Gravitational waves from spinning binary black holes at the leading post-Newtonian orders at all orders in spin

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We determine the binding energy, the total gravitational wave energy flux, and the gravitational wave modes for a binary of rapidly spinning black holes, working in linearized gravity and at leading orders in the orbital velocity, but to all orders in the black holes' spins. Though the spins are treated nonperturbatively, surprisingly, the binding energy and the flux are given by simple analytical expressions which are finite (respectively third- and fifth-order) polynomials in the spins. Our final results are restricted to the important case of quasi-circular orbits with the black holes' spins aligned with the orbital angular momentum.

I. INTRODUCTION

The general relativistic two-body problem, in particular the description of compact binaries as one of the most important sources of gravitational waves (GWs) detectable from Earth, poses a large challenge for analytical as well as numerical calculations. Accurate waveform models and templates generated by such calculations are utilized in matched filtering techniques to confidently assign gravitational wave signals to compact binaries, extract information about the source, and investigate possible deviations from the predictions of general relativity (GR). Recent detections of such signals [1–7] have demonstrated the success of these approaches, but the need for more accurate and more general descriptions of binary dynamics in GR persists and will grow with future more sensitive GW detectors.

Several analytical approaches to the two-body problem in GR have been developed over the past century. The predominant method for describing the dynamics and predicting the form of the emitted radiation for arbitrary-mass-ratio compact binaries employs the post-Newtonian (PN) approximation [8, 9]. The PN approach seeks to extract information from full GR by perturbatively expanding about the slow-motion, weak-field regime. Applied to a binary system, the computed dynamics and waveforms are expected to provide accurate predictions only in the early inspiral phase. However, analytic results from the PN approximation, together with numerical descriptions of the late inspiral, plunge, and merger, can be combined synergetically in effective-one-body (EOB) models [10, 11]. In such, the binary's evolution can be accurately modeled from the early inspiral stage through the merger and ringdown of the remnant. In order for EOB models to efficiently extract maximal information from detected gravitational wave signals, accurate analytical and numerical input must be provided. Thus, efforts continue to push PN theory to ever increasing accuracy.

The PN approximation is an expansion of a compact binary's full general relativistic dynamics in the dimensionless small parameter $\epsilon_{\rm PN} \sim v^2/c^2 \sim Gm/c^2r$.[1] This amounts to using $1/c^2$ as the formal expansion parameter; an $\mathcal{O}(c^{-2n})$ contribution is said to be at nPN order. The approximation has been pushed to 4PN order in the conservative sector [12, 13] and to 3PN for the gravitational wave modes [8, 14] for binaries with constituents represented as structureless (monopole) point particles, which could describe non-spinning binary black holes (BBHs). In the seminal work of Mathisson, Papapetrou [15–17], and later Dixon [18, 19], finite-size effects for an extended body in GR have been modeled in the form of a multipolar structure associated with the body. Subsequently, spin is incorporated in PN theory through spin-dependent multipole moments attached to a point-particle representation of the body. Spin, besides the field strength and velocity, is typically treated perturbatively in PN calculations, expanding in the dimensionless small parameter $\epsilon_{\rm spin} \sim Gm\chi/rc^2$.[2] For rapidly rotating black holes, one finds that $\epsilon_{\rm spin} \sim \epsilon_{\rm PN}$. Previously, the conservative dynamics of spinning BBHs was computed up to 4PN order [20]; for the leading orders at various orders in spin see Refs. [21–33]. For spin-dependent leading orders in the radiative sector see Refs. [26, 32, 34, 35]. The traditional approach has been to truncate the expansion in $\epsilon_{\rm spin}$ as accords with the order counting made natural by the relation $\epsilon_{\rm spin} \sim \epsilon_{\rm PN}$. At 4PN order, this corresponds to a truncation at fourth order in the spins.

In this paper, we treat $\epsilon_{\rm PN}$ and $\epsilon_{\rm spin}$ as independent expansion parameters, and we determine the conservative and radiative dynamics of an arbitrary-mass-ratio BBH at leading PN order (leading order in $\epsilon_{\rm PN}$), but to *all orders in spin* (no expansion in $\epsilon_{\rm spin}$) [33]. Therefore, we obtain terms which are, according to the traditional PN counting in $1/c^2$, of *arbitrarily large PN order*. Our final results are valid for circular orbits and spins aligned with

[1] As usual, G is Newton's constant, c the speed of light, m the binary's mass scale, v its orbital velocity, and r the orbital distance.

[2] Here, $\chi = cS/Gm^2$ is the dimensionless spin parameter for a body of mass m and spin S, with $\chi \in [0, 1)$ for a black hole.

the orbital angular momentum (though several intermediate results are applicable to more general situations). This is an important case, since waveforms for aligned (nonprecessing) spins can approximate *precessing* waveforms very well when viewed from a certain frame [36] and hence can form the basis for precessing waveform models. The nonperturbative aspect of our results can improve the synergetic EOB waveform model by suggesting improved resummations of perturbative PN results in future work.

In the framework of an effective action principle, we model the BBH as two point-particles with infinite sets of spin-induced multipole moments, fixed by matching to the Kerr metric. We consider the coupling of this multipolar structure to gravity at first post-Minkowskian (PM) order (i.e. in linearized gravity), in Sec. II. In Sec. III, the near zone field equations are solved at leading post-Newtonian order (at leading orders in the orbital velocity) and the binding energy of the binary is computed for circular orbits and spins aligned with the orbital angular momentum. Lastly, for the same orbital configuration and at the same level of approximation, the source multipole moments of the complete system are computed and employed to determine the GW modes and total GW energy flux emitted by the BBH in Sec. IV.

