NOTES AND CORRESPONDENCE

The Level of No Motion in an Ideal Fluid

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ABSTRACT

The level of no motion plays a central role in the classical dynamic method and the more advanced diagnostic schemes of the β-spiral (e.g., Stommel and Schott) and inverse method (Wunsch) to calculate the absolute velocity in the ocean. Following simple arguments, each velocity component should vanish on separate surfaces in the fluid and the absolute velocity vector vanishes on the intersection of these surfaces, i.e., on curves in the fluid. It has been suggested, however, that besides these simple configurations there may be surfaces in the fluid on which the velocity vector vanishes. Killworth has based a diagnostic scheme on this concept which is different from the β-spiral approach and the inverse method. In this note we examine the possible configuration of the level of no-motion in a fluid using ideal fluid theory. It is shown that stagnation surfaces in the fluid, i.e. surfaces on which the velocity vector vanishes, normally do not exist.

1. Introduction

The concept of the level of no motion is a basic ingredient of the classical dynamical method to infer absolute velocities from hydrographic data. A discussion of early speculations on the level of no motion can be found in a recent paper by Wunsch and Grant (1982). With the β-spiral approach of Stommel and Schott (1977) and the inverse technique approach of Wunsch (1978) there are now objective methods to calculate the level of no motion from data. Whereas these methods use only assumptions on the dynamics and conservation properties of tracers, Killworth (1980) has advanced a method that additionally involves Ekman pumping. This approach allows one to infer absolute velocities from a single NS-section which obviously is a great advantage. However, Killworth's method strongly rests on an assumption about the configuration of the level of no motion: the method always yields a surface where the velocity vector vanishes, i.e., $u = v = w = 0$. This corresponds to a stagnation surface in the fluid and for an ideal fluid there would be no communication of properties across this surface. Although such a configuration does not contradict the dynamics of an ideal fluid the existence of a stagnation surface in the ocean is at least questionable. One rather would expect the velocity vector to vanish on a curve (where the surfaces $u = 0$, $v = 0$ and $w = 0$ intersect) than on an entire surface. However, to confirm this concept, one has to dive deeply into the dynamics. In this note we advance the opinion that vanishing of the velocity vector is a rather exceptional case compared to a level of no motion in only one velocity component. This contrasts Killworth's (1980) statement that vanishing of the velocity vector normally occurs where the east velocity vanishes.

For convenience we denote a simple level of no-motion, i.e., a zero in either the u, v or w profiles, by LNM. If the velocity vector vanishes at some point, i.e., $u = v = w = 0$, this level will be called LNAM (level of no absolute motion).

The ideal fluid equations and the functional approach of Welander (1971a) are reviewed in Section 2. In Section 3 we classify the different configurations of the levels of no-motion and attempt to infer their spatial structure by considering a specific solution of the ideal fluid equations. Section 4 gives a discussion of some diagnostic treatments of the ideal fluid equations in which the density field is regarded as given.

The ideal fluid equations conserve density, potential vorticity and the Bernoulli function along streamlines. This implies a functional relation between these quantities in the fluid. We should point out that the analysis in this paper excludes the degenerate case in which density is a function of potential vorticity alone.

2. The ideal fluid equations

Steady motion of an incompressible, ideal, rotating fluid of low Rossby number is described by the geostrophic and hydrostatic balances

$$\mathbf{f} \times \mathbf{u} = -\nabla p - g\rho\mathbf{k}, \tag{2.1}$$

the conservation of density

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$$\mathbf{u}\cdot\nabla\rho = 0 \quad (2.2)$$

and the continuity equation

$$\nabla\cdot\mathbf{u} = 0. \quad (2.3)$$

Eqs. (2.1) and (2.3) imply a vorticity equation in the form

$$w_z = \frac{\beta}{f}v. \quad (2.4)$$

Two convenient conservation laws can be derived (Welander, 1971a). Eqs. (2.1) and (2.2) imply the conservation of the Bernoulli function $B = p + g\rho z$ along a streamline

$$\mathbf{u}\cdot\nabla B = 0, \quad (2.5)$$

and the complete set (2.1) to (2.3) leads to conservation of the potential vorticity $Q = f\rho_z$

$$\mathbf{u}\cdot\nabla Q = 0. \quad (2.6)$$

These equations have been used in models of the oceanic thermocline (e.g., Welander, 1959, 1971a,b; Needler, 1967, 1971; Luyten *et al.*, 1983) or, with given density field from observations, in diagnostic calculations of the oceanic velocity field (e.g., Stommel and Schott, 1977; Killworth, 1980; Olbers *et al.*, 1984; note that in some of these papers potential density is used).

The problem of boundary conditions for the ideal fluid equations has recently been discussed by Killworth (1983a). The normal velocity must vanish at rigid boundaries. Further we would like to prescribe the vertical velocity at the bottom of the mixed layer by the Ekman pumping velocity. Hence, on the vertical boundaries we have

$$\left.\begin{array}{lll} w = w_0(x, y) & \text{at} & z = 0 \\ w = 0 & \text{at} & z = -h \end{array}\right\}. \quad (2.7)$$

It also seems reasonable to prescribe the surface density on points of inflow (e.g., Luyten *et al.*, 1983). Thus, at the top

$$\rho = \rho_0(x, y) \quad \text{at} \quad z = 0, \quad \text{if} \quad w_0(x, y) < 0. \quad (2.8)$$

This presumes that the equations are hyperbolic though, as Killworth (1983a) states, no such proof exists. Corresponding boundary conditions should hold at open lateral boundaries.

