A Formal Theory of Internal Wave Scattering with Applications to Ocean Fronts

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ABSTRACT

A concise theory for scattering of internal waves at localized inhomogeneities (i.e., topographic features, baroclinicity in the density field, variations of the mean sea level, jetlike currents) in the oceanic waveguide is presented within the formal framework of quantum mechanical scattering theory. The equations of motion of the wave system are reduced to a form resembling the Schrödinger equation with an interaction operator describing the effect of the ambient inhomogeneities. By standard Green's function techniques integral equations for the scattered field and its Fourier transform (which relates to the amplitudes of the scattered waves) are derived, both for a scattering region of finite extent (representing a two-dimensional scattering problem) and a "wall-like" scattering region of infinite extent (representing a one-dimensional scattering problem). As an example, the theory is applied to the scattering at a straight geostrophic front. The far-field is described in the Born approximation valid for $(U/c)(kL_s) \ll 1$, where U is the speed of the geostrophic current of width L_s, and c and k are the phase speed and wavenumber of the incident wave. It is found that the scattering process has a significant directional signature while modal redistribution appears to be weak.

1. Introduction

Wave theory generally assumes that the wave supporting background is homogeneous in the spatial directions (the propagation space) where the waves have a propagating character. Inhomogeneities may occur in the directions orthogonal to the propagation space which generally may be handled by an expansion in terms of eigenfunctions appropriate to the problem. Weak variations of the medium in the propagation space can then be treated by a WKB theory of wave propagation. The theory of oceanic waves is formulated within this framework (see, e.g., Le Blond and Mysak, 1978). Vertical variations of the stratification (in particular, at top and bottom) are taken into account but horizontally the ocean is assumed homogeneous (in a WKB sense).

In recent years this traditional model has increasingly fallen into disfavor with the accumulation of overwhelming experimental evidence which demonstrates that the real ocean is by no means a smoothly varying stationary medium. Ocean fields show significant variability on all scales ranging from the dimensions of the ocean basins down to molecular dissipation scales and only in rare cases are the spectra of oceanic waves separated by spectral gaps from motions of different dynamics. Thus, when dealing with a theory of waves in the real ocean one has to include in some way interactions of the wave field with ambient background variations with scales not covered by a WKB approach. Obviously, this is a quite complex problem. However, since the notion of waves implies that the coupling to the environment be weak, approximation methods can be used which, loosely speaking, expand the problem of wave interactions about the basic state of the traditional horizontally homogeneous ocean background. For the case of the oceanic internal wave field this scenario has been presented by Müller and Olbers (1975) and recently been reviewed by Garrett and Munk (1979) and Olbers (1982).

The purpose of this paper is to develop a unified formulation of a scattering theory of internal waves at localized oceanic inhomogeneities with scales comparable to the wavelength. The approach to this problem presented here is general enough to treat the scattering of waves at topographic features such as seamounts, ridges, canyons or the continental shelf, as well as scattering at baroclinic disturbances of the mean density field, variations of the mean sea level or jetlike currents. Scattering at topographic features has been extensively investigated by various authors (e.g., Cox and Sandstrom, 1962; Baines, 1971; Rattray *et al.*, 1969) using different approaches. The author is not aware of any detailed study of the scattering at baroclinic or barotropic frontal zones. In fact, this lack was the first motivation for the present research.

The scattering of waves has been investigated in many branches of theoretical physics (Morse and Feshbach, 1953) and formal mathematical ap-

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proaches have particularly been developed in quantum mechanics (e.g., Schweber, 1961; Landau and Lifschitz, 1967). Our scattering problem bears some resemblance to the quantum mechanical scattering of particles at a potential described by the Schrödinger equation. Having transformed the equations of motion of internal waves in a inhomogeneous ocean in the form of the Schrödinger equation and establishing the structure of free wave solutions (Section 2) we are able to translate the quantum mechanical scattering theory to the internal wave problem (Section 3). As a result the amplitudes of the scattered waves arising from the interaction of an incident wave with a disturbance of the ambient state are expressed in a condensed form by the matrix elements of the interaction operator corresponding to that disturbance. We should emphasize here that the analogy to potential scattering cannot be carried over to the energetics of the process since, in general, the wave energy is not conserved by interactions with frontal zones (Appendix C). The formal scattering theory is applied to straight, geostrophic fronts (Section 4) by working out the appropriate interaction operator and describing the scattered field in the Born approximation.

2. Equations of motion

The scattering of waves at some disturbance of the wave-supporting ambient medium is a process which has been studied in many branches of theoretical physics. The mathematical treatment has been developed in particular in quantum mechanics where the waves are described by the Schrödinger equation. We shall devise our theory of internal wave scattering as far as possible along the quantum mechanical scattering theory. It is therefore desirable to cast the equations of motion describing oceanic internal wave motion into a form similar to the Schrödinger equation. This is the aim of this section.

a. Derivation of the Schrödinger equation

The ocean is described as an inviscid, incompressible, stratified, rotating fluid of infinite horizontal extent bounded by a rigid bottom and a free surface. The scattering theory which we present in this paper uses the Boussinesq as well as the quasi-hydrostatic approximation of the equations of motion. The mean state supporting the linear wave motion is defined differently inside and outside the scattering region. In Cartesian coordinates $\mathbf{x} = (x_1, x_2)$, $x_3 = z$, where x_1 points toward east, x_2 toward north, and x_3 upward, the equations of motion take the form

$$\partial_t u_\alpha - f\epsilon_{\alpha\beta}u_\beta + \partial_\alpha \pi = S_\alpha, \quad \alpha = 1, 2, \tag{2.1a}$$

$$-b + \partial_3 \pi = S_3, \tag{2.1b}$$

$$\partial_t b + N^2 u_3 = S_4, \tag{2.1c}$$

$$\partial_j u_j = 0, \tag{2.1d}$$

with boundary conditions

$$\partial_t \zeta - u_3 = S_5 \quad \text{at} \quad x_3 = 0, \tag{2.1e}$$

$$\pi - g\zeta = S_6 \quad \text{at} \quad x_3 = 0, \tag{2.1f}$$

$$u_3 = S_7 \quad \text{at} \quad x_3 = -H, \tag{2.1g}$$

which have been expanded about the mean sea level $x_3 = 0$ and the mean ocean depth $x_3 = -H$, respectively. The Coriolis parameter f is considered as constant. The terms S_n ($n = 1, \ldots, 7$) are defined as follows.

Far away from the scattering region the waveguide is characterized by the state of no motion (a barotropic mean current may trivially be incorporated but a shear current is prohibited), by the horizontally constant stability frequency $N(z)$ and the constant ocean depth H. The corresponding equations of motion are given by (2.1) with vanishing S_n. In the scattering region the waveguide is perturbed, for instance, by the presence of a mean shear flow $U_j(\mathbf{x}, z)$, a baroclinic buoyancy field $B(\mathbf{x}, z)$, a mean pressure field $P(\mathbf{x}, z)$, a variation $Z(\mathbf{x})$ of the mean sea level or a topographic deformation $h(\mathbf{x})$ of the bottom. The interaction of incident waves with these perturbation fields is described by the functionals

$$S_\alpha = -\partial_j(u_j U_\alpha + U_j u_\alpha), \quad \alpha = 1, 2, \tag{2.2a}$$

$$S_3 = 0, \tag{2.2b}$$

$$S_4 = -\partial_j(u_j B + U_j b), \tag{2.2c}$$

$$S_5 = -u_\alpha \partial_\alpha Z - U_\alpha \partial_\alpha \zeta + Z\partial_3 u_3 + \zeta \partial_3 U_3 + \cdots, \tag{2.2d}$$

$$S_6 = -\zeta\partial_3 P - Z\partial_3 \pi - N^2 \zeta Z + \cdots, \tag{2.2e}$$

$$S_7 = -(U_\alpha + u_\alpha)\partial_\alpha h + h\partial_3(U_3 + u_3) + \cdots. \tag{2.2f}$$

The particular form of these functionals is not relevant in this section. The only property used is their linearity in the field variables.

The hydrostatic approximation neglects the rate of change of the vertical velocity in (2.1), thus restricting the theory to frequencies which are small compared to the maximum of the stability frequency. Notice that we have omitted the local rate of change $\partial_t u_3$ as well as the advection term $\partial_j(U_j u_3 + u_j U_3)$ by putting $S_3 = 0$. The consistency of this procedure is revealed when combining (2.1b) and the complete equation of motion for u_3 by elimination of the buoyancy field. Eqs. (2.1) may be simplified further replacing the free surface conditions (2.1e,f) by the rigid-lid approximation $u_3 = 0$ at $x_3 = 0$. This approximation, which is good for baroclinic modes (see Appendix A) will partly be utilized in Section 4. Formally it may be obtained from (2.1) and all the derived equations below by letting g tend to infinity while keeping N^2 fixed.

In the present form the equations of motion involve the six field components u_i, b, π and ζ compared to only four prognostic equations. The state of the fluid should thus be describable by a state vector of four or even less dimensions. This suggests a reduction of the system (2.1). The following reduction which is basically due to Hasselmann (1970), and Frankignoul and Müller (1979) uses the three-dimensional state vector ψ_a defined by

$$\left.\begin{aligned} \psi_1 &= \partial_\alpha u_\alpha \\ \psi_2 &= \epsilon_{\alpha\beta}\partial_\alpha u_\beta \\ \psi_3 &= f^{-1}\partial_\alpha\partial_\alpha\pi \end{aligned}\right\} . \tag{2.3}$$

This vector completely specifies the state of the flow since the six field components are uniquely determined by the relations

$$u_1 = L^{-1}(\partial_1\psi_1 - \partial_2\psi_2), \tag{2.4a}$$

$$u_2 = L^{-1}(\partial_1\psi_2 + \partial_2\psi_1), \tag{2.4b}$$

$$u_3 = -\int_{-H}^{x_3} dx_3'\psi_1 + S_7, \tag{2.4c}$$

$$\pi = fL^{-1}\psi_3, \tag{2.4d}$$

$$b = fL^{-1}\partial_3\psi_3, \tag{2.4e}$$

$$\zeta = (f/g)(L^{-1}\psi_3 - S_6). \tag{2.4f}$$

These relations involve either vertical integration or solution of a Poisson equation with appropriate horizontal boundary conditions since L is the horizontal Laplace operator

$$L = \partial_\alpha\partial_\alpha. \tag{2.5}$$

The state vector ψ_a is governed by equations of motion of the form

$$i\partial_t\psi_a - H_{ab}\psi_b = q_a, \tag{2.6}$$

where the linear operator H_{ab} arises from the left-hand side of (2.1) and the functional q_a from the right-hand side. Explicitly, one finds

$$H_{ab} = if\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ LM & 0 & 0 \end{pmatrix}, \tag{2.7}$$

with

$$M = \frac{1}{f^2}\left[\int_{x_3}^{0} dx_3' N^2(x_3') \int_{-H}^{x_3'} dx_3'' + g\int_{-H}^{0} dx_3''\right] \tag{2.8}$$

acting only on the vertical dependence. In terms of the S_n, the functionals q_a are given by

$$\left.\begin{aligned} q_1 &= i\partial_\alpha S_\alpha, \\ q_2 &= i\epsilon_{\alpha\beta}\partial_\alpha S_\beta, \\ q_3 &= (i/f)L\Big[g(S_5 + S_7) + \partial_t S_6 \\ &\qquad - \int_{x_3}^{0} dx_3'(S_4 - N^2S_7)\Big] \end{aligned}\right\} . \tag{2.9}$$

These may be expressed as linear functionals of the state vector

$$q_a = V_{ab}\psi_b \tag{2.10}$$

by introducing (2.4) into (2.2). The complicated operator V_{ab} arising in this lengthly manipulation will be considered in Section 4a.

The field equations (2.6) appear to be far more complex than the original equations of motion (2.1) since the evolution of the state vector ψ_a at one point in the fluid depends (through the occurrence of the integral operator M) on its state in the entire water column to that point. In quantum mechanical terminology the representation ψ_a of the flow is governed by a non-local field theory. Its advantage is the simple formal structure of the equations (2.6) as compared to (2.1). Especially for scattering problems, standard techniques have been developed for the form (2.6).

The general solution of the linearized system (2.1) is a superposition of internal waves and a geostrophic shear flow. This well-known result will be recovered in the next section where we derive the normal modes of the system. It is possible to decompose the motion into the internal wave branch and the geostrophic flow branch without the use of the horizontal and vertical mode structure of the system. This is performed by a transformation of the state vector ψ_a into a base where the operator H_{ab} is diagonal. In analogy to ordinary matrix algebra the diagonalization of H_{ab} may be performed by computing the right and left (or adjoint) eigenvectors with corresponding eigenvalues which here, however, are integro-differential operators.

The solution of the eigenvalue problem

$$H_{ab}\beta_b^p = \Omega^p\beta_a^p \tag{2.11}$$

yields the pth eigenvector β_a^p defined by

$$\left.\begin{aligned} \beta_1^p &= ip\Omega/f \\ \beta_2^p &= 1 \\ \beta_3^p &= 1 - (p\Omega/f)^2 \end{aligned}\right\} , \tag{2.12}$$

with the eigenvalue $\Omega^p = p\Omega$ ($p = +, -, 0$), where

$$\Omega = f(1 - LM)^{1/2}. \quad (2.13)$$

The adjoint problem

$$\tilde{\beta}_a^p H_{ab} = \Omega^p \tilde{\beta}_b^p \quad (2.14)$$

yields the adjoint eigenvectors $\tilde{\beta}_a^p$ ($p = +, -, 0$) defined by

$$\left.\begin{array}{l}\tilde{\beta}_1^p \\ \tilde{\beta}_2^p \\ \tilde{\beta}_3^p\end{array}\right\} = \tfrac{1}{2}(\Omega/f)^{-2}\left\{\begin{array}{l}-ip\Omega/f \\ 1 \\ -1\end{array}\right. \quad \text{for} \quad p = +, -, \quad \begin{array}{l}\tilde{\beta}_1^0 \\ \tilde{\beta}_2^0 = (\Omega/f)^{-2} \\ \tilde{\beta}_3^0\end{array}\left\{\begin{array}{l}0 \\ (\Omega/f)^2 - 1. \\ 1\end{array}\right. \quad (2.15)$$

Notice that H_{ab} (though being non-Hermitian) possesses real eigenvalues. From $(H_{ab})^* = -H_{ab}$ one deduces that $(\beta_a^p)^*$ and $(\tilde{\beta}_a^p)^*$ are eigenvectors with eigenvalue $-\Omega^p = \Omega^{-p}$. Indeed, (2.12) and (2.15) satisfy the relations

$$\left.\begin{array}{l}(\beta_a^p)^* = \beta_a^{-p} \\ (\tilde{\beta}_a^p)^* = \tilde{\beta}_a^{-p}\end{array}\right. \quad (p = +, -, 0). \quad (2.16)$$

The orthogonality relation

$$\tilde{\beta}_a^p \beta_a^q = \delta^{pq} \quad (2.17)$$

may be derived from (2.11) and (2.14). The completeness relation

$$\beta_a^p \tilde{\beta}_b^p = \delta_{ab} \quad (2.18)$$

follows from the definition (2.12) and (2.14). These relations state that the matrices β_a^p and $\tilde{\beta}_a^p$ are mutually inverse of each other.

