The Propagation of Internal Waves in a Geostrophic Current

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(Manuscript received 18 December 1980, in final form 2 June 1981)

ABSTRACT

The paper presents the WKB theory of internal wave propagation in a large-scale geostrophic mean flow with vertical as well as horizontal shears. As an application a mean flow with isopycnals having constant slope but arbitrary spacing is considered and the behavior of waves at turning points and critical layers is discussed. In particular, it is shown that horizontal variations of the mean flow shift the critical layer to the interior of the wave guide, i.e., away from $\omega_0^2 = f^2$, where ω_0 is the intrinsic frequency, and produces a valve effect at the critical layer which can be penetrated by a wave incident from one side while incidence from the other side results in absorption.

1. Introduction

The kinematical structure of internal waves is determined by the mean buoyancy and current fields and the bottom topography. Only variations of these mean oceanic fields have an essential influence: the reflection properties of a flat horizontal bottom are fairly simple, a constant current merely implies a Doppler shift, and the waves feel the buoyancy field only through its gradients. Horizontal gradients of buoyancy and currents generally are small compared to the vertical gradients and therefore are neglected in the traditional kinematical models based on a horizontally homogeneous buoyancy frequency and shear current. However, observations have shown quite clearly that significant horizontal variations of these fields locally occur in the ocean (Woods, 1982). It may be expected that such local inhomogeneities of the wave-supporting background may have a profound influence on the structure of the wave field.

The effect of vertical and horizontal inhomogeneities on the oceanic internal wave field has been investigated only for very special cases. Ivanov and Morozov (1974) considered the propagation of a wave through a barotropic jet with no horizontal variations in the density field and constant buoyancy frequency. Samodurov (1974), Miropol'skiy (1974) and Miropol'skiy *et al.* (1976) studied the propagation effects in a density front with no jet and a vertically constant but horizontally varying buoyancy frequency. These authors found critical layer properties of the wave at the point on the ray where the wave frequency equals the local buoyancy frequency. There are other authors (e.g., Jones, 1969; Mooers, 1975a) investigating the propagation in a medium with constant horizontal and vertical gradients of the shear flow or considering merely normal incidence onto the jet (Magaard, 1969; Healey and LeBlond, 1969; Mooers, 1975b). The structure of valve-like critical layers in a non-rotating flow with vertical and horizontal shears has been demonstrated by Acheson (1973).

If the spectral scales of the inhomogeneities are large compared to the wavelength, the interaction of the waves with the mean flow may be treated within the framework of geometric optics (the WKB theory). This theory is reviewed in many textbooks in a general context (see, e.g., Lighthill, 1978; LeBlond and Mysak, 1978). In this paper we discuss the propagation of internal waves in a general baroclinic density field with an associated geostrophic current. In Section 2 the WKB theory for internal waves in a slowly varying mean flow is developed. A brief review of existing solutions of the ray equations is presented in Section 3. In Section 4 we present a general solution for a geostrophic flow with constantly sloping isopycnals. This model represents a generalization of the cases studied so far. The structure of the wave guide, turning and critical layers of the rays, and the behavior of energy are determined.

2. Evolution of wave parameters in the WKB approximation

Linear wave motion in a slowly varying mean flow is governed by the equations of motion ($i = 1, 2, 3$)

$$\left.\begin{aligned} (\partial_t + U_j\partial_j)u_i - \epsilon_{ij}fu_j \\ - \delta_{i3}b + \partial_i p - u_j\partial_j U_i = 0 \\ (\partial_t + U_j\partial_j)b + u_j\partial_j B = 0 \\ \partial_j u_j = 0 \end{aligned}\right\}, \quad (2.1)$$

0022-3670/81/091224-10$06.50

where U_i is the mean current and B the mean buoyancy field. Evidently the mean flow affects the waves via the local current U_i and its gradients $\partial_j U_i$ while the buoyancy field enters only by its gradient $\partial_j B$. The waves do not "feel" the local buoyancy or density but only the local buoyancy or density gradient. This is essential in the WKB scaling discussed below. The wave problem posed by Eqs. (2.1) has been treated among others by Mooers (1975a) and Müller (1976) by WKB methods. We can follow Müller's analysis except for some minor changes. Mooers' treatment, however, is inconsistent with a WKB solution. He retains all terms in (2.1) (except for special geometry of the mean flow), i.e., in particular the current and its gradients, for deriving the dispersion relation while in the propagation and refraction equations of wave groups second-order gradients are neglected. This is in strong contrast to a WKB concept where in the dispersion relation only the local fields are retained and gradients enter the propagation and refraction equations. Waves feel the spatial variability of the mean flow by propagation and refraction and not locally, i.e., in the dispersion relation.

The WKB approximation is a solution by an asymptotic expansion in a small parameter ϵ. In most applications only the two lowest orders of the expansion are considered. In a wave problem the lowest order yields the local wave structure, i.e., the representation of the wave state and the dispersion relation, and the second order determines the evolution of the parameters describing the local state, i.e., the amplitude a, the frequency ω and the wave vector $\mathbf{k}$. Let the mean flow (U_i, B) and the wave state $(a, \omega, \mathbf{k})$ vary on a scale of order unity, but let the wavelength and period be $O(\epsilon)$, $\epsilon \ll 1$. If we required that Eqs. (2.1) describe internal wave motion (including rotation and density stratification) we must assume the scaling $f = O(\epsilon^{-1})$, $b = O(\epsilon^{-1}u)$ and $\partial_3 B = O(\epsilon^{-2})$. Hence the following WKB expansion is appropriate:

$$(u_i, b, p) = \sum_{n=0}^{\infty} \epsilon^n a(u_i^{(n)}, \quad \epsilon^{-1}b^{(n)}, \quad p^{(n)})e^{i\theta/\epsilon} + \text{c.c.}, \tag{2.2}$$

where the amplitude a, the amplitude factors $u_i^{(n)}$, $b^{(n)}$ and $p^{(n)}$ and the phase function θ vary on the scale unity. Gradients of u_i, b and p are thus dominated by the local frequency and wave vector, i.e.,

$$\left.\begin{aligned} \omega &= -\partial_t(\theta/\epsilon) \\ k_i &= \partial_i(\theta/\epsilon) \end{aligned}\right\}. \tag{2.3}$$

