partial}{\partial v_{\|}} \left(D \frac{\partial n}{\partial v_{\|}} \right) = 0 . \tag{1}$$

D is the parallel velocity-diffusion coefficient which is proportional to the pitch-angle diffusion coefficient (cf. Jokipii 1966, or Hasselmann and Wibberenz 1968). If the time dependence of the magnetic field is neglected, the D of first order always approaches zero for $v_{\|}$ approaching zero (cf. Alpers, Hasselmann, and Kunstmann 1975, 1977):

$$D(v_{\|} \to 0) \sim |v_{\|}|^p \to 0 . \tag{2}$$

The index $p > 0$ is a measure of the effectivity of diffusion near $v_{\|} = 0$. It is shown below that the relation (2) is valid in this context in spite of the δ-like function in D at $v_{\|} = 0$, which is found, e.g., for isotropic fluctuations by Fisk *et al.* (1974) and Goldstein, Klimas, and Sandri (1975).

In the usual derivation of a spatial diffusion equation one considers a density averaged over the total $v_{\|}$-range:

$$\rho(x_{\|}, t) = \int_{-U}^{+U} dv_{\|} n(x_{\|}, v_{\|}, t) . \tag{3a}$$

U is the constant total particle velocity. The density ρ appears, however, not to be a useful quantity if D approaches zero too rapidly, since this prohibits the necessary diffusive coupling across $v_{\|} = 0$. In this case it is more useful to consider instead the averaged densities ρ_+ and ρ_- defined by

$$\rho_{\pm}(x_{\|}, t) = \int_{L(\pm U)} dv_{\|} n(x_{\|}, v_{\|}, t) , \tag{3b}$$

where $L(+U)$ defines an integration from 0 to $+U$ and $L(-U)$ from $-U$ to 0. For these "$\pm$" densities there is no integration across the critical point $v_{\|} = 0$.

The methods applied previously, e.g., by Jokipii (1966) or Hasselmann and Wibberenz (1968, 1970), to the isotropic density ρ can be applied correspondingly to the analysis of $\rho_{\pm}$. Integrating the Fokker-Planck equation (1) either from $-U$ to $+U$ or over $L(\pm U)$, one obtains the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_{\|}} \int_{-U}^{+U} dv_{\|} v_{\|} n = 0 \tag{4a}$$

and

$$\frac{\partial \rho_{\pm}}{\partial t} + \frac{\partial}{\partial x_{\|}} \int_{L(\pm U)} dv_{\|} v_{\|} n = \pm j_0 , \tag{4b}$$

where

$$j_0 = -D \frac{\partial n}{\partial v_{\|}} \bigg|_{v_{\|}=0} \tag{5}$$

is the diffusive flux in phase space through $v_{\|} = 0$. In the following we make the additional assumption for the $\rho_{\pm}$ case, that the diffusive flux j_0 through $v_{\|} = 0$ is zero. Below we will further discuss the range of validity for this crucial assumption. It has been illustrated above that this assumption is plausible for a velocity diffusion coefficient D approaching zero rapidly enough for $v_{\|} \to 0$.

Following the analysis of Hasselmann and Wibberenz (1968, 1970), it makes sense to split the total density into the average isotropic part ρ and an anisotropic part n',

$$n(x_{\|}, v_{\|}, t) = \frac{\rho(x_{\|}, t)}{2U} + n'(x_{\|}, v_{\|}, t) \tag{6a}$$

for equation (3a), and correspondingly,

$$n(x_{\|}, v_{\|}, t) = \frac{\rho_{\pm}(x_{\|}, t)}{U} + n_{\pm}'(x_{\|}, v_{\|}, t) \tag{6b}$$

for (3b). The anisotropies are normalized (see eqs. [3a, b] and [6a, b]) by

$$\int_{-U}^{U} dv_{\|} n' = 0 \quad \text{and} \quad \int_{L(\pm U)} dv_{\|} n_{\pm}' = 0 , \tag{7}$$

respectively. We now make the basic assumption of the usual parallel diffusion theory that the particle distributions adjust very rapidly to a quasi-equlibrium isotropic state through pitch-angle diffusion. Under this assumption and with equation (1), (4a, b), and (6a, b), differential equations for the anisotropy are obtained, which to first order read

$$\frac{\partial}{\partial v_{\|}}\left(D\frac{\partial n'}{\partial v_{\|}}\right) = \frac{\partial \rho}{\partial x_{\|}}\frac{v_{\|}}{2U} \tag{8a}$$

with the solution

$$n' = -\frac{1}{4U}\frac{\partial \rho}{\partial x_{\|}}\left(\int^{v_{\|}} du_{\|}\frac{u_{\perp}^2}{D} - N\right) \tag{9a}$$

(cf. Hasselmann and Wibberenz 1968, 1970). The perpendicular velocity $u_{\perp}$ is given by $u_{\perp}^2 = U^2 - u_{\|}^2$.