Throughout the paper, Greek letters $\mu,\nu,\alpha,\beta,...$ are used as spacetime (abstract or coordinate-basis) indices. After Sec. II, Latin letters $i,j,k,a,b,...$ are used as spatial indices. Various other types of Latin indices are used as indicated in the text. We exploit the multi-index notation $L := \mu_1\dots\mu_\ell$ for ℓ tensorial powers of a vector v^μ, such that $v^L = v^{\mu_1\cdots\mu_\ell} := v^{\mu_1}\dots v^{\mu_\ell}$, as well as $L-1 = \mu_l\dots\mu_{\ell-1}$, etc., both for spacetime indices as here and for spatial indices, $L = i_1\dots i_\ell$, the distinction being clear from the context. Our sign convention for the volume form is such that $\epsilon_{0123} = +1$ in a local Minkowskian basis, and our sign convention for the Riemann tensor is such that $2\nabla_{[\mu}\nabla_{\nu]}w_\alpha = R_{\mu\nu\alpha}{}^\beta w_\beta$.

II. EFFECTIVE ACTION FOR A SPINNING BLACK HOLE

In this section, we review the construction of an effective action functional for localized spinning body coupled to gravity, assuming the body has only translational and rotational degrees of freedom and only spin-induced multipole moments, as is appropriate for a spinning black hole at the orders considered in this paper. We begin in Sec. II A with a general such spinning body, seeing how its translational and rotational kinematics are linked to its universal (i.e. body-independent) monopole and dipole couplings to gravity. In Sec. II B, we consider the couplings of the body's higher-order spin-induced multipoles (quadrupole, octupole, etc.) to the spacetime curvature, working at linear order in the curvature, and then specialize to the multipole structure of a spinning (Kerr) black hole. While those two subsections work in what could be (in principle) a general curved spacetime, maintaining general covariance, we specialize to the case of first-post-Minkowskian spacetime (a linear perturbation of flat spacetime) in Sec. II C.

A. Effective action of a spinning point-particle

The effective translational and rotational degrees of freedom for a spinning point-particle can be taken to be an arbitrarily parametrized worldline $x = z(\lambda)$, with tangent $u^\mu = dz^\mu/d\lambda$, and a "body-fixed" tetrad $\varepsilon_A{}^\mu(\lambda)$ (defined only along the worldline) with orthonormality conditions

$$\varepsilon_A{}^\mu\varepsilon^{A\nu} = g^{\mu\nu}, \qquad \varepsilon_{A\mu}\varepsilon_B{}^\mu = \eta_{AB}, \tag{1}$$

where η_{AB} is a frame Minkowski metric, and $g^{\mu\nu}$ is the spacetime (inverse) metric (evaluated at $x = z$) which is used to raise or lower all spacetime indices in this subsection and the next. For later convenience, we also define throughout spacetime a "global" tetrad field $e_a{}^\mu$ with analogous orthonormality conditions. The two tetrads are related by a local Lorentz transformation at each point on the worldline: $\varepsilon_A{}^\mu = \Lambda_A{}^a e_a{}^\mu$. The angular velocity tensor,

$$\Omega^{\mu\nu} = \varepsilon_A{}^\mu\frac{D\varepsilon^{A\nu}}{d\lambda}, \tag{2}$$

with $\Omega^{\mu\nu} = -\Omega^{\nu\mu}$, serves as a measure of the particle's rotation along the worldline. Of the six (local Lorentz transformation) degrees of freedom in the body-fixed tetrad $\varepsilon_A{}^\mu$, we expect three to be physical, describing the body's (spatial) rotation, while the other three (boosts) are redundant with the translational degrees of freedom. We can remove this redundancy by fixing the timelike vector $\varepsilon_0{}^\mu$ to be the direction of the worldline tangent, or velocity,

$$\varepsilon_0{}^\mu = U^\mu := u^\mu/\sqrt{-u_\rho u^\rho}. \tag{3}$$

This constraint on the tetrad will translate into a corresponding constraint on the body's spin tensor.

An effective action for the spinning point-particle can be given as a functional of the worldline $z(\lambda)$, the body-fixed tetrad $\varepsilon_A{}^\mu(\lambda)$, and the spin tensor $S_{\mu\nu}(\lambda)$ conjugate to the tetrad, by [37–39]

$$S_{\text{p.p.}}[z,\varepsilon,S] = \int d\lambda\left\{-m\sqrt{-u_\mu u^\mu} + \frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} + \mathcal{L}_c\right\}, \tag{4}$$

where m is the conserved (bare) rest mass. Here, the first two terms serve as translational and rotational "kinetic terms," and they also implicitly encode the monopole- and dipole-type couplings to the spacetime geometry, through $u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu$ and through the covariant derivative in (2), respectively; the dipole couplings are explicitly extracted at the end of this subsection. The couplings between the body's higher multipoles and the spacetime curvature are contained in $\mathcal{L}_c(z, U, S)$ as specified in the following subsection.