If we require that the lowest order yielding the local wave state includes modification by the mean current we must assume $U_i = O(1)$. Since this also yields $\partial_j U_i = O(1)$ the gradients of the current drop out of the lowest order and thus do not affect the local wave state. We still have the freedom to fix the scale of the lateral buoyancy gradients $\partial_\alpha B(\alpha = 1, 2)$. They would enter the lowest order if $\partial_\alpha B = O(\epsilon^{-2})$ as the vertical gradient $\partial_3 B$. However, in case of a geostrophically balanced mean flow the thermal wind relation

$$f\epsilon_{\alpha\beta}\partial_3 U_\beta = \partial_\alpha B \tag{2.4}$$

imposes the restriction $\partial_\alpha B = O(\epsilon^{-1})$. Note that this scaling implies a large Richardson number $\mathrm{Ri} = N^2/(\partial_3 U_\alpha)^2 = \epsilon^2 \gg 1$. The scaling slightly differs from Müller's scaling who only retains the vertical gradient of the equilibrium stratification in the lowest order and not the one associated with the mean flow. We assume that the vertical mean current U_3 vanishes so that the horizontal current is nondivergent, i.e.,

$$\partial_\alpha U_\alpha = 0. \tag{2.5}$$

The local dispersion relation obtained by this scaling procedure is given by

$$\omega = \Omega(\mathbf{k}, \mathbf{x}) = \omega_0 + k_\alpha U_\alpha, \tag{2.6}$$

with

$$\omega_0 = \Omega_0(\mathbf{k}, \mathbf{x}) = \pm k^{-1}[N^2(\mathbf{x})\alpha^2 + f^2\beta^2]^{1/2} \tag{2.7a}$$

or equivalently,

$$(\beta/\alpha)^2 = \frac{N^2(\mathbf{x}) - \omega_0^2}{\omega_0^2 - f^2}, \tag{2.7b}$$

where $\alpha^2 = k_\alpha k_\alpha$, $\beta^2 = k_3^2$ and $k^2 = \alpha^2 + \beta^2$.

Further

$$N(\mathbf{x}) = (\partial_3 B)^{1/2} \tag{2.8}$$

is the buoyancy frequency associated with the mean buoyancy field. Eq. (2.6) relates the local frequency ω to the local wave vector $\mathbf{k}$. The intrinsic frequency ω_0 given by (2.7) is the frequency measured by an observer moving with the mean flow (not in the absence of the mean flow). The local amplitude factors are given by

$$\begin{Bmatrix} u_1^{(0)} \\ u_2^{(0)} \\ u_3^{(0)} \\ b^{(0)} \\ p^{(0)} \end{Bmatrix} = C \begin{Bmatrix} \omega_0 k_1 + ifk_2 \\ \omega_0 k_2 - ifk_1 \\ -\dfrac{\omega_0^2 - f^2}{N^2 - \omega_0^2}\omega_0 k_3 \\ iN^2\dfrac{\omega_0^2 - f^2}{N^2 - \omega_0^2}k_3 \\ \omega_0^2 - f^2 \end{Bmatrix} \tag{2.9}$$

and the total wave energy

$$E = \tfrac{1}{2}(u_i u_i + N^{-2}b^2) = 2aa^*|C\omega_0\alpha k\beta^{-1}|^2, \tag{2.10}$$

where C is an arbitrary normalization factor.

Global changes of the wave parameters are governed by the propagations equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i} = v_i + U_i \tag{2.11}$$

determining the ray along which the wave travels, and the refraction equations

$$\frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i} = r_i - k_j \frac{\partial U_j}{\partial x_i}. \tag{2.12}$$

Here v_i is the intrinsic group velocity

$$v_i = \frac{\partial \Omega_0}{\partial k_i} = \begin{cases} \dfrac{N^2 - \omega_0^2}{\omega_0 k^2} k_i, & i = 1, 2 \\ -\dfrac{\omega_0^2 - f^2}{\omega_0 k^2} k_3, & i = 3, \end{cases} \tag{2.13}$$

and r_i is the intrinsic rate of refraction

$$r_i = -\frac{\partial \Omega_0}{\partial x_i} = -\frac{N}{\omega_0}\left(\frac{\omega_0^2 - f^2}{N^2 - f^2}\right)\frac{\partial N}{\partial x_i}. \tag{2.14}$$

Furthermore, changes of frequency are given by

$$\frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}, \tag{2.15}$$

which in our case states the constancy of frequency of encounter $\omega = \omega_0 + k_\alpha U_\alpha$ along the ray.

The ray equations (2.11) and (2.12) show that the current directly affects the wave propagation in two ways: the rays are advected by the local current U_i while the wave vector is distorted by the local gradients of U_i. Well-known properties of the wave-group propagation are that the intrinsic stretching or shrinking of the wave vector always occurs along the gradient of the buoyancy frequency, and that phase and group propagation of a wave are orthogonal ($\mathbf{v} \cdot \mathbf{k} = 0$). Also, the local particle motion takes place in the plane orthogonal to $\mathbf{k}$ as a consequence of incompressibility.

Changes of the (complex) amplitudes along the ray follows from the first order of the WKB approximation. If the evolution of the phase of the amplitude is not of interest one may conveniently use the conservation of wave action (Whitham, 1965; Bretherton, 1968) expressed by

$$\frac{\partial}{\partial t}(E/\omega_0) + \frac{\partial}{\partial x_i}[(v_i + U_i)(E/\omega_0)] = 0, \tag{2.16}$$

which has been derived from the Eulerian equations of motion (2.1) by Grimshaw (1975) and Müller (1976). Eq. (2.16) states that the wave action

$$\int_{\nu(t)} d^3x (E/\omega_0)$$

contained in a volume $\nu(t)$ moving with the local group velocity dx_i/dt remains constant. If waves of a fixed frequency are considered so that the wave pattern at a fixed point in space is stationary, the conservation equation (2.16) takes the form

$$\frac{\partial}{\partial x_i}[(v_i + U_i)(E/\omega_0)] = 0, \tag{2.17}$$

expressing the fact that the wave action flux $(v_i + U_i)(E/\omega_0)$ varies along ray tubes in inverse portion to their cross-sectional area, i.e.,

$$(v_i + U_i) n_i \delta F (E/\omega_0) = \text{constant} \tag{2.18}$$

for an infinitesimal cross section δF with normal unit vector n_i.

Eqs. (2.11) and (2.12) represent a system of six ordinary differential equations which determine the rays and variation of the wave vector along the rays. They yield parametrical ray equations $x_i(t)$ and $k_i(t)$ if at an initial point $x_i^0 = x_i(0)$ the initial wave vector $k_i^0 = k_i(0)$ is specified. Solution of (2.17) or evaluation of (2.18) then requires knowledge about neighboring solutions to determine the partial derivatives $\partial v_i/\partial x_i$ or δF. Thus, though rays can be traced individually, tracing of energy involves solution in a larger domain.