There are various ways to reduce the set of equations (2.1) to (2.3) to one equation for either pressure (e.g., Needler, 1967) or other functionals of density (e.g., Welander, 1959, 1971b). We will make use of another framework (Welander, 1971a) in which the conservation of ρ, Q and B on each streamline implies a functional relationship between these variables in the fluid, say

$$B = F(\rho, Q). \quad (2.9)$$

The function F is in principle determined by the boundary conditions since these are responsible for the flow pattern in the fluid. However, the detailed functional correspondence between F and the boundary values is unclear. If the form of F is prescribed *a priori* to generate solutions to the ideal fluid equations (e.g., Welander, 1971a, and Section 3 of this paper) it is impossible to fulfill all physical boundary conditions as (2.7) and (2.8). We will return later to this problem.

In this paper we assume that the functional relation between B, ρ and Q is not degenerate, in particular that ρ is not a function of Q alone. As Needler (1971) pointed out this latter case is degenerate in so far as the solution in a laterally unbounded domain contains an arbitrary barotropic flow. For this reason the β-spiral method does not work either. Moreover, we will assume that $F(\rho, Q)$ is differentiable though it is not at all clear to us which class of boundary conditions (if any at all) is compatible with such a property.

With $B = p + g\rho z$ and the hydrostatic balance from (2.1) there are two first-order ordinary differential equations

$$\left.\begin{array}{l} p + g\rho z = F(\rho, Q) \\ p_z = -g\rho \end{array}\right\} \quad (2.10)$$

for the pressure field and the density field. If the density field is known, the solution is completed by computing the pressure from the first equation in (2.10), the horizontal velocities from the geostrophic balance in (2.1) and the vertical velocity from the density conservation (2.2). This yields

$$\left.\begin{array}{l} u = -\dfrac{1}{f}[(F_\rho - gz)\rho_y + F_Q Q_y] \\ v = \dfrac{1}{f}[(F_\rho - gz)\rho_x + F_Q Q_x] \\ w = \dfrac{F_Q}{Q}(\rho_x Q_y - \rho_y Q_x) \end{array}\right\}. \quad (2.11)$$

It is easy to show that the continuity constraint (2.3) is satisfied by these expressions.

If $F(\rho, Q)$ is differentiable, the two first-order equations (2.10) can be replaced by a second-order equation

$$F_Q f\rho_{zz} + (F_\rho - gz)\rho_z = 0 \quad (2.12)$$

for the density field. Since (2.12) [as well as (2.10)] does not contain any horizontal derivatives one is tempted to consider it as a local differential problem for the profile $\rho(z)$ where (x, y) enter as parameters only. This is indeed the way how the approach has been exploited to construct solutions to the ideal fluid equations (Welander, 1971a): the functional $F(\rho, Q)$ was prescribed and (2.11) was solved with boundary condition for ρ and ρ_z at a level $z = 0$, i.e.,

$$\left.\begin{array}{l} \rho = \rho_0(x, y) \\ Q = Q_0(x, y) \end{array}\right\} \quad \text{at} \quad z = 0. \quad (2.13)$$

Somehow, these local conditions—or more general a linear combination of ρ and Q at two different levels—seem to be natural for a local treatment of (2.11). The true problem, however, is nonlocal and it can be demonstrated that with a prescribed $F(\rho, Q)$

one cannot expect to satisfy the three boundary conditions (2.7) and (2.8) for any choice of ρ_0 and Q_0. Suppose we are in a region of downwelling and the surface density ρ_0 would be prescribed. Then, because of the upper boundary condition (2.7) for the vertical velocity, the potential vorticity Q_0 at $z = 0$ cannot be prescribed but depends on the function $F(\rho, Q)$ and on $w_0(x, y)$ since from (2.11)

$$w_0(x, y) = \frac{F_Q(\rho_0, Q_0)}{Q_0}(\rho_{0x}Q_{0y} - \rho_{0y}Q_{0x}). \quad (2.14)$$

Integration along each isopycnal ρ_0 = constant with a given initial value $\hat{Q}_0[\rho_0]$ determines $Q_0(x, y)$ on the entire surface $z = 0$ in terms of the boundary fields $\rho_0(x, y)$ and $w_0(x, y)$ and the one-dimensional manifold $\hat{Q}_0[\rho_0]$. Suppose now that we have solved (2.12) subject to the boundary conditions (2.13) for the downwelling region. For w to vanish at the bottom $z = -h$, density lines ρ_{-h} = constant and potential vorticity lines Q_{-h} = constant must be parallel, i.e., ρ_{-h} must be function of Q_{-h}. It is evident that this cannot be achieved for the two-dimensional domain by tuning of the one-dimensional manifold $\hat{Q}_0[\rho_0]$. The property of ρ_{-h} and Q_{-h} to be parallel must thus be contained in the correct choice of $F(\rho, Q)$.

We have pointed out the shortcomings of the F-function approach to the ideal fluid problem. Nevertheless, it is a very convenient tool to generate analytic solutions to the ideal fluid equations. We will proceed this way in Section 3 to present our presumably strongest argument against the possibility of a stagnation surface in the fluid.

3. Properties of the fields near a LNAM

If we insert (2.12) into (2.11) we obtain a representation of the velocity field

$$\mathbf{u} = \alpha\nabla\rho \times \nabla Q = \alpha(J_{yz}, -J_{xz}, J_{xy}), \quad (3.1)$$

where

$$\alpha = \frac{F_Q}{Q} \quad (3.2)$$

and J_{yz} is the y–z Jacobian of ρ and Q, etc. Apart from the functional relation (3.2) expressing α in terms of ρ and Q the form (3.1) can be inferred from the conservation of ρ and Q alone: since the intersection of ρ = const and Q = const must be a streamline the velocity vector $\mathbf{u}$ must be parallel to $\nabla\rho \times \nabla Q$.