The transformation of the (real) state vector ψ_a given by

$$\psi^p = \tilde{\beta}_a^p \psi_a = (\psi^{-p})^* \quad (2.19)$$

and its inverse

$$\psi_a = \beta_a^p \psi^p \quad (2.20)$$

lead to the desired diagonalization of H_{ab}. From (2.11) and (2.17) we find

$$H^{pq} = \tilde{\beta}_a^p H_{ab} \beta_b^q = \delta^{pq}\Omega^p = p\Omega\delta^{pq}, \quad (2.21)$$

so that ψ^p is governed by

$$(i\partial_t - p\Omega)\psi^p = V^{pq}\psi^q, \quad (2.22)$$

with

$$V^{pq} = \tilde{\beta}_a^p V_{ab} \beta_b^q = -(V^{-p-q})^*. \quad (2.23)$$

The representation ψ^p of the state vector is the canonical decomposition of the system (2.1). If there are no interactions, i.e., $S_n \equiv 0$ and thus $V^{pq} \equiv 0$, they are decoupled and evolve harmonically in time

$$\psi^p(\mathbf{x}, z, t) = e^{-ip\Omega t}\psi^p(\mathbf{x}, z, 0), \quad (2.24)$$

thus defining the normal mode branches of the system. We show in the next section that in the free state (i.e., $V^{pq} \equiv 0$) of the system the fields ψ^+ and ψ^- represent the internal wave part of the flow while ψ^0 represents the geostrophic part. The transformation $\tilde{\beta}_a^p$ ($p = +, -, 0$) which generate this separation of the state vector ψ_a may be referred to as decomposition operators. In view of (2.20), the operators β_a^p ($p = +, -, 0$) may be regarded as polarization operators of the representation ψ_a. In the general case when the V^{pq} does not vanish identically the decomposition into the branches $p = +, -, 0$ is not necessarily a separation of the motion into internal waves and a geostrophic flow. Yet, it is unique and invertable and, moreover, it transforms the problem (2.1) into a more tractable form.

The formal analogy of the equations of motion in the form (2.6) or (2.22) with the Schrödinger equation in quantum mechanics will be exploited in the presentation of the scattering theory in Section 3. In vectorial notation the equations of motion are

$$i\partial_t\psi = \mathcal{H}\psi = (H + V)\psi. \quad (2.25)$$

We will refer to H as the free wave operator and V as the interaction operator. The representation of the state vector which makes the free wave operator H diagonal, i.e.,

$$\left.\begin{array}{l}\psi = (\psi^+, \psi^-, \psi^0) \\ H = (H^{pq}) = (p\Omega\delta^{pq}) \\ V = (V^{pq})\end{array}\right\} \quad (2.26)$$

will be called the normal branch representation.

b. Normal modes

Normal modes of the system are free wave solutions described by (2.24) if the initial state is an eigenfunction of Ω. Thus, they are of the form

$$\psi^p(y, t) = \phi(y)e^{-ip\omega t}, \quad (2.27)$$

where $\phi(y)$ satisfies the eigenvalue problem

$$\Omega\phi(y) = \omega\phi(y). \quad (2.28)$$

For the sake of neatness we have introduced the notation y for the three-dimensional position vector $(\mathbf{x}, z)$ with z restricted to $0 \leqslant z \leqslant -H$.

Separable eigenfunctions of Ω may be constructed from the eigenfunctions $(\tfrac{1}{2}\pi)\exp(i\mathbf{k}\cdot\mathbf{x})$ of the Laplace operator L and the eigenfunctions $\varphi_l(z)$ of the operator M. The integral eigenvalue problem

$$M\varphi_l = \lambda_l\varphi_l \quad (2.29)$$

is equivalent to the usual differential form

$$(f^2N^{-2}\varphi_l')' + \lambda_l^{-1}\varphi_l = 0, \quad (2.30a)$$

with boundary conditions

$$\left.\begin{aligned} g\varphi_l' + N^2\varphi_l = 0 \quad \text{at} \quad z = 0 \\ \varphi_l' = 0 \quad \text{at} \quad z = -H \end{aligned}\right\} . \qquad (2.30b)$$

The prime denotes differentiation with respect to z. Eq. (2.30) defines the vertical normal modes of the quasi-hydrostatic approximation of oceanic waves (see, e.g., Le Blond and Mysak, 1978). It is a Sturm-Liouville problem and thus the eigenfunctions have the well-known property of forming an denumerable set of modes $\varphi_l(z)$, $l = 0, 1, 2, \cdots$ which satisfy the orthogonality relation

$$\int_{-H}^{0} dz \varphi_l(z)\varphi_m(z) = \delta_{lm} \qquad (2.31)$$

and the completeness relation

$$\sum_l \varphi_l(z)\varphi_l(z') = \delta(z - z'). \qquad (2.32)$$

Further properties are discussed in Appendix A. Using (2.29) eigenfunctions of Ω are found to be

$$\phi(y) = \phi_{l\mathbf{k}}(\mathbf{x}, z) = (2\pi)^{-1}e^{i\mathbf{kx}}\varphi_l(z) \qquad (2.33)$$

with eigenvalues

$$\omega = \omega_{l\mathbf{k}} = f(1 + \lambda_l k^2)^{1/2}. \qquad (2.34)$$

Notice that $\phi_{l\mathbf{k}}$ and $\phi_{l\mathbf{k}}^* = \phi_{l-\mathbf{k}}$ have the same eigenvalue $\omega_{l\mathbf{k}} = \omega_{l-\mathbf{k}}$. For any eigenfunction ϕ of Ω with eigenvalue ω the vector functions $\beta_a^p\phi$ and $\delta^{sp}\phi$ are eigenfunctions of H_{ab} and H^{rs}, respectively. Both have the eigenvalue $p\omega$. We shall use the eigenfunctions

$$\phi_a^\lambda(y) = \beta_a^p\phi_{l\mathbf{k}}(y) = \begin{pmatrix} ip\omega_{l\mathbf{k}}/f \\ 1 \\ 1 - (p\omega_{l\mathbf{k}}/f)^2 \end{pmatrix} \phi_{l\mathbf{k}}(y) \qquad (2.35)$$

of H_{ab} and

$$\phi^{s\lambda}(y) = \tilde{\beta}_a^s\phi_a^\lambda = \delta^{sp}\phi_{l\mathbf{k}}(y) \qquad (2.36)$$

of H^{rs}, where λ represents the quadruplet $(p, l, \mathbf{k})$. These eigenfunctions obey the reality conditions

$$\left.\begin{aligned} (\phi_a^\lambda)^* = \phi_a^{-\lambda} \\ (\phi^{s\lambda})^* = \phi^{-s-\lambda} \end{aligned}\right\} , \qquad (2.37)$$

with $-\lambda = (-p, l, -\mathbf{k})$. The normal mode $p = 0$ is time-independent. Its polarization satisfies $\phi_1 = 0$, $\phi_2 = \phi_3$, which defines a geostrophic flow. The normal modes $p = \pm$ with eigenfrequencies $\pm\omega = \pm f(1 + \lambda_l k^2)^{1/2}$ represent internal waves in the quasi-hydrostatic approximation. Specifically, $p = +$ describes a wave propagating into the direction of the wave vector $\mathbf{k}$ and $p = -$ describes a wave propagating opposite to $\mathbf{k}$.

The eigenfunctions of Ω are orthogonal and complete. The relations

$$\left.\begin{aligned} \int dy \phi_{l\mathbf{k}}(y)\phi_{m\mathbf{k}'}^*(y) = \delta_{lm}\delta(\mathbf{k} - \mathbf{k}') \\ \sum_l \int d^2k \phi_{l\mathbf{k}}(y)\phi_{l\mathbf{k}}^*(y') = \delta(y - y') \end{aligned}\right\} \qquad (2.38)$$

follow from the corresponding properties of the factorial constituents of ϕ. To express orthogonality and completeness of the eigenfunctions ϕ_a^λ we define adjoint eigenfunctions by

$$\tilde{\phi}_a^\lambda(y) = \tilde{\beta}_a^p\phi_{l\mathbf{k}}^*(y) = \tilde{\beta}_a^\lambda\phi_{l\mathbf{k}}^*(y), \qquad (2.39)$$

where the vector $\tilde{\beta}_a^\lambda$ is given by (2.15) replacing the operator Ω by its eigenvalue $\omega = \omega_{l\mathbf{k}}$. Orthogonality and completeness are then expressed by

$$\left.\begin{aligned} \int dy \tilde{\phi}_a^\lambda(y)\phi_a^\mu(y) = \delta^{pq}\delta_{lm}\delta(\mathbf{k} - \mathbf{k}') \\ \sum_\lambda \phi_a^\lambda(y)\tilde{\phi}_b^\lambda(y') = \delta_{ab}\delta(y - y') \end{aligned}\right\} , \qquad (2.40)$$

where $\mu = (q, m, \mathbf{k}')$. The sum over λ abbreviates a sum over the branches p as well as over the vertical mode number l and an integral with respect to the wavevector $\mathbf{k}$. Orthogonality and completeness of the eigenfunctions of H^{rs} are expressed by

$$\left.\begin{aligned} \int dy \phi^{-s-\lambda}(y)\phi^{s\mu}(y) = \delta^{pq}\delta_{lm}\delta(\mathbf{k} - \mathbf{k}') \\ \sum_\lambda \phi^{-s-\lambda}(y)\phi^{r\lambda}(y') = \delta^{sr}\delta(y - y') \end{aligned}\right\} . \qquad (2.41)$$

The relations (2.38) and (2.40) may be utilized to derive a representation of operators $C(\Omega)$ such as the evolution operator $\exp(-ip\Omega t)$ and the vector operators $\tilde{\beta}_a^p$ and β_a^p introduced. Expanding a function $\zeta(y, t)$ in terms of the set $\phi_{l\mathbf{k}}(y)$, we have

$$\zeta(y, t) = \sum_l \int d^2k \phi_{l\mathbf{k}}(y) \int dy' \phi_{l\mathbf{k}}^*(y')\zeta(y', t) \qquad (2.42)$$

and thus the representation

$$C(\Omega) = \sum_l \int d^2k \phi_{l\mathbf{k}}(y)C(\omega_{l\mathbf{k}}) \int dy' \phi_{l\mathbf{k}}^*(y'). \qquad (2.43)$$

Obviously, $C(\Omega)$ is self-adjoint since from (2.42) we derive

$$\int dy u^*(y)C(\Omega)v(y) = \int dy v(y)C(\Omega)u^*(y) \qquad (2.44)$$

for any u and v since $\phi_{l\mathbf{k}}^* = \phi_{l-\mathbf{k}}$ and $\omega_{l\mathbf{k}} = \omega_{l-\mathbf{k}}$. Particular examples of (2.42) are the representations

$$\tilde{\beta}_a^p = \sum_l \int d^2k \phi_{l\mathbf{k}}(y) \int dy' \phi_a^\lambda(y'), \qquad (2.45)$$

$$\beta_a^p = \sum_l \int d^2k \phi_a^\lambda(y) \int dy' \phi_{l\mathbf{k}}^*(y'), \qquad (2.46)$$

for the decomposition operator and the polarization operator of the normal mode branches.

3. A formal theory of internal wave scattering

We are interested in describing the process in which an internal wave packet propagates toward the scattering region, interacts with it, and generates a perturbation of the flow called the scattered field. Far away from the scattering region the scattered field takes the form of outward radiating waves. The prediction of the relative amounts of incident and outward radiating energy is the problem to be solved. Formal theories and techniques of such a scattering problem have especially been developed in quantum mechanics (see, e.g., Schweber, 1961; Landau and Lifschitz, 1967). In this section we will translate these methods to our particular problem. First, we treat the case of a scattering region of finite extent in all directions. This poses a two-dimensional scattering problem. Then we continue with the idealized model in which the ambient fields forming the scattering region depend on only one of the horizontal coordinates. This effectively one-dimensional problem will be considered in more detail in a particular example in Section 4.

We consider a wave packet which at time $t = 0$ is located around $\mathbf{x}_0$ far away from the scattering region having a local wave vector $\mathbf{k}_0$ and a vertical mode m. We may describe it by a state vector $\psi(y, 0)$. It may be expanded in the eigenfunctions of the free wave operator H, i.e.,

$$\psi(y, 0) = \sum_q \int d^2k' a^\mu \phi^\mu(y), \tag{3.1}$$

where $\mu = (q, m, \mathbf{k}')$ and $\phi^\mu(y)$ is the normal mode vector given by (2.35) or (2.36). To represent the described packet the amplitudes are of the form

$$a^\mu = a^q_{m\mathbf{k}'} = \begin{cases} a_m(\mathbf{k}' - \mathbf{k}_0)e^{-i\mathbf{k}'\mathbf{x}_0}, & q = + \\ a_m(-\mathbf{k}' - \mathbf{k}_0)e^{i\mathbf{k}'\mathbf{x}_0}, & q = - \\ 0, & q = 0 \end{cases} \tag{3.2}$$

where $a_m(\mathbf{k}')$ is a real function peaked around $\mathbf{k}' = 0$. The evolution of this packet is governed by equation (2.25). For times $t > 0$ when the packet has not yet reached the scattering region it evolves according to the normal mode form (2.24). The free wave evolution operator $\exp(-ip\Omega t)$ attaches the harmonic time-dependence $\exp(-ip\omega_{m\mathbf{k}'}t)$ to each of the constituents of ψ. The motion of the packet can be obtained by the method of stationary phase. The position of maximum amplitude of ψ^+ occurs at $\mathbf{x}$ if

$$\nabla_{\mathbf{k}'}\{-\mathbf{k}'\cdot\mathbf{x}_0 + \mathbf{k}'\cdot\mathbf{x} - \omega_{m\mathbf{k}'}t\}_{\mathbf{k}'=\mathbf{k}_0} = 0 \tag{3.3}$$

or

$$\mathbf{x} = \mathbf{x}_0 + t\boldsymbol{\nu}_{m\mathbf{k}_0}. \tag{3.4}$$

The packet thus propagates with the group velocity

$$\boldsymbol{\nu}_{m\mathbf{k}} = \nabla_{\mathbf{k}}\omega_{m\mathbf{k}} = \lambda_m(f^2/\omega_{m\mathbf{k}})\cdot\mathbf{k}. \tag{3.5}$$

of the mode m at wave vector $\mathbf{k} = \mathbf{k}_0$. The packet moves into the direction of $\mathbf{k}_0$ toward the scattering region. Here we must include the scattering operator V in the time evolution. Then the formal solution is

$$\psi(y, t) = e^{-i\mathscr{H}t}\psi(y, 0). \tag{3.6}$$

To evaluate this expression it will be convenient to expand the initial state $\psi(y, 0)$ into eigenfunctions of $\mathscr{H}$. Because of the physical reasoning we anticipate that $\psi(y, t)$ has the form of the incident wave packet before the impingement and the form of diverging waves after the scattering. Therefore, we shall consider eigenfunctions of $\mathscr{H}$ with a particular asymptotic behavior. The mathematical formulation of this behavior differs slightly for two- and one-dimensional scattering regions. These cases will be treated separately.