Eq. (2.17) can be cast into a form which is more convenient for applications. Suppose $dx_1/dt \neq 0$ so that rays are not confined to planes perpendicular to the x_1-axis. Then x_1 may be used as independent variable instead of the time t. The ray passing the point x_i^0 ($i = 1, 2, 3$) is then given by

$$\left.\begin{aligned} x_2 &= x_2(x_1, x_i^0) \\ x_3 &= x_3(x_1, x_i^0) \end{aligned}\right\}. \tag{2.19}$$

A small element δF^0 in the plane $x_1 = x_1^0$ at x_2^0, x_3^0 is mapped by the transformation (2.19) into an element δF with area

$$\delta F = \delta F^0 \frac{\partial(x_2, x_3)}{\partial(x_2^0, x_3^0)}. \tag{2.20}$$

Thus applying (2.18)

$$[(v_1 + U_1)(E/\omega_0)]_{x_1} \frac{\partial(x_2 x_3)}{\partial(x_2^0 x_3^0)} = [(v_1 + U_1)(E/\omega_0)]_{x_1^0}, \tag{2.21}$$

from which the variation of energy along the ray (2.19) can be computed. This is simple if the Jacobian is unity.

In physical terms Eq. (2.21) as well as (2.18) state that the wave action flux through any cross section of a ray tube is constant. Following (2.21) the energy density E varies along rays if there are changes in the group velocity and intrinsic frequency or if the ray tubes are squeezed. Variation of wave energy is due to interaction with the mean flow by the wave-induced Reynolds stresses. For a nonrotating fluid the reader is referred to Lighthill's (1978) textbook. In case of rotation the problem is more entangled (see Müller, 1976).

3. A review of solutions of the ray equations

The ray equations (2.11) and (2.12) and the action conservation obviously are too complex to find a general solution. In the literature only the most trivial cases are treated, in particular wave propagation in a horizontally homogeneous ocean with zero or constant mean flow which is the basic state of a WKB internal wave field (see, e.g., Müller and Olbers, 1975). Refraction of internal waves by a horizontally constant flow with vertical shear has been studied by Phillips (1966), Müller (1977) and others.

In a horizontally homogeneous ocean with a constant horizontal mean current (the traditional kinematical model) integration of (2.11), (2.12) and (2.17) yields the constancy of the horizontal wave vector k_α and the intrinsic frequency ω_0, whereas the vertical wavenumber and energy change according to

$$\left.\begin{aligned} k_3(z) &\approx [N^2(z) - \omega_0^2]^{1/2} \\ E(z) &\approx 1/v_3 \approx \frac{N^2(z) - f^2}{[N^2(z) - \omega_0^2]^{1/2}} \end{aligned}\right\}, \quad (3.1)$$

where $\approx$ denotes proportionality.

Thus, when a wave group propagates toward a region of lower $N(z)$ the vertical wavenumber and the group velocity tend to zero. Close to the level z_T defined by $N(z_T) = \omega_0$ the vertical group velocity behaves as $v_3 \approx |z - z_T|^{1/2}$ so that the group still reaches the depth z_T (turning depth) in a finite time and internal reflection occurs. Near this depth the WKB solution becomes invalid and should be replaced by solutions in terms of Airy functions (Desaubies, 1973). These show that the energy $E(z)$ possesses a finite maximum at the turning depth rather than the weak singularity given by (3.1).

By allowing the current to have a vertical shear, i.e., $\mathbf{U} = \mathbf{U}(z)$, another important kinematical feature is introduced. Here we still have $(k_1, k_2) =$ constant, but the intrinsic frequency $\omega_0(z) = \omega - \mathbf{k}\mathbf{U}(z)$ now varies with depth and Eqs. (2.6), (2.7) and (2.17) yield

$$\left.\begin{aligned} k_3(z) &\approx \left[\frac{N^2(z) - \omega_0^2(z)}{\omega_0^2(z) - f^2}\right]^{1/2} \\ E(z) &\approx \omega_0/v_3 \\ &\approx \frac{\omega_0^2(z)[N^2(z) - f^2]}{(\omega_0^2(z) - f^2)^{3/2}[N^2(z) - \omega_0^2(z)]^{1/2}} \end{aligned}\right\}. \quad (3.2)$$

A wave group propagating toward increasing $\mathbf{k}\cdot\mathbf{U}(z)$ and thus decreasing ω_0 may encounter a level where ω_0 approaches f. Here k_3 and E tend to infinity, the wave group shrinks, its vertical shear increases, and the group velocity tends toward the horizontal. However, in contrast to the turning depth behavior, the group never reaches the level z_C, where $\omega_0(z_C) = f$ (Bretherton, 1966) since $v_3 \approx |z - z_C|^{3/2}$ close to $\omega_0 = f$. Dynamical considerations suggest that near this critical layer where the wave shear becomes very large there will be substantial dissipation and the wave will be absorbed by the mean flow (Booker and Bretherton, 1967).

Horizontally homogeneous models are of limited value, in particular, when discussing internal waves in the upper ocean where N and $\mathbf{U}$ are known to vary in the horizontal direction on a broad range of scales. In this paper, inhomogeneities of large scale are considered, which may be treated in the WKB approximation. These may have a profound influence on the structure of the internal wave field, since not only do they distort the rays, but they also may introduce enhancement of the wave energy similar to critical-layer trapping even with zero mean current (Samodurov, 1974; Miropol'skiy, 1974; Miropol'skiy *et al.*, 1976). For illustration we consider an internal wave impinging on a frontal zone with horizontally varying but vertically constant $N(x_1)$ associated with a geostrophic jet (Fig. 1). For normal incidence where ω_0 remains constant the ray equations (2.11), (2.12) and (2.17) integrate to yield

$$\left.\begin{aligned} k_1(x_1) &\approx [N^2(x_1) - \omega_0^2]^{-1/2} \\ E(x_1) &\approx 1/v_1 \approx \frac{N^2(x_1) - f^2}{[N^2(x_1) - \omega_0^2]^{3/2}} \end{aligned}\right\}. \quad (3.3)$$

A schematic display of the ray paths and the energy change along the rays is given in Fig. 1. Waves proceeding toward decreasing $N(x_1)$ with a frequency ω_0 larger than the minimum buoyancy frequency (a case which leads to reflection in the corresponding vertical propagation problem) will be refracted into the vertical plane $x_1 = x_{C'}$ defined by $N(x_1) = \omega_0$. Approaching this plane the total intrinsic group velocity $\mathbf{v}$ tends to zero and the wave pattern becomes increasingly frozen in the mean current. Since $v_1 \approx |x_1 - x_{C'}|^{3/2}$ the wave does not reach the plane $x_1 = x_{C'}$. Energy and also the shears $k_i u_j$ become infinite and absorption of the wave by the frontal jet will occur. Thus, in contrast to vertical variations of the buoyancy frequency in a horizontally homogeneous ocean, which act as a spatial filter to the wave field prohibiting penetration beyond the level $N = \omega_0$, horizontal variations of N may additionally have a filtering effect in frequency space because of absorption occurring at the surface $\omega_0 = N(x_1)$. Evidently, there is some similarity with the critical