The relation (3.1) allows to give a classification of the possible configurations of the levels of no motion in the fluid: the velocity vector should vanish if either

$$J_{yz} = J_{xz} = J_{xy} = 0 \quad \text{(LNAM of the first kind)} \quad (3.3)$$

or

$$\alpha = 0 \quad \text{(LNAM of the second kind)}. \quad (3.4)$$

If only one of the Jacobians vanishes only one component will have a LNM. For a LNM in, say, the east component we have

$$J_{yz} = 0 \quad \text{(LNM in } u). \quad (3.5)$$

A way to judge which of the cases (3.3) to (3.5) is "normal" is to consider the spatial dimensions defined by these relations. Obviously, (3.5) defines a two-dimensional surface. In contrast, (3.3) is the intersection of two surfaces (note that the vanishing of two Jacobians implies vanishing of the third). Hence, a LNAM of the first kind generally occurs only along the intersection of two surfaces; i.e., it is a curve and thus an exceptional case compared to a LNM as given by (3.5).

The condition $\alpha = F_Q(\rho, Q)/Q = 0$ defines a curve in the (ρ, Q)-domain that corresponds to a surface in the fluid. However, $\alpha = 0$ is only sufficient for the vanishing of $\mathbf{u}$ if the Jacobians remain regular on this surface given by $\alpha = 0$. The investigation of this configuration is the main goal of this section.

To investigate the structure of the density field close to a point where $F_Q = 0$ we consider the profile equation (2.12). A point $z = z_*$ where F_Q vanishes is a singular point of this differential equation and we expect in general, i.e. for two independent boundary conditions as e.g. (2.13), that ρ_{zz} becomes singular at this level. In this case the Jacobians J_{yz} and J_{xz} are singular, too, and the criterion (3.4), i.e. $\alpha = 0$, is not sufficient for the existence of a LNAM. Thus the absolute velocity does not necessarily vanish on the entire surface given by $\alpha = F_Q/Q = 0$.

As obvious from (2.14), the singular behavior of the density field $\rho(x, y, z)$ on the surface $z = z_*(x, y)$ can only be prevented if $F_\rho - gz = 0$ on this surface, more strictly the ratio $(F_\rho - gz)/F_Q$ must attain a finite value as the surface $F_Q = 0$ is approached. An example is given below where the surfaces $z = z_*(x, y)$ defined by $F_Q = 0$ and $z = \hat{z}(x, y)$ defined by $F_\rho = gz$ are different unless the boundary conditions are chosen in a very specific way.

The linear function $B = F(\rho, Q) = a\rho + bQ + c$ of Welander's (1971a) thermocline solution yields $F_Q = b \neq 0$. This model exhibits only LNMs of the kind (3.5) and LNAMs of the first kind (3.3). The simplest function with a zero in F_Q must be quadratic in Q. Thus we take

$$F(\rho, Q) = ag\rho + b(Q - Q_*)^2 + c, \quad (3.6)$$

where a, b, c and Q_* are constants. Then

$$\alpha = 2b(Q - Q_*)/Q \quad (3.7)$$

vanishes for $Q = Q_*$ at, say, $z = z_*$. This point will be determined by the boundary conditions as shown below. Eq. (2.12) can be solved exactly for Q (any form which is linear in ρ can be solved exactly as shown by Welander, 1971a). Defining

$$\phi = p + g\rho(z - a) - c, \quad (3.8)$$

the potential vorticity is just

$$Q = \frac{f\phi_z}{g(z - a)}, \quad (3.9)$$

and the first equation of (2.10) reduces to a first-order problem in ϕ; namely

$$\frac{f\phi_z}{g(z - a)} = Q_* + \left(\frac{\phi}{b}\right)^{1/2} \quad (3.10)$$

which may be integrated to yield

$$K[(z - a)^2 - (z_1 - a)^2] = \left(\frac{\phi}{b}\right)^{1/2} - \left(\frac{\phi_1}{b}\right)^{1/2} - Q_* \ln \frac{Q_* + (\phi/b)^{1/2}}{Q_* + (\phi_1/b)^{1/2}} \quad (3.11)$$

or, using (3.9) and (3.10) again,

$$Q - Q_1 - Q_* \ln \frac{Q}{Q_1} = K[(z - a)^2 - (z_1 - a)^2], \quad (3.12)$$

where $Q = Q_1(x, y)$ at a level $z = z_1$ and $K = g/(4bf)$. For $Q = Q_*$ this equation determines the surface $z = z_*(x, y)$, where F_Q vanishes. The function z_* is thus solution of the quadratic equation

$$Q_* - Q_1 - Q_* \ln \frac{Q_*}{Q_1} = K[(z_* - a)^2 - (z_1 - a)^2]. \quad (3.13)$$

The (x, y)-dependence of z_* enters through the integration constant $Q_1(x, y)$ and the y-dependence of K. Eliminating Q_1 from (3.12) and (3.13) yields

$$Q - Q_* - Q_* \ln\left(\frac{Q}{Q_*}\right) = K[(z - a)^2 - (z_* - a)^2]. \quad (3.14)$$

It is easy to show from this first integral of the profile equation (2.12) that the vanishing of α on the surface $z = z_*(x, y)$ is insufficient for a LNAM. The gradients of Q follow from (3.14) in the implicit form

$$\nabla Q = \frac{Q}{Q - Q_*} \nabla\{K[(z - a)^2 - (z_* - a)^2]\}. \quad (3.15)$$

In particular,

$$Q_z = f\rho_{zz} = \frac{Q}{Q - Q_*} 2K(z - a). \quad (3.16)$$

The second-order gradients ∇Q of the density field apparently become singular for $Q = Q_*$, i.e., on $z = z_*(x, y)$. The exceptional case $z_* = a$ is considered below. From the expression (2.11) for the velocity vector one notices that the zero in α is cancelled exactly by the singularity in ∇Q, so that

$$\mathbf{u} = 2b\nabla\rho \times \nabla\{K[(z - a)^2 - (z_* - a)^2]\} \quad (3.17)$$

is generally nonzero on $z = z_*(x, y)$.