The constraint (3) on the tetrad translates [37, 40] into the following constraint on the spin tensor,

$$S^{\mu\nu}U_\nu = 0, \tag{5}$$

serving as a spin supplementary condition (SSC) [15, 16, 41], which we will refer to as the *covariant SSC*.[3] In order to simplify our computations below, we implement a change of variables from the rotational degrees of freedom as appearing in the action (4). As in [37, 40, 44], we apply a local Lorentz transformation to the body-fixed tetrad,

$$\varepsilon_A{}^\nu \to \tilde{\varepsilon}_A{}^\mu = L^\mu{}_\nu \varepsilon_A{}^\nu, \tag{6}$$

such that the timelike vector $\varepsilon_0{}^\mu = U^\mu$ as in (3) is boosted into the direction of the timelike vector $e_0{}^\mu$ of the global tetrad: $\tilde{\varepsilon}_0{}^\mu = L^\mu{}_\nu \varepsilon_0{}^\nu = e_0{}^\mu$. This is accomplished with the standard boost

$$L^\mu{}_\nu = \delta^\mu_\nu - 2e_0{}^\mu U_\nu + \frac{\omega^\mu \omega_\nu}{-U_\rho \omega^\rho}, \quad \omega^\mu = U^\mu + e_0{}^\mu. \tag{7}$$

Under (6) with (7), the second term of (4) becomes [37]

$$\frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} = \frac{1}{2}\tilde{S}_{\mu\nu}\tilde{\Omega}^{\mu\nu} + \tilde{S}_{\mu\nu}U^\mu \frac{DU^\nu}{d\lambda}, \tag{8}$$

where $\tilde{\Omega}^{\mu\nu} = \tilde{\varepsilon}_A{}^\mu (D\tilde{\varepsilon}^{A\nu}/d\lambda)$, and where the new spin tensor $\tilde{S}^{\mu\nu}$ is defined by

$$S^{\mu\nu} = \mathcal{P}^\mu_\alpha \mathcal{P}^\nu_\beta \tilde{S}^{\alpha\beta}, \tag{9}$$

$$0 = \tilde{S}^{\mu\nu}(U_\nu + e_{0\nu}), \tag{10}$$

with

$$\mathcal{P}^\mu_\nu = \delta^\mu_\nu + U^\mu U_\nu \tag{11}$$

being the projector orthogonal to U^μ.

We can explicitly extract the dipole coupling to the spacetime geometry by switching from the coordinate basis to the global tetrad basis and from covariant derivatives to ordinary derivatives, recalling $\varepsilon_A{}^\mu = \Lambda_A{}^a e_a{}^\mu$,

$$\tilde{S}_{\mu\nu}\tilde{\Omega}^{\mu\nu} = \tilde{S}_{ab}\tilde{\Lambda}_A{}^a \frac{d\tilde{\Lambda}^{Ab}}{d\lambda} + \omega_\mu{}^{ab}\tilde{S}_{ab}u^\mu, \tag{12}$$

$$\frac{DU^\nu}{d\lambda} = \left[\frac{dU^a}{d\lambda} + \omega_\beta{}^{ba}U_b u^\beta\right] e_a{}^\nu, \tag{13}$$

where $\omega_\mu{}^{ab} = e^b{}_\alpha \nabla_\mu e^{a\alpha} = e^b{}_\alpha \partial_\mu e^{a\alpha} + e^b{}_\alpha \Gamma^\alpha{}_{\mu\beta} e^{a\beta}$ are the Ricci rotation coefficients for the global tetrad. Finally, using (8)–(13), the action (4) reads

$$S_{\text{p.p.}} = \int d\lambda \left\{ -m\sqrt{-u_\mu u^\mu} + \frac{1}{2}\tilde{S}_{ab}\tilde{\Lambda}_A{}^a \frac{d\tilde{\Lambda}^{Ab}}{d\lambda} + \tilde{S}_{ab}U^a \frac{dU^b}{d\lambda} + \frac{1}{2}\omega_\mu{}^{ab} S_{ab}u^\mu + \mathcal{L}_c \right\}. \tag{14}$$

Starting from the following subsection, we drop the tildes used here to indicate the boosted variables; the use of the un-boosted variables is restricted to this subsection.

Given (5) and (9)–(10), the full components of both spin tensors, $S_{\mu\nu}$ and $\tilde{S}_{\mu\nu}$, are determined by U^μ, $e_0{}^\mu$, and the *covariant spin vector*

$$S_\mu := U^\nu (*S)_{\nu\mu} = U^\nu (*\tilde{S})_{\nu\mu}, \tag{15}$$

satisfying $S_\mu U^\mu = 0$, where $(*S)_{\nu\mu} = \frac{1}{2}\epsilon_{\nu\mu}{}^{\alpha\beta}S_{\alpha\beta}$ is the spin tensor's dual, and similarly for $(*\tilde{S})_{\nu\mu}$, with $\epsilon_{\mu\nu\alpha\beta}$ being the volume form.

B. Multipole-curvature couplings

So far, only the monopole and dipole couplings, arising from the first two terms in (4), have been fixed. Here we fix the couplings of the higher-order spin-induced multipoles to the spacetime curvature, in the $\mathcal{L}_c$ term in (4), by considering all possible combinations of the relevant degrees of freedom which would contribute in a first post-Minkowskian approximation (but not yet making that approximation)—this corresponds to keeping only terms linear in the Riemann tensor and its derivatives.

In constructing the curvature couplings, it is convenient to decompose the (vacuum) Riemann (or Weyl) tensor of the (external) gravitational field into its electric (even under parity) and magnetic (odd under parity) components with respect to the normalized velocity U^μ,

$$\begin{aligned} \mathcal{E}_{\alpha\beta} &= R_{\alpha\mu\beta\nu}U^\mu U^\nu, \\ \mathcal{B}_{\alpha\beta} &= (*R)_{\alpha\mu\beta\nu}U^\mu U^\nu, \end{aligned} \tag{16}$$

where $(*R)_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu\gamma\delta}$ is the dual of the Riemann tensor.

