J. Fluid Mech. (1980), *vol.* 97, *part* 1, *pp.* 215–224
Printed in Great Britain

A similarity relation for the nonlinear energy transfer in a finite-depth gravity-wave spectrum

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(Received 21 September 1977 and in revised form 26 June 1979)

The energy transfer in a finite-depth gravity-wave spectrum is investigated in the approximation of a narrow spectrum. It is shown that for ocean depths larger than approximately one tenth of the wavelength ($kh \geqslant 0{\cdot}7$) the finite-depth case can be reduced to Longuet-Higgins' (1976) result for an infinitely deep ocean by a similarity transformation involving changes in scale of the angular spreading function and the transfer rate. For shallower water ($kh < 0{\cdot}7$) Longuet-Higgins' expansion technique is no longer applicable without modification, as the nonlinear coupling coefficient develops a discontinuity at the origin of the expansion. In the range $kh \geqslant 0{\cdot}7$ both the magnitude and the two-dimensional frequency-directional distribution of the energy transfer are found not to differ significantly (to within variations by a factor of 2) from the case of an infinitely deep ocean. The transformation rules relating the infinite-depth and finite-depth cases may provide a useful guide for constructing parametrizations of the nonlinear transfer for finite-depth wave prediction models.

1. Introduction

A number of recent experiments (Mitsuyasu 1968, 1969; Hasselmann *et al.* 1973, 1976) suggest that the shape and evolution of a wind-wave spectrum is largely controlled by nonlinear wave–wave interactions. These conclusions were drawn from a comparison of spectral growth measurements with numerical computations of the nonlinear energy transfer for infinite-depth spectra. Although Hasselmann's expression (1961, 1963*b*) for the nonlinear energy transfer was originally derived for the general finite-depth case, similar calculations for finite-depth waves have not yet been carried out. Such computations would clearly be desirable not only for an improved understanding of the energy balance of gravity-waves of finite depth, but also for the construction of numerical wave models for shallow-water areas, where the demand for improved wave forecasts and wave-climate statistics has steadily increased through the expansion in off-shore activities.

Numerical calculations of the nonlinear energy transfer for the general case of an arbitrary finite-depth spectrum may be expected to be considerably more time consuming than in the infinite-depth case. Various simplifications arising from the homogeneity of the coupling coefficients and the dispersion relation with respect to wavenumber, which enables the transfer rates for different wavenumbers to be related by scaling factors, are no longer applicable. More importantly, to derive parametrized transfer expressions for use in numerical wave models, a large series of computations,

0022-1120/80/4400-8470 $02.00

including the dependence on the wavelength-to-depth ratio as additional external parameter, needs to be carried out. For this reason it appears appropriate to restrict the investigation of finite-depth influences first to the case of a very narrow spectrum, which can be treated more simply using the approximations of Longuet-Higgins (1976). An investigation of this limiting case may then provide some theoretical guidance for incorporating the depth dependence into general parametrical expressions of the nonlinear transfer for an arbitrary spectrum.

The Boltzmann integral for the energy transfer due to resonant third-order wave–wave interactions has the general form (Hasselmann 1961, 1963*b*)

$$\begin{aligned}\frac{\partial F}{\partial t}(\mathbf{k}_4, h) = \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\, \pi\omega_4 \left(\frac{3g^2 D}{2\omega_1\omega_2\omega_3\omega_4}\right)^2 \\ \times (\omega_4 F_1 F_2 F_3 + \omega_3 F_1 F_2 F_4 - \omega_2 F_1 F_3 F_4 - \omega_1 F_2 F_3 F_4) \\ \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\, \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \end{aligned} \quad (1)$$

in which the rate of change of the spectrum at the wavenumber $\mathbf{k}_4$ is determined by the integral over all third-order interactions with wave components $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ satisfying the resonance conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \quad \omega_1 + \omega_2 = \omega_3 + \omega_4,$$

with wave frequencies ω_i given by the dispersion relation

$$\omega_i^2 = gk_i \tanh(k_i h).$$

Here g is the acceleration of gravity, h the ocean depth, $F_i = F(\mathbf{k}_i)$ the variance spectrum of the surface displacement and D an interaction coefficient which is given in Hasselmann (1961) (a recalculation yielded two additional terms which vanish in the infinite-depth limit but cannot be neglected for finite depth, cf. appendix B).