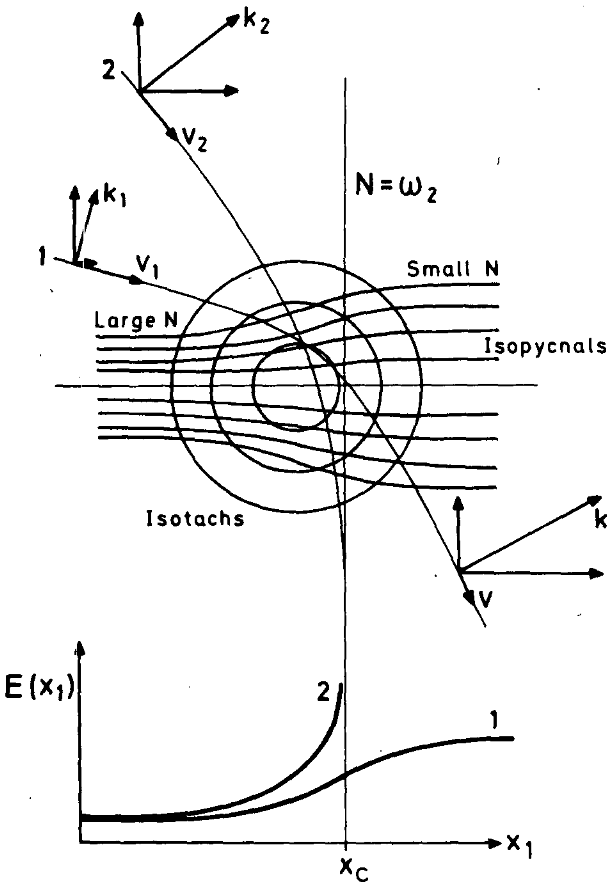

FIG. 1. Sketch of rays and behavior of energy in a fluid with horizontally varying stability frequency (decreasing in the positive x_1 direction). Isopycnals and isotachs of the geostrophic flow are indicated. The waves move normal to the current. Wave 1 with a frequency larger than the minimum N penetrates the jet while wave 2 with a frequency smaller than the minimum N encounters a critical layer.

layer case discussed above. We will investigate this process in a broader context in the next section.

Jones (1969) and Mooers (1975a) considered refraction in a flow with constant vertical and horizontal current shears. In these papers the intrinsic rate of refraction has been neglected in (2.11) thus restricting the results to a buoyancy field with constant gradients. For constant N and constant shear $\partial_i U_\alpha$ a solution of (2.12) can readily be obtained. Except for special conditions an exponential growth ultimately dominates the behavior of all components of the wave vector and the intrinsic frequency ω_0 approaches some limiting value asymptotically (Jones, 1969). If the mean flow is unidirectional, say, $U_2(x_1, x_3)$, then the wavenumber component k_2 in this direction remains constant. If $k_2 \neq 0$ the moduli of the other two components grow linearly in time, i.e.,

$$k_i \sim -tk_2 \frac{\partial U_2}{\partial x_i}, \quad i = 1, 3. \tag{3.4}$$

Following Mooers (1975a), the intrinsic frequency approaches the value

$$\omega_\infty = \pm \left[\frac{N^2 \left(\frac{\partial U_2}{\partial x_1}\right)^2 + f^2 \left(\frac{\partial U_2}{\partial x_3}\right)^2}{\left(\frac{\partial U_2}{\partial x_1}\right)^2 + \left(\frac{\partial U_2}{\partial x_3}\right)^2} \right]^{1/2} \tag{3.5}$$

and the intrinsic group velocity v_i tends to zero. The cascade of energy to high wavenumbers implied by (3.4) [termed the "shrinking catastrophe" by Jones (1969)] thus is accompanied by a refraction of the wave toward the mean flow. From this the author concludes that the surface

$$\omega_0 = \omega - k_2 U_2(x_1, x_3) = \omega_\infty \tag{3.6}$$

acts as a critical layer to the wave. However, the problem is somewhat more entangled. We will show below that the layer $\omega_0 = \omega_\infty$ exhibits a valve effect where a wave approaching from one side may penetrate while approaching from the other side results in critical layer absorption. The structure of this process has been revealed by Acheson (1973) for the case of wave propagation in a nonrotating shear flow.

4. Reflection and critical layers in a geostrophic mean flow

In this section we discuss the propagation of internal waves in a geostrophic current. As a model of the mean flow we consider a mean buoyancy field $B(x_1, x_3)$ which is constant in the x_2 direction, and a mean current $U_2(x_1, x_3)$. In fact, this configuration can always be achieved by suitable rotation of the coordinate system since for a horizontal geostrophic current the thermal wind equations (2.4) and the conservation of heat, expressed by $U_\alpha \partial_\alpha B = 0$, implies that the mean current must be unidirectional. The ray equations (2.11) then take the form

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \frac{k_1}{\alpha} \frac{(N^2 - \omega_0^2)(\omega_0^2 - f^2)}{\alpha\omega_0(N^2 - f^2)} \\ \frac{dx_2}{dt} &= \frac{k_2}{\alpha} \frac{(N^2 - \omega_0^2)(\omega_0^2 - f^2)}{\alpha\omega_0(N^2 - f^2)} + U_2 \\ \frac{dx_3}{dt} &= -\frac{k_3}{\alpha} \frac{(\omega_0^2 - f^2)^2}{\alpha\omega_0(N^2 - f^2)} \end{aligned}\right\} . \tag{4.1}$$

The refraction equations become ($i = 1, 2, 3$)

$$\frac{dk_i}{dt} = -\frac{N}{\omega_0}\frac{\omega_0^2 - f^2}{N^2 - f^2}\frac{\partial N}{\partial x_i} - k_2\frac{\partial U_2}{\partial x_i}. \quad (4.2)$$

The second component yields

$$k_2 = \text{constant}. \quad (4.3)$$

Since ω remains constant for a given wave, the intrinsic frequency ω_0 is a completely determined function of position, frequency of encounter ω and wavenumber k_2

$$\omega_0 = \omega_0(x_1, x_3; \omega, k_2) = \omega - k_2 U_2(x_1, x_3). \quad (4.4)$$

This also applies to the vector of total refraction

$$\mathbf{R} = \frac{d\mathbf{k}}{dt} = \mathbf{R}(x_1, x_3; \omega, k_2) \quad (4.5)$$

which represents the local rate of change of $\mathbf{k}$ for a wave with the constants ω and k_2. Thus for a wave with initial values ω and k_2 the scalar field of the intrinsic frequency ω_0 and the vector field of the reflection rate $\mathbf{R}$ can be regarded as given functions of position (x_1, x_3). Wave propagation is allowed in the region defined by $f^2 \leq \omega_0^2(x_1, x_3) \leq N^2(x_1, x_3)$. The vector field $\mathbf{R}(x_1, x_3)$ is perpendicular on the limiting curves $\omega_0 = \pm N$ or, equivalently,

$$\omega = T(x_1, x_3) := \pm N(x_1, x_3) + k_2 U_2(x_1, x_3) \quad (4.6)$$

($a := b$ means a is defined by b) and $\omega_0 = \pm f$ or, equivalently,

$$\omega = C(x_1, x_3) := \pm f + k_2 U_2(x_1, x_3), \quad (4.7)$$

since $\mathbf{R} = -\nabla T$ for $\omega_0 = \pm N$ and $\mathbf{R} = -\nabla C$ for $\omega_0 = \pm f$. In general, however, the curves normal to $\mathbf{R}$ defined by $dx_1 R_1 + dx_3 R_3 = 0$ do not coincide with neither any curve $T = \text{constant} \neq \omega$ nor $C = \text{constant} \neq \omega$.