We can now build the linear-in-curvature couplings in $\mathcal{L}_c$ out of $\mathcal{E}_{\mu\nu}$ and $\mathcal{B}_{\mu\nu}$ and their derivatives and the available point-particle degrees of freedom, namely, only the normalized tangent U^μ and the spin-vector S_μ, recalling $S^\mu u_\mu = 0$. We require reparametrization invariance, which means that the tangent u^μ can enter only through its normalized version U^μ, and invariance under internal rotations of the body-fixed tetrad $\Lambda_A{}^a$, which means that $\Lambda_A{}^a$ cannot enter explicitly. We also require invariance under parity transformations, noting that U^μ is even while S^μ is odd. Then, all possible linear-in-curvature

[3] At the order considered in this paper, this SSC is equivalent to the condition $S^{\mu\nu}p_\nu = 0$ due to Tulczyjew, where p_ν is the linear momentum appearing in the Mathisson-Papapetrou-Dixon equations [15–19], which in general leads to a better behaved motion of the center [42, 43]. The time evolution defined by the action does not automatically preserve the SSC. The SSC can either be added to the action with a Lagrange multiplier, or one can insert a solution of the SSC into the action.

couplings are given by [37] (see also Refs. [45, 46])

$$\mathcal{L}_{\rm c} = -\sqrt{-u_\rho u^\rho}\Bigg\{ \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\ell)!}\frac{C_{\mathcal{E},\ell}}{m^{2\ell-1}} S^{2L}\nabla_{2L-2}\mathcal{E}_{\mu_{2\ell-1}\mu_{2\ell}} - \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!}\frac{C_{\mathcal{B},\ell}}{m^{2\ell}} S^{2L+1}\nabla_{2L-1}\mathcal{B}_{\mu_{2\ell}\mu_{2\ell+1}} \Bigg\}, \tag{17}$$

where $C_{\mathcal{B},\ell}$ and $C_{\mathcal{E},\ell}$ are dimensionless constants (Wilson coefficients), the spacetime multi-indices are written out as $2L = \mu_1 \dots \mu_{2\ell}$, $2L \pm 1 = \mu_1 \dots \mu_{2\ell\pm1}$, etc., and it is understood that ∇_μ does not act on the factors of U_μ in (16). Note, the monopole (no-spin) and dipole (spin-orbit) couplings, i.e. the $\ell = 0$ terms, are not included here, since those are already provided outside of $\mathcal{L}_c$ in (14).

As written in (17), $\mathcal{L}_c$ describes the higher-multipole couplings of an arbitrary body with spin-induced multipole moments. In order to specialize to the case where the multipoles match those of a spinning (Kerr) black hole, we set all the C coefficients to unity: $C_{\mathcal{B},\ell} = C_{\mathcal{E},\ell} = 1$ for all ℓ. This can be justified retrospectively, e.g., by matching the binding energy of a BBH with one spinning and one nonspinning black hole to the binding energy for a geodesic in the Kerr spacetime [47] at leading PN order.

C. Linear approximation

We now specialize to a first-post-Minkowskian (or linearized) approximation, writing the spacetime metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(G^2), \tag{18}$$

where $h_{\mu\nu} \sim \mathcal{O}(G)$ is the linear metric perturbation, using G as a formal expansion parameter. Ultimately, we aim for a leading-order post-*Newtonian* (PN) approximation for the binary dynamics in the near zone of the source, which will be obtained in the following section by starting from the post-Minkowskian results discussed here and then re-expanding at leading orders in the orbital velocity.

When the effective point-particle action (for now, for just one black hole) is added to the Einstein-Hilbert action for the gravitational field, using the post-Minkowskian expansion (18) of the metric, we obtain a total effective action of the form

$$S_{\rm eff}[\mathbf{h},\mathbf{T}] = S_{\rm G}[\mathbf{h}] + S_{\rm kin}[\mathbf{T}] + S_{\rm int}[\mathbf{h},\mathbf{T}] + \mathcal{O}(G^2), \tag{19}$$

where $\mathbf{h}$ represents the degrees of freedom of the gravitational field, the metric perturbation $h_{\mu\nu}(x)$, and $\mathbf{T} = \{m, z(\lambda), \Lambda_A{}^a(\lambda), S^\mu(\lambda)\}$ represents the spinning point-particle degrees of freedom. The term $S_{\rm G}[\mathbf{h}]$ is the Einstein-Hilbert action at leading order in $\mathbf{h}$, which can be written as

$$\mathcal{S}_G[\mathbf{h}] = -\frac{1}{64\pi G}\int d^4x\, \partial_\rho h_{\mu\nu} P^{\mu\nu\alpha\beta}\partial^\rho h_{\alpha\beta}, \tag{20}$$

while enforcing the harmonic gauge condition,

$$\partial_\mu \bar{h}^{\mu\nu} = 0, \tag{21}$$

where $\bar{h}^{\mu\nu}$ is the trace-reversed metric perturbation,

$$\begin{aligned} \bar{h}^{\mu\nu} &:= P^{\mu\nu\alpha\beta}h_{\alpha\beta}, \\ P^{\mu\nu\alpha\beta} &:= \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\nu\alpha}\eta^{\mu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}). \end{aligned} \tag{22}$$

The *kinematic* term $S_{\rm kin}[\mathbf{T}]$ and the *interaction* term $S_{\rm int}[\mathbf{h},\mathbf{T}]$ in (19) are respectively the zeroth- and first-order terms in the expansion in $\mathbf{h}$ of the spinning point-particle action functional from Sec. II A,

$$S_{\rm p.p.}[\mathbf{h},\mathbf{T}] = S_{\rm kin}[\mathbf{T}] + S_{\rm int}[\mathbf{h},\mathbf{T}] + \mathcal{O}(h^2). \tag{23}$$

We now proceed to find explicit forms for these terms, while establishing appropriate conventions.