For a general surface-wave spectrum the integral (1) can be evaluated only numerically (Hasselmann 1963*b*; Sell & Hasselmann 1973; Webb 1978). However, analytical results can be derived in the limiting case of a very narrow spectrum (Longuet-Higgins 1976; Fox 1976). In this case all interactions are concentrated in a limited region around the peak wavenumber $\mathbf{k}_p$, $\mathbf{k}_1 \approx \mathbf{k}_2 \approx \mathbf{k}_3 \approx \mathbf{k}_4 \approx \mathbf{k}_p$. The interaction coefficient, if continuous, can then be regarded as constant, $D = D_0$, and taken outside the integral. The frequency δ-function can also be expanded around the peak frequency, and the interaction diagrams in the wavenumber plane (cf. Hasselmann 1963*b*, figure 6) reduce to a set of hyperbolas centred at $\frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) = \frac{1}{2}(\mathbf{k}_3 + \mathbf{k}_4)$.

A complication of the finite-depth case is that D is in fact not continuous at the expansion origin $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4 = \mathbf{k}_p$. However, as discussed below, for $k_p h \gtrsim 0{\cdot}7$ the discontinuous contribution is small and can be neglected. In the limit of an infinitely deep ocean, D is continuous everywhere and its value at the expansion origin is given by $D_0 = -\frac{4}{3}k_p^4$ (appendix B; Hasselmann 1963*b*). The coefficient D_0 is related to Longuet-Higgins' (1976) interaction coefficient G_0 through $G_0 = \pi(\frac{3}{2}D_0)^2 = 4\pi$, where the units are chosen such that $g = k_p = 1$.

For the infinite-depth case, Longuet-Higgins found a positive energy transfer in the directions $\alpha = \pm \arctan(\pm 1/\surd 2)$ relative to the expansion origin $\mathbf{k}_p$. Using the same approximation, Fox (1976) calculated the energy transfer in the vicinity of $\mathbf{k}_p$ for a variety of differently shaped, narrow spectra. For a symmetrical peak the maximal positive transfer values occur a short distance on either side of the peak along Longuet-Higgins' directions α. Between these directions, along the axes, the transfer is negative, with a maximum negative value at $\mathbf{k}_p$. An asymmetry of the spectrum relative to the peak frequency along the frequency axis reduces the negative trough on the steeper side of the peak.

This is in general agreement with numerical calculations for the complete Boltzmann integral (Hasselmann 1963*b*; Sell & Hasselmann 1973; Webb 1978), in which a strong asymmetry of the peak is found to shift the energy transfer pattern towards the flatter side of the peak (higher frequencies). This leads to a positive rather then negative energy transfer on the steep side of the peak, and explains the observed shift of the peak of a growing wind–sea spectrum towards lower frequencies. For a rather broad spectrum, such as a fully developed Pierson–Moskowitz spectrum, the full calculations show that the positive growth region encompasses the peak itself, and tends to sharpen the peak as opposed to the peak broadening found for a very sharp spectrum. The sensitive dependence of the nonlinear transfer on the spectral shape appears to be responsible for the self-generation and stabilization of the spectral shape of a growing wave spectrum in the form observed.

As many of the principal features of the complete calculations are reproduced, at least qualitatively, by the narrow-peak approximation it is significant that the principal result of the present study is that the narrow-peak results for infinite-depth waves can be carried over directly, for $k_p h \gtrsim 0{\cdot}7$, to the finite-depth case by straightforward scale transformations. The similarity relations between the finite-depth and infinite-depth cases may be expected to apply also, at least approximately, for more general spectra, thereby providing a basis for constructing parametrizations of the nonlinear transfer for the general finite-depth case once the simpler infinite-depth case has been parametrized. Such an approximation would of course need to be tested and possibly modified by independent calculations of the complete nonlinear transfer for realistic finite-depth spectra.

2. The narrow-peak approximation in the case of finite-depth waves

The method of Longuet-Higgins and Fox is applicable also to the finite-depth case, provided the terms of the integrand in (1) which do not involve the wave spectrum are continuous at the expansion origin $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4 = \mathbf{k}_p$. In this section we ignore for the present the fact that the coefficient D is in reality discontinuous for finite h at the expansion origin, and assume that the non-spectral terms in (1) can all be expanded for finite h. In this case we show that there is no need to repeat the calculations of Longuet-Higgins and Fox, since the results of their analysis can be carried over directly to the finite-depth case by suitable scale transformations.