The surface $z = \hat{z}(x, y)$ where $F_\rho = gz$ is simply the level surface $z = a$ in this example. It is evident from (3.15) that ∇Q is regular on this surface, in particular $Q_z = 0$, except on the curve where $z = a$ and $z = z_*(x, y)$ intersect. Approaching this curve on the level surface $z = a$, Eq. (3.16) yields a vanishing Q_z, whereas approaching on the surface $z = z_*(x, y)$, we find an infinite value. Hence the density is still singular on this curve.

The surfaces $z = z_*(x, y)$ and $z = a$ can be made identical by a specific choice of Q_0. According to (3.13) we must simply take Q_1 as solution of

$$Q_* - Q_1 - Q_* \ln \frac{Q_*}{Q_1} = -K(z_1 - a)^2. \quad (3.18)$$

For this specific $Q_1 = Q_1(y)$ one finds regular behaviour $Q - Q_* = (2KQ_*)^{1/2}(z - a)$ close to $z = a$ with $Q_x \equiv 0$, $Q_y \sim 0$ and $Q_z \sim (2KQ_*)^{1/2}$ as $z \sim a$. The ratio $(F_\rho - gz)/F_Q$ remains finite and indeed $u = v = w = 0$ at $z = a$, i.e., a LNAM of the second kind is obtained.

What is the normal case? One could make the hypothesis that the specific form (3.18) for Q_1 is the only one compatible with the form (3.6) of the F-function. One may express Q_1 in terms of the boundary value Q_0 (or simply identify z_1 with the top of the fluid, $z = 0$) and proceed to investigate the compatibility with the remaining boundary conditions. On the other hand, if one believes that (3.6) is compatible with more general boundary conditions the above described singularities occur. We believe that such singularities do not happen in the ocean and conclude that α does not vanish within the oceanic water body or, equivalently, that the relation $B = F(\rho, Q)$ between B, ρ and Q is such that $F_Q \neq 0$ within the range of oceanic density and potential vorticity values. As the only possibility for a LNAM then remains the configuration of the LNAM of the first kind characterized by the simultaneous vanishing of all three Jacobians J_{yz}, J_{xz} and J_{xy}. As explained above, this generally defines a curve in the fluid. Again, only a specific choice of the boundary conditions (2.13) will produce solutions to the profile equation (2.12) that have the three Jacobians vanishing simultaneously on an entire surface. It is unlikely that such boundary conditions should just happen to be realized in the ocean. We conclude that stagnation surfaces are unlikely to exist in the ocean.

4. Inference about the LNAM from diagnostic approaches

In the last section we attempted to construct complete solutions for the full set of ideal fluid equations.

These equations have also found wide applications in diagnostic calculations of the oceanic velocity field from observed density. There are various ways to relate the velocity components to the density field and exploit such relations for a diagnostic calculation. Here we will briefly investigate three of these approaches with respect to their treatment of the level of no-motion problem. We emphasize again that in this section the density field is regarded as given, e.g., from observations and as such will not necessarily be compatible with any solution of the ideal fluid equations.

The diagnostic methods considered below share the property of inferring the local (i.e., at each horizontal position separately) velocity profile from the profiles of the horizontal and vertical density gradients. The methods are based on the thermal wind equations

$$\left.\begin{aligned} u_z &= \left(\frac{g}{f}\right)\rho_y \\ v_z &= -\left(\frac{g}{f}\right)\rho_x \end{aligned}\right\}, \qquad (4.1)$$

the conservation of density (2.2) and the local form of the vorticity equation (2.4) or, equivalently, the potential vorticity equation (2.6). Constraints that relate the velocities from horizontally separate points, i.e., the continuity constraint, have been eliminated. Consequently, any velocity field inferred from this set of equations (4.1), (2.2), and (2.4) using a given density field will generally not satisfy continuity (unless, of course, the density field has been taken from a complete solution of the ideal fluid equations).

a. *The β-spiral approach*

The β-spiral equation

$$uJ_{xz} + vJ_{yz} = 0 \qquad (4.2)$$

can be obtained from (3.1) by elimination of α. Then integrating (4.1) in the form

$$\left.\begin{aligned} u &= u_0 + \left(\frac{g}{f}\right)\int_0^z dz'\rho_y \\ v &= v_0 - \left(\frac{g}{f}\right)\int_0^z dz'\rho_x \end{aligned}\right\}, \qquad (4.3)$$

the unknown reference velocities u_0 and y_0 are determined in principle by considering (4.2) at two different levels. In practice many levels are used to solve (4.2) by a least squares procedure. The vertical component w then follows from (3.1) or equivalently, (2.2). Notice that the equivalence does not hold if the solution is obtained from noisy data. Then the solution depends on the assumption where the noise enters into the equations. In the above scheme which is used, e.g., by Stommel and Schott (1977), the density conservation is satisfied exactly whereas the vorticity and potential vorticity equations contain noise and are satisfied only in a least squares sense. It is possible to place the noise elsewhere which yields different formulations of the β-spiral. Integrating the vorticity equation (2.4)

$$w = w_0 + \left(\frac{\beta}{f}\right)\int_0^z dz'v \qquad (4.4)$$

and inserting this and (4.3) into the density conservation (2.2) yields a β-spiral equation from which the three reference velocities u_0, v_0, and w_0 may be determined by a least squares fit. The w-profile then follows from (4.4) so that the vorticity equation is exactly satisfied but the density conservation contains the noise. Details of this approach are described in Olbers *et al.* (1984).