We choose to parameterize the worldline with coordinate time, $\lambda = t$, in an asymptotically Minkowskian coordinate system $x^\mu = (x^0, x^i) = (t, x^i)$. This implies $u^0 = 1$, as well as $u^i = v^i := dx^i/dt$ for the 3-velocity. We define the usual Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{-u_\rho u^\rho}} + \mathcal{O}(h), \tag{24}$$

where, here and *henceforth*, the Minkowski metric $\eta_{\alpha\beta}$ is used to raise and lower all spacetime indices; note also $v^2 = v^i v^j \delta_{ij}$. The directional derivative along the particle's worldline is denoted with a dot

$$\frac{d}{d\lambda} \to \frac{d}{dt} =: \dot{} \,. \tag{25}$$

We fix the freedom in the choice of the global tetrad $e_a{}^\mu$ be taking it to be the symmetric square root of the metric,

$$e_a{}^\mu = \delta_a^\nu\left(\delta_\nu^\mu - \frac{1}{2}h^\mu{}_\nu\right) + \mathcal{O}(h^2). \tag{26}$$

With these conventions, taking the $\mathcal{O}(h^0)$ part of $S_{\rm p.p.}$ from (14) (with tildes removed) to obtain $S_{\rm kin}$ in (23), noting that $\Lambda_0{}^\mu = \delta_0^\mu$ as follows from (3) and (6)–(7), we find

$$S_{\rm kin}[\mathbf{T}] = \int dt \Bigg\{ -\frac{m}{\gamma} + \frac{1}{2}S_{ij}\Lambda_K{}^i\dot{\Lambda}^{Kj} + \frac{1}{2}S_{ij}v^i\dot{v}^j \Bigg\}, \tag{27}$$

with $K = 1, 2, 3$.

For the $\mathcal{O}(h^1)$ interaction part of $S_{\rm p.p.}$ in (23), we write $S_{\rm int} = \int dt\,(\mathcal{L}^{\rm pole\text{-}}_{\rm dipole} + \mathcal{L}_{\rm c})$, and the pole-dipole terms, arising from the expansions of the first and fourth terms in (14) are found to be

$$\mathcal{L}^{\rm pole\text{-}}_{\rm dipole} = \frac{m}{2}U^\mu U^\nu \left[h_{\mu\nu} + \epsilon_{\nu\rho}{}^{\alpha\beta}a^\rho\partial_\alpha h_{\beta\mu}\right] + \mathcal{O}(\dot{h}), \tag{28}$$

where we use the mass-rescaled spin vector

$$a^\mu := \frac{S^\mu}{m}. \tag{29}$$

Notice, we neglect here time derivatives of $h_{\mu\nu}$; via integration-by-parts, noting $\dot{U}^\mu = \mathcal{O}(h)$ and $\dot{a}^\mu = \mathcal{O}(h)$, these become $O(h^2)$ terms and total time derivative terms. (Neglecting the total time derivatives corresponds to an implicit redefinition of variables [48].)

Finally, for the $\mathcal{O}(h)$ part of the higher-multipole couplings in $\mathcal{L}_c$ (17), we can use the expansions

$$\begin{aligned}\mathcal{E}_{\alpha\beta} &= -\frac{1}{2}U^\mu U^\nu \partial_\alpha\partial_\beta h_{\mu\nu} + \mathcal{O}(\dot{h}, h^2),\\ \mathcal{B}_{\alpha\beta} &= \frac{1}{2}U^\mu U^\nu \epsilon_{\nu(\beta}{}^{\gamma\tau}\partial_{\alpha)}\partial_\gamma h_{\tau\mu} + \mathcal{O}(\dot{h}, h^2)\end{aligned} \tag{30}$$

of the curvature tensors. We then find that it is natural to combine the contributions from $\mathcal{L}^{\text{pole-}}_{\text{dipole}}$ and $\mathcal{L}_c$ by extending the sums in (17) to $\ell = 0$. Having fixed $C_{\mathcal{B},\ell} = C_{\mathcal{E},\ell} = 1$ to ensure matching with the Kerr spacetime, as outlined above, the full interaction term is expressed as

$$\begin{aligned}S_{\text{int}}[\mathbf{h}, \mathbf{T}] = \int dt \, \Bigg\{ \sum_{\ell=0}^{\infty} \frac{mU^\mu U^\nu}{2\gamma\, \ell!} \text{Re}\Big[i^\ell a^L \partial_L h_{\mu\nu} \\ + i^{\ell-1} a^\mu \epsilon_{\nu\mu}{}^{\alpha\beta}\partial_\alpha a^{L-1}\partial_{L-1} h_{\beta\mu}\Big]\Bigg\},\end{aligned} \tag{31}$$

which matches a result in [33]. Having completed the relevant post-Minkowskian expansions, we henceforth set $G = 1$.

III. CONSERVATIVE DYNAMICS

The building blocks from the previous section are now combined to give an effective description of a BBH at leading post-Newtonian order, but to all orders in spin. The underlying field equations are derived from the full effective BBH action

$$\begin{aligned}S^{\text{BBH}}_{\text{eff}}[\mathbf{h}, \mathbf{T}_1, \mathbf{T}_2] &= S_G[\mathbf{h}] \\ &+ \{S_{\text{kin}}[\mathbf{T}_1] + S_{\text{int}}[\mathbf{h}, \mathbf{T}_1] + (1 \leftrightarrow 2)\},\end{aligned} \tag{32}$$

with two copies of the kinetic and interaction terms as discussed above, for each of the black holes, 1 and 2. A slow-motion approximation is achieved by expansion in the orbital velocity v up to linear order. Our near-zone (NZ) solution $h^{\mu\nu}_{\text{NZ}}$, obtained in the no-retardation limit, agrees with results presented in [49, 50]. We follow the Fokker-action approach used in [51] to derive the equation of motion of the orbital parameters. A Hamiltonian encoding equivalent equations of motion has been obtained using effective field theoretic tools in [33]. Finally, we put the equations of motion into explicit form, solve them, under the above assumption of circular spin-aligned motion, and obtain the conserved energy and conserved orbital angular momentum of the BBH to all orders in spin and at the leading PN orders.