In the narrow-peak approximation equation (1) may be written

$$\frac{\partial F}{\partial t}(\mathbf{k}_4, h) = \pi\left(\frac{3g^2D_0}{2\omega_p^3}\right)^2 \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\, [F_1 F_2(F_3+F_4) - (F_1+F_2)F_3F_4)] \times \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4). \quad (2)$$

Introducing difference wavenumbers $\mathbf{k}_i' = \mathbf{k}_i - \mathbf{k}_p \equiv (\lambda', \mu')$ and frequencies

$$\omega_i' = \omega_i - \omega_p,$$

where $w_p = (gk_p \tanh(k_p h))^{\frac{1}{2}}$, the dispersion relation can be expanded in the neighbourhood of the peak in the form

$$\omega' = \frac{c_1}{2}\frac{g}{\omega_p}\lambda' + \frac{c_2}{8}\frac{g^2}{\omega_p^3}(-c_3\lambda'^2 + 2\mu'^2) + \ldots, \tag{3}$$

where c_1, c_2 and c_3 are functions of the ocean depth which are given in appendix A. In the infinite-depth limit, $k_p h \to \infty$, $c_1, c_2, c_3 \to 1$.

For the evaluation of the frequency δ-function in (2) only the quadratic terms in (3) are relevant, since the sum of the linear terms $\lambda'_1 + \lambda'_2 - \lambda'_3 - \lambda'_4$ vanishes through the interaction condition for the wavenumbers. The quadratic terms can be made identical, except for a common factor, with the corresponding expression in the infinite-depth case by the scale transformations

$$\lambda'' = c_4 \lambda', \quad \mu'' = \frac{c_4}{\sqrt{c_3}}\mu', \tag{4}$$

where the scaling factor c_4 at this point is arbitrary and will be determined later. Equation (3) then becomes

$$\begin{aligned}\omega'' &= \frac{c_1}{2c_4}\frac{g}{\omega_p}\lambda'' + \frac{c_2 c_3}{8c_4^2}\frac{g^2}{w_p^3}(-\lambda''^2 + 2\mu''^2) + \ldots \\ &= \frac{c_1}{c_4}\alpha(\lambda'') + \frac{c_2 c_3}{c_4^2}\beta(\lambda'', \mu'') + \ldots,\end{aligned}$$

where the functions

$$\alpha(\lambda'') = \frac{1}{2}\frac{g}{\omega_p}\lambda''$$

and

$$\beta(\lambda'', \mu'') = \frac{1}{8}\frac{g^2}{\omega_p^3}(-\lambda''^2 + 2\mu''^2)$$

represent the infinite-depth relations, the depth dependence of ω'' being collected in the coefficients c_1, c_2, c_3 and c_4.

Substituting these transformations into equation (2), we obtain

$$\begin{aligned}\frac{\partial F''}{\partial t}(\mathbf{k}''_4, h) &= \frac{c_4^2}{c_2 c_3} f_D^2 \pi \left(\frac{3g^2 D(\infty)}{2\omega_p^3}\right)^2 \\ &\quad \times \int d\mathbf{k}''_1\, d\mathbf{k}''_2\, d\mathbf{k}''_3 [F''_1 F''_2 (F''_3 + F''_4) - (F''_1 + F''_2) F''_3 F''_4) \\ &\quad \times \delta(\mathbf{k}''_1 + \mathbf{k}''_2 - \mathbf{k}''_3 - \mathbf{k}''_4)\, \delta(\beta_1 + \beta_2 - \beta_3 - \beta_4),\end{aligned} \tag{5}$$

where f_D is defined by $D(k_p h) = f_D(k_p h) D(\infty)$ and $D(\infty) = -\frac{4}{3}\omega_p^8/g^4$ is the interaction coefficient for infinite depth (at the same frequency ω_p). We have made use of the relation $\delta(c\Omega) = (1/|c|)\,\delta(\Omega)$ in factoring out the depth-dependent coefficients in the δ-function. The spectra F''_i are defined as densities with respect to the wavenumber space (λ'', μ''), i.e. $F''\, d\lambda''\, d\mu'' = F\, d\mathbf{k}$.