This latter procedure has been used to calculate the geostrophic velocities in the North Atlantic from the Levitus (1982) atlas. Fig. 1 displays u, v, and w for two sections which should exemplify the complexity of the structure of the levels of no-motion in the ocean. Actually, the velocity field was not derived from the ideal fluid equations but using conservation of potential density instead of *in situ* density. The reference velocities were determined from data below the mixed layer down to 2000 m depth. The thick curves in Fig. 1 indicate the LNMs for each component. They are reproduced in Fig. 2. Apparently, on these sections the LNMs of the different components do not coincide, even regarding possible error bounds. Generally the pair-wise intersection occurs close together (the LNMs do not perfectly intersect in points since the conservation equation for density is satisfied only in a least squares sense). Hence, this structure is more likely consistent with LNAMs of the first kind and not with those of the second kind.

b. *The* w*-equation*

The w-equation (Killworth, 1980) resides on the same subset of the ideal fluid equations as the β-spiral. By straightforward elimination in (4.1), (2.2) and (2.4), one finds a second-order differential equation for w

$$-uw_{zz} + u_z w_z + \frac{g\beta}{f^2}\rho_z w = 0. \qquad (4.5)$$

Given $u(z)$ and $\rho_z(z)$, the solution of this equation subject to the boundary conditions $w = 0$ at the bottom and $w = w_E$ (the Ekman pumping velocity) at the surface would yield the profile of the vertical velocity. Killworth (1980), however, uses (4.5) to determine the LNM in u and hence the absolute velocity profile. We give a brief discussion of this concept which is entirely based on the mathematical properties of (4.5).

The properties of w in the vicinity of a zero in u (say, $u = 0$ at $z = z_0$) are obtained by the local analysis of (4.5). One expands u and ρ_z about $z = z_0$