A. Field equations and near zone solution

In the full effective BBH action, the metric perturbation was considered at leading post-Minkowskian order. The linearized field equations, in harmonic coordinates, are obtained by varying (32) with respect to the fields $h_{\mu\nu}$. Integrating by parts, and dropping vanishing boundary terms, yields the field equations

$$\begin{aligned}\Box\bar{h}^{\mu\nu}_A &= -16\pi T^{\mu\nu}_A,\\ T^{\mu\nu}_A &= m_A\gamma_A \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\text{Re}\Big[i^l u^\mu_A u^\nu_A a^L_A \partial_L \delta_A \\ &\quad + i^{l-1}u^\sigma_A a^\rho_A \epsilon_{\sigma\rho}{}^{\alpha(\mu}u^{\nu)}_A a^{L-1}_A \partial_{\alpha L-1}\delta_A\Big],\end{aligned} \tag{33}$$

for black holes $A = 1, 2$, where we split $\bar{h}^{\mu\nu} = \bar{h}^{\mu\nu}_1 + \bar{h}^{\mu\nu}_2$. We have used $\int dt = \int d^4x\, \delta_A$, with $\delta_A := \delta[\mathbf{x} - \mathbf{z}_A(t)]$ in (31).

The general solution to the inhomogeneous wave equation is well-known. The retarded inverse d'Alembertian integral operator, defined by $\bar{h}^{\mu\nu} = -16\pi\Box^{-1}_{\text{ret}}T^{\mu\nu}$, reduces as $(\Box^{-1}_{\text{ret}}T^{\mu\nu})(\mathbf{x}, t) \to (\Delta^{-1}T^{\mu\nu})(\mathbf{x}, t)$ at leading PN order in the NZ, where [8]

$$(\Delta^{-1}T^{\mu\nu})(\mathbf{x}, t) = -\frac{1}{4\pi}\int d^3x' \frac{T^{\mu\nu}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}. \tag{34}$$

Retardation effects would contribute only at next-to-leading PN orders. Applying Δ^{-1} to the effective energy-momentum tensor in (33), the NZ linearized gravitational field of the Ath black hole in the binary is explicitly given by

$$\begin{aligned}\bar{h}^{\mu\nu}_{\text{NZ},A}(t, \mathbf{x}) &= 4m_A\gamma_A \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\text{Re}\Big[i^l u^\mu_A u^\nu_A a^L_A \partial_L r^{-1}_A \\ &\quad + i^{l-1}u^\sigma_A a^\rho_A \epsilon_{\sigma\rho}{}^{\alpha(\mu}u^{\nu)}_A a^{L-1}_A \partial_{\alpha L-1} r^{-1}_A\Big],\end{aligned} \tag{35}$$

with $r_A := |\mathbf{x} - \mathbf{z}_A(t)|$. This solution has been obtained in the linear post-Minkowskian approximation, but it still contains non-linear-in-velocity contributions at each order in spin.

The leading-order slow-motion approximation is achieved by truncating the NZ solutions (35) after linear-in-v terms. This yields a leading PN expansion at each order in spin. Carefully excluding the higher-order v terms (e.g., noticing that $\gamma = 1 + \mathcal{O}(v^2)$ and $a^0 = \mathcal{O}(v)$), the trace of the solution $h^{\mu\nu}_{\text{NZ},A} = P^{\mu\nu}{}_{\alpha\beta}\bar{h}^{\alpha\beta}_{\text{NZ},A}$ is given by

$$\begin{aligned}h_{\text{NZ},A} &= \bar{h}^{00}_{\text{NZ},A} - \bar{h}^{ij}_{\text{NZ},A}\delta_{ij}\\ &= 4m_A\mathcal{D}_C[\mathbf{a}_A]r^{-1}_A + \mathcal{O}(v^2).\end{aligned} \tag{36}$$

Here, we made use of the differential operator

$$\begin{aligned}\mathcal{D}_C[\mathbf{a}] &:= \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{(2\ell)!}a^{2L}\partial_{2L}\\ &= \cosh(\mathbf{a}\times\nabla),\end{aligned} \tag{37}$$

and define for later convenience

$$\begin{aligned}\mathcal{D}_S^i[\mathbf{a}] :=& \; - a^j \epsilon_j{}^{ik} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)!} a^{2L} \partial_{2Lk} \\ =& \; [\sinh(\mathbf{a}\times\nabla)]^i .\end{aligned} \tag{38}$$

Here $\nabla = \partial/\partial\mathbf{x}$, and the index i in (38) labels the components of the operator with respect to the chosen basis (i.e., can be raised and lowered with δ_{ij}). Additionally, both operators obey the usual hyperbolic trigonometric identities.