a. *Two-dimensional scattering*

Let the interaction operator V be a function of both horizontal coordinates x_1 and x_2 such that it vanishes outside a finite region. Thus placing the origin of the coordinate system into this region we assume that $V \equiv 0$ for $R = |\mathbf{x}| > L_s$ which defines the extent of the scattering region. We consider the eigenvalue problem

$$\mathscr{H}\Psi^\mu = \omega^\mu\Psi^\mu \tag{3.7}$$

or, equivalently,

$$(\omega^\mu - H)\Psi^\mu = V\Psi^\mu, \tag{3.8}$$

with the asymptotic boundary conditions

$$\Psi^\mu(y) \sim \phi^\mu(y) + F^\mu(R, \theta, z) \tag{3.9}$$

as $R \sim \infty$. Here $\mu = (q, m, \mathbf{k}')$ as before. Further R and θ are polar coordinates of the horizontal position vector $\mathbf{x}$. The vector function $F^\mu(R, \theta, z)$ represents a superposition of circular waves. In the normal branch representation it is given by

$$F^{r\mu}(R, \theta, z) = \begin{cases} \delta^{rq} \sum_l \varphi_l(z)\ R^{-1/2} \exp(iqk_{lm}R) f^\mu_l(\theta), & q = +, - \\ 0, & q = 0 \end{cases} \tag{3.10}$$

with radial wavenumbers

$$k_{lm} = (\lambda_m/\lambda_l)^{1/2}k' = \left[\frac{(\omega_{m\mathbf{k}'})^2 - f^2}{\lambda_l f^2}\right]^{1/2}. \tag{3.11}$$

As a matter of fact, solutions of (3.8) with this prescribed asymptotic behavior exist and we may even show that the eigenvalue ω^μ is identical to the eigenvalue $q\omega_{m\mathbf{k}'}$ of the eigenfunction $\phi^\mu(y)$ of the free wave operator H, i.e.,

$$\omega^\mu = q\omega_{m\mathbf{k}'}, \quad (3.12)$$

which justifies characterization of the eigenfunctions Ψ^μ and eigenvalues ω^μ of $\mathscr{H}$ by the triplet μ. To prove (3.12) consider the eigenvalue problem (3.8) for large R where it reduces to

$$(\omega^\mu - H)\Psi^\mu = 0. \quad (3.13)$$

This is satisfied by the normal mode term $\phi^\mu(y)$ and the second term yields

$$(\omega^\mu - r\Omega)F^{r\mu}(R,\theta,z) = \delta^{rq}\sum_l [\omega^\mu - rf(1 + \lambda_l k_{lm}^2)^{1/2}] \times \varphi_l(z)\, R^{-1/2}\exp(iqk_{lm}R)f_l^\mu(\theta) + O(R^{-3/2}) \quad (3.14)$$

for $q \neq 0$. Thus the eigenvalue problem is satisfied asymptotically to $O(R^{-3/2})$. We will require that the amplitude $f_l^\mu(\theta)$ of the circular waves satisfy

$$(f_l^\mu(\theta))^* = f_l^{-\mu}(\theta), \quad (3.15)$$

with $-\mu = (-q, m, -\mathbf{k}')$ defined as before.

The time evolution of Ψ^μ given by

$$e^{-i\mathscr{H}t}\Psi^\mu = \Psi^\mu e^{-i\omega^\mu t} \quad (3.16)$$

indicates that asymptotically the eigenvector with $q = +$ describes a normal mode propagating into the direction of $\mathbf{k}'$ plus circularly expanding waves, whereas the vector with $q = -$ describes a normal mode propagating opposite to $\mathbf{k}'$ plus incoming circular waves. We shall refer to $f_l^\mu(\theta)$ as the scattering amplitude of the circular wave component with vertical mode number l and radial wavenumber k_{lm}.

It will be convenient to rewrite (3.8) for $q = +, -$ and the boundary condition (3.9) in an integral form. Eq. (3.8) is satisfied by solutions of the integral equation

$$\Psi^\mu(y) = \phi^\mu(y) + \int dy' G^\mu(y|y')V(y')\Psi^\mu(y'); \quad (3.17)$$

where $G(y|y')$ is the Green's operator (a matrix) defined as a solution of

$$(\omega^\mu - H)G^\mu(y|y') = \mathbb{1}\delta(y - y'), \quad (3.18)$$

where $\mathbb{1}$ is the unit matrix. By expansion into the eigenfunctions of H it is easily shown that the matrix G^μ is diagonal in the normal branch representation

$$G^{rs\mu}(y|y') = \sum_\lambda \frac{\phi^{r\lambda}(y)\phi^{-s-\lambda}(y')}{\omega^\mu - p\omega_{l\mathbf{k}}} = \delta^{rs}\sum_l \int d^2k\, \frac{\phi_{l\mathbf{k}}(y)\phi_{l\mathbf{k}}^*(y')}{\omega^\mu - r\omega_{l\mathbf{k}}}. \quad (3.19)$$

The diagonal elements will be denoted by $G^{(r)\mu}(y|y')$. Obviously,

$$G^{(0)\mu}(y|y') = (\omega^\mu)^{-1}\delta(y - y'), \quad (3.20)$$

which follows from (3.18) or the completeness relation (2.38).

For $r = \pm$ the integral in (3.19) is as yet undefined and we have to employ the usual rules of handling the poles to make the Green's function either the incoming or outgoing wave solution of (3.18). Proper choice of the integration path then will lead to a complete equivalence of the integral equation (3.17) with (3.8) and the boundary condition (3.9).

To show this we rewrite (3.19) for $r = \pm$ in the form

$$G^{(r)\mu}(y|y') = f^{-2}\sum_l \varphi_l(z)\varphi_l(z')\lambda_l^{-1} \times \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \times \frac{\omega^\mu + r\omega_{l\mathbf{k}}}{k_{lm}^2 - k_1^2 - k_2^2}\exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}')]. \quad (3.21)$$

For $q = +$ we obviously need the outgoing wave solution and, consequently, choose the k_2 path as indicated in Fig. 1. The k_2 integration may be performed to yield

$$-\frac{1 + r}{2}\frac{i\omega_{m\mathbf{k}'}}{2\pi}\int_{-\infty}^{+\infty} dk_1 \times \frac{\exp[ik_1(x_1 - x_1') + i(k_{lm}^2 - k_1^2)^{1/2}|x_2 - x_2'|]}{(k_{lm}^2 - k_1^2)^{1/2}} \quad (3.22)$$

for the integral in (3.21). This remaining integral may be expressed by $H_0^{(1)}$, the Hankel function of the first kind and zero order (cf. Morse and Feshbach, 1953, p. 823) so that finally

$$G_{m\mathbf{k}'}^{(r)+}(y|y') = -\delta^{r+}\frac{\omega_{m\mathbf{k}'}}{2\pi f^2} \times \sum_l \frac{\varphi_l(z)\varphi_l(z')}{\lambda_l} i\pi H_0^{(1)}(k_{lm}|\mathbf{x} - \mathbf{x}'|). \quad (3.23)$$

Similarly, with the k_2 path shown in Fig. 1 to perform the k_2 integration for incoming wave solution $q = -$ we find the representation

$$G_{m-\mathbf{k}'}^{(r)-}(y|y') = -\delta^{r-}\frac{\omega_{m\mathbf{k}'}}{2\pi f^2} \times \sum_l \frac{\varphi_l(z)\varphi_l(z')}{\lambda_l} i\pi H_0^{(2)}(k_{lm}|\mathbf{x} - \mathbf{x}'|) \quad (3.24)$$

in terms of the Hankel function of the second kind.

The asymptotic behavior of the Hankel functions

$$\left.\begin{aligned} i\pi H_0^{(1)}(kR) &\sim \left(\frac{2\pi}{kR}\right)^{1/2} e^{i(kR+\pi/4)} \\ i\pi H_0^{(2)}(kR) &\sim \left(\frac{2\pi}{kR}\right)^{1/2} ie^{-i(kR-\pi/4)} \end{aligned}\right\}, \quad (3.25)$$

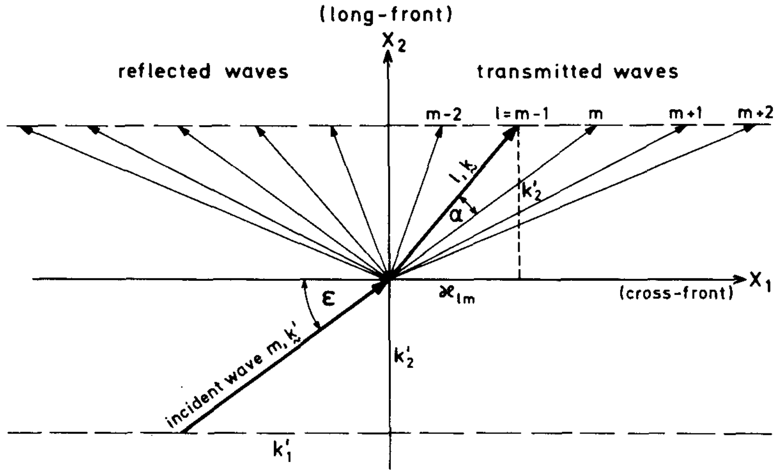

FIG. 1. Integration paths for the Green's function (3.21): upper panel, outgoing solution; lower panel, incoming solution.

will be used to recover the boundary conditions (3.9) from (3.17). Since contributions to the integral in (3.17) only arise from $|\mathbf{x}'| < L_s$ we may write for large R

$$|\mathbf{x} - \mathbf{x}'| = R - x' \cos\alpha + \cdots, \qquad (3.26)$$

where α is the angle between $\mathbf{x}$ and $\mathbf{x}'$. Hence, as $R \sim \infty$

$$\Psi^{r\mu}(y) \sim \phi^{r\mu}(y) - \delta^{rq}\omega^{\mu}f^{-2}\frac{1}{2}(1+iq)\sum_l \frac{\varphi_l(z)}{\lambda_l} \times \left(\frac{2\pi}{k_{lm}}\right)^{1/2} R^{-1/2}\exp(iqk_{lm}R) \times \frac{1}{2\pi}\int dy'\varphi_l(z')\exp(-iqk_{lm}x'\cos\alpha) \times V^{qs}\Psi^{s\mu}(y'), \qquad (3.27)$$

so that (3.15) is satisfied with the scattering amplitude

$$f_l^{\mu}(\theta) = -\frac{\omega^{\mu}}{\lambda_l f^2}\left(\frac{2\pi}{k_{lm}}\right)^{1/2}\frac{1}{2}(1+iq)(2\pi)^{-1} \times \int dy'\phi^{*}_{l\mathbf{k}^{\mu}}V^{qs}\Psi^{s\mu} \qquad (3.28a)$$

or, expressed in a base-free form,

$$f_l^{\mu}(\theta) = -\frac{\omega^{\mu}}{\lambda_l f^2}\left(\frac{2\pi}{k_{lm}}\right)^{1/2}\frac{1}{2}(1+iq) \times \int dy'(\phi^{\sigma})^{*}V\Psi^{\mu}, \qquad (3.28b)$$

with $\sigma = (q, l, \mathbf{k}^{\mu})$. The vector

$$\mathbf{k}^{\mu} = qk_{lm}\mathbf{x}R^{-1} \qquad (3.29)$$

has been introduced for convenience. For $q = +$ this vector has polar coordinates (k_{lm}, θ) and points into the direction of the point of observation $\mathbf{x}$. Using (2.23) it is immediately shown that (3.28) satisfies relation (3.15).

We now return to the evaluation of (3.6) and the expansion of the initial state of $\psi(y, 0)$ into eigenfunctions of $\mathscr{H}$. Since this operator may have eigenfunctions not covered by (3.10), the set of functions $\Psi^{\mu}(y)$ may not be complete. Usually the additional eigenfunctions of $\mathscr{H}$ are "bound states" which fall off exponentially for $R \gg L_s$. Then at large distances from the scattering region the Ψ^{μ}'s may be regarded as complete and the initial state $\psi(y, 0)$ has an expansion

$$\psi(y, 0) = \sum_{\nu} b^{\nu}\Psi^{\nu}(y) \qquad (3.30)$$

with $\nu = (r, n, \mathbf{k}'')$. Since initially the packet is located far away from the scattering region, we may replace $\Psi^{\nu}(y)$ by its asymptotic value and find

$$\left.\begin{aligned} \psi^r(y,0) &= \sum_n \int d^2k''b^{\nu}[\phi_{n\mathbf{k}''}(y) + \sum_l \varphi_l(z) \\ &\quad \times R^{-1/2}\exp(irk_{ln}R)f_l^{\nu}(\theta)], \quad r = +, - \\ \psi^0(y,0) &= \sum_n \int d^2k''b^0_{n\mathbf{k}''}\phi_{n\mathbf{k}''}(y) \end{aligned}\right\}, \qquad (3.31)$$

where k''_{lm} is defined by (3.11) with k' replaced by k''.

Since the geostrophic component of the initial state must vanish we put $b^0_{n\mathbf{k}''} \equiv 0$. For $r = \pm$ the function b^{ν} must be chosen such that (3.31) represents a wave packet of mode m localized at $\mathbf{x}_0$ with wave vector $\mathbf{k}_0$. Thus $b^{\nu} \sim \delta_{nm}$ for $r = \pm$. In view of the free wave expansion (3.2) we anticipate that the second term in the brackets of (3.31) gives a vanishing contribution. To obtain the position of the maximum we use the method of stationary phase.