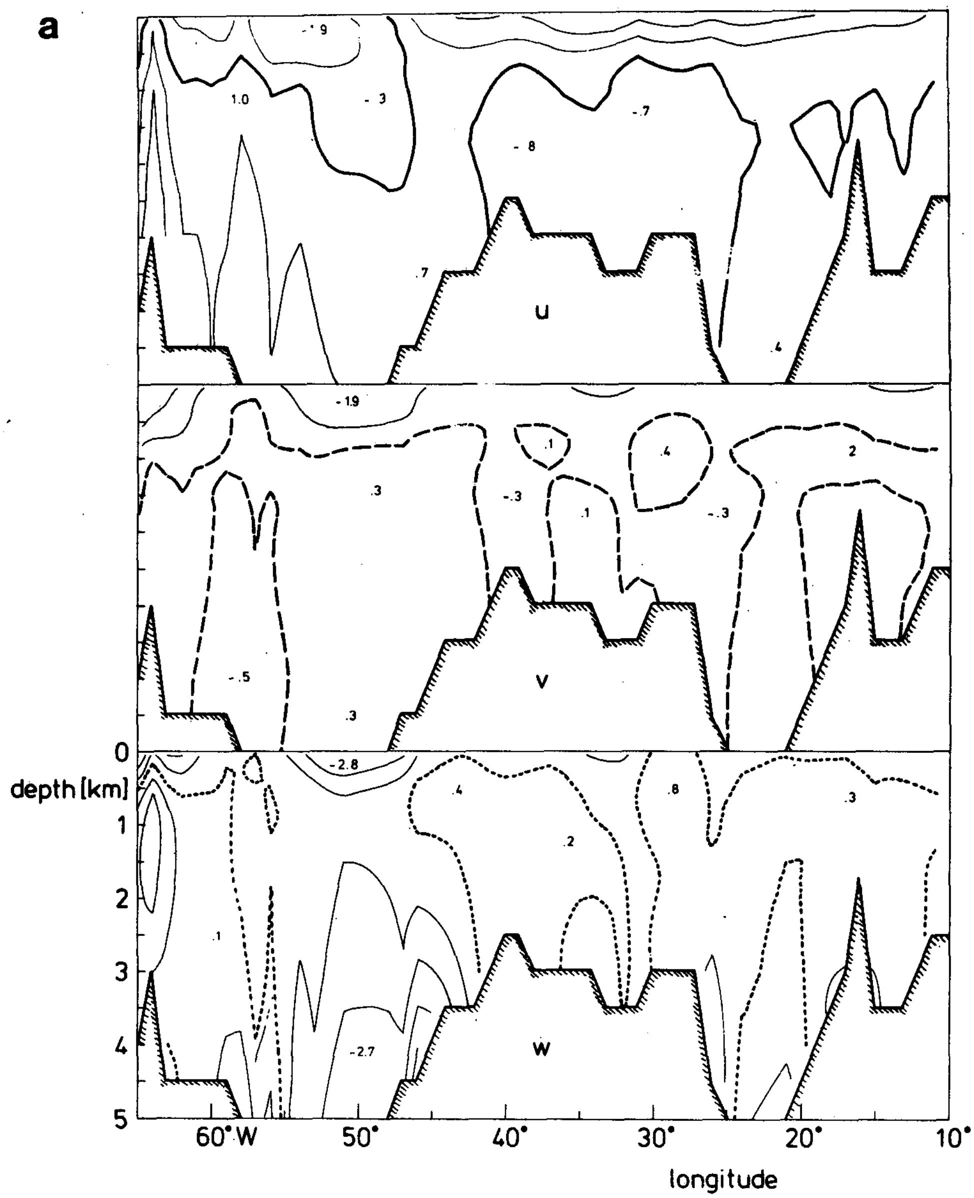

FIG. 1. Sections of u, v and w along (a) 32.5°N and (b) 30.5°W in the North Atlantic. Units are cm s^{-1} for u and v and 10^{-4} cm s^{-1} for w. Heavy lines indicate LNM of the velocity components.

$$\left.\begin{aligned} u &= \frac{g}{f}[\rho_y(z_0)(z - z_0) + \tfrac{1}{2}\rho_{yz}(z_0)(z - z_0)^2 + \cdots \\ \rho_z &= \rho_z(z_0) + \rho_{zz}(z_0)(z - z_0) + \cdots \end{aligned}\right\} \tag{4.6}$$

and derives the recurrence relations for the Frobenius solution

$$w = (z - z_0)^\gamma \sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad a_0 \neq 0, \tag{4.7}$$

by inserting (4.6) and (4.7) into (4.5). The indical equation $\gamma(\gamma - 2) = 0$ has solutions $\gamma = 2$ and $\gamma = 0$. The Frobenius solution for $\gamma = 2$ always exists [this is $w_1 \sim (z - z_0)^2$]. For the lower exponent $\gamma = 0$ the recurrence relations break down if $J_{yz} \neq 0$ at $z = z_0$. In this case any solution independent to w_1 contain a logarithmic term

$$w_2 = J_{yz}(z_0)(z - z_0)^2 \ln(z - z_0) \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{4.8}$$

so that the general solution will produce a singular w_{zz} and hence ρ_x. This, however, is not the case if $J_{yz} = 0$ at $z = z_0$. Then the recurrence relations remain consistent and yield a second solution which is the sum of w_1 and an independent regular solution w_2 which does not vanish at $z = z_0$. The complicated

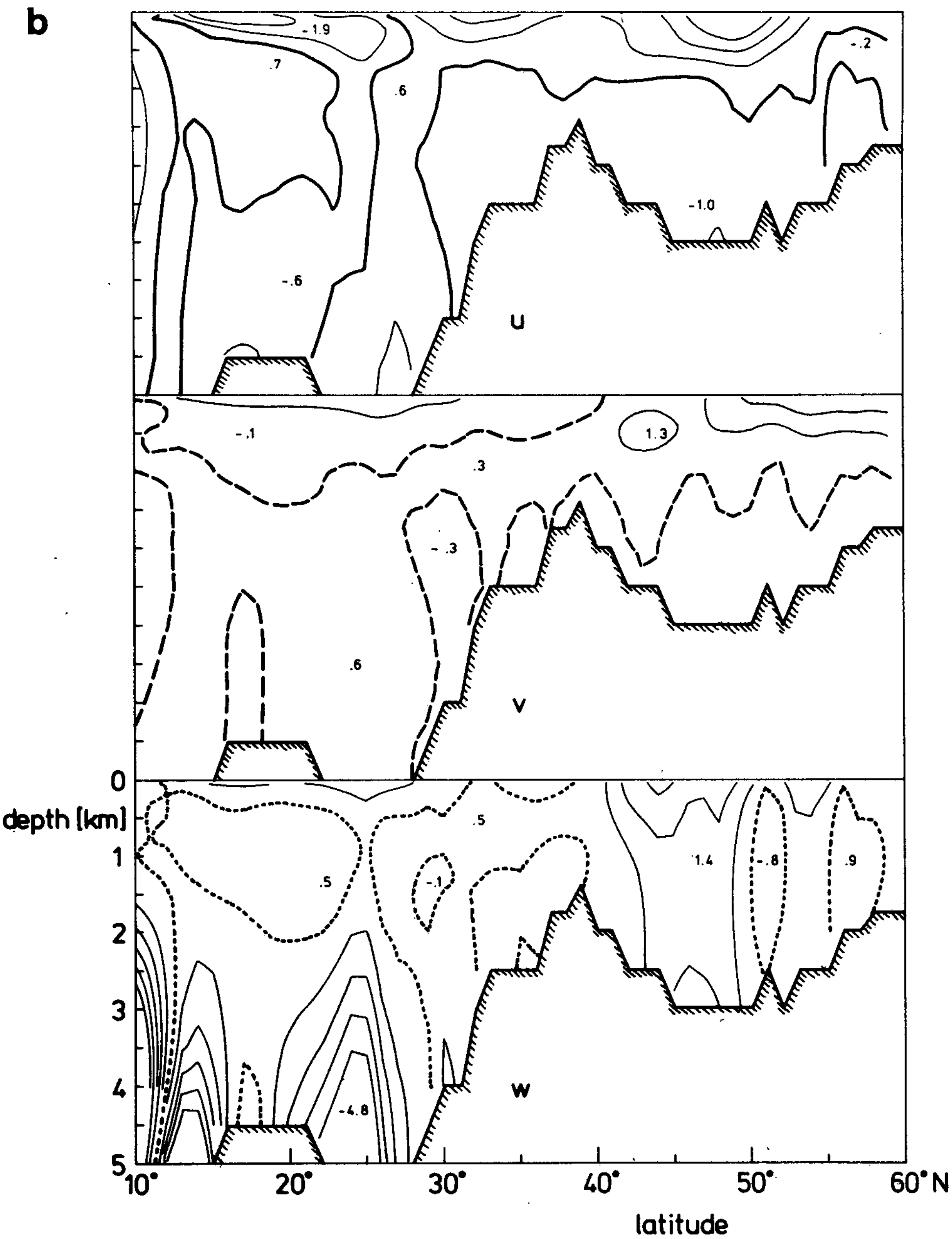

FIG. 1. (*Continued*)

analysis of the singular w-equation thus recovers of course the configuration of the singular behavior if $J_{yz} \neq 0$ or the occurrence of a simple LNM in u if $J_{yz} = 0$.