The NZ gravitational fields $h^{\mu\nu}_{\mathrm{NZ},A}$ of a spinning black hole at 1PM order is

$$h^{00}_{\mathrm{NZ},A} = -2\phi_A + \mathcal{O}(v^2), \qquad \phi_A := \left\{ -\mathcal{D}_C[\mathbf{a}_A] + 2v_i^{(A)} \mathcal{D}_S^i[\mathbf{a}_A] \right\} \frac{m_A}{r_A}, \tag{39a}$$

$$h^{0j}_{\mathrm{NZ},A} = A^j_A + \mathcal{O}(v^2), \qquad A^i_A := \left\{ 4v^i_A \mathcal{D}_C[\mathbf{a}_A] - 2\mathcal{D}_S^i[\mathbf{a}_A] \right\} \frac{m_A}{r_A}, \tag{39b}$$

$$h^{ij}_{\mathrm{NZ},A} = \frac{1}{2} h_{\mathrm{NZ},A}\delta^{ij} + \sigma^{ij}_A + \mathcal{O}(v^2), \qquad \sigma^{ij}_A := -4 v_A^{(j} \mathcal{D}_S^{i)}[\mathbf{a}_A] \frac{m_A}{r_A}. \tag{39c}$$

The complete leading-PN-order NZ solution of the BBH

$$h^{\mathrm{NZ,BBH}}_{\mu\nu} = h^{\mathrm{NZ},1}_{\mu\nu} + h^{\mathrm{NZ},2}_{\mu\nu} + \mathcal{O}(v^2) \tag{40}$$

is obtained by superposing the gravitational field of both black holes.

B. Equations of motion for separation vector

In the following, the focus lies on the derivation of the set of equations of motion describing the binary's separation vector $\boldsymbol{r} := \mathbf{z}_1 - \mathbf{z}_2$, with $r = |\boldsymbol{r}|$. From this, the spin corrections to the Newtonian orbital parameter ω, the angular velocity, are obtained. These equations for $\boldsymbol{r}$ result from the kinematic behavior of the black holes in the time-dependent near-zone field (40). This behavior is encoded in

$$\begin{aligned} S^{\mathrm{EoM}}_{\mathrm{eff}}[\mathbf{h}^{\mathrm{NZ},1}, \mathbf{h}^{\mathrm{NZ},2}, \mathbf{T}_1, \mathbf{T}_2] = \frac{1}{2} S_{\mathrm{int}}[\mathbf{h}^{\mathrm{NZ},2}, \mathbf{T}_1] \\ + S_{\mathrm{kin}}[\mathbf{T}_1] + (1 \leftrightarrow 2), \end{aligned} \tag{41}$$

where we made again use of the functionals (27) and (31). Note, we made use of $S_G[\mathbf{h}^{\mathrm{NZ},1} + \mathbf{h}^{\mathrm{NZ},2}] = -1/2 S_{\mathrm{int}}[\mathbf{h}^{\mathrm{NZ},1}, \mathbf{T}_2] + (1 \leftrightarrow 2)$, plus (divergent) h_1^2 and h_2^2 terms which do not influence the equations of motion. The equation relating the frequency ω to the radius r will be obtained from the equations of motion for $\boldsymbol{r}$, restricted to circular spin-aligned motion.

One of the interaction terms in (41) is given by

$$\begin{aligned} S_{\mathrm{int}}[\mathbf{h}^{\mathrm{NZ},2}, \mathbf{T}_1] = \frac{m_1\gamma_1}{2} \int dt \sum_{l=0}^{\infty} \frac{1}{l!} \mathrm{Re}\Big[i^l u_1^\mu u_1^\nu a_1^L \partial_L h^{\mathrm{NZ},2}_{\mu\nu}\Big|_{\mathbf{z}_1} \\ + i^{l-1} u_1^\sigma a_1^\rho \epsilon_{\sigma\rho}{}^{\alpha\mu} u_1^\nu a_1^{L-1} \partial_{\alpha L-1} h^{\mathrm{NZ},2}_{\mu\nu}\Big|_{\mathbf{z}_1} \Big]. \end{aligned} \tag{42}$$

By redefining $\nabla := \partial/\partial\mathbf{z}_1$ (which we use throughout the rest of the paper), and noting that the differential operators $\mathcal{D}_C[\mathbf{a}]$ and $\mathcal{D}^i_S[\mathbf{a}]$ in (37) and (38) change accordingly, the interaction terms can be combined into

$$\begin{aligned} S_{\mathrm{int}}[\mathbf{h}^{\mathrm{NZ},2}, \mathbf{T}_1] + S^{\mathrm{BH}}_{\mathrm{int}}[\mathbf{h}^{\mathrm{NZ},1}, \mathbf{T}_2] = \\ \int dt \left\{ \Big[2\mathcal{D}_C[\mathbf{a}_0] + 4v_i \mathcal{D}^i_S[\mathbf{a}_0] \Big] \frac{m_1 m_2}{r} + \mathcal{O}(v^2) \right\}, \end{aligned} \tag{43}$$

Hyperbolic trigonometric identities have been used to combine the operators, and we have defined

$$\begin{aligned} \mathbf{a}_0 &:= \mathbf{a}_1 + \mathbf{a}_2, \\ \boldsymbol{v} &:= \boldsymbol{v}_1 - \boldsymbol{v}_2. \end{aligned} \tag{44}$$

The kinetic terms follow directly from (27). With $S_{ij} = \epsilon_{ijk} S^k$, both kinetic terms in (41) read

$$\begin{aligned} S_{\mathrm{kin}}[\mathbf{T}_1] + S_{\mathrm{kin}}[\mathbf{T}_2] = \int dt \Big\{ -m_1 \\ + \frac{m_1}{2} v_1^2 + \frac{1}{2} \mathbf{S}_1 \cdot (\boldsymbol{v}_1 \times \ddot{\mathbf{z}}_1) + (1 \leftrightarrow 2) \Big\}. \end{aligned} \tag{45}$$

Then, the effective action (41), together with (43) and (45), provides the full conservative dynamics of the binary.