Equation (5) is identical to the integral considered by Longuet-Higgins and Fox for an infinite-depth spectrum $F''(\mathbf{k}'')$ except for the additional factor

$$R = \frac{c_4^2}{c_2 c_3} f_D^2. \tag{6}$$

It may be remarked that to relate the finite-depth and infinite-depth cases we could have normalized the coupling coefficient D and the second-order frequency expression β with respect to either the peak frequency or the peak wavenumber. We have chosen the peak frequency rather than wavenumber, since it remains invariant as the waves propagate into shallow water. For the same reason we shall derive the general relation between the finite-depth and infinite-depth cases in terms of the frequency spectra rather than wavenumber spectra.

We may now prescribe the free scaling factor c_4 such that within the narrow-peak approximation not only the peak frequency, but also the one-dimensional frequency distribution $f(\omega)$ of the finite-depth spectrum and the transformed equivalent infinite-depth spectrum are identical. The two-dimensional frequency-directional spectra for finite and infinite depth are related by

$$f(\omega)\,S(\omega,\,\Theta)\,d\omega\,d\Theta = F(\mathbf{k})\,d\mathbf{k} = F''(\mathbf{k}'')\,d\mathbf{k}'' = f_\infty(\omega_\infty)\,S_\infty(\omega_\infty,\,\Theta_\infty)\,d\omega_\infty\,d\Theta_\infty,$$

where

$$\omega = \omega_p + \frac{c_1}{2}\frac{g}{\omega_p}\lambda' + \ldots = \omega_p + \frac{c_1}{2c_4}\frac{g}{\omega_p}\lambda'' + \ldots, \quad \omega_\infty = \omega_p + \frac{g}{2\omega_p}\lambda'' + \ldots,$$

$$\Theta = \frac{\mu'}{k_p} + \ldots = \frac{c_3^{\frac{1}{2}}}{c_4}\frac{\mu''}{k_p} + \ldots, \quad \Theta_\infty = \frac{\mu''}{k_\infty} + \ldots,$$

and S, S_∞ denote spreading functions which are normalized such that their integrals over the directions Θ, Θ_∞ are unity. The requirement $f(\omega) = f_\infty(\omega_\infty)$ is clearly satisfied if we set $c_4 = c_1$. The spreading functions are then related by

$$S_\infty(\Theta_\infty) = \gamma S(\Theta) \quad \text{with} \quad \Theta = \gamma\Theta_\infty \quad \text{and} \quad \gamma = \frac{c_3^{\frac{1}{2}}}{c_1}\frac{k_\infty}{k_p}. \tag{7}$$

Noting that both the left- and right-hand side of equation (5) are expressed in terms of the transformed variables, the rules for deriving the nonlinear transfer for a finite-depth spectrum may therefore be summarized as follows:

(1) replace the spreading function S by the broader distribution S_∞ according to the transformation $\Theta_\infty = \Theta/\gamma$ (maintaining normalization of the spreading function in accordance with (7));

(2) compute the nonlinear transfer for the new frequency-directional spectrum as though the depth were infinite;

(3) multiply the resultant transfer rate by the factor R;

(4) transform the result back into the original angle variable $\Theta = \gamma\Theta_\infty$ (again conserving angular integrals in accordance with (7)).

3. The discontinuity of the interaction coefficient D at $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4 = \mathbf{k}_p$

The above results need to be modified by consideration of the discontinuity of the coefficient D at the expansion origin $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \mathbf{k}_4 = \mathbf{k}_p$. Formally, the discontinuity is associated with second-order difference interactions between the components $\mathbf{k}_2$ and $\mathbf{k}_3$ or $\mathbf{k}_1$ and $\mathbf{k}_3$ which create a resonant wave at wavenumber zero. It arises in the first term in (B 2), after permutation of the wavenumbers as required for the first and second terms in (B 1). As $\mathbf{k}_j \to \mathbf{k}_p$, the terms take the indefinite form