The general solution of (4.5) is of the form $w = a_1 w_1 + a_2 w_2$ where a_1 and a_2 are determined by the two boundary conditions. Killworth (1980) now argues that the vanishing of the Jacobian J_{yz} at $z = z_0$ is very unlikely and, moreover, that the ocean will not develop singularities so that *a priori* $a_2 = 0$. The two boundary conditions then enable the determination of another parameter such as the reference velocity u_0. Further, with $w_1 \sim (z - z_0)^2$ one obtains from (2.4) $v = 0$ at $z = z_0$. Hence a LNM in u would necessarily imply the existence of a LNAM. In essence, this concept attempts to determine the absolute velocity from the demand to prevent singular behavior which is somehow selfgenerated by the assumption $J_{yz} \neq 0$. In contrast we would rather tend to conclude that in general $J_{yz} = 0$ where $u = 0$ and that the singular behavior of a LNAM of the second kind should not occur at all.

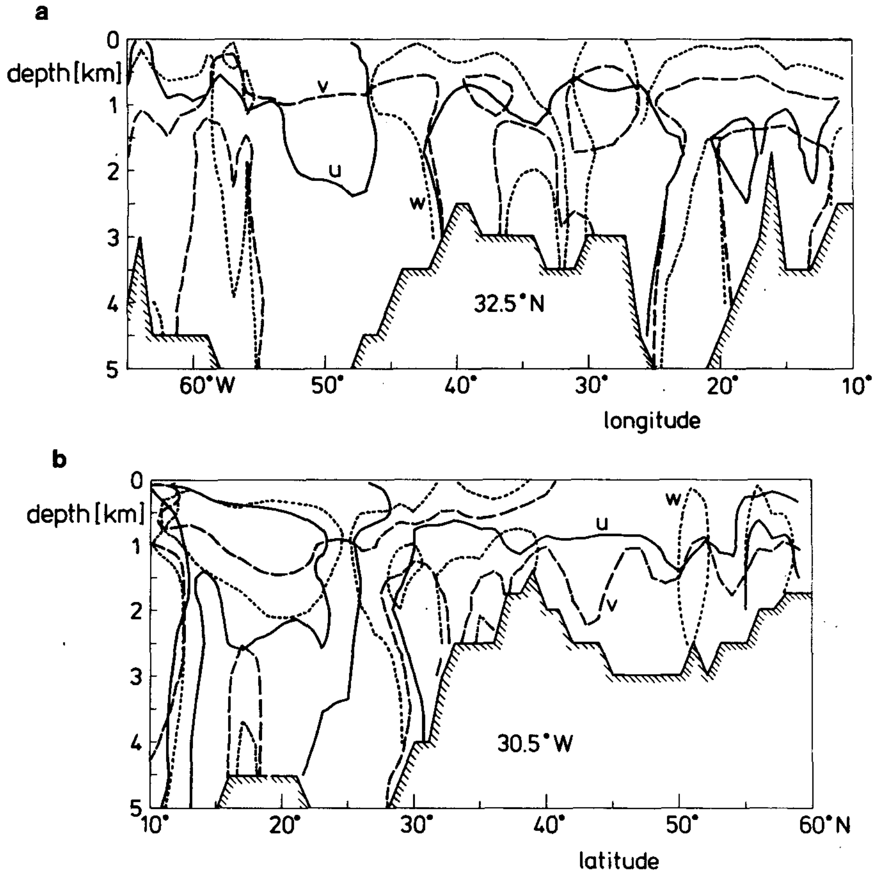

FIG. 2. Level of no motion structure of the three velocity components for the sections shown in Fig. 1. Fat curves indicate LNM for u, dashed lines for v, and dotted lines for w.

c. Needler's formula

An explicit expression in terms of density for the proportionality factor α in (3.1) can be derived by differentiating the conservation equation (2.6) for potential vorticity with respect to z and using the relations (2.4) and (4.1) to eliminate the velocity shear. This yields

$$\mathbf{u}\cdot\nabla(fQ_z) = gJ_{xy}. \qquad (4.9)$$

With (4.2) one obtains for α the expression

$$\alpha = g\frac{\mathbf{k}\cdot(\nabla\rho\times\nabla Q)}{(\nabla\rho\times\nabla Q)\cdot\nabla(fQ_z)} = \frac{gJ_{xy}}{J_{yz}(fQ_z)_x - J_{xz}(fQ_z)_y + J_{xy}(fQ_z)_z} \qquad (4.10)$$

which involves third-order derivatives of density (Needler, 1982).