In a general geostrophic flow determined by the buoyancy field $B(x_1, x_3)$ the frequency of encounter ω and the wavenumber k_2 in the direction of the current appear to be the only constants of the wave motion, and analytical solutions of the ray equations (4.1) and (4.2) seem to be impossible. However, there are more general, analytically tractable configurations of the mean flow than the cases of horizontal or vertical homogeneity reviewed above. Consider the local orthogonal transformation

$$\left.\begin{aligned} k_\parallel &= \rho_1 k_1 + \rho_3 k_3 \\ k_\perp &= \rho_3 k_1 - \rho_1 k_3 \end{aligned}\right\} \quad (4.8)$$

of the wave vector (k_1, k_3) with $\boldsymbol{\rho} = \boldsymbol{\rho}(x_1, x_3)$ and $\rho^2 = \rho_1^2 + \rho_3^2 = 1$. We may ask for the constraints to be imposed upon $B(x_1, x_3)$ leading to the constancy of one of the components $k_\parallel$ or $k_\perp$. From (4.2) we infer

$$\left.\begin{aligned} \frac{d}{dt}k_\parallel &= (\rho_1 R_1 + \rho_3 R_3) + k_\perp\left(\rho_3\frac{d\rho_1}{dt} - \rho_1\frac{d\rho_3}{dt}\right) \\ \frac{d}{dt}k_\perp &= (\rho_3 R_1 - \rho_1 R_3) - k_\parallel\left(\rho_3\frac{d\rho_1}{dt} - \rho_1\frac{d\rho_3}{dt}\right) \end{aligned}\right\}. \quad (4.9)$$

Obviously, $k_\perp$ (taking without any restriction the component orthogonal to $\boldsymbol{\rho}$) remains constant if and only if the parentheses on the right-hand side vanish separately. Hence $\boldsymbol{\rho}$ must be a constant vector parallel to $\mathbf{R}$ which can only be satisfied for a uni-directional vector field $\mathbf{R}$. This requires a buoyancy field of the form

$$B(x_1, x_3) = \chi(\rho_1 x_1 + \rho_3 x_3), \quad (4.10a)$$

$$N^2(x_1, x_3) = \rho_3 \chi'(\rho_1 x_1 + \rho_3 x_3), \quad (4.10b)$$

$$U_2(x_1, x_3) = (\rho_1/f\rho_3)\chi(\rho_1 x_1 + \rho_3 x_3) + \text{constant},$$

for which

$$R_i = -\rho_i \frac{N}{\omega_0}\frac{\omega_0^2 - f^2}{N^2 - f^2} \times [\rho_3\chi'' + k_2(\rho_1/f\rho_3)\chi']. \quad (4.11)$$

Here $\chi(\eta)$ is an arbitrary function and the prime denotes its derivate with respect to η. The isopycnals of (4.10) are sloping straight lines with normal $\boldsymbol{\rho}$ and spacing determined by $\chi(\eta)$. This model obviously generalizes the vertically and the horizontally homogeneous cases discussed in Section 3.

The following analysis will be confined to the mean flow (4.10) and (4.11). From (4.8) and (2.7) we find the dispersion relation in terms of $k_\parallel$ and $k_\perp$:

$$(L^2 - \omega_0^2)k_\parallel^2 + (M^2 - \omega_0^2)k_\perp^2 + (N^2 - \omega_0^2)k_2^2 + 2(N^2 - f^2)\rho_1\rho_3 k_\parallel k_\perp = 0, \quad (4.12)$$

where we have introduced the frequencies L and M by

$$\left.\begin{aligned} L^2 &= N^2\rho_1^2 + f^2\rho_3^2 \\ M^2 &= N^2\rho_3^2 + f^3\rho_1^2 \end{aligned}\right\} \quad (4.13)$$

satisfying $f^2 \leq L^2$, $M^2 \leq N^2$. Note that L is the generalization of the expression (3.5). Eq. (4.13) determines the local value of $k_\parallel$ in terms of the constant ω, k_2 and $k_\perp$.

It is straightforward to solve (4.12) for $k_\parallel$ and transform the group velocity (v_1, v_3) to the components $v_\parallel$ and $v_\perp$ parallel and normal to $\boldsymbol{\rho}$. For $\omega_0^2 \neq L^2$ one finds

$$\left.\begin{aligned} k_\parallel &= \frac{-(N^2-f^2)\rho_1\rho_3k_\perp \pm (N^2-\omega_0^2)^{1/2}(\omega_0^2-\omega_c^2)^{1/2}\kappa}{L^2-\omega_0^2} \\ v_\parallel &= \pm\frac{\kappa(N^2-\omega_0^2)^{1/2}(\omega_0^2-\omega_c^2)^{1/2}(\omega_0^2-f^2)}{\alpha^2\omega_0(N^2-f^2)} \\ v_\perp &= \frac{(\omega_0^2-f^2)(N^2-\omega_0^2)^{1/2}}{\alpha^2\omega_0(N^2-f^2)(L^2-\omega_0^2)}[-k_\perp(\omega_0^2-f^2)(N^2-\omega_0^2)^{1/2} \pm \rho_1\rho_3\kappa(\omega_0^2-\omega_c^2)^{1/2}(N^2-f^2)] \end{aligned}\right\}, \quad (4.14)$$

with

$$\left.\begin{aligned} \kappa^2 &= k_\perp^2 + k_2^2 = \text{constant} \\ \alpha^2 &= k_1^2 + k_2^2 = \frac{\omega_0^2-f^2}{N^2-\omega_0^2}(\rho_3k_\parallel - \rho_1k_\perp)^2 \end{aligned}\right\}. \quad (4.15)$$

and

$$\omega_c^2 = \frac{L^2k_2^2 + f^2k_\perp^2}{\kappa^2}. \quad (4.16)$$

Notice that $f^2 \leqslant \omega_c^2 \leqslant L^2$. The two solutions in (4.14) correspond to the two possible wavevectors in the (k_1, k_3) plane for given ω_0, k_2 and $k_\perp$. This is illustrated in Fig. 2. The inclination of the refraction vector **R** (or $\boldsymbol{\rho}$) may be inferred from (4.13) yielding $(\rho_3/\rho_1)^2 = (N^2 - L^2)/(L^2 - f^2)$. This may be used to obtain the orientation of **R** relative to the asymptotic cone $k_3 = \pm k_1[(N^2-\omega_0^2)/(\omega_0^2-f^2)]^{1/2}$ of the dispersion curve defined by $\omega_0^2 = \Omega_0^2(k_1, k_2, k_3)$ with fixed ω_0 and k_2. For $\omega_0^2 > L^2$, the refraction vector **R** lies inside the asymptotic cone of the dispersion curve and the two solutions (4.14) have different vertical but equal horizontal direction of propagation. This behavior is reversed for $\omega_0^2 < L^2$, where **R** lies outside the cone. The case $\omega_0^2 = L^2$, where **R** lies on the asymptotic cone is discussed below.