The first term has its maximum where

$$\nabla_{\mathbf{k}''}[\beta(\mathbf{k}'') + \mathbf{k}''\cdot\mathbf{x}]_{\mathbf{k}''=\mathbf{k}_0} = 0, \qquad (3.32)$$

with $\beta(\mathbf{k}'') = \arg(b_m^+(\mathbf{k}''))$. Thus, the maximum occurs at

$$\mathbf{x}_{\max} = \nabla_{\mathbf{k}_0}\beta(\mathbf{k}_0). \qquad (3.33)$$

The second term then yields

$$\nabla_{\mathbf{k}''}[\chi(\mathbf{k}'') + \beta(\mathbf{k}'') + k''_{lm}R]_{\mathbf{k}''=\mathbf{k}_0} = 0 \qquad (3.34)$$

or

$$\nabla_{\mathbf{k}_0}\chi - \mathbf{x}_{\max} + (\lambda_m/\lambda_l)^{1/2}R\mathbf{k}_0/k_0 = 0, \qquad (3.35)$$

where $\chi(\mathbf{k}'')$ is the argument of the scattering amplitude. Using the representation (3.28) we notice that $\nabla_{\mathbf{k}_0}\chi$ is a length characterizing the range of the scattering region and, consequently, we may omit this term. Now if we choose $\mathbf{x}_{\max}$ to be the position $\mathbf{x}_0$ of the wave packet, Eq. (3.35) cannot be fulfilled for positive R since $\mathbf{k}_0$ and $\mathbf{x}_0$ have opposite directions, i.e., the diverging wave part of (3.31) is negligibly small due to cancellation by interference and we get

$$\psi(y, 0) = \sum_q \int d^2k' b^\mu \phi^\mu(y). \qquad (3.36)$$

Hence, comparing with (3.1), we see that

$$b^\mu = a^\mu. \qquad (3.37)$$

The importance of this simple result must be emphasized since it enables the complete reduction of the time-dependent scattering problem (3.6) to the solution of the stationary problem (3.8) and (3.11) or, equivalently, (3.17). Since the expansion of $\psi(y, 0)$ into normal modes has the same expansion coefficients as in an expansion into the eigenfunctions of the operator $\mathscr{H}$ the solution of the scattering problem for the wave packet (3.1) is simply given by

$$\psi(y, t) = \sum_q \int d^2k' a^\mu \Psi^\mu(y) e^{-i\omega^\mu t}. \qquad (3.38)$$

The simplicity of this result, of course, is due to the proper choice of the asymptotic behavior of the functions $\Psi^\mu(y)$.

For sufficiently large R we may replace Ψ^μ in (3.38) by its asymptotic value and get

$$\psi(y, t) \sim \psi_0(y, t) + \psi_s(y, t), \qquad (3.39)$$

where the first term

$$\psi_0(y, t) = \sum_q \int d^2k' a^\mu \phi^\mu(y) e^{-i\omega^\mu t} = e^{-iHt}\psi(y, 0), \qquad (3.40)$$

corresponds to the unscattered wave packet. The second term

$$\psi_s(y, t) = \sum_q \int d^2k' a^\mu F^\mu(R, \theta, z) e^{-i\omega^\mu t} \qquad (3.41)$$

describes outward propagating circular waves. In the normal branch representation $\psi_s^0 \equiv 0$ and

$$\psi_s^+(y, t) = \sum_l \varphi_l(z) \int d^2k' a_{m\mathbf{k}'}^+ f_{lm\mathbf{k}'}^+(\theta) \times \exp[i(k_{lm}R - \omega_{m\mathbf{k}'}t)]R^{-1/2}. \qquad (3.42)$$

Again the method of stationary phase may be invoked to yield the positions R, where ψ_s^+ attains its maximum. Similar to (3.34) we find

$$-\mathbf{x}_0 + (\lambda_m/\lambda_l)^{1/2}R\mathbf{k}_0/k_0 - t\boldsymbol{\nu}_{m\mathbf{k}_0} = 0. \qquad (3.43)$$

From the above condition we get

$$R = (\lambda_m/\lambda_l)^{1/2}(x_0 + \nu_0 t) \qquad (3.44)$$

with $x_0 = \mathbf{k}_0\cdot\mathbf{x}_0/k_0$ (which is negative since $\mathbf{k}_0$ and $\mathbf{x}_0$ have opposite directions) and $\nu_0 = \boldsymbol{\nu}_{m\mathbf{k}_0}\cdot\mathbf{k}_0/k_0 = \lambda_m f^2/\omega_{m\mathbf{k}_0}$. Since R is positive, Eq. (3.44) only holds for $t > -x_0/\nu_0$. Thus the function ψ_s^+ is different from zero only after the time x_0/ν_0 has elapsed, i.e., after the impingement of the packet. Then, in addition to the unscattered wave, circular waves of vertical modenumber l and radial wavenumber k_{lm} are diverging from the scattering region with relative amplitudes $f_{lm\mathbf{k}'}^+(\theta)$ depending on the direction θ.

In most scattering problems one is not interested in a complete solution for the state vector for all times, but only in the outcome of the scattering event at large distances from the scatterer, or more precisely, in the rate of conversion of energy in the incident mode m to the scattered modes l. As will be demonstrated in Appendix C, this quantity is completely determined by the scattering amplitudes $f_l^\mu(\theta)$ characterizing the asymptotic behavior of the solution of the stationary scattering problem (3.17). Therefore, it seems sensible to derive an equation which yields $f_l^\mu(\theta)$ directly rather than solving (3.17) and studying the asymptotic form of the solution. We shall derive an integral equation for what is called the reaction matrix in quantum mechanical scattering. This is defined by

$$R^{\lambda\mu} = \int dy(\phi^\lambda)^* V \Psi^\mu = \int dy \phi_{l\mathbf{k}}^* V^{ps} \Psi_{m\mathbf{k}'}^{sq} \qquad (3.45)$$

for $q = +, -$ and is related to the scattering amplitude by

$$f_l^\mu(\theta) = -\frac{\omega^\mu}{\lambda_l f^2}\left(\frac{2\pi}{k_{lm}}\right)^{1/2} \frac{1}{2}(1 + iq)R_{l\mathbf{k}^\mu m\mathbf{k}'}^{q\ q} \qquad (3.46)$$

where there is no sum over q. As before we use the abbreviation $\lambda = (p, l, \mathbf{k})$ and $\mu = (q, m, \mathbf{k}')$. To keep the notation short, we write the Green's function in the condensed form

$$G^{rs\mu}(y|y') = \delta^{rs} \sum_l \int d^2k \frac{\phi_{l\mathbf{k}}(y)\phi_{l\mathbf{k}}^*(y')}{\omega^\mu - r\omega_{l\mathbf{k}} + ri0}. \qquad (3.47)$$

In this frequently used notation the integration paths have been retained as the real axis but the poles have been shifted by $ri0$ (with infinitesimal 0) into the complex plane. This, of course, is equivalent to the previous analysis. As an application of (2.43) we may then express the integral equation (3.17) as

$$\Psi^{r\mu} = \phi^{r\mu} + \delta^{rs}(\omega^\mu - r\Omega + ri0)^{-1}V^{st}\Psi^{t\mu}. \quad (3.48)$$

Again, using (2.43) it is straightforward to derive the integral equation

$$R^{\lambda\mu} = V^{\lambda\mu} + \sum_\nu \frac{V^{\lambda\nu}R^{\nu\mu}}{\omega^\mu - \omega^\nu + ri0}, \quad (3.49)$$

where $\nu = (r, n, \mathbf{k}'')$ and $\omega^\nu = r\omega_{n\mathbf{k}''}$. In this equation we have introduced the matrix elements.

$$V^{\lambda\mu} = \int dy(\phi^\lambda)^* V\phi^\mu = \int dy\phi^*_{l\mathbf{k}} V^{pq}\phi_{m\mathbf{k}'} \quad (3.50)$$

of the interaction operator V. Eq. (3.49) has the advantage over the original equation (3.48) in that it directly deals with the scattering amplitudes.

Eq. (3.49) may be solved by classical iteration perturbation techniques. Simple iteration yields

$$R^{\lambda\mu} = V^{\lambda\mu} + \sum_\nu \frac{V^{\lambda\nu}V^{\nu\mu}}{\omega^\mu - \omega^\nu + ri0} + \cdots, \quad (3.51)$$

defining the successive orders of the Born approximation to the scattering problem. More involved perturbation expansions are considered in standard text books such as Morse and Feshbach (1953).

b. One-dimensional scattering

We now consider the scattering problem for an interaction operator V which is a function of only one of the horizontal coordinates, say, x_1. Further, let V vanish for $|x_1| > L_s$ defining the extent of the scattering region. This has the form of a "wall" of thickness $2L_s$ oriented along the $x_1 = 0$ plane. This mathematical problem may be an adequate idealization of the scattering at oceanic features like long ridges or shelf zones and baroclinic perturbations of the density field with a frontal structure. In such a model the spatial extent of the incident wave packet must be small compared to the true physical length scale of the scattering region. Anticipating the application of this model to the scattering at oceanic fronts treated in Section 4 we will here refer to the scattering region as the front and the x_1 and x_2 directions as the cross-front and long-front directions, respectively. The mathematical formulation of the problem is completely analogous to the two-dimensional case.

Though being two-dimensional in physical space, the scattering process may simply be described by a one-dimensional theory. The eigenfunctions of $\mathscr{H}$ with eigenvalues $\omega^\mu = q\omega_{m\mathbf{k}'}$ are obviously of the form

$$\Psi^\mu(y) = \left(\frac{1}{2\pi}\right)^{1/2} \exp(ik_2'x_2)\chi^\mu(s), \quad (3.52)$$

where the vector χ^μ depends on $s = (x_1, z)$ only. The asymptotic boundary coordinates imposed on $\chi^\mu(s)$ state that far away from the front the vector should represent plane waves radiating outward. Thus we demand the asymptotic form

$$\chi^\mu(s) \sim \eta^\mu(s) + E^\mu(s) \quad (3.53)$$

as $|x_1| \sim \infty$, where η^μ is the normal mode vector defined by

$$\phi^\mu(y) = \left(\frac{1}{2\pi}\right)^{1/2} \exp(ik_2'x_2)\eta^\mu(s) \quad (3.54)$$

and $E^\mu(s)$ describes the superposition of plane waves

$$E^\mu(s) = \delta^{rq}\left(\frac{1}{2\pi}\right)^{1/2} \sum_l \varphi_l(z) \times \begin{cases} A_l^\mu \exp(-iq\kappa_{lm}x_1), & x_1 \sim -\infty \\ B_l^\mu \exp(iq\kappa_{lm}x_1), & x_1 \sim +\infty \end{cases} \quad (3.55)$$

for $q = +, -$ and $E^\mu \equiv 0$ for $q = 0$.

Eigenfunctions with this prescribed form have the eigenvalue ω^μ if the wavenumber κ_{lm} satisfies the condition

$$\omega_{m\mathbf{k}'} = f[1 + \lambda_l(\kappa_{lm}^2 + (k_2')^2]^{1/2} \quad (3.56)$$

or

$$\kappa_{lm} = \left[\frac{\lambda_m}{\lambda_l}(k')^2 - (k_2')^2\right]^{1/2}. \quad (3.57)$$

Denoting by ϵ the angle of incidence of the wave onto the front ($\epsilon = 0$ being normal incidence), we may express κ_{lm} in the form

$$\kappa_{lm} = k'\left[\frac{\lambda_m}{\lambda_l} - \sin^2\epsilon\right]^{1/2} \quad (3.58)$$

and note that only terms in the sum (3.55) with

$$\sin^2\epsilon < \lambda_m/\lambda_l \quad (3.59)$$

have real κ_{lm}. These correspond to waves propagating in the cross-front direction. Those with $\sin^2\epsilon > \lambda_m/\lambda_l$ decay exponentially away from the front. In general, the eigenvalues λ_l of the vertical eigenvalue problem (2.30) decrease with increasing modenumber l. Thus the asymptotic form of $\chi^\mu(s)$ only contains modes l larger than some minimum. A schematic graph of the spatial distribution of the complete wave vectors $(\pm\kappa_{lm}, k_2')$ for a given incident wavevector $\mathbf{k}' = (k_1', k_2')$ is given in Fig. 2.

The mathematical treatment of the evolution of the initial wave packet by the method of stationary phase is completely analogous to the two-dimensional case. One immediately finds that the expansion coefficients of $\psi(y, 0)$ in the normal mode

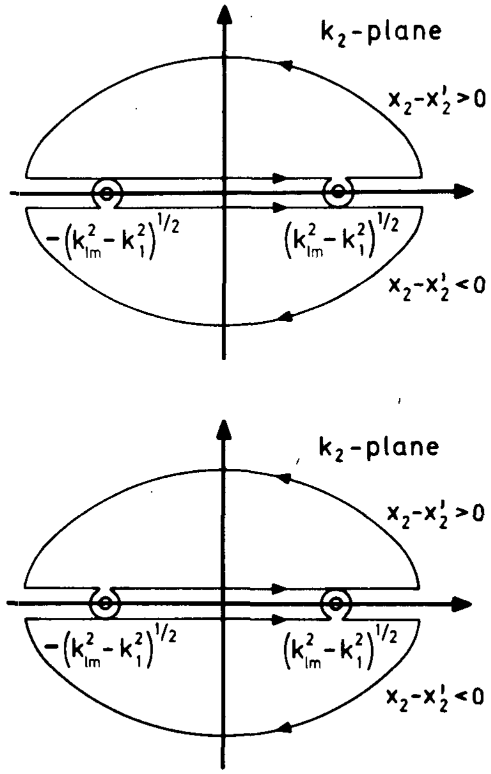

FIG. 2. Geometry of the scattered waves for a straight front.

expansion (3.1) are identical to those in the eigenfunctions Ψ^μ defined by (3.52) to (3.55). Separating the along-front dependence by Fourier expansion

$$\psi(y,t) = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dk_2' \exp(ik_2'x_2)\psi_{k_2}(s,t) \quad (3.60)$$

and omitting the subscript k_2' the solution for the long-front Fourier component is thus given by

$$\psi(s,t) = \sum_q \int dk_1' a^\mu \chi^\mu(s) e^{-i\omega^\mu t}. \quad (3.61)$$

At large $|x_1|$ the asymptotic form of χ^μ yields the decomposition

$$\psi(s,t) \sim \psi_0(s,t) + \psi_s(s,t) \quad (3.62)$$

into the two contributions

$$\begin{Bmatrix} \psi_0(s,t) \\ \psi_s(s,t) \end{Bmatrix} = \sum_q \int_{-\infty}^{+\infty} dk_1' a^\mu \begin{Bmatrix} \eta^\mu(s) \\ E^\mu(s) \end{Bmatrix} e^{-i\omega^\mu t}. \quad (3.63)$$

The part ψ_0 corresponds to the initial packet passing unchanged through the front, whereas ψ_s represents the scattered field. With (3.55) we get

$$\psi_s^\pm(s,t) = \left(\frac{1}{2\pi}\right)^{1/2} \sum_l \varphi_l(z) \int dk_1' a_{m\mathbf{k}'}^+ \begin{Bmatrix} A_{lm\mathbf{k}'}^+ \\ B_{lm\mathbf{k}'}^+ \end{Bmatrix} \times \exp[i(\mp\kappa_{lm}x_1 - \omega_{m\mathbf{k}'}t)], \quad x_1 \sim \mp\infty. \quad (3.64)$$

If the wave support a^μ is sufficiently peaked at k_0, the method of stationary phase yields for the position of the maximum of $\psi_s^+(s,t)$

$$x_{\max} = \left[(-x_0 - \nu_1 t)\left(\frac{d\kappa_{lm}}{dk_1'}\right)^{-1}\right]_{k_1'=k_0} \quad (3.65)$$

for large negative x_1, and

$$x_{\max} = \left[(x_0 + \nu_1 t)\left(\frac{d\kappa_{lm}}{dk_1'}\right)^{-1}\right]_{k_1'=k_0} \quad (3.66)$$

for large positive x_1. Here $\nu_1 = d\omega_{m\mathbf{k}'}/dk_1'$ so that, evidently, the maxima propagate with

$$\left[\nu_1\left(\frac{d\kappa_{lm}}{dk_1'}\right)^{-1}\right]_{k_1=k_0} = \lambda_l k_0 f^2/\omega_{l\mathbf{k}_0}, \quad (3.67)$$

which is the group velocity appropriate to mode l and wave vector $\mathbf{k}_0 = (k_0, k_2')$. Since x_0, the initial position of the packet on the x_1 axis, is negative the scattered field is zero for times $t < |x_0|/\nu_1$, i.e., before the impingement. When this time has elapsed wave packets of modes l diverge from the front to $+\infty$ and $-\infty$ which we shall refer to as transmitted and reflected wave packets. The amplitude function A_l^μ and B_l^μ modify the initial support function a^μ to represent the support of the transmitted and reflected packets, respectively. These functions, called the scattering amplitudes, completely determine the solution of the problem. As with the two-dimensional case, we will derive an integral equation for the scattering amplitudes.