For $\rho_{\pm}$ the same quasi-equilibrium assumption as mentioned above is made for each half-space separately. Then it follows that

$$\frac{\partial}{\partial v_{\|}}\left(D\frac{\partial n_{\pm}'}{\partial v_{\|}}\right) = \pm\frac{\partial \rho_{\pm}}{\partial x_{\|}}\left(\frac{v_{\|}}{\pm U} - \frac{1}{2}\right) + \frac{j_0}{\pm U}\,. \tag{8b}$$

A special solution for $j_0 \equiv 0$ is

$$n_{\pm}' = \pm\frac{1}{2}\frac{\partial \rho_{\pm}}{\partial x_{\|}}\left(\frac{1}{\pm U}\int^{v_{\|}} du_{\|}\frac{u_{\|}^2}{D} - \int^{v_{\|}} du_{\|}\frac{u_{\|}}{D} \pm N_{\pm}\right) = -\frac{1}{2}\frac{\partial \rho_{\pm}}{\partial x_{\|}}\left[\int^{v_{\|}} du_{\|}\frac{|u_{\|}|}{D}\left(1 - \frac{|u_{\|}|}{U}\right) - N_{\pm}\right]. \tag{9b}$$

Here N and $N_{\pm}$ are constants determined by the normalizations (7). The lower limits of the integrations in (9a, b) can, in general, be chosen as zero. In the first order equation (8a) n' is neglected compared with ρ in $\partial/\partial x_{\|}$ and $\partial/\partial t$. As $n' \sim \partial\rho/\partial x_{\|}$ it is actually assumed that the density distribution is rather smooth in the quasi-equilibrium state. The same applies to $\rho_{\pm}$ and $n_{\pm}'$.

The total density $n(x_{\|}, v_{\|}, t)$ has been split in formulae (6a, b) into an average isotropic and an anisotropic part. This n is substituted into the integrals of equations (4a, b). For the isotropic density ρ, which was averaged from $-U$ to $+U$, the well-known diffusion equation

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x_{\|}}\left(K_{\|}\frac{\partial \rho}{\partial x_{\|}}\right) = 0 \tag{10a}$$

with

$$K_{\|} = -\int_{-U}^{+U} dv_{\|}v_{\|}n'\left(\frac{\partial \rho}{\partial x_{\|}}\right)^{-1} = \frac{1}{8U}\int_{-U}^{+U} dv_{\|}\frac{v_{\perp}^4}{D} \tag{11a}$$

results (cf. Hasselmann and Wibberenz 1968, 1970, or Jokipii 1966, 1971). The second form of $K_{\|}$ is obtained by inserting n' from (9a) and employing partial integration. For $\rho_{\pm}$, which is the density averaged from 0 to $\pm U$, we obtain an equation which describes convection plus diffusion:

$$\frac{\partial \rho_{\pm}}{\partial t} \pm \frac{\partial}{\partial x_{\|}}\left(\rho_{\pm}\frac{U}{2}\right) - \frac{\partial}{\partial x_{\|}}\left(K_{\|}^{\pm}\frac{\partial \rho_{\pm}}{\partial x_{\|}}\right) = 0 \tag{10b}$$

with

$$K_{\|}^{\pm} = -\int_{L(\pm U)} dv_{\|}v_{\|}n_{\pm}'\left(\frac{\partial \rho}{\partial x_{\|}}\right)^{-1} = \frac{U}{4}\int_{L(\pm U)} dv_{\|}v_{\|}^2\frac{(1 - |v_{\|}|/U)^2}{D}\,. \tag{11b}$$

The name "convection" is conventionally associated in cosmic ray physics with a sweeping along of particles with the solar wind and not, as is usual, with a collective motion of particle bunches. So we follow Earl (1974*a*) and call the particle transport described by equation (10b) "coherent propagation" instead of convection plus diffusion.

The equation (10b) describing coherent propagation can be transformed to the equation (10a), which describes pure diffusion, by $x_{\|} \rightarrow x_{\|}' = x_{\|} \mp Vt$ where $V = U/2$. Coherent solutions can thus be obtained from the well known purely diffusive solutions of (10a). Thus a coherent solution is, e.g.,

$$\rho_{\pm} \sim (K_{\|}^{\pm}t)^{-1/2}\exp\left[-(x_{\|} \mp Vt)^2/4K_{\|}^{\pm}t\right]$$

(cf. Earl 1973*b*, 1974*a*, *b*). It describes the "coherent" propagation of a pulse of particles, whose center moves with a velocity $V = U/2$ into either the positive or negative direction. This solution is, however, restricted to an initial δ-like injection of particles.