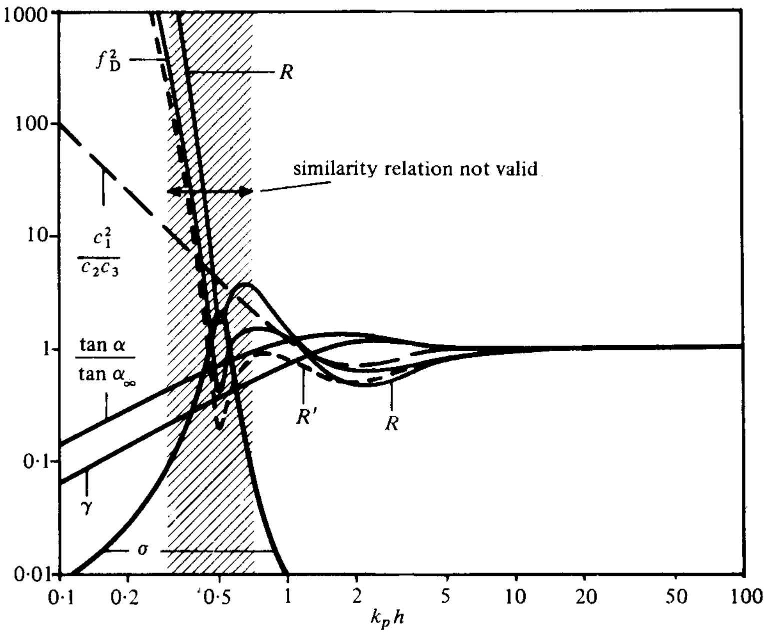

FIGURE 1. Depth dependence of the ratios R and γ of the transfer rate and spreading angle, respectively, in the finite and infinite-depth cases. R consists of the two factors f_D^2 and $c_1^2/c_2\,c_3$ representing the contributions from the coupling strength and the resonance phase volume respectively. The effective ratio of the transfer rates for comparable spreading angles is of order $R' = R\gamma^2$. The ratio of the angles α of the transfer pattern relative to an origin in the wavenumber plane at $\frac{1}{2}(\mathbf{k}_1+\mathbf{k}_2) = \frac{1}{2}(\mathbf{k}_3+\mathbf{k}_4)$ is given by the curve $\tan\alpha/\tan\alpha_\infty$. σ is the normalized contribution of the discontinuous term of the interaction coefficient. The similarity relations are valid for $k_p h \gtrsim 0{\cdot}7$ and $k_p h \lesssim 0{\cdot}3$ ($\sigma \lesssim 0{\cdot}1$).

0/0. For given fixed directions α_1, α_3 of the difference wavenumbers $\Delta\mathbf{k}_1 = \mathbf{k}_1 - \mathbf{k}_m$ $(= -\Delta\mathbf{k}_2)$, and $\Delta\mathbf{k}_3 = \mathbf{k}_3 - \mathbf{k}_m (= -\Delta\mathbf{k}_4)$, where $\mathbf{k}_m = \frac{1}{2}(\mathbf{k}_1+\mathbf{k}_2) = \frac{1}{2}(\mathbf{k}_3+\mathbf{k}_4)$, a finite limit exists as $\Delta\mathbf{k}_1, \Delta\mathbf{k}_3 \to 0$, but its value depends on α_1 and α_3.

The term in the denominator of (B2) is readily seen to be proportional to the square of the group velocity at zero wavenumber. Since this is infinite for infinite-depth waves, the discontinuity vanishes for an infinite-depth ocean. Thus the corrections due to the discontinuity will become negligible for sufficiently large h.

To determine the region in which the discontinuity becomes important, the interaction coefficient may be written in the form $D = D_c + D_p$, where D_c is continuous and D_p contains the discontinuous contribution. The separation is made unique by defining the angular average of the discontinuous term to be zero,

$$\langle D_p \rangle = \frac{1}{(2\pi)^2}\iint d\alpha_1\, d\alpha_3\, D_p = 0.$$

Noting that D occurs quadratically in (1), we may then consider the ratio

$$\sigma = \langle (D^2 - \langle D^2\rangle)^2\rangle / \langle D^2\rangle^2 \quad \text{at} \quad \mathbf{k}_1 \approx \mathbf{k}_2 \approx \mathbf{k}_3 \approx \mathbf{k}_4 \approx \mathbf{k}_p$$

as a characteristic index of the error incurred by neglecting the discontinuous term D_p in the narrow-peak analysis. Computed values of σ are plotted in figure 1 as a function of $k_p h$. The errors incurred by ignoring the discontinuity are seen to be negligible ($\sigma < 0{\cdot}1$) for $k_p h \gtrsim 0{\cdot}7$ and $k_p h \leqslant 0{\cdot}3$, but can become appreciable near $k_p h \approx 0{\cdot}5$ ($\sigma \approx 20$). In practice, the side condition $h \geqslant 0{\cdot}7/k_p \approx \lambda/10$ for the validity of the similarity relations should not be too restrictive, since we are not interested here in very shallow water in which strongly nonlinear processes occur, but rather in intermediate regions, such as continental shelves, where the basic radiative transfer description of a slowly varying wave spectrum is still applicable.