The integral formulation of the stationary scattering problem (3.8) and (3.53) is given by

$$\chi^\mu(s) = \eta^\mu(s) + \int ds' G^\mu(s|s')V(s')\chi^\mu(s') \quad (3.68)$$

for $q = +, -$. Here V has been written for the matrix element

$$V_{k_2'} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx_2 \exp(-ik_2'x_2)V \times \exp(ik_2'x_2). \quad (3.69)$$

In the normal branch representation Green's operator again turns out to be diagonal with diagonal elements

$$G^{(r)\mu}(s|s') = \frac{1}{2\pi}\sum_l \int_{-\infty}^{+\infty} dk_1 \times \frac{\varphi_l(z)\varphi_l(z') \exp[ik_1(x_1 - x_1')]}{\omega^\mu - r\omega_{l\mathbf{k}} + ri0}, \quad (3.70)$$

where $k = (k_1, k_2')$. Explicit expressions are

$$G^{(0)\mu}(s|s') = (\omega^\mu)^{-1}\delta(s-s')$$

$$G^{(r)\mu}(s|s') = i\delta^{rq}\frac{\omega^\mu}{f^2}\sum_l \frac{\varphi_l(z)\varphi_l(z')}{\lambda_l\kappa_{lm}} \times \exp[iq\kappa_{lm}|x_1-x_1'|], \quad r=+,-. \quad (3.71)$$

For the wave components $r = \pm$ the integral equation (3.68) may then be written

$$\chi^{r\mu}(s) = \eta^{r\mu}(s) - i\delta^{rq}\frac{\omega^\mu}{f^2}\sum_l \frac{\varphi_l(z)}{\lambda_l\kappa_{lm}} \times \left\{ e^{iq\kappa_{lm}x_1}\int_{-\infty}^{x_1} dx_1' \int_{-H}^{0} dz' \times \exp(-iq\kappa_{lm}x_1')\varphi_l(z')V^{qs}(s')\chi^{s\mu}(s') + e^{-iq\kappa_{lm}x_1}\int_{x_1}^{+\infty} dx_1' \int_{-H}^{0} dz' \times \exp(iq\kappa_{lm}x_1')\varphi_l(z')V^{qs}(s')\chi^{s\mu}(s')\right\}. \quad (3.72)$$

This form immediately verifies the proper choice of the Green's function since the boundary conditions (3.53) are satisfied with scattering amplitudes

$$\left.\begin{matrix} A_l^\mu \\ B_l^\mu \end{matrix}\right\} = -i\frac{\omega^\mu}{f^2}\frac{(2\pi)^{1/2}}{\lambda_l\kappa_{lm}}\int ds \times \exp(\pm iq\kappa_{lm}x_1)\varphi_l(z)V^{qs}\chi^{s\mu}(s). \quad (3.73)$$

Defining the reaction matrix for the one-dimensional problem by

$$R^{\lambda\mu} = \int ds(\eta^\lambda)^* V\chi^\mu, \quad (3.74)$$

the scattering amplitudes may be written

$$\left.\begin{matrix} A_l^\mu \\ B_l^\mu \end{matrix}\right\} = -i\frac{2\pi\omega^\mu}{f_2\lambda_l\kappa_{lm}}R^{\sigma\mu}. \quad (3.75)$$

Here $\sigma = (q, l, \mp q\kappa_{lm})$, where the upper sign applies to A and the lower sign to B.

The integral equation for the one-dimensional reaction matrix takes the same form as (3.49) but the matrix element $V^{\lambda\mu}$ is taken to be the diagonal element with respect to k_2' and the sum over ν included only integration with respect to k_1''.

4. Scattering at ocean fronts

In this section we apply the scattering theory to ocean fronts. As front we will consider a geostrophic flow with a density field which has widely different scales in the two horizontal directions. This structure which we refer to as a straight front is described by a buoyancy field $B(x_1,z)$ with associated pressure field $P(x_1, z)$ and geostrophic current $U_2(x_1, z)$. For simplicity we assume that there is no variation in the mean sea level, i.e., $Z(x_1) \equiv 0$, which implies the vanishing of the barotropic component of the jet. As a further restriction we will impose the rigid-lid approximation for the wave part of the motion and restrict the analysis to the scattering of baroclinic waves.

The theory of internal wave scattering presented in Section 3b is immediately applicable to this problem. The only constraints are the homogeneity of the stratification and the vanishing of the interaction operator outside the front. These requirements hardly sound restrictive, but when combined with the hydrostatic and the geostrophic balance they confine the theory to fronts with rather special properties. Let us briefly elucidate this problem.

The theory developed in Section 3 must assume an identical density gradient on both sides of the front in order to get a homogeneous $N(z)$ outside the frontal zone. Hence, the buoyancy field $B(x_1, z)$ must have vanishing gradients outside the front, a requirement which exactly is the condition for the interaction operator to be confined to the frontal region. This will be shown in the next section. The jump ΔB of the buoyancy field across the front is thus constant and the hydrostatic and the geostrophic balances imply

$$\Delta P = f\int_{-L_s}^{L_s} dx_1 U_2(x_1, z) = z\Delta B + \text{constant}. \quad (4.1)$$

The constant must vanish if the mean sea level is flat. The cross-front integral of the current (along-front transport at each level) is thus constrained to have a linear vertical shear. Though this does not apply to the current itself we will use a jet with a linear shear in the examples discussed in Section 4b. The interaction operator will be derived in Section 4a for the general case.

a. The interaction operator

The analysis will be clearer if we make no explicit use of the "straight" structure of the front and consider for the moment a general mean flow which is in hydrostatic and geostrophic balance so that

$$\left.\begin{aligned} P &= -\int_z^0 dz'B \\ U_\alpha &= f^{-1}\epsilon_{\alpha\beta}\partial_\beta\int_z^0 dz'B \end{aligned}\right\}. \quad (4.2)$$

The only nonvanishing source terms S_n in the equations of motion (2.1) are S_1, S_2 and S_4. With (4.2) these may entirely be expressed by the buoyancy field in the form

$$\left.\begin{aligned} S_\alpha &= -f^{-1}\Big[u_j\partial_j\Big(\epsilon_{\alpha\gamma}\partial_\gamma\int_z^0 dz'B\Big) \\ &\qquad + \Big(\epsilon_{\beta\gamma}\partial_\gamma\int_z^0 dz'B\Big)\partial_\beta u_\alpha\Big] \\ S_4 &= -\Big[u_j\partial_j B + \Big(\epsilon_{\beta\gamma}\partial_\gamma\int_z^0 dz'B\Big)\partial_\beta b\Big] \end{aligned}\right\} . \quad (4.3)$$

We write the internal wave current field as

$$u_j = Q_{ja}\psi_a, \quad (4.4)$$

where the operator Q_{ja} follows from (2.4a–c), i.e.,

$$Q_{ja} = \begin{pmatrix} L^{-1}\partial_1 & -L^{-1}\partial_2 & 0 \\ L^{-1}\partial_2 & L^{-1}\partial_1 & 0 \\ -\int_{-H}^{z} dz' & 0 & 0 \end{pmatrix} \quad (4.5)$$

and use (2.4e) to eliminate the wave buoyancy field b. Inserting this into the definition (2.9) and (2.10) of the interaction operator V_{ab} we find after some algebra

$$\left.\begin{aligned} V_{1a} &= -if^{-1}\partial_\alpha\Big[\Big(\epsilon_{\alpha\gamma}\int_z^0 dz'B_{\gamma\beta}\Big)Q_{\beta a} + \Big(\epsilon_{\beta\gamma}\int_z^0 dz'B_\gamma\Big)\partial_\beta Q_{\alpha a} - \epsilon_{\alpha\gamma}B_\gamma Q_{3a}\Big] \\ V_{2a} &= if^{-1}\partial_\gamma\Big[\Big(\int_z^0 dz'B_{\beta\gamma}\Big)Q_{\beta a} + \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\Big(\int_z^0 dz'B_\delta\Big)\partial_\beta Q_{\alpha a} - B_\gamma Q_{3a}\Big] \\ V_{3a} &= if^{-1}L\Big[\int_z^0 dz'B_\alpha[Q_{\alpha a} - \epsilon_{\alpha\beta}\partial_\beta L^{-1}\delta_{3a}] + \Big(\int_z^0 dz'B_\alpha\Big)\epsilon_{\alpha\beta}\partial_\beta L^{-1}\delta_{3a} + \int_z^0 dz'B_3 Q_{3a}\Big] \end{aligned}\right\} , \quad (4.6)$$

where B_γ represents $\partial_\gamma B$, etc. Note that V_{ab} is nonlocal in the vertical as well as in the horizontal domain.

To meet the constraints discussed above we decompose the buoyancy field

$$B(x_1, z) = A(x_1, z) + B(x_1) \quad (4.7)$$

into a field $B(x_1)$ with linear vertical current shear and nonzero transport, i.e., $B(L_s) - B(-L_s) = \Delta B \neq 0$, and into a zero-transport field $A(x_1, z)$, i.e., $A(x_1, z) \sim 0$ as $x_1 \sim \pm L_s$. We shall present here only the analysis of the linear shear field which gives a reasonable model for typical ocean fronts. For a linear shear flow the expressions (4.6) slightly simplify, in particular the last term in V_{3a} vanishes.

b. The scattered field in the Born approximation

Following the frame work of the scattering theory we have to compute the matrix elements

$$V^{\lambda\mu} = \int dy\,\tilde{\phi}_a^\lambda V_{ab}\phi_b^\mu. \quad (4.8)$$

This straightforward but lengthly computation is outlined in Appendix B. The expansion parameter of the Born approximation [Eq. (3.51)] given by k/ω times the order of magnitude of $V^{\lambda\mu}$ follows from the scaling of the intraction matrix in the form

$$V^{\lambda\mu} = U\cdot k'\cdot L_s v^{\lambda\mu}, \quad (4.9)$$

where $v^{\lambda\mu}$ is a dimensional function of order unity and $U = H\Delta B/(fL_s)$ the velocity scale of the frontal jet. Thus (3.51) is an expansion with respect to $(U/c)(k'L_s)$ involving the ratio of current speed U and phase speed c of the incident wave and the ratio of the frontal width L_s and the wavelength $2\pi/k'$. For most cases of interest $k'L_s$ is of order unity or smaller and $U/c \ll 1$ so that the lowest order of the Born approximation should be sufficient. Then following (3.51), we only need the interaction matrix evaluated for the resonant case $\omega^\lambda = \omega^\mu$ given by

$$v^{\lambda\mu} = \gamma(k_1 - k_1')J_{lm}E^\pm(\nu, \omega/f, \epsilon), \quad (4.10)$$

with

$$\lambda = (+, l, \pm\kappa_{lm}), \quad \mu = (+, m, k_1')$$

and

$$k_1 - k_1' = k'[\pm|\nu^2 - \sin^2\epsilon|^{1/2} - \cos\epsilon], \quad (4.11)$$

with

$$\nu = (\lambda_m/\lambda_l)^{1/2}.$$

The first factor in (4.10)

$$\gamma(k_1'') = \frac{1}{\Delta B}\Big(\frac{1}{2\pi}\Big)^{1/2} \times \int_{-\infty}^{+\infty} dx_1 \frac{dB}{dx_1}\exp(-ik_1''x_1) \quad (4.12)$$

expresses the dependence on the frontal shape while the other two factors,

$$J_{lm} = H^{-1}\int_{-H}^0 dz\varphi_l\varphi_m\cdot z \quad (4.13)$$

and $E^\pm(\nu, \omega/f, \epsilon)$, describe the coupling of the incident and outgoing wave components to the Fourier component of the linear shear current. The function $E^\pm$ is given by (B7) and (B8).

A view of the spatial distribution of the interacting components has been presented in Fig. 2. It can be seen that (for oblique incidence $\epsilon \neq 0$) low modes of the scattering field are aligned more toward the along-front direction than high modes. More strictly, modes $l < m$ propagate with angles $< \pi/2 - \epsilon$ relative to the along-front direction and modes $l > m$ propagate in the complementary sectors. This feature follows entirely from the resonance condition and is independent of the frontal shape. According to (3.75) the lowest order of the Born approximation yields the scattering amplitudes

$$T_{lm}^{\pm}(\omega, \epsilon) = -i \frac{2\pi\omega}{f^2 \lambda_l \kappa_{lm}} V^{\lambda\mu}$$

$$= \left(2\pi \frac{U}{fL_s} \frac{L_s^2}{\lambda_m}\right)\left[\frac{-i(\omega/f)\nu^2 v^{\lambda\mu}}{|\nu^2 - \sin^2\epsilon|^{1/2}}\right] \quad (4.14)$$

for transmitted (T_{lm}^{+}) and reflected (T_{lm}^{-}) waves.

We will discuss this expression for some simple models of the front, i.e., buoyancy fields $B(x_1)$, since these cases already demonstrate the principal features of the interaction. However, it is straightforward to evaluate the scattering amplitudes for any reasonable frontal model.

To get a wave response in modes $l \leqslant m$ from an incident mode m, the angle of incidence ϵ must be in the range $-\epsilon_c \leqslant \epsilon \leqslant \epsilon_c$ with

$$\epsilon_c = \arcsin \nu^2. \quad (4.15)$$

Note that this range is very small for the barotropic channel $l = 0$ since $\lambda_0^{-1} = O(f^2/gH) \ll 1$. This channel will thus only respond for orthogonal impingement. At $\epsilon = \pm\epsilon_c$ there occurs a singularity in the scattering amplitudes (4.14). The corresponding resonant triplet is $\mathbf{k}' = (k_1', k_2')$, $\mathbf{k}'' = (-k_1', 0)$ and $\mathbf{k} = (0, k_2')$, so that formally the scattered waves

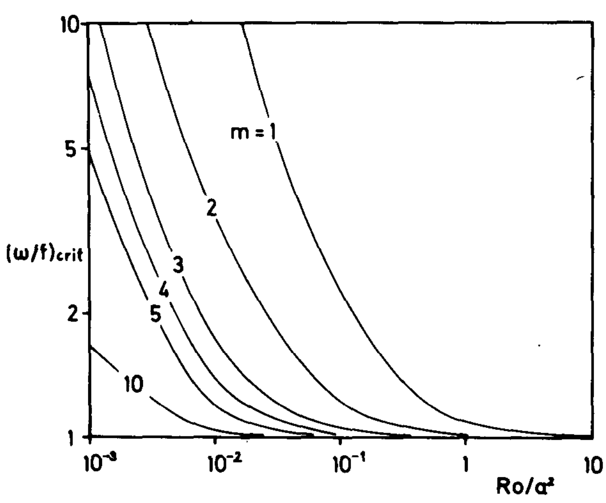

FIG. 3. $(\omega/f)_{\rm crit}$ as a function of Ro/a^2 for different modes m.