According to formula (11a), $K_{\parallel}$ starts to diverge for $p \to 1$. An approximated form of $K_{\parallel}$ proposed by Jokipii (1968) instead of (11a) avoids this divergence and has often been used. It is shown, however, in Hasselmann and Wibberenz (1970) and Earl (1973*a*) that the expression (11a) given in Jokipii (1966) and in Hasselmann and Wibberenz (1968) is the more correct one. The approximated form remains finite for $p \to 1$, but is then no longer a relevant approximation. In fact, the divergence of $K_{\parallel}$ according to equation (11a) is meaningful as it corresponds to situations in which the tendency of the particle distribution to isotropy is retarded so strongly through the weak diffusion near $v_{\parallel} = 0$ that the first order approximation (8a), which assumes the rapid establishment of a quasi-isotropic local equilibrium, is no longer applicable. Coherent instead of pure diffusive transport modes result.

III. RANGE OF VALIDITY

The existence of the coherent transport mode instead of pure diffusion (see eqs. [10a, b]) crucially depends on the assumption of a zero diffusive flux j_0 through $v_{\parallel} = 0$. This condition has been discussed until now only in a qualitative sense. According to formula (9a), j_0 is nonzero for the anisotropy n' so that the assumption of $j_0 \equiv 0$ could be justified only for those cases where the definition of ρ and the resulting diffusion coefficient $K_{\parallel}$ are not meaningful. Consider a density n which linearly increases from $v_{\parallel,1} = -\epsilon/2$ to $v_{\parallel,2} = +\epsilon/2$ by an amount Δn. If $p > 1$, the diffusive flux

$$j_0 = -D\frac{\partial n}{\partial v_{\parallel}} \sim \epsilon^p \frac{\Delta n}{\epsilon} = \epsilon^{p-1}\Delta n \to 0 \tag{12}$$

will approach zero for $\epsilon \to 0$ in spite of the infinite gradient. For $p < 1$ the j_0 is nonzero in general (see eq. [9a]).

The assumption $j_0 \equiv 0$ is thus justified for $p > 1$. A particle distribution with an initial discontinuous step at $v_{\parallel} = 0$ is stable for $p < 1$, and it is smeared out by diffusion for $p < 1$. Another way of showing this is given in Kunstmann and Alpers (1975).

To test the long time stability of the coherent solution, one has to add a small perturbation and to analyze its evolution. This corresponds to higher order solutions which are treated in the Appendix. The result is that to highest order (i.e., on a long-time basis) only $p < 2$ is allowed and not the larger range $p \leq 3$, which is obtained from evaluating the integral (11b) at $v_{\parallel} \approx 0$. We note that the anisotropy $n_{\pm}'$ (9b) of first order also diverges for $p > 2$.

In addition to the above conditions on the steepness parameter p, which must be satisfied in order that essentially the flux j_0 vanishes at $v_{\parallel} = 0$ and the particle distributions n' remain finite, a basic two timing approximation must be valid. The convection time $t_c = \rho(U\partial\rho/\partial x_{\parallel})^{-1}$ has to be large compared with the pitch-angle relaxation time $\tau = O(U^2/D)$ in order for the particles to have time to adjust locally to the near isotropic equilibrium. On the other hand, the convection time must not be too large. This upper bound enters the problem because of the assumption that the velocity diffusion coefficient is accurately described by (2) which implies $D \to 0$ for $v_{\parallel} \to 0$. This, however, holds only to first order. In higher orders there is a finite rate of scattering through $v_{\parallel} = 0$ (see, e.g., Jokipii 1971; Völk 1973; or Alpers, Hasselmann, and Kunstmann 1977). Thus for very large times t the particle distribution will always approach isotropy in the full space. Whether or not the normal diffusion theory or the coherent theory is valid depends on whether $\tau \ll t \ll t_c$ or $\tau \ll t_c \leq t$.

There exists a δ-like function in D at $v_{\parallel} = 0$ for quite general magnetic field models including, e.g., the isotropic model considered below (see Fisk *et al.* 1974; Goldstein, Klimas, and Sandri 1975). This would invalidate equation (2), i.e., $D \to 0$ for $v_{\parallel} \to 0$. Because of the nearly infinite sharpness, however, this δ-function has no effect on the assumption of a zero diffusive flux through the region $v_{\parallel} \approx 0$, which was essential for the existence of the coherent transport mode (cf. Alpers, Hasselmann, and Kunstmann 1977). The nonrelevance can also be seen from the formulae (11a, b) for $K_{\parallel}$ and $K_{\parallel}^{\pm}$ in which a δ-like singularity with a vanishing width has a negligible effect on the integral.