According to (4.14) the parallel wavenumber is real if

$$\omega_c^2 \leqslant \omega_0^2 \leqslant N^2 \quad (4.17)$$

which defines the spatial structure of the waveguide for a wave with constant ω, k_2 and $k_\perp$. Note that $\omega_c^2 \sim f^2$ if $k_2/k_\perp \sim 0$ and $\omega_c^2 \sim L^2$ if $k_2/k_\perp \sim \infty$. Further, we find $L^2 \sim f^2$ if $\rho_3/\rho_1 \sim \infty$ and $L^2 \sim N^2$ if $\rho_3/\rho_1 \sim 0$.

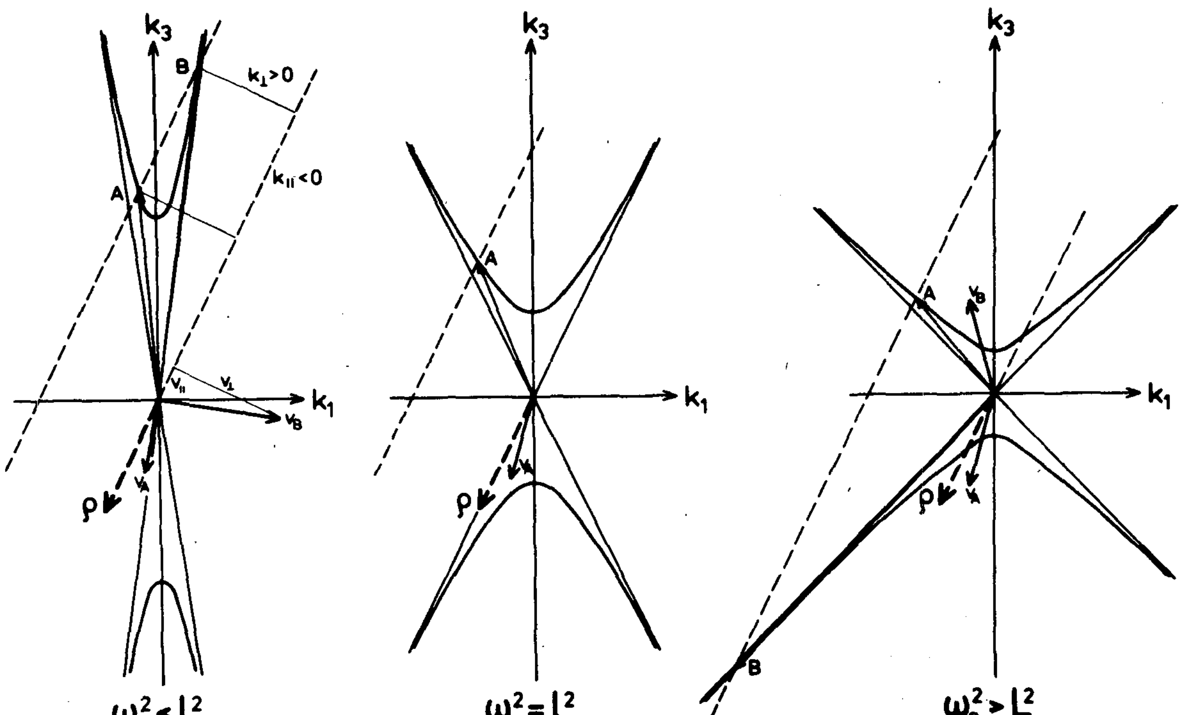

FIG. 2. The two solutions for the wave vector and group velocity for given ω_0, k_2 and $k_\perp$ according to (4.14) for $\omega_0^2 > \omega_c^2$ and the three regimes (a) $\omega_0^2 < L^2$, (b) $\omega_0^2 = L^2$ and (c) $\omega_0^2 > L^2$. Corresponding wave vector and group velocity are labeled with the same letter. The vector $\boldsymbol{\rho}$ is normal to the isopycnals. For $\omega_0^2 = \omega_c^2$ the dashed line in (a) becomes tangent to the dispersion curve and A and B coincide with the group velocity orthogonal to $\boldsymbol{\rho}$.

We now investigate the behavior of a wave propagation in this waveguide. Of particular interest is the behavior close to layers where the normal component of the intrinsic group velocity vanishes. It is obvious from (4.14) that this occurs at $\omega_0 = \omega_c$, $\omega_0 = L$ and $\omega_0 = N$. From the examples discussed in Section 3 we expect that at layers where $v_{\|} \sim 0$ the wave energy density tends to infinity. We shall use a simple criterion to decide whether this behavior is just an artifact of the WKB approximation, which can be "healed" by an extended version of the theory (Lighthill, 1978) or whether the behavior points toward some dynamically important process, as critical layer absorption, which is not covered by WKB theory.

Suppose the group velocity component $v_{\|}$ normal to a layer of interest vanishes as η^{μ}, where η is the distance to the layer measured along the normal $\boldsymbol{\rho}$. From (2.21) the conservation of action takes the form

$$E(\eta) = \text{constant} \times \frac{\omega_0(\eta)}{v_{\|}(\eta)} \approx \eta^{-\mu}. \quad (4.18)$$

Introducing $v_{\|}$ from (4.14) it is seen that (4.18) recovers (3.1), (3.2) and (3.3) for either ρ_1 or ρ_3 equal to zero. The time required to proceed from η_1 to η_2 toward the layer at $\eta = 0$ is

$$\Delta t = \begin{cases} \dfrac{\text{constant}}{1-\mu} \times \eta^{1-\mu}\Big|_1^2 & \text{for} \quad \mu \neq 1 \\ \text{constant} \times \ln\eta\Big|_1^2 & \text{for} \quad \mu = 1. \end{cases} \quad (4.19)$$