It is evident from this derivation that other representations of α in terms of ρ exist which involve higher order derivatives: further differentiation of (4.9) with respect to z yields

$$\mathbf{u}\cdot\nabla(f^2Q_{zz}) = gf(J_{xy})_z + gJ_{xy}(\rho, fQ_z) \qquad (4.11)$$

which results in an expression for α with fourth order derivatives of the density field

$$\alpha = g\frac{f(J_{xy})_z + J_{xy}(\rho, fQ_z)}{J_{yz}(f^2Q_{zz})_x - J_{xz}(f^2Q_{zz})_y + J_{xy}(f^2Q_{zz})_z}. \qquad (4.12)$$

We may proceed along this line to generate more representations of α with increasing order of derivatives. The possibility to derive closed representations for $\mathbf{u}$ in terms of the density field simply by successive vertical differentiation of (2.2) and use of (2.4) and (4.1) was implicit in the considerations of Killworth (1979) but only recently Needler (1982) has drawn attention to it.

What can we learn from Needler's formula (4.10) for the configuration of the LNAM of the second kind? If we assume that ρ has finite gradients α can only vanish if $J_{xy} = 0$ which is a surface in the fluid. If the denominator in (4.11) does not happen to vanish on this surface too, we would conclude that α vanishes on the surface $J_{xy} = 0$ which then would represent a stagnation surface in the fluid. However, the situation is not that simple as revealed by consideration of (4.12). It is not at all evident from this representation that the vanishing of J_{xy} implies $\alpha = 0$. The vanishing of the numerator of (4.12) generally defines an entirely different surface. Apparently one cannot consider the behavior of the numerators of these expressions apart from the behavior of the denominators. The complete dynamics identify α with F_Q/Q [Eq. (3.2)]. This indeed shows that the vanishing of J_{xy} is not sufficient for a zero in α.

So far no attempts have been made to apply Needler's formula to the observed density field. It may be of interest, however, to point out a property of the representation (4.10) which is not readily apparent from the derivation. The β-spiral approach makes use of both thermal wind relations (4.1) and the resulting current profile will satisfy both these relations for any given density field. Needler's representation (4.10) does not have this property. Only the component of the thermal wind orthogonal to the horizontal velocity $(u, v, 0)$ is used in the derivation of (4.11), i.e., the relation

$$J_{xz}u_z + J_{yz}v_z = \frac{g}{f}(\rho_y J_{xz} - \rho_x J_{yz}) \qquad (4.13)$$

will hold for any density field if u_z and v_z are computed from (4.10). The component of the thermal wind parallel to $(u, v, 0)$ has not been used so that the relation

$$J_{yz}u_z - J_{xz}v_z = \frac{g}{f}(\rho_y J_{yz} + \rho_x J_{xz}) \qquad (4.14)$$

will generally not be reproduced with an arbitrary density field. Notice, however, that Needler's formula simultaneously conserves density and potential vorticity which is not the case for the β-spiral approach.

5. Summary

We have collected some aspects of ideal fluid theory with the aim to investigate the structure of the flow close to a level of no motion. Since density ρ and potential vorticity $Q = f\rho_z$ are conserved the intersections of the surfaces ρ = constant and Q = constant define the streamlines of the flow and the direction of the velocity vector is given by $\nabla\rho \times \nabla Q$. If this vector happens to be orthogonal to a coordinate axis at one point the corresponding velocity component must vanish there. The condition for the occurrence for such a level of no motion in one velocity component (a LNM) may thus be expressed in terms of the vanishing of a Jacobian of density and potential vorticity. The LNMs for one component generally form a surface and the interaction of such surfaces for different components generally form a curve where the fluid is entirely motionless. Thus, on purely dimensional grounds, such a configuration (a LNAM of the first kind in this paper) where the velocity vector vanishes will be rare compared to a LNM in one component.

These simple types of levels of no-motion entirely derive from the directional properties of the velocity vector. Any other type must be associated with the vanishing of the factor α in the representation $\mathbf{u} = \alpha\nabla\rho \times \nabla Q$ and hence be a zero for the velocity vector. Since α must be a function of position, $\alpha = 0$ would generally define a surface where the fluid is stagnant. We have searched for evidence for the occurrence of such a configuration (termed a LNAM of the second kind in this paper), in diagnostic models and in exact solutions of the ideal fluid equations.

As a convenient way to generate exact solutions of the ideal fluid equation we have used Welander's (1971a) approach that is based on a functional relation between density ρ, potential vorticity Q and Bernoulli function B in the fluid, say $B = F(\rho, Q)$. The ideal fluid equations then identify α with a function of ρ and Q, i.e., $\alpha = F_Q(\rho, Q)/Q$. We have argued within the general framework of the F-function approach and exemplified by an exact solution for a specific $F(\rho, Q)$ that on a fluid surface defined by $F_Q/Q = 0$ the density field is most likely to become singular. Only very specific boundary conditions can prevent a singular behaviour on the surface $F_Q/Q = 0$. Thus, $\mathbf{u} = (F_Q/Q)\nabla\rho \times \nabla Q$ does not necessarily vanish where F_Q/Q is zero. Since we do not believe in a singular flow field in the ocean we conclude that the oceanic $F(\rho, Q)$ would be such that F_Q/Q does not become zero. Then the only levels of no-motion for the entire velocity vector are those described above as LNAM of the first kind.

The result from β-spiral calculations are consistent with the sole occurrence of the LNAMs of the first kind. The w-equation approach of Killworth (1980) presumes the existence of a LNAM of the second kind and determines the absolute velocity profile by explicitly suppressing the singular behaviour of the flow which originates from the *a priori* assumption of a LNAM of the second kind. Needler's formula (Needler, 1982) expresses α in terms of the density field but gives no clear statement on the vanishing of α.

To summarize we found no convincing evidence that the ideal fluid equations with sufficiently general boundary conditions are compatible with a stagnation surface in the fluid.

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