IV. MAGNETIC FIELD FLUCTUATION MODELS

In the following we apply the above general formulas to three often discussed models of magnetic field fluctuations.

a) *Slab Model*

For the slab model the fluctuations are assumed to depend on the parallel coordinate only. The resulting velocity diffusion coefficient D is given by

$$D(v_{\parallel}) = \beta v_{\perp}^2 |v_{\parallel}|^{-q-1}[1 - \sigma \operatorname{sign}(v_{\parallel})] \tag{13}$$

with a constant $\beta > 0$ and a polarization σ with $|\sigma| \leq 1$ (cf. Jokipii 1966; Hasselmann and Wibberenz 1968; Alpers, Hasselmann, and Kunstmann 1977). It is assumed in equation (13) that the scalar part of the spectrum is represented by a power law

$$P(k) \sim |k|^q, \tag{14}$$

TABLE 1

FUNCTIONAL FORM OF THE EXPRESSION DEFINED IN EQUATION (17) FOR THE ANISOTROPIES $n_\pm'$ FOR THE SLAB MODEL

q	A	N	F	$K_\parallel^\pm$
−1..........	$\ln(1+\lvert\mu\rvert)-\lvert\mu\rvert$	$2[\ln(2)-\frac{3}{4}]$	$-2U$	$2U^2[\ln(2)-\frac{2}{3}]E_\pm > 0$
−2..........	$-\ln(1+\lvert\mu\rvert)$	$-2[\ln(2)-\frac{1}{2}]$	-2	$2U[\frac{3}{4}-\ln(2)]E_\pm > 0$
−3..........	$\ln\left(\frac{1+\lvert\mu\rvert}{\lvert\mu\rvert}\right)$	$2\ln(2)$	$-2/U$	$2[\ln(2)-\frac{1}{2}]E_\pm > 0$
−4..........	$-\ln\left(\frac{1+\lvert\mu\rvert}{\lvert\mu\rvert}\right)+\frac{1}{\lvert\mu\rvert}$	∞	$-2/U^2$	

NOTE.—The constant $E_\pm$ is defined by $E_\pm = \pm(4\beta)^{-1}(1-\sigma^2)^{-1}(\pm 1+\sigma)$ and μ by $\mu = v_\parallel/U$.

where $\boldsymbol{k}$ is the wavenumber. The index q is restricted to $q < -1$ as the spectral magnetic energy density, which is given by $\int^{k_0} P(\boldsymbol{k})d\boldsymbol{k}$, must remain finite for $k_0 \to \infty$. A comparison of formulae (2) and (13) shows that $p = -q - 1$. With D given by (13) the anisotropy n' can be calculated for $-2 < q \le -1$:

$$n' = -\frac{1}{4U\beta}\frac{1}{1-\sigma^2}\frac{\partial\rho}{\partial x_\parallel}\frac{U^{q+2}}{q+2}\left[\left|\frac{v_\parallel}{U}\right|^{q+2}[\operatorname{sign}(v_\parallel)+\sigma] - \frac{\sigma}{q+3}\right]. \tag{15}$$

The parallel diffusion coefficient (11a) is given by

$$K_\parallel = \frac{1}{2\beta}\frac{1}{1-\sigma^2}\frac{U^{q+3}}{(q+2)(q+4)} \tag{16}$$

(cf. Jokipii 1966 or Hasselmann and Wibberenz 1968). Figure 1 shows the anisotropy n' for the parameters $\sigma = 0$ and $\sigma = 0.5$. For $q \to -2$, $K_\parallel$ diverges.

In the range $-4 < q \le -2$ the coherent propagation mode applies. One obtains

$$n_\pm' = -\frac{1}{4U\beta}\frac{1}{1-\sigma^2}\frac{\partial\rho_\pm}{\partial x_\parallel}F[\pm 1+\sigma][A(v_\parallel) - N], \tag{17}$$

where the constants F and N and the function $A(v_\parallel)$ are given in Table 1 for $q = -1, -2, -3$ and -4. The solutions for nonintegral q cannot be expressed by simple formulae as in Table 1. The analytical expression is

$$n_\pm' = -\frac{1}{4U\beta}\frac{1}{1-\sigma^2}\frac{\partial\rho_\pm}{\partial x_\parallel}2U^{q+2}[\pm 1+\sigma]\left[\int^{|v_\parallel/U|} dx\frac{x^{q+2}}{1+x} - N_\pm\right]. \tag{18}$$

The graphs of n_+' (positive $v_\parallel$ only) are shown in Figure 2 for some q. The quantity $n_\pm'$ is finite for $-3 < q \le -1$. For $-4 < q \le -3$ there are integrable infinities at $v_\parallel = 0$. For $q \le -4$ no normalization exists.