Thus the wave will not reach $\eta = 0$ if $\mu \geq 1$ and asymptotically freezes in the mean current. At such critical layers the wave energy density given by (4.18) increases in a nonintegrable way. Across the layer strong gradients in the wave-induced Reynolds stresses build up and lead to an exchange of momentum with the mean flow. If $\mu < 1$, the layer will be reached in a finite time and we expect some regular behavior (penetration or reflection if the layer limits the wave guide). Here the wave energy density increases in an integrable way.

a. The behavior at $\omega_0 = \pm N$

We now consider a wave approaching $\omega_0 = \pm N$. In section 3 we demonstrated for the horizontally homogeneous ocean (i.e., $\rho_1 \equiv 0$) that at $\omega_0 = \pm N$ the wave is subject to internal reflection. If $\rho_1\rho_3 \neq 0$, we learn from (4.14) that

$$k_{\|} \sim \frac{N^2 - f^2}{N^2 - L^2}\rho_1\rho_3 k_{\perp} = (\rho_1/\rho_3)k_{\perp} \quad (4.20)$$

as $\omega_0^2 \sim N^2$. From (4.9) we then find $k_3 \sim 0$ and $k_1 \sim k_{\perp}/\rho_3$. Thus, the ray turns to the vertical as in the horizontally homogeneous case. The group velocity tends to zero, i.e.,

$$\left.\begin{aligned} v_{\|} &\approx (N^2 - \omega_0^2)^{1/2} \approx \eta^{1/2} \\ v_{\perp} &\approx (N^2 - \omega_0^2) \approx \eta \\ v_2 &\approx (N^2 - \omega_0^2) \approx \eta \end{aligned}\right\}, \quad (4.21)$$

so that $\mu = 1/2$, which attributes to the planes $\omega_0 = \pm N$ the behavior of reflection.

The situation changes if the singular case $\rho_3 \equiv 0$ is considered (i.e., a vertically homogeneous ocean). Then $L^2 \equiv N^2$ and (4.14) yields

$$k_{\|} = \rho_1 k_1 = \pm\left(\frac{\omega_0^2 - \omega_c^2}{N^2 - \omega_0^2}\right)_k^{1/2} \quad (4.22)$$

which tends to infinity at $\omega_0^2 = N^2$. Since $k_3 = \rho_1 k_{\perp}$ is constant, the ray must turn to the vertical as well. However, the parallel component of the group velocity behaves as

$$v_{\|} = \rho_1 v_1 \approx (N^2 - \omega_0^2)^{3/2} \approx \eta^{3/2}, \quad (4.23)$$

so that $\mu = 3/2$. Hence, in a vertically homogeneous ocean (which is a rather unrealistic model since in general $\rho_3 \gg \rho_1$), the planes $\omega_0 = \pm N$ act as critical layers on the wave propagation. This case has been illustrated in Fig. 1.

b. The behavior at $\omega_0 = \pm\omega_c$

Our next concern is the behavior at the planes $\omega_0 = \pm\omega_c$. The case $k_{\perp} = 0$, where $\omega_c^2 = L^2$ is considered in Section 4c. Thus, let $k_{\perp} \neq 0$ and further $k_2 \neq 0$ and $\rho_1 \neq 0$ so that $\omega_c^2 \neq f^2$. At $\omega_0^2 = \omega_c^2$ the two solutions (4.14) of $k_{\|}$ become identical:

$$k_{\|} = k_{\perp}\rho_1\rho_3 \frac{N^2 - f^2}{\omega_c^2 - L^2} = -\frac{\rho_3}{\rho_1}\frac{\kappa^2}{k_{\perp}}. \quad (4.24)$$

and the group velocity

$$v_{\|} \approx (\omega_0^2 - \omega_c^2)^{1/2} \approx \eta^{1/2} \quad (4.25)$$

tends to zero with $\mu = 1/2$ so that the layers $\omega_0 = \pm\omega_c$ will be reached in a finite time and internal reflection must occur. The components $v_{\perp}$ and v_2 remain finite.

If k_2 tends to zero ω_c^2 tends to f^2, but the curves $\omega_0^2 = \omega_c^2$ and $\omega_0^2 = f^2$ disappear from the fluid. If ρ_1 tends to zero, we recover the well-known critical layer case discussed in section 3: since $L^2 = \omega_c^2 = f^2$ for $\rho_1 = 0$, Eq. (4.14) yields

$$\begin{aligned} k_{\|} &= \rho_3 k_3 \approx (\omega_0^2 - f^2)^{-1/2} \sim \infty, \\ v_{\|} &= \rho_3 v_3 \approx (\omega_0^2 - f^2)^{3/2} \sim 0, \end{aligned} \quad (4.26)$$

which is identical to (3.2).

c. The behavior at $\omega_0 = \pm L$

Finally, we consider the behavior at $\omega_0 = \pm L$ where according to Mooers (1975a) critical layer properties should occur. The situation for $k_{\perp} \neq 0$, close to $\omega_0 = \pm L$, can be inferred from Fig. 2 showing the two possible waves for given $k_{\perp}$ above and

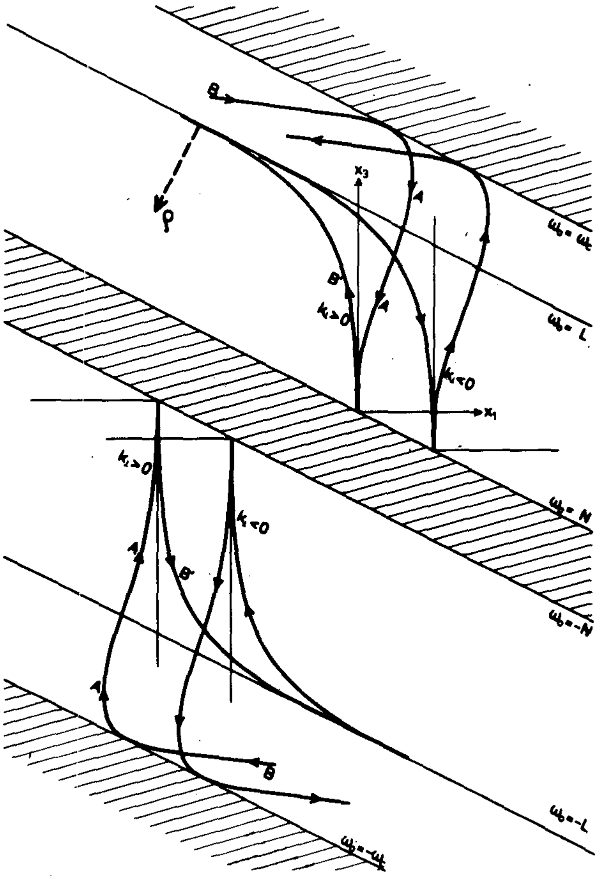

FIG. 3. Sketch of the rays for a wave $(\omega, k_2, k_\parallel)$ projected on the (x_1, x_3) plane. The forbidden region for wave propagation is hatched. The labels of the rays refer to Fig. 2.