TABLE 1. Typical scales and dimensionless parameters of large- and small-scale ocean fronts.

	Large-scale front	Small-scale front
U (m s^{-1})	0.5	0.1
L_s (m)	10^5	10^4
f (s^{-1})	7×10^{-5}	
N_0 (s^{-1})	5×10^{-3}	3×10^{-2}
b (m)	10^3	10^2
H (m)	5×10^3	5×10^2
H/b	5	5
$a = N_0 b/(fL_s)$	0.7	4.3
Ro $= U/(fL_s)$	5×10^{-2}	0.1
T_0	1.5	8×10^{-2}
Ro/a^2	10^{-1}	5×10^{-3}

propagate along the front with infinite amplitudes. This singularity also occurs in the conversion ratios

$$C_{lm}^{\pm} = \nu^{-4}\left(\frac{\nu^2 - \sin^2\epsilon}{\cos^2\epsilon}\right)^{1/2} \left\{ \begin{matrix} |\delta_{lm} + T_{lm}^{+}|^2 \\ |T_{lm}^{-}|^2 \end{matrix} \right\}, \quad (4.16)$$

which represent the ratio of the energy fluxes in the scattered mode l and the incident mode m (see Appendix C). However, this singularity is not severe: physically meaningful and measurable quantities such as the response due an incident wave packet remains finite. This is immediately verified by inserting (4.14) into (3.64) with a continuous initial wave support. The mathematical reason is that the integral of $(\nu^2 - \sin^2\epsilon)^{-1/2}$ times a smooth function of ϵ is finite if $\nu^2 < 1$. For $\nu^2 = 1$, i.e., $l = m$, the singular behavior occurs at $\epsilon = \pm\pi/2$, i.e., for incident waves propagating along the front which are excluded from our idealized frontal model.

For the further discussion we rewrite $T_{lm}^{\pm}$ in the form

$$T_{lm}^{\pm} = T_0(2\pi)^{1/2}\gamma(k_1 - k_1')\rho_{lm}^{\pm}(\omega/f, \epsilon), \quad (4.17)$$

with the scale factor

$$T_0 = (2\pi)^{1/2} \frac{U}{fL_s} \frac{L_s^2}{\lambda_1} \quad (4.18)$$

and

$$\rho_{lm}^{\pm}(\omega/f, \epsilon)$$

$$= (\lambda_1/\lambda_m)(\omega/f)\nu^2 J_{lm}\left[\frac{-iE^{\pm}(\nu, \omega/f, \epsilon)}{|\nu^2 - \sin^2\epsilon|^{1/2}}\right]. \quad (4.19)$$

This function describes the response to a step in the density field, i.e., $dB/dx_1 = \Delta B\delta(x_1)$. If the scattered field $\psi_{\rm step}^{+}$ is evaluated from (3.64) for this case the actual wave pattern due to a realistic buoyancy field $B(x_1)$ can be found by convolution, i.e.,

$$\psi_s^{+}(x_1, z, t)$$

$$= \int_{-\infty}^{+\infty} dx_1' \psi_{\rm step}^{+}(x_1', z, t) B(x_1 - x_1'). \quad (4.20)$$

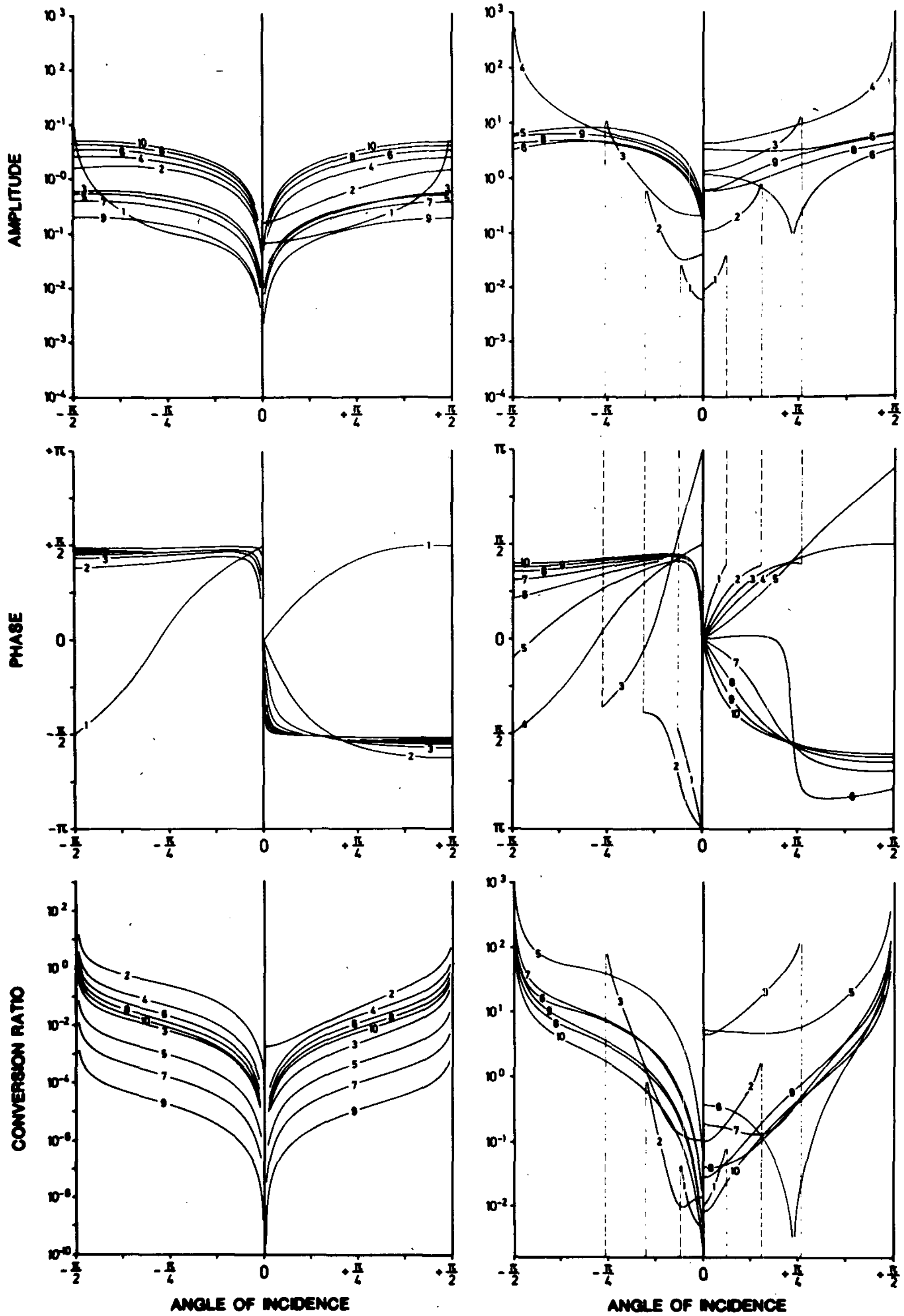

FIG. 4. Modulus and phase (upper panels) of the response function (4.19) for incident modes $m = 1$ (left column) and $m = 4$ (right column) and $H/b = 5$ and $\omega/f = 2$. The two lowest panels display the conversion ratios for a density step front. In each figure the negative range of ϵ refers to the reflected part and the positive range of ϵ to the transmitted part.

However, we will not follow this line here but rather confine the discussion to the scattering amplitudes.

The response function $\rho_{lm}^{\pm}$ involves as a factor the integral J_{lm} which depends on the overlapping of the vertical eigenfunctions of the incident and outgoing modes with the shear current. In Appendix A

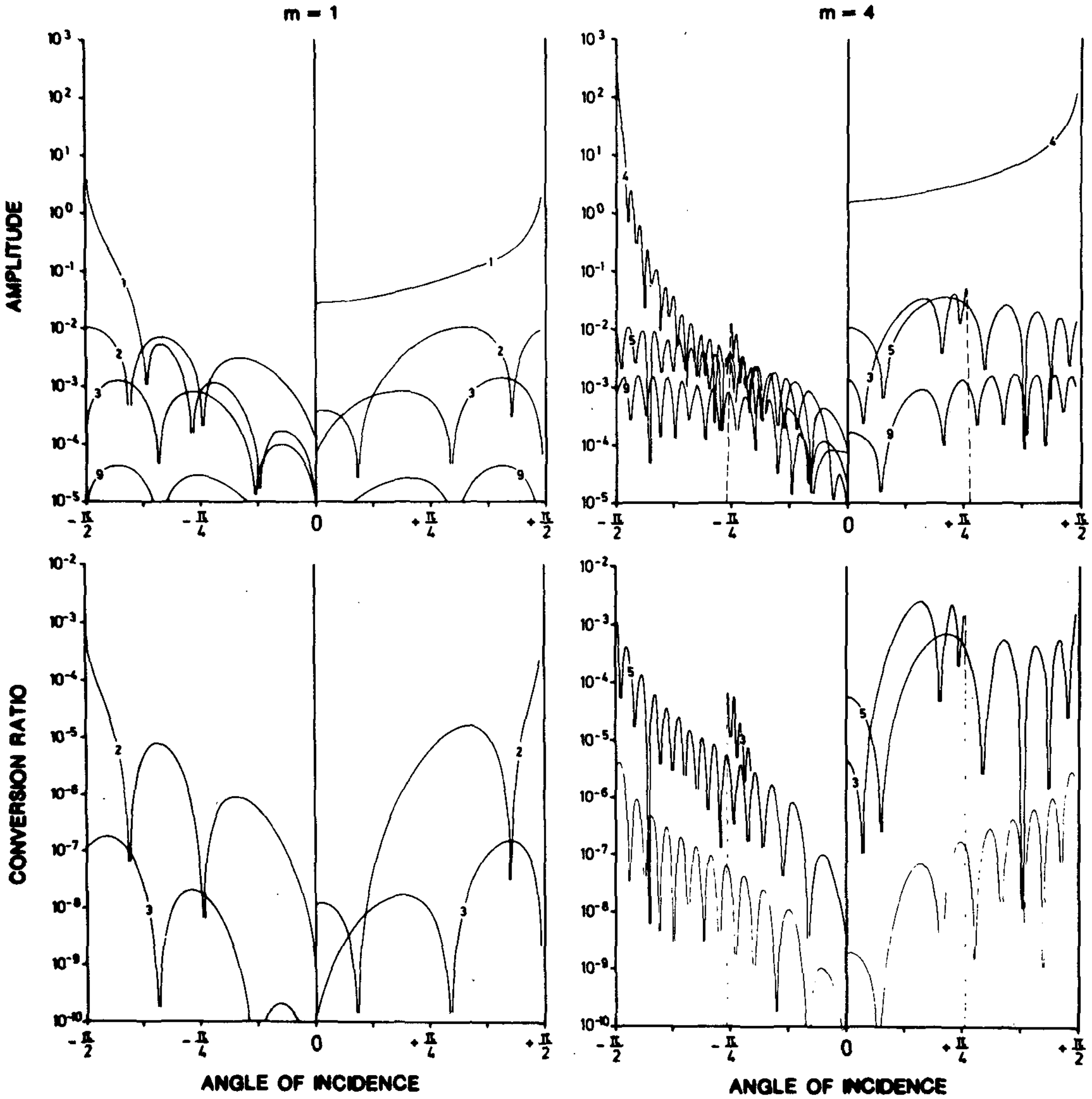

FIG. 5. The modulus of scattering amplitudes $T^{\pm}_{lm}/T_0$ and the scaled conversion ratios $C^{\pm}_{lm}/T_0^2$ of the front front [(4.22)] for $a = 0.7$.

we have evaluated J_{lm} for an N = constant model and a more realistic exponential model $N = N_0 \exp \times (z/b)$. Both models yield similar results: the modulus of J_{lm} falls off rapidly away from the diagonal $l = m$ restricting the coupling essentially to the diagonal and the first few off-diagonal rows. There is a difference in sign of J_{lm} in the off-diagonal elements for the two models which causes a difference in the phase pattern of the scattered field. For the discussion to come we will use the exponential model.