4. Discussion

Figure 1 shows the depth dependence of the transfer-rate scale factor R and the angular scale factor γ. Also shown is the scale factor $c_3^{\frac{1}{2}}$ relating the angles α and α_∞ in the wavenumber plane relative to an origin at $\mathbf{k}_m = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2) = \frac{1}{2}(\mathbf{k}_3 + \mathbf{k}_4)$ (cf. equation (4)),

$$\tan\alpha = \frac{\mu'}{\lambda'} = c_3^{\frac{1}{2}}\frac{\mu''}{\lambda''} = c_3^{\frac{1}{2}}\tan\alpha_\infty.$$

The factor R contains two contributions, the term $f_D^2 = (D(h)/D(\infty))^2$ representing the change in nonlinear coupling, and the term $c_1^2/c_2 c_3$ arising from the depth-dependence of the dispersion relation in the frequency resonance condition. With decreasing depth, the near-resonance region in wavenumber phase space (defined, for example, by the region $|\omega_1 + \omega_2 - \omega_3 - \omega_4| < \epsilon$, where ϵ is the some fixed but small number) first decreases and then increases. In the limit $h \to 0$, second-order resonance becomes possible, and the third-order resonance region approaches infinity. Thus $c_1^2/c_2 c_3 \to \infty$ as $k_p h \to 0$. The strength of the coupling f_D^2 also increases to infinity as $k_p h \to 0$.

In assessing the strength of the energy transfer for finite-depth waves it should be noted that the reference spectrum in the infinite-depth case has a spreading function broadened by the factor $1/\gamma$ (or narrowed, if $\gamma > 1$). Directional broadening reduces the energy transfer by a factor of approximately γ^2 (Longuet-Higgins 1976), so that the energy transfer for finite-depth waves is stronger than in the infinite-depth case by a net factor of approximately $R' = R\gamma^2$.

In the shaded region of the figure, $0{\cdot}3 < k_p h < 0{\cdot}7$, the similarity relations no longer hold, since the discontinuous part of the coupling coefficient is no longer negligible. However, the function R should still give an indication of the order of magnitude of the transfer rate, although the detailed distribution of the transfer will no longer correspond to the infinite-depth case.

For sufficiently small depths the theory will ultimately break down because the nonlinear transfer becomes too strong for application of the weak-interaction approximation. A necessary requirement of the theory is that the characteristic nonlinear transfer time is large compared with the time needed to resolve the spectrum – in the present case, about ten times the inverse peak width. This is essentially the same two-timing condition which is needed generally for the description of the wave field as a quasi-homogeneous, quasi-stationary process governed by a spectral transport equation.

The similarity relation between the finite-depth and infinite-depth energy transfer implies that the basic nonlinear mechanisms that control the evolution of an infinite-depth wind-wave spectrum should act similarly in the finite-depth case. An increase in the net transfer rate by a factor R' implies that the equilibrium between the wind input and the nonlinear transfer in the central region of the spectrum will be established more rapidly, and the equilibrium level of the spectrum will be lower by a factor of order $1/R'^{\frac{1}{2}}$ (assuming a wind input proportional to the spectrum – cf. Hasselmann *et al.* 1976). However, the rate of shift of the peak frequency at this lower energy level remains approximately the same as in the infinite-depth case: the stronger transfer rate proportional to R' is offset by the lower spectral level, which reduces the rate of shift by the inverse factor $(1/R'^{\frac{1}{2}})^2 = 1/R'$. Thus changes in the nonlinear transfer rate should not affect the rate of shift of the peak frequency, or the shape of the spectrum. This is in accordance with the observations of Bouws (1978), who found good agreement of the shape of the energy-containing region of shallow-water wind–sea spectra with the JONSWAP form for deep-water waves. However, both Bouws' figures and the measurements of Kitaigorodskii, Krasitskii & Zaslavskii (1975) indicate that the spectrum falls off less steeply than ω^{-5} at higher frequencies ($\omega \gtrsim 2\omega_p$). In this range the narrow-peak approximation is clearly no longer applicable. It would be interesting to investigate the energy balance of the complete spectrum by computing the full non-linear transfer expression for some typical observed finite-depth spectra. However, the present analysis suggests that for the energy-containing range of the spectrum ($\omega \lesssim 2\omega_p$) existing concepts on the growth and quasi-equilibrium shape of wind-sea spectra, as developed for infinite-depth waves (and incorporated in simplified prediction models, cf. Hasselmann *et al.* 1976), should be applicable, with only minor modifications, also to the finite-depth case.