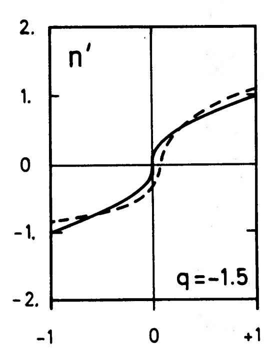

FIG. 1

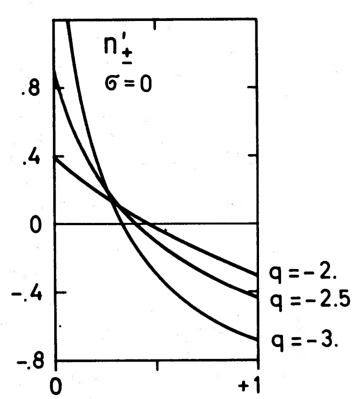

FIG. 2

FIG. 1.—n' anisotropies for the slab model as a function of $v_\parallel/U$ in units of the last bracket of (15). *Solid line*, $\sigma = 0$; *dashed lines*, $\sigma = 0.5$.

FIG. 2.—$n_\pm'$ anisotropies for the slab model as a function of $v_\parallel/U$ in units of the last brackets of (17) and (18). In these units $n_+' = n_-'$.

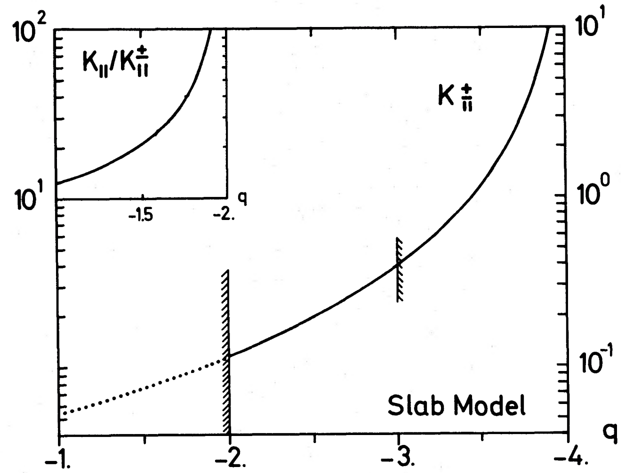

FIG. 3.—Parallel diffusion coefficient $K_\|{}^\pm$ as a function of q for the slab model in units of $\pm(4\beta)^{-1}[(\pm 1 + \sigma)/(1 - \sigma^2)]U^{q+3}$. In the upper insert the relation $K_\|/K_\|{}^\pm$ is shown for $-2 < q \leq 1$ and $\sigma = 0$.

The diffusion coefficients $K_\|{}^\pm$ computed from equations (11b) and (13) are also given in Table 1. For $\sigma = 0$ one obtains

$$K_\|{}^\pm = \frac{U^{q+3}}{4\beta}\left[\frac{1}{q+4} - 2\left(\frac{1}{q+5} - \frac{1}{q+6} + \cdots\right)\right]. \tag{19}$$

Values of $K_\|{}^\pm$ are shown in Figure 3. To first order the q-range is restricted to $-4 < q \leq -2$. To highest order it would be restricted to $-3 < q \leq -2$, which is marked by the right hatched line in Figure 3. Also shown is the relation $K_\|/K_\|{}^\pm$ for $-2 < q \leq -1$, which can be calculated formally although $K_\|$ and $K_\|{}^\pm$ are valid on different q-regimes. $K_\|$ clearly dominates, so an identification of a measured diffusion coefficient with $K_\|$ instead of $K_\|{}^\pm$ would heavily distort practical results (cf. Earl 1974*a*).

In summary, for a slowly varying particle distribution situated in random magnetic fields of the slab model type the propagation parallel to the mean field is purely diffusive if the steepness q of the magnetic spectrum lies in the range $-2 < q \leq -1$. For $q \to -2$ there is the known divergence which is physically meaningful and implies that the net particle propagation now becomes coherent rather than purely diffusive. The coherent transport exists for $-4 < q \leq -2$ and applies separately for the positive and negative velocity hemispheres which are not coupled with each other.

b) *Isotropic Fluctuations*

We now consider isotropic fluctuations for which D is given by

$$D(v_\|) \sim v_\perp{}^2[A(v_\|) + \sigma B(v_\|)], \tag{20}$$

A and B are sums over integral expressions given by, e.g., Goldstein, Klimas, and Sandri (1975) or Alpers, Hasselmann, and Kunstmann (1977). A and B depend on q, where q is again the power-law index of the scalar part of the spectrum (see eq. [14]). It can be shown that for formula (20) one obtains

$$D(v_\| \to 0) \sim |v_\||^{-q}. \tag{21}$$

A comparison with (2) gives $p = -q$.