The scattering amplitudes depend on the scale quantities U, L_s of the jet and f, N_0, H and b of the ambient background only through the parameters $\mathrm{Ro} = U/(fL_s)$ (the frontal Rossby number) as a factor, the ratio H/b of ocean depth to scale depth of the stratification (via J_{lm}), the aspect ratio $a = (N_0/f) \times (b/L_s)$ via the k dependence in the frontal shape function γ and the factor L_s^2/λ_1. The Born approximation is valid for $(U/c)(k'L_s) \ll 1$ which for the exponential model may be rewritten as

$$\omega/f \ll (\omega/f)_{\mathrm{crit}} = \frac{1 + (4\mu^2 + 1)^{1/2}}{2\mu}, \quad (4.21)$$

with $\mu = (N_0 b/f)^2 \lambda_m^{-1} \mathrm{Ro}/a^2$. As may be inferred from Fig. 3 showing $(\omega/f)_{\mathrm{crit}}$ as function of Ro/a^2 the approximation seems to be particularly good for fronts with a small transport UL_s in a background with strong, deep reaching stratification. Further, incident waves with low modes and low frequencies are favorably treated by the Born approximation. We shall

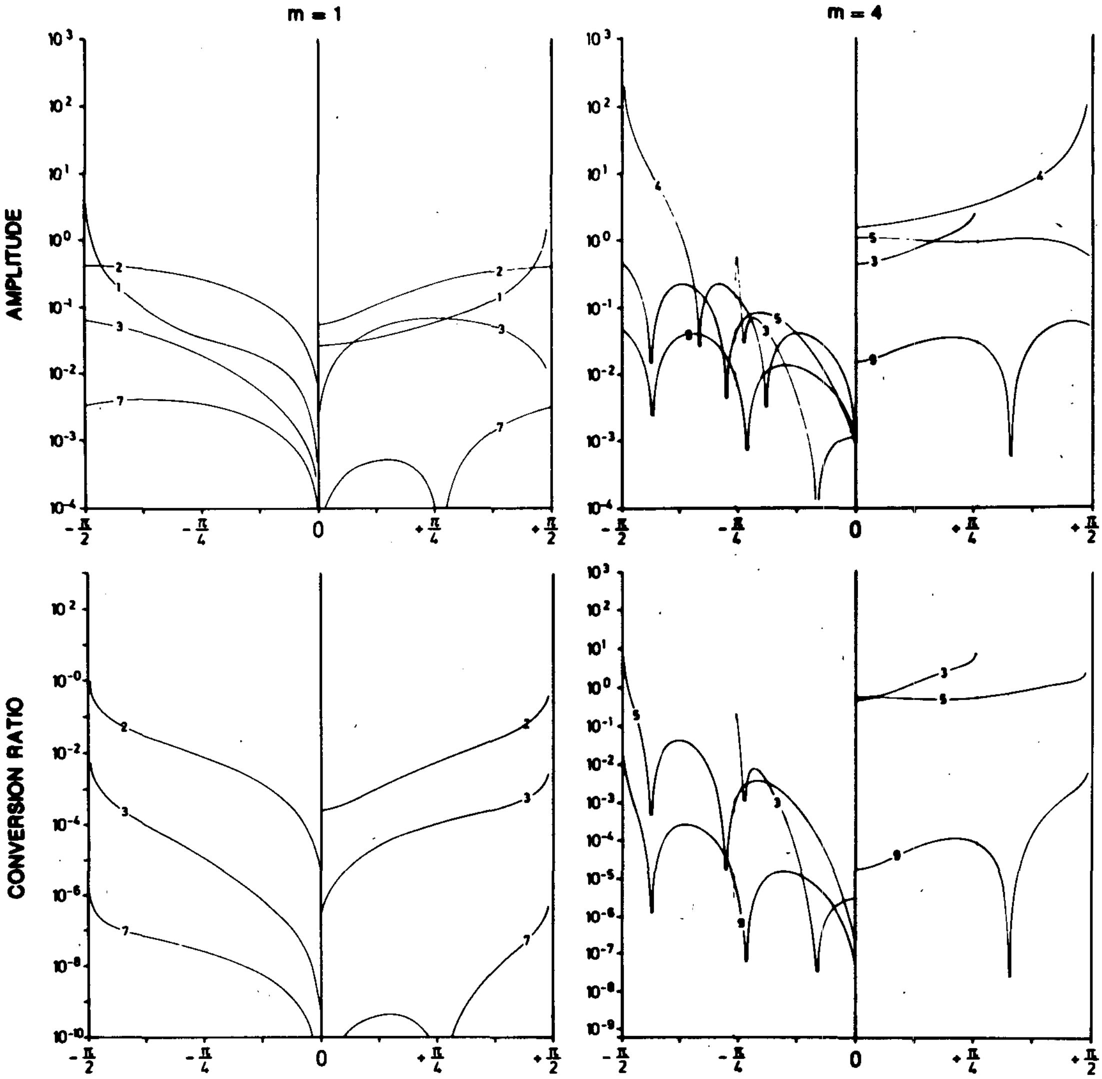

FIG. 6. As in Fig. 5 except for $a = 4.3$.

discuss the scattering amplitudes for the density step model and for two models with a finite frontal width. Table 1 summarizes the parameters of a large-scale, deep reaching front and for a smaller scale front confined to the upper ocean. Notice, however, that from the point of view of the shape parameters H/b and a there is no significant difference between the two cases: a value of $N_0/f = 10^2$ which is as well appropriate to small-scale fronts (say, with $N_0 = 10^{-2}\ \mathrm{s}^{-1}$ and $f = 10^{-4}\ \mathrm{s}^{-1}$) which yields an aspects ratio of $a = 1.0$.

The dependence of the response function $\rho_{lm}^{\pm}(\omega/f, \epsilon)$ on the output channel l and the angle of incidence ϵ is shown in Fig. 4. This displays the modulus of $\rho_{lm}^{\pm}$ which is even in ϵ, and the phase of $\rho_{lm}^{\pm}$ which is odd in ϵ. Not included is the barotropic response $\rho_{lm}^{\pm}$ which is nonzero only close to normal incidence $\epsilon = 0$ but still negligible there compared to the baroclinic channels as may be inferred from (A.7), (B.7) and (4.19). We mentioned above that $T_0\rho_{lm}^{\pm}$ represents the scattering amplitudes for a jet of infinitesimal width but finite transport UL_s. The amplitudes of the scattered field show no monotonic behavior in their magnitudes but the related conversion ratios shown in the lowest panels of Fig. 4 indicate a predominant conversion to mode numbers close to the incident mode number. Waves with slanting incidence generate larger scattered amplitudes than those with near-normal incidence. The diagonal conversion ratio $C_{mm}^{\pm}$ has not been included in the figure since it involves the second order of the approximation (3.51) which has not been calculated (cf. Appendix C).

The next set of figures displays the scattering amplitudes for a more realistic front of nonzero width. The buoyancy field used for the results of Figs. 5 and 6 is

$$B(x_1) = \Delta B \begin{cases} 0, & \text{for } x_1 \leqslant -L_s \\ \frac{1}{2}\left(1 + \sin \frac{\pi x_1}{2L_s}\right), & \text{for } |x_1| < L_s \\ 1, & \text{for } x_1 \geqslant L_s. \end{cases} \quad (4.22)$$

Figs. 5 and 6 show the scattering amplitudes and conversion ratios for aspect ratios $a = 0.7$ and $a = 4.3$. Phases are not presented since for the symmetric model (4.22) they are identical to those of the response function.

The most outstanding property of the scattering amplitudes is the dominance of the diagonal channel $l = m$. This is particularly obvious at low values of the aspect ratio (Fig. 5) where in the entire range of ϵ the transmitted diagonal channel is about an order of magnitude larger than the non-diagonal channels. There is a marked increase toward slanting incidence. The diagonal channel of the reflected field is dominant only at slanting angles of incidence. The off-diagonal amplitudes show a general decrease away from $l = m$ but their behavior is rather complex because of the frequent occurence of zeroes [for the specific front (4.22) the scattering amplitudes vanish if $k''L_s = (2n + 1)(\pi/2)$ for $n \geqslant 1$].

The behavior of the conversion ratios $l \neq m$ (Fig. 6) can be described similarly: strong decrease away from $l = m \pm 1$ and slight dominance of the transmitted field. All these gross features are relatively stable against modest changes of the ambient buoyancy field: a model with a linear increase in $B(x_1)$ (a top-hat jet) yielded similar results. Also, the stratification model with constant N produces a similar pattern of the scattered field. Here the near-diagonal channels $l = m \pm 1$ also get importance and, in some cases, dominate the diagonal channel.

The functions $\rho^{\mp}_{lm}$, $T^{\mp}_{lm}$ and $C^{\mp}_{lm}$ are displayed in Figs. 4, 5 and 6 for $\omega/f = 2$. The dependence on ω/f is only weak and will not be considered here.

c. *Summary and conclusions*

To summarize the structure of the scattered field we found that for a given incident mode 1) low modes are more aligned with the front than high modes; 2) the diagonal channel represents the dominant response, in particular at slanting incidence; 3) the response in the off-diagonal channels decreases away from the diagonal; and 4) the transmitted and reflected field is asymmetric with respect to the front. The most pronounced asymmetry occurs at near-normal incidence in the diagonal channel where transmitted amplitudes are large compared to reflected amplitudes which here are as large as off-diagonal amplitudes.

The first statement is a general result since it is a pure geometrical constraint of the resonance condition between incident and scattered waves. The other statements have been derived for the frontal shape (4.22) and the exponential ambient stability frequency. However, as studies with different models have shown that also these statements have a more general relevance.

The structure of the scattered field revealed in the last section suggests that scattering of internal waves at frontal zone may play an important role in shaping the wave spectrum in the ocean. While the redistribution in the modal domain appears to be weak there should be a stronger influence on the directional pattern of the wave field. Waves with slanting incidence on a front are enhanced compared to those with near-normal incidence. We therefore expect that the wave spectrum in the vicinity of a front shows an anisotropy with more energy along the frontal axis than normal to it.

APPENDIX A

Some Properties of the Vertical Eigenfunctions

This appendix deals with the rigid-lid approximation of the vertical eigenvalue problem, its solution for two models of the stratification, and the evaluation of the integral J_{lm} given by (4.13) for these models.

1. Expansion of the eigenvalue problem

Scaling the vertical coordinate z by the ocean depth H the eigenvalue problem (2.30) becomes

$$\left.\begin{aligned} \left(\frac{f^2}{N^2H^2}\varphi_l'\right)' + \frac{1}{\lambda_l}\varphi_l = 0 \\ \varphi_l' + \frac{N^2H}{g}\varphi_l = 0 \quad \text{at} \quad z = 0 \\ \varphi_l' = 0 \quad \text{at} \quad z = -H \end{aligned}\right\}, \quad \text{(A1)}$$

where the dash now denotes the derivative with respect to z/H. The parameter

$$\delta = N_0^2H/g \quad \text{(A2)}$$

formed with a reference value N_0 [e.g., the surface value of $N(z)$] generally is very small for oceanic conditions. It may be used to yield a solution of (A1) in form of a power series

$$\left.\begin{aligned} \varphi_l = \varphi_l^{(0)} + \delta\varphi_l^{(1)} + \cdots \\ \frac{1}{\lambda_l} = \frac{1}{\lambda_l^{(0)}} + \delta\frac{1}{\lambda_l^{(1)}} + \cdots \end{aligned}\right\}, \quad \text{(A3)}$$

where the zero order function $\varphi_l^{(0)}$ obeys the simpler rigid-lid condition

$$(\varphi_l^{(0)})' = 0 \quad \text{at} \quad z = 0 \quad \text{(A4)}$$

at the surface. One obvious solution of the zero-order equations is

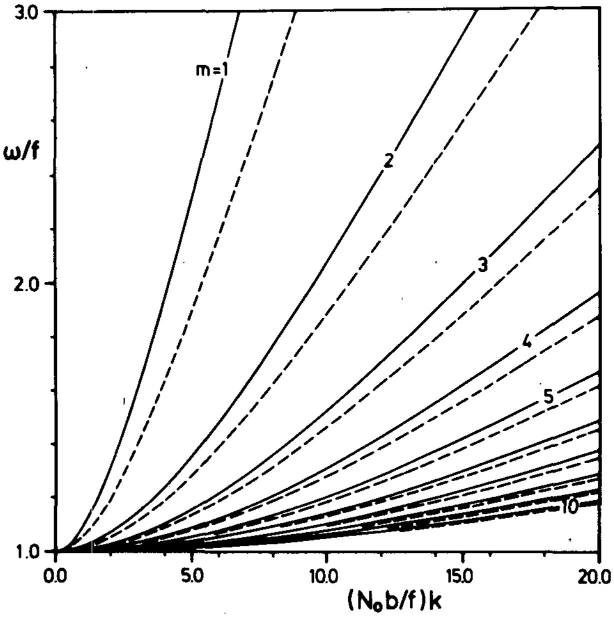

FIG. 7. Dispersion relation for the constant (dashed line) and exponential (full line) stability frequency ($b = H$ for the case N = constant).

$$\left.\begin{aligned} \varphi_0^{(0)} &= \text{constant} \\ \frac{1}{\lambda_0^{(0)}} &= 0 \end{aligned}\right\} . \quad \text{(A5)}$$

This defines the barotropic mode $l = 0$ which does not depend on the specific form of the stratification. Whereas the O(δ) corrections are small for the baroclinic modes, $l \neq 0$ this is not the case for the barotropic mode since $\lambda_0^{-1} = O(\delta)$ as indicated by (A5). The O(δ) term follows from the first-order barotropic equation

$$\left.\begin{aligned} \left[\frac{f^2}{N^2H^2}(\varphi_0^{(1)})'\right]' &= -\frac{1}{\lambda_0^{(1)}}\varphi_0^{(0)} \\ (\varphi_0^{(1)})' &= -\varphi_0^{(0)} \quad \text{at} \quad z = 0 \\ (\varphi_0^{(1)})' &= 0 \quad \text{at} \quad z = -H \end{aligned}\right\} , \quad \text{(A6)}$$

which may be integrated to yield

$$\delta\frac{1}{\lambda_0^{(1)}} = \frac{f^2}{gH} . \quad \text{(A7)}$$

The determination of the O(δ) corrections of the eigenfunction becomes unique by requiring that the lowest order be normalized according to (2.31). We will entirely neglect the higher orders and use the rigid-lid approximation.

2. N = constant model

For $N = N_0$ = constant, the eigenvalue problem is solved by

$$\left.\begin{aligned} \varphi_0 &= (1/H)^{1/2}, & 1/\lambda_0 &= 0 \\ \varphi_l &= (2/H)^{1/2}\cos(l\pi z/H), & 1/\lambda_l &= \left(\frac{fl\pi}{N_0H}\right)^2 \end{aligned}\right\} \quad \text{(A8)}$$

where $l = 1, 2, \ldots$.

3. Exponential N

For

$$N(z) = N_0e^{z/b} \quad \text{(A9)}$$

with $b > 0$, we find a solution for the baroclinic modes in the terms of Bessel functions

$$\mathrm{J}_n(\zeta) = J_n(\zeta)Y_0(\alpha_l) - Y_n(\zeta)J_0(\alpha_l), \quad \text{(A10)}$$

with

$$\alpha_l = \frac{N_0b}{f\lambda_l^{1/2}} . \quad \text{(A11)}$$

The eigenfunctions are

$$\varphi_l(z) = D_le^{z/b}\mathrm{J}_1(\alpha_le^{z/b}), \quad \text{(A12)}$$

with the normalization constant

$$D_l = (2/b)^{1/2}[\mathrm{J}_1^2(\alpha_l) - \mathrm{J}_1^2(\alpha_le^{-H/b})]^{-1/2} \quad \text{(A13)}$$

and the α_l follows from the dispersion relation

$$\mathrm{J}_0(\alpha_le^{-H/b}) = 0. \quad \text{(A14)}$$

If $e^{-H/b} \ll 1$, these expressions may be approximated by

$$\left.\begin{aligned} \varphi_l(z) &= D_l'e^{z/b}J_1(\alpha_le^{z/b}) \\ J_0(\alpha_l) &= 0 \\ D_l' &= (2/b)^{1/2}|J_1(\alpha_l)|^{-1} \end{aligned}\right\} . \quad \text{(A15)}$$

The dispersion relations for the two models of stratification are displayed in Fig. 7.

4. The integral J_{lm}

For the N = constant case the integral J_{lm} defined by (4.13) can analytically be evaluated which yields

$$\left.\begin{aligned} J_{ll} &= -1/2, \\ J_{0m} &= 2^{1/2}[1 - (-1)^m]m^{-2}, \quad m \neq 0, \\ J_{lm} &= 2[1 - (-1)^{l+m}] \\ &\times \frac{l^2 + m^2}{(l - m)^2(l + m)^2} , \quad l \neq m \end{aligned}\right\} . \quad \text{(A16)}$$

Obviously, $|J_{lm}|$ drops rapidly away from the diagonal (see Table 1). For large $l + m$ = constant one finds $J_{lm} = 0[2/(l - m)^2]$. Thus, only the first few off-diagonals show essential coupling between l and m. The strongest coupling occurs between the modes m and $l = m \pm 1$.