Appendix A

The dispersion relation for finite-depth waves is given by

$$\omega = (gk \tanh kh)^{\frac{1}{2}},$$

where

$$k = [(k_p+\lambda')^2+\mu'^2]^{\frac{1}{2}}.$$

Expansion in a Taylor series around $\omega_p = (gk_p \tanh k_p h)^{\frac{1}{2}}$ yields, correct to second order in (λ', μ'),

$$\begin{aligned}\omega = \omega_p &+ \frac{g}{2\omega_p}\tanh k_p h\left(1+\frac{k_p h}{\sinh k_p h \cosh k_p h}\right)\lambda' \\ &- \frac{g^2}{8\omega_p^3}\tanh^2 k_p h\left[\left(1-\frac{k_p h}{\sinh k_p h\cosh k_p h}\right)^2+\left(\frac{2k_p h}{\cosh k_p h}\right)^2\right]\lambda'^2 \\ &+ \frac{g^2}{4\omega_p^3}\tanh^2 k_p h\left(1+\frac{k_p h}{\sinh k_p h \cosh k_p h}\right)\mu'^2.\end{aligned}$$

Comparison with equation (3) shows that

$$c_1 = \tanh k_p h(1+k_p h/\sinh k_p h \cosh k_p h), \quad c_2 = c_1 \tanh k_p h,$$

$$c_3 = \frac{(1-k_p h/\sinh k_p h\cosh k_p h)^2+(2k_p h/\cosh k_p h)^2}{1+k_p h/\sinh k_p h \cosh k_p h}.$$

Limiting values are

$$c_1, c_2, c_3 = \begin{cases} 0 & \text{for } h = 0, \\ 1 & \text{for } h = \infty. \end{cases}$$

Appendix B

The interaction coefficient $D(h)$ of equation (1) is given by Hasselmann (1961)

$$D = \tfrac{1}{3}(D^{++-}_{\mathbf{k}_1\mathbf{k}_2-\mathbf{k}_3} + D^{+-+}_{\mathbf{k}_2-\mathbf{k}_3\mathbf{k}_1} + D^{-++}_{-\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2}), \tag{B 1}$$

where

$$\begin{aligned} D^{s_1s_2s_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = {} & \frac{iD^{s_2s_3}_{\mathbf{k}_2\mathbf{k}_3}}{\omega^2_{\mathbf{k}_2+\mathbf{k}_3} - (\omega_2+\omega_3)^2} \cdot \left\{ 2(\omega_1+\omega_2+\omega_3)\left(\frac{\omega_1^2\,\omega^2_{\mathbf{k}_2+\mathbf{k}_3}}{g^2} - \mathbf{k}_1 . (\mathbf{k}_2+\mathbf{k}_3)\right)\right. \\ & \left. - \frac{\omega_1(\mathbf{k}_2+\mathbf{k}_3)^2}{\cosh^2|\mathbf{k}_2+\mathbf{k}_3|\,h} - \left[\frac{(\omega_2+\omega_3)\,k_1^2}{\cosh^2 k_1 h}\right]\right\} - iD^{s_2s_3}_{\mathbf{k}_2\mathbf{k}_3}\frac{\omega_1}{g^2}(\omega_1^2+\omega^2_{\mathbf{k}_2+\mathbf{k}_3}) \\ & + E^{s_2s_3}_{\mathbf{k}_2\mathbf{k}_3}\left\{\frac{\omega_1^3(\omega_2+\omega_3)}{g} - g\mathbf{k}_1 . (\mathbf{k}_2+\mathbf{k}_3) - \left[\frac{gk_1^2}{\cosh^2 k_1 h}\right]\right\} \\ & + \frac{\omega_1}{2g^2}(\mathbf{k}_2 . \mathbf{k}_3)\{(\omega_1+\omega_2+\omega_3)(\omega_2^2+\omega_3^2) + \omega_2\,\omega_3(\omega_2+\omega_3)\} \\ & - \frac{\omega_1\,\omega_2^2\,k_3^2}{2g^2}(\omega_1+\omega_2+2\omega_3) - \frac{\omega_1\,\omega_3^2\,k_2^2}{2g^2}(\omega_1+2\omega_2+\omega_3). \end{aligned} \tag{B 2}$$