The integration (11a) for $K_\|$ diverges for $v_\| \to 0$ as $q < -1$. Thus for isotropic fluctuations there is no pure diffusion. Fisk *et al.* (1974) have stated this in a numerical approximation for low-rigidity particles and another form of the scalar spectrum and have taken this as evidence for rejecting the expression (11a) for the parallel diffusion coefficient (see § III).

For the '$\pm$' cases,

$$n_\pm'(v_\|) \sim \int_{\pm U}^{v_\|} d\left(\frac{u_\|}{U}\right) \frac{(u_\|/U)}{1 + |u_\|/U|} \frac{1}{A(u_\|) + \sigma B(u_\|)} - N_\pm \tag{22}$$

and

$$K_\|{}^\pm \sim \int_{L(\pm U)} d\left(\frac{u_\|}{U}\right)\left(\frac{u_\|}{U}\right)^2 \frac{1 - |u_\|/U|}{1 + |u_\|/U|} \frac{1}{A(u_\|) + \sigma B(u_\|)} \tag{23}$$

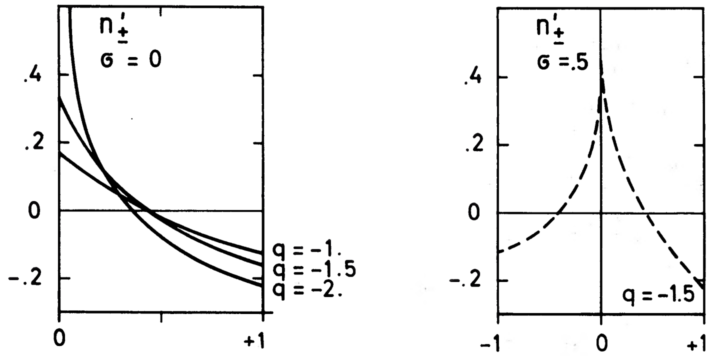

FIG. 4

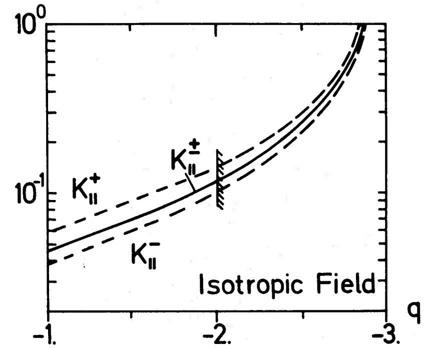

FIG. 5

FIG. 4.—$n_\pm'$ anisotropies for isotropic fluctuations in units of formula (22). The plot on the left refers to $\sigma = 0$ (*solid lines*), where n_+' and the n_-' are equal. The plot on the right refers to $\sigma = 0.5$ (dashed lines), where n_+' and n_-' are different.

FIG. 5.—Parallel diffusion coefficient $K_\parallel^\pm$ in units of formula (23) for the isotropic spectrum. For notations see Fig. 4.

have to be considered. The quantity $n_\pm'$ is finite for $q > -2$, but diverges for $v_\parallel \to 0$ and $q \le -2$. For $q > -3$ this divergence is still integrable, which ensures the existence of the normalization $N_\pm$. The integration for $K_\parallel^\pm$ converges for $q > -3$. The resulting anisotropies $n_\pm'$ are shown in Figure 4 and diffusion coefficients $K_\parallel^\pm$ for some σ and q in Figure 5.

In summary, particles propagating in isotropic magnetic field fluctuations do not undergo pure diffusion for any spectral index q. There is not enough power in the high-wavenumber range of isotropic fluctuations to produce a diffusive coupling across $v_\parallel = 0$. The coherent transport mode exists, however, for $-3 < q \le -1$.

The q range is restricted to $-2 < q < -1$ if going to highest order (see III).

c) *Alfvénic Fluctuations*

For the axisymmetric Alfvén wave model of fluctuations the velocity diffusion coefficient D is characterized by

$$D(v_\parallel \to 0) \sim |v_\parallel|^{-q+2} ,$$

when again a power law for the scalar spectrum is assumed (cf. Alpers, Hasselmann, and Kunstmann 1977). Again q is limited by $q < -1$. It follows from equation (11a) that no $K_\parallel$ exists, so there is no pure diffusion. Even $K_\parallel^\pm$ diverges, so for Alfvén waves there is no coherent transport either.