In the approximation (A15) the integrals for the exponential model are

$$\left.\begin{aligned} J_{0m} &= 2^{1/2}(b/H)^{3/2}\int_{-H/b}^{0} dx e^{x} x \\ &\quad\times\left[\frac{J_1(\alpha_m e^x)}{|J_1(\alpha_m)|}\right], \qquad m \neq 0 \\ J_{lm} &= 2(b/H)\int_{-H/b}^{0} dx e^{2x} x \\ &\quad\times\left[\frac{J_1(\alpha_l e^x) J_1(\alpha_m e^x)}{|J_1(\alpha_l) J_1(\alpha_m)|}\right], \quad l, m \neq 0 \end{aligned}\right\}. \tag{A17}$$

These have been evaluated numerically for different values of H/b. Again, $|J_{lm}|$ falls off rapidly away from the diagonal with strongest coupling along the diagonal. The pattern changes only slightly by varying H/b.

APPENDIX B

The Matrix Elements of $\mathbf{V}_{ab}$ for a Linear Shear Flow

The derivation of

$$V^{\lambda\mu} = \int dy \tilde{\phi}_a^{\lambda} V_{ab} \phi_b^{\mu} \tag{B1}$$

is an elementary task. It may obviously be expressed in terms of the Fourier transform of dB/dx_1, defined by

$$\Gamma(k_1'') = \left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{+\infty} dx_1 \frac{dB}{dx_1}\exp(-ik_1''x_1) \tag{B2}$$

and some integrals involving the vertical eigenfunctions $\varphi_l(z)$ and $\varphi_m(z)$. These are

$$\left.\begin{aligned} A_{lm} &= \int_{-H}^{0} dz \varphi_l \varphi_m z \\ B_{lm} &= \int_{H}^{0} dz \varphi_l \int_{-H}^{z} dz' \varphi_m \\ C_{lm} &= \int_{-H}^{0} dz \varphi_l \int_{z}^{0} dz' \varphi_m \end{aligned}\right\}. \tag{B3}$$

The lengthly expressions for $V^{\lambda\mu}$ may extremely be reduced by using the vertical eigenvalue problem (2.30) to relate B_{lm} and C_{lm} to the symmetric matrix A_{lm}. These relations are

$$\left.\begin{aligned} B_{lm} &= \frac{\lambda_l - \lambda_m}{\lambda_l + \lambda_m} A_{lm} \\ C_{lm} &= -B_{lm} = \frac{\lambda_m - \lambda_l}{\lambda_l - \lambda_m} A_{lm} \end{aligned}\right\}. \tag{B4}$$

These relations hold for baroclinic modes l, m in the rigid-lid approximation. The case $l = 0$, $m \neq 0$ is obtained by letting λ_l tend to ∞ in the expression (B4). With (B4) the matrix elements (B1) take the form

$$V^{\lambda\mu} = \Gamma(k_1 - k_1')\frac{k' A_{lm}}{f} E^{\lambda\mu}, \tag{B5}$$

where E is a simple dimensionless algebraic function of $k, k', \lambda_l, \lambda_m, \omega^{\lambda}$ and ω^{μ}. The first order of the Born approximation only uses $E^{\lambda\mu}$ with $\lambda = (+, l, \pm\kappa_{lm})$ and $\mu = (+, m, k_1')$ whence $\omega^{\lambda} = \omega^{\mu} = \omega$. This resonant coefficient is given by

$$(E^{\lambda\mu})_{\text{res}} = E^{\pm}\left(\nu, \frac{\omega}{f}, \epsilon\right) = R^{\pm} + iJ^{\pm}, \tag{B6}$$

with

$$R^{\pm} = \sin\epsilon\left\{\sin^2\epsilon + \frac{1}{1+\nu^2}\left[1 - \nu^4 + \nu^2\frac{f^2}{\omega^2}(1 - \tfrac{3}{2}\nu^2)\right] \pm \cos\epsilon\left|\nu^2 - \sin^2\epsilon\right|^{1/2}\left(1 + \frac{f^2}{\omega^2}\right)\right\}. \tag{B7}$$

$$J^{\pm} = -f\omega^{-1}\cos\epsilon\left[\frac{\nu^2}{1+\nu^2} + \sin^2\epsilon \pm \left|\frac{\nu^2 - \sin^2\epsilon}{\cos^2\epsilon}\right|^{1/2} \times\left(\frac{1}{1+\nu^2} - 2\sin^2\epsilon\right)\right], \tag{B8}$$

where ϵ is the angle of incidence and

$$\nu = (\lambda_m/\lambda_l)^{1/2}.$$

The upper sign applies to transmitted waves, the lower sign to reflected waves. Further, the wavenumber $k_1'' = k_1 - k_1'$ picked out of the front by this resonance may be written in terms of k' and ϵ as

$$\begin{aligned} k_1'' = k_1 - k_1' &= \pm\kappa_{lm} - k_1' \\ &= k'[\pm|\nu^2 - \sin^2\epsilon|^{1/2} - \cos\epsilon]. \end{aligned} \tag{B9}$$

Notice the symmetry relation of the interaction coefficients

$$\left.\begin{aligned} R^{\pm}(-\epsilon) &= -R^{\pm}(\epsilon) \\ J^{\pm}(-\epsilon) &= J^{\pm}(\epsilon) \\ E^{\pm}(-\epsilon) &= -[E^{\pm}(\epsilon)]^* \end{aligned}\right\}. \tag{B10}$$

For normal incidence, $\epsilon = 0$, Eq. (B7) yields $R^{\pm} = 0$ and $E^{\pm}$ reduces to the simple expression

$$E^{\pm} = -i\frac{f}{\omega}\left[\frac{\nu(\nu \pm 1)}{\nu^2 + 1}\right]. \tag{B11}$$

APPENDIX C

Some Comments on the Energetics

The scattering theory of internal waves presented in Section 3 has been developed along the potential scattering theory in quantum mechanics. Turning to the energetics of the problem, the formal analogy with potential scattering cannot be carried any further since the flow described by the equations of motion (2.1) in general does not conserve energy. The energy equation

$$\partial_t\left[\int_{-H}^{0} dz(\tfrac{1}{2}u_\alpha u_\alpha + \tfrac{1}{2}N^{-2}b^2) + \tfrac{1}{2}g\zeta^2\right] + \partial_\alpha\left[\int_{-H}^{0} dz u_\alpha \pi + \pi h(U_\alpha + u_\alpha)\big|_{-H}\right] = \int_{-H}^{0} dz(u_j S_j + N^{-2}bS_4) + (\partial_t\zeta)S_6 + g\zeta S_5 \quad \text{(C1)}$$

derived from (2.1) shows that energy can be exchanged with the ambient fields which form the scattering region. Thus, though the interaction operator V has been regarded as given, it does not play the role of a potential. If not maintained by some external mechanisms V will change through the coupling with the incident wave—generally on a much slower time scale than those of internal waves.

But even if we disregard the energy exchange with the ambient fields—it may be small or incidentally vanish—we may not expect conservation of wave energy in the scattering process since in addition to the outward-radiating waves the incident wave also generates a time-dependent geostrophic perturbation ψ^0 of the flow as given by (2.22). The $p = 0$ component of this equation

$$i\partial_t\psi^0 - V^{00}\psi^0 = \sum_{r=+,-} V^{0r}\psi^r \quad \text{(C2)}$$

shows that ψ^0 remains in its initial state $\psi^0 = 0$, where V vanishes, i.e., outside the scattering region. But inside this region a nonzero ψ^0 is forced by the wave branches $\psi^\pm$. Since these will have propagated outward after a sufficiently large time Δ the geostrophic response trapped in the scattering region is of the form

$$\psi^0(t) = -i\exp(-iV^{00}t)\int_0^\Delta dt' \times \exp(iV^{00}t')\sum_{r=+,-} V^{0r}\psi^r(t'). \quad \text{(C3)}$$

To establish the energetics of the scattering process one may try to pursue the more practical way of disregarding the complex process in the scattering region as far as possible and concentrate on the relation of the energy fluxes in the incident wave and the scattered far-field. The difference between incident and outgoing energy must have been exchanged with the ambient fields or stored in the trapped geostrophic flow in the scattering region. This concept for obtaining the energetics of the far-field is presented below in the stationary formulation of the scattering process for the simpler case of the one-dimensional scattering problem where incident *and* scattered waves are plane waves.

Using (2.4), (2.13) and (2.21) one finds that the energy flux vector (the horizontal average over some wavelengths is denoted by angle brackets)

$$F_\alpha = \left\langle \int_{-H}^{0} dz u_\alpha \pi \right\rangle \quad \text{(C4)}$$

of a field of plane waves

$$\psi^+(y, t) = \sum_l \int d^2k A_l(\mathbf{k})\phi_{l\mathbf{k}}(y)\exp(-i\omega_{l\mathbf{k}}t) \quad \text{(C5)}$$

has the form

$$F_\alpha = \sum_l \int d^2k F_{\alpha l}(\mathbf{k}) = \sum_l \int d^2k |A_l(\mathbf{k})|^2 2\lambda_l\omega_{l\mathbf{k}}k^{-2}k_\alpha. \quad \text{(C6)}$$

The total energy of (C5) is

$$E = \left\langle \int_{-H}^{0} dz(\tfrac{1}{2}u_\alpha u_\alpha + \tfrac{1}{2}N^{-2}b^2) + \tfrac{1}{2}g\zeta^2 \right\rangle = \sum_l \int d^2k E_l(\mathbf{k}) = \sum_l \int d^2k |A_l(\mathbf{k})|^2 \frac{2\omega_{l\mathbf{k}}^2}{fk^2}, \quad \text{(C7)}$$

so that, as generally in wave problems, the flux vector $F_{\alpha l}(\mathbf{k})$ of each individual wave component is the product of its group velocity and its energy, i.e.,

$$F_{\alpha l}(\mathbf{k}) = \nu_{\alpha l}(\mathbf{k})E_l(\mathbf{k}). \quad \text{(C8)}$$

With (C6) the energy flux per unit length of the front of the incident wave component $(m, \mathbf{k}')$ is given by

$$F_{\text{in}} = 2\frac{\lambda_m\omega_{m\mathbf{k}}k_1'}{(k')^2}|A_m(\mathbf{k}')|^2 \quad \text{(C9)}$$

with $A_m(\mathbf{k}') = a_m(\mathbf{k}' - \mathbf{k}_0)\exp(-i\mathbf{k}'\cdot\mathbf{x}_0)$. The outgoing flux per unit length of the front may be computed from the asymptotic form (3.55). Denoting by $F_{\text{out}}^\pm$ the flux to $\pm\infty$ we find

$$F_{\text{out}}^\pm = \pm F_{\text{in}}\sum_l C_{lm}^\pm \quad \text{(C10)}$$

with a conversion ratio $C_{lm}^\pm$ representing the ratio of energy fluxes in the outgoing mode l and the incident mode m. It is given by

$$C_{lm}^\pm = \left(\frac{\lambda_l}{\lambda_m}\right)^2\left[\frac{\lambda_m/\lambda_l - \sin^2\epsilon}{\cos^2\epsilon}\right]^{1/2} \times \begin{cases} |\delta_{lm} + B_{lm\mathbf{k}'}^+|^2 \\ |A_{lm\mathbf{k}'}^+|^2. \end{cases} \quad \text{(C11)}$$

Then the total outward flux may be written

$$F_{\rm out} = F^{+}_{\rm out} - F^{-}_{\rm out} = F_{\rm in} \sum_{l} (C^{+}_{lm} + C^{-}_{lm}). \quad \text{(C12)}$$

In most applications the scattering amplitudes cannot be determined exactly and approximation methods must be invoked. Whereas for the description of the scattered field the lowest order approximation might be sufficient, the energetics of the scattering process generally require the calculation of higher order approximations. Considering the Born approximation (3.51) the conversion ratios $C^{\pm}_{lm}$ become

$$\left.\begin{aligned} C^{-}_{lm} &= \alpha_{lm}\left|\gamma_{lm}V^{\sigma\mu}\right|^2 + \cdots \\ C^{+}_{lm} &= \alpha_{lm}\Big|\delta_{lm} + \gamma_{lm}V^{\sigma\mu} \\ &\quad + \gamma_{lm}\sum_{\nu}\frac{V^{\sigma\nu}V^{\nu\mu}}{\omega^{\mu} - \omega^{\nu} + ri0} + \cdots\Big|^2 \end{aligned}\right\}, \quad \text{(C13)}$$

with

$$\left.\begin{aligned} \alpha_{lm} &= \left(\frac{\lambda_l}{\lambda_m}\right)^2\left(\frac{\lambda_m/\lambda_l - \sin^2\epsilon}{\cos^2\epsilon}\right)^{1/2} \\ \gamma_{lm} &= -i\,\frac{2\pi\omega_{m\mathbf{k}'}}{\kappa_{lm}f^2} \end{aligned}\right\}. \quad \text{(C14)}$$

Eq. (C13) gives C^{-}_{lm} correct to the second in the interaction operator and involves only the first-order Born approximation. However, to evaluate the conversion ratio C^{+}_{lm} describing the transmitted waves correct to the second order, the second-order Born approximation is needed since from (C13) we find

$$C^{+}_{lm} = \alpha_{lm}\Big[\delta_{lm} + \delta_{lm}2\,\mathrm{Re}(\gamma_{lm}V^{\sigma\mu}) + \left|\gamma_{lm}V^{\sigma\mu}\right|^2 + \delta_{lm}2\,\mathrm{Re}\Big(\gamma_{lm}\sum_{\nu}\frac{V^{\sigma\nu}V^{\nu\mu}}{\omega^{\mu} - \omega^{\nu} + ri0}\Big) + \cdots\Big]. \quad \text{(C15)}$$

As a general property of resonant interaction coefficients, the second term in the square brackets vanishes. Thus, to determine the energetics to the lowest nontrivial order one has to determine the scattering amplitude $B^{+}_{mm\mathbf{k}'}$ for the transmitted wave in the incident made channel m to the second order in V.

APPENDIX D

List of Symbols

We use the sum convention for all indices.

$\alpha, \beta, \gamma, \cdots$	horizontal Cartesian components of vector variables [=1, 2]
$i, j, \cdots$	Cartesian components of vector variables [=1, 2, 3]
$a, b, c, \cdots$	components of state variables [=1, 2, 3]
$p, q, r, \cdots$	normal branch representation of state variables [=+, −, 0]
λ	$(p, l, \mathbf{k})$ normal mode indices
μ	$(q, m, \mathbf{k}')$ normal mode indices
ν	$(r, n, \mathbf{k}'')$ normal mode indices
$l, m, n, \cdots$	index for vertical eigenfunctions [=0, 1, 2, ···]
$\mathbf{k}$	horizontal wave vector (k_1, k_2)
$\mathbf{x}$	horizontal position vector (x_1, x_2)
z	vertical coordinate x_3
y	$(\mathbf{x}, z)$
s	(x_1, z)

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