below the layers $\omega_0 = \pm L$. In each case one of the two waves propagates toward the layer $\omega_0 = L$ (or $\omega_0 = -L$), while the other one moves away (remember that $\mathbf{R}$ is orthogonal to the layer). At $\omega_0 = L$ (or $\omega_0 = -L$) the refraction vector $\mathbf{R}$ becomes parallel to one of the asymptotics of the dispersion curve. Wave A gets a finite wavevector and nonzero group velocity at $\omega_0 = \pm L$. This contrasts Mooers' arguments presented in Section 3. Evidently, wave A reaches the surface $\omega_0 = L$ (or $\omega_0 = -L$) in a finite time and not in the asymptotic limit $t \sim \infty$ as assumed by Mooers, and thus penetrates the layer. The other wave (B at $\omega_0^2 < L^2$ and B′ at $\omega_0^2 > L^2$) gets an infinite component $k_\parallel$ as ω_0^2 tends to L^2. We will show below that this wave runs into a critical layer if it approaches $\omega_0 = L$ (or $\omega_0 = -L$). This asymmetry in the wave propagation whereby one of the two possible waves is permitted to penetrate while the other one is absorbed at the critical layer is referred to as valve effect (Acheson, 1973; Grimshaw, 1975, 1979). A sketch of this behavior is shown in Fig. 3. Note that the reflection at $\omega_0 = \pm N$ will not appear cusplike in three dimensions unless the mean current U_2 vanishes at these layers.

Using the dispersion relation in the form (4.12) it is straightforward to show that the infinite solution at $\omega_0^2 = L^2$ behaves as

$$k_\parallel \sim -\frac{2(N^2 - f^2)\rho_1\rho_3 k_\perp}{L^2 - \omega_0^2} \sim \infty \tag{4.27}$$

if $k_\perp \neq 0$. Then, from Eqs. (4.1), (4.14) and (4.15)

$$\left.\begin{aligned} \alpha^2 &\approx (L^2 - \omega_0^2)^{-2} \approx \eta^{-2} \\ v_\parallel, v_2 &\approx (L^2 - \omega_0^2)^2 \approx \eta^2 \\ v_\perp &\approx (L^2 - \omega_0^2) \approx \eta \end{aligned}\right\}. \tag{4.28}$$

Thus such a wave will be refracted into the direction of the mean current. The component $v_\parallel$ normal to the surface $\omega_0 = \pm L$ behaves as η^2 which reveals the critical layer property.

It is of interest to consider the transition of these results to pure horizontal and pure vertical inhomogeneity of the mean flow. If ρ_3 tends to zero L^2 tends to N^2 and the regions $\pm N > \omega_0 > \pm L$ collapse. The surfaces $\omega_0 = \pm N$ then attain critical layer properties as discussed in (4.22), (4.23) and (3.3). If ρ_1 vanishes we have $L^2 = \omega_c^2 = f^2$ and the regions $\pm\omega_c < \omega_0 < \pm L$ shrink to the critical layers at $\omega_0 = \pm f$ discussed in (4.26) and (3.2).

Now, if $k_\perp = 0$ we have $L^2 = \omega_c^2$ so that waves cannot exist in the regions $\omega_0 < L$ and $\omega_0 > -L$. From (4.12) we find for $\omega_0^2 \sim L^2$

$$k_\parallel^2 = \frac{N^2 - \omega_0^2}{\omega_0^2 - L^2} k_2^2 \sim \infty \tag{4.29}$$

and further, from (4.1), (4.14) and (4.15)

$$\left.\begin{aligned} \alpha^2 &\approx (\omega_0^2 - L^2)^{-1} \approx \eta^{-1} \\ v_\parallel &\approx (\omega_0^2 - L^2)^{3/2} \approx \eta^{3/2} \\ v_2 &\approx (\omega_0^2 - L^2) \approx \eta \\ v_\perp &\approx (\omega_0^2 - L^2)^{1/2} \approx \eta^{1/2} \end{aligned}\right\}. \tag{4.30}$$

Since $v_\parallel \approx \eta^{3/2}$ waves with $k_\perp = 0$ have critical layers at $\omega_0 = \pm\omega_c$.

5. Summary and discussion

The preceding analysis of internal wave propagation in a geostrophic flow is based on the WKB theory which assumes a small ratio of the wave scales to the scales of mean flow. This condition restricts the mean flow to large Richardson number. A further limitation of the theory has been introduced by the specific model of the mean flow used in this study, i.e., a mean buoyancy field with straight isopycnals. This model generalizes the analytical approaches to wave propagation in a mean flow presented so far.

Though the WKB formalism is known to fail close to layers where the intrinsic group velocity vanishes and the wave energy shows a singular behavior, this elementary approach still allows a qualitative discussion of the processes anticipated in a more rigorous treatment. The main result of this paper is then the discussion of the structure of waveguide and its critical layers. It is shown that for nonzero finite slope of the isopycnals the critical layers reside within the waveguide and exhibit a valve-like effect, being transparent to a wave approaching from one side while incidence from the other side results in absorption. The basic structure of such a behavior was pointed out by Acheson (1973). If the slope of the isopycnals tends to zero the critical layer shifts to the horizontal level where the intrinsic frequency equals the Coriolis frequency and limits the waveguide on one side. This well-known case of critical layers in a horizontally homogeneous fluid was initially studied by Bretherton (1966). For infinite slope, i.e., a vertically homogeneous fluid, the critical layer shifts to the other limit of the waveguide where the intrinsic frequency equals the stability frequency, which is here a vertical plane. This limiting case has been investigated by Samodurov (1974) and others. The present paper does not dwell on the physical behavior of the critical layers except that the energy is shown to become non-integrally singular so that significant interaction with the mean flow is anticipated resulting in absorption of the wave near such layers. It may be worth mentioning that the behavior near the critical layers may be quite different for small Richardson numbers where it has been shown in a horizontally homogeneous flow that waves may substantially be reflected or even overreflected extracting energy and momentum from the mean flow (Jones, 1968).

Regarding the restriction to straight isopycnals the applicability of the theory to wave propagation in the ocean appears to be limited. Certainly, the theory describes the local propagation features of waves in the large-scale circulation and represents in this context an improvement over the horizontally homogeneous models. However, the more interesting problem of ray tracing through baroclinic structures as fronts with significant curving of the isopycnals remains unsolved and seems to be accessible only by numerical integration of the ray equations. In such an effort the present analysis may be of value to sort out the structure of the waveguide.

Acknowledgments. The comments of one referee, which helped to sort out a mistake in the first manuscript and introduced the term valve effect to me for the phenomenon presented in this paper are gratefully acknowledged.

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