V. RELATIONSHIP WITH EARL'S RESULTS

Earl (1973*b*, 1974*a*, *b*) has investigated the coherent propagation only for the slab model. The method applied by Earl is to approximate the scattering eigenfunctions and eigenvalues by using a trial function technique for a first estimate. This method, however, is restricted to the slab model, where D is given by a power-law. It cannot be generalized to other magnetic field models, e.g., to the isotropic model, where D has a different structure. By applying the perturbation approach, however, the transport equations (10a, b) and the transport coefficients $K_\parallel$ and $K_\parallel^\pm$ (11a, b) can be derived in a straightforward manner and in a closed form for any model of magnetic field fluctuations. By the parallelism of reasoning the ranges of validity for pure diffusion and for coherent propagation could be discussed.

Our result (19) for $K_\parallel^\pm$ specialized for the slab model can be compared with the result of the corresponding "coefficient of dispersion," D_*, given by Earl (1974*a*—see his eq. [84] as the first term of a Taylor expansion). The ratio $D_*/K_\parallel^\pm$ is plotted in Figure 6. For $q \approx -2.5$, D_* is only by a factor of 2 larger than $K_\parallel^\pm$. The ratio diverges, however, for $q \to -3$ as D_* diverges. $K_\parallel^\pm$ still exists up to $q \ge -4$. $K_\parallel^\pm$, which is calculated in a perturbation expansion, is arbitrarily accurate provided the expansion parameter is small enough. A trial function technique, however, depends on the number of iterations and how well the first estimate is chosen. In this sense $K_\parallel^\pm$ is more accurate and is also a generalization of Earl's coefficient of dispersion.

VI. DISCUSSION

We have derived solutions of the Fokker-Planck equation by applying the known perturbation method of splitting the total particle density into an averaged isotropic and a small anisotropic component. In addition to the usual averaging for velocities from $-U$ to $+U$ we have also introduced densities averaged from 0 to $\pm U$. Thus it becomes possible to derive in a very similar way to the derivation of spatial diffusion the mode of transport which exists for very steep magnetic spectra. The density averaged over half the velocity space is governed by a convection-diffusion equation and not only by the well-known pure diffusion equation. The particle transport switches

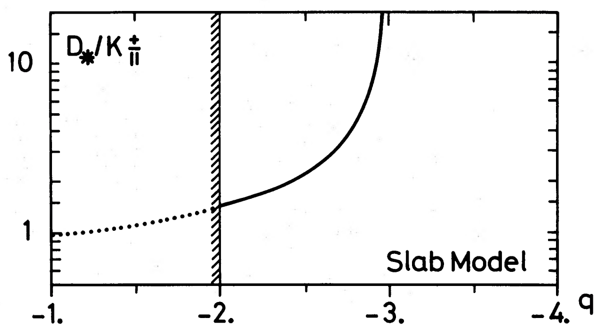

FIG. 6.—Ratio between Earl's coefficient of dispersion D_* and the corresponding transport coefficients $K_{\parallel}^{\pm}$ for the slab model as a function of q.

over from pure diffusion to coherent propagation for magnetic field spectra which are steep enough that the interconnection between the $v_{\parallel} > 0$ and $v_{\parallel} < 0$ hemispheres becomes ineffective.

The coherent transport mode exists for the *slab model* of magnetic fluctuations if the power-law index q is between -4 and -2, whereas the pure diffusive mode is valid for $-2 < q \leq -1$. For the more general *isotropic* magnetic field spectrum the coherent transport mode is derived for $-3 < q \leq -1$. There is, however, no pure diffusion for the isotropic spectrum, since there is not enough power in the high-wavenumber range of these fluctuations to produce the necessary diffusive coupling across $v_{\parallel} = 0$. For *Alfvén* wave fluctuations neither the pure diffusion nor the coherent propagation exists. In general the coherent mode exists if the velocity diffusion coefficient D approaches zero at least as fast as $|v_{\parallel}|^1$, but not faster than $|v_{\parallel}|^3$. In short: coherent propagation exists when the diffusive connection between the $v_{\parallel} > 0$ and $v_{\parallel} < 0$ hemispheres is ineffective, i.e., when $j_0 \approx 0$. If D approaches zero even faster than $|v_{\parallel}|^3$, other types of streaming processes will become important.

In interplanetary space the measured magnetic fluctuation spectrum has a mean slope of $q = -1.7$ (Sari 1975), but several occasions of steeper spectra have been observed. The problem is then to determine which mode of transport is present for a specific index q actually measured in interplanetary space. The magnetic field data are normally measured by *one* satellite only, so that the fluctuations have to be interpreted according to a special model, which itself cannot unambiguously be determined from the data (cf. Jokipii 1971). This implies that, e.g., for $q = -1.7$, pure diffusion is applicable if the slab model is adopted, but the totally different coherent mode results if an isotropic model is assumed.