$$\begin{aligned} D^{s_2s_3}_{\mathbf{k}_2\mathbf{k}_3} = {} & i(\omega_2+\omega_3)(k_2 k_3 \tanh(k_2 h)\tanh(k_3 h) - \mathbf{k}_2 . \mathbf{k}_3) \\ & - \tfrac{1}{2}i(\omega_2 k_3^2/\cosh^2 k_3 h + \omega_3 k_2^2/\cosh^2 k_2 h, \end{aligned}$$

$$E^{s_2s_3}_{\mathbf{k}_2\mathbf{k}_3} = \frac{1}{2g}\left\{\mathbf{k}_2 . \mathbf{k}_3 - \frac{\omega_2\,\omega_3}{g^2}(\omega_2^2+\omega_3^2+\omega_2\,\omega_3)\right\}$$

and $\omega_i = \omega_{\mathbf{k}i}\, s_i$, $s_i = \pm 1$.

Hasselmann's (1961) result has been corrected by the two additional terms indicated by square brackets in the expression for $D^{s_1s_2s_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$. These were discovered after testing the invariance of D under permutations of the wavenumbers, as required for the conservation of energy and momentum (Hasselmann 1963*a*). The terms vanish for $kh \to \infty$ and therefore do not affect previously published results for infinite-depth waves.

In the limit $\mathbf{k}_1 \approx \mathbf{k}_2 \approx \mathbf{k}_3 \approx \mathbf{k}_4 \approx \mathbf{k}_p$ we obtain

$$D \approx D_c + D_p,$$

where $\quad D_c = D_a + \langle D_b\rangle, \quad D_p = D_b - \langle D_b\rangle, \quad \langle D_b\rangle = \dfrac{1}{(2\pi)^2}\displaystyle\iint d\alpha_1\, d\alpha_3\, D_b$

and

$$\begin{aligned} D_a = \frac{\omega_p^8}{g^4}\frac{1}{3}\left\{\frac{\cosh^2 k_p h}{\sinh^4 k_p h}\left[\tfrac{1}{2}(3+\tanh^2 k_p h) + 6\tanh k_p h \times \tanh 2k_p h\right.\right. \\ \left.\left. - 4\sin^2 k_p h + \frac{3[2(\tanh k_p h \times 2\tanh 2k_p h + 2) + 4/\cosh^2 2k_p h - 2/\cosh^2 k_p h]}{2\tanh 2k_p h/\tanh k_p h - 4}\right]\right\}, \end{aligned}$$

$$D_b = \frac{\alpha}{3}\left(\frac{1}{\beta\lambda_+ - 1} + \frac{1}{\beta\lambda_- - 1}\right),$$

with

$$\alpha = \frac{k_p^4}{2\cosh^4 k_p h(1 + k_p h/\sinh k_p h \cos k_p h)^2}\left[4\cosh^2 k_p h + 1 + \frac{k_p h}{\sinh k_p h \cosh k_p h}\right]^2,$$

$$\beta = \frac{4k_p h}{\tanh k_p h(1 + k_p h/\sinh k_p h \cosh k_p h)^2},$$

$$\lambda_\pm = \frac{(\delta\mathbf{k}_1 \pm \delta\mathbf{k}_3)^2}{(\delta k_{1x} \pm \delta k_{3x})^2}, \quad \delta\mathbf{k}_i = \mathbf{k}_i - \tfrac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2).$$

Limiting values are

$$D_a \to -\infty, \quad D_b \to \infty \quad \text{for} \quad h \to 0;$$

$$D_a \to -\frac{4}{3}\frac{\omega_p^8}{g^4}, \quad D_b \to 0 \quad \text{for} \quad h \to \infty.$$

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