The geometry underlying the velocity diffusion coefficient D quoted for the model fluctuations considered here is that of a constant mean magnetic field and not a divergent magnetic field, as is actually present in interplanetary space. Because of the adiabatic focusing of the particles to $v_{\parallel} = +U$ in this more realistic geometry, the assumption $j_0 \approx 0$ of no backscattering will be a good one for the particles coming from the Sun, so the coherent propagation may play an important role for the transport of charged particles in interplanetary space, especially in the early phase of a solar particle event. The generalization of the perturbation approach to divergent magnetic fields is straightforward.

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APPENDIX

HIGHER ORDER EXPANSIONS OF THE ANISOTROPIES $n_{\pm}'$

For higher order solutions of the Fokker-Planck equation (1) the density n is split into

$$n = \frac{\rho_{\pm}}{U} + \sum_{i=1} n_{\pm}^{(i)} \tag{A1}$$

(see eq. [6b]), where $n_{\pm}^{(1)}$ is the anisotropy of first order given by (9b) and $n_{\pm}^{(i)}$ that of order i. Using equations (1), (4b), and (8b), one obtains

$$\sum_{i=2} \frac{\partial}{\partial v_{\parallel}}\left(D \frac{\partial n_{\pm}^{(i)}}{\partial v_{\parallel}}\right) = -\frac{1}{U} \sum_{i=1} \frac{\partial}{\partial x_{\parallel}} \int_{L(\pm U)} du_{\parallel} u_{\parallel} n_{\pm}^{(i)} + \sum_{i=1} \left(\frac{\partial}{\partial t} + v_{\parallel} \frac{\partial}{\partial x_{\parallel}}\right) n_{\pm}^{(i)} . \tag{A2}$$

By comparing the orders in $\partial/\partial x_{\|}$ and/or $\partial/\partial t$ the following equations result:

$$\frac{\partial}{\partial v_{\|}}\left(D\frac{\partial n_{\pm}^{(i+1)}}{\partial v_{\|}}\right) = -\frac{1}{U}\frac{\partial}{\partial x_{\|}}\int_{L(\pm U)} du_{\|}u_{\|}n_{\pm}^{(i)} + \left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial x_{\|}}\right)n_{\pm}^{(i)} \tag{A3}$$

with $i = 1, 2, \ldots$. An exponent α_{i+1} is defined by $n_{\pm}^{(i+1)}(v_{\|} \to 0) \sim |v_{\|}|^{\alpha_{i+1}}$. By comparing the exponents of $v_{\|}$ on the left side of the equation with the second term on the right side of (A3) the conditions $\alpha_{i+1} = \alpha_i - p + 2$ are obtained. Because of $\alpha_1 = 2 - p$ (see eq. [9b]) this condition is identical to

$$\alpha_{i+1} = (i + 1)(2 - p)\,. \tag{A4}$$

The condition (7) of normalization for $n_{\pm}^{(i+1)}$ means that the integral $\int n_{\pm}^{(i+1)}dv_{\|}$ has to exist, which is critical for small $v_{\|}$. Thus the following inequality must be fulfilled:

$$2 + \frac{1}{i+1} - p > 0\,. \tag{A5}$$

In second order $p < 2.5$, and for the highest order $p < 2$ results. (For the first and third terms on the right-hand side of eq. [A3] $p < 3$ is obtained.) This must be interpreted in the sense that for $p \geq 2$ there are terms in the *exact transport equation* for $\rho_{\pm}$, which on a long time scale become dominant compared with the first order coherent mode.

Equations corresponding to (A1)–(A3) can also be derived for the anisotropy $n^{(i)}$ instead of $n_{\pm}^{(i)}$. The first two orders $n^{(1)}$ and $n^{(2)}$ introduced into (4a) give the second order transport equation, which reads

$$\frac{\partial \rho}{\partial t} + \frac{K_{\|}}{V_0^2}\frac{\partial^2 \rho}{\partial t^2} = K_{\|}\frac{\partial^2 \rho}{\partial x_{\|}^2}\,, \tag{A6}$$

where, for the slab model,

$$V_0^2 = U^2\frac{2q + 5}{(q + 4)^2}\,. \tag{A7}$$

Equation (A6) is the telegrapher's equation which embodies both diffusion, characterized by the coefficient $K_{\|}$, and wavelike propagation at velocity V_0 (cf. Earl 1974*a*).

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