At this stage it is convenient to change the coordinate system on the spatial slices. The coordinate origin is chosen to be the center of mass of the BBH. This amounts to setting the conserved quantity associated with boost symmetry of (41) to zero, resulting in the coordinate transformation

$$\begin{aligned} \mathbf{z}_1 = \frac{m_2}{M}\boldsymbol{r} - \boldsymbol{b}, \qquad \mathbf{z}_2 = -\frac{m_1}{M}\boldsymbol{r} - \boldsymbol{b}, \\ \boldsymbol{b} = \frac{1}{M}(\boldsymbol{v}_1 \times \mathbf{S}_1 + \boldsymbol{v}_2 \times \mathbf{S}_2), \end{aligned} \tag{46}$$

where $M = m_1 + m_2$ is the total mass of the system, and $\mu = M^{-1} m_1 m_2$ the reduced mass. Similar relations are

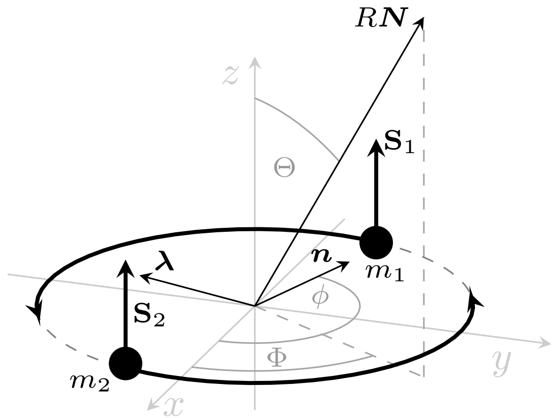

FIG. 1. The BBH configuration is illustrated for the case of $m_1 = m_2$ and $\mathbf{S}_1 = \mathbf{S}_2$. The coordinates are chosen such that the orbital plane coincides with the (x, y)-plane: $\hat{z} \equiv \boldsymbol{\ell}$. Finally, $\boldsymbol{N}$, introduced together with R in (68), is pointing radially outwards, parameterized by Θ and Φ in the usual way.

found for the velocities $\boldsymbol{v}_A = \dot{\mathbf{z}}_A$. The unit vectors

$$\boldsymbol{n} := \frac{\boldsymbol{r}}{r}, \qquad \boldsymbol{\lambda} := \frac{\boldsymbol{v}}{v}, \tag{47}$$

with $v = |\boldsymbol{v}|$, span the orbital plane. Finally, a third vector $\boldsymbol{\ell}$ is constructed to be orthonormal to the orbital plane. Then the three vectors $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ pose a positively-oriented triad, $\boldsymbol{n} \times \boldsymbol{\lambda} = \boldsymbol{\ell}$. In FIG. 1 the BBH configuration in the center of mass frame is depicted for the case of equal masses and spin vectors.

Applying this transformation at the level of the action, and utilizing $df/dt = (\boldsymbol{v} \cdot \nabla) f$, for any $f(\boldsymbol{r})$, the variation of (41) with respect to the wordline of the center of mass yields

$$\begin{aligned} \ddot{r}^j = \; & \Big\{ \partial^j \mathcal{D}_C[\mathbf{a}_0] + v^i \sigma^{*k} \epsilon_{kl}{}^j \partial_i \partial^l \mathcal{D}_C[\mathbf{a}_0] \\ & - 2\Big[\delta^j_i v^k \partial_k - \partial^j v_i\Big] \mathcal{D}^i_S[\mathbf{a}_0] \Big\} \frac{M}{r}, \end{aligned} \tag{48}$$

the equation of motion for the seperation vector. Here we followed the conventions of [33], defined

$$M \boldsymbol{\sigma}^* = m_2 \mathbf{a}_1 + m_1 \mathbf{a}_2 \tag{49}$$

and used $\partial_i = \partial / \partial z^i_1$. The action of the differential operators $\mathcal{D}_S[\mathbf{a}_0]$ and $\mathcal{D}^i_S[\mathbf{a}_0]$ on r^{-1} is presented in detail in the Appendix. Utilizing the results shown there, the implicit equations of motion (48) simplify to the explicit form

$$\ddot{\boldsymbol{r}} = -\boldsymbol{r} \left[\frac{M}{(r^2 - a_0^2)^{3/2}} - \frac{vM(\sigma^* + 2a_0)}{r(r^2 - a_0^2)^{3/2}} \right], \tag{50}$$

where $a_0 = \boldsymbol{\ell} \cdot \mathbf{a}_0$ and $\sigma^* = \boldsymbol{\ell} \cdot \boldsymbol{\sigma}^*$.

C. Orbital parameter and conserved energy

The equations of motion (50) allow a wide variety of different solutions for the separation vector $\boldsymbol{r}$. However, in this paper we only focus on the special case of circular motion with aligned spins. In this case, the spins are constant vectors (as can be verified from their equations of motion, not presented here), and the acceleration $\ddot{\boldsymbol{r}}$ is directly proportional to $\boldsymbol{r}$, with the squared angular velocity serving as proportionality constant:

$$\ddot{\boldsymbol{r}} = -\omega^2 \boldsymbol{r}. \tag{51}$$

Comparing (50) and (51), we find

$$\omega^2 = \frac{M}{r} \frac{r - v(2a_0 + \sigma^*)}{(r^2 - a_0^2)^{3/2}}, \tag{52}$$

for the radius-frequency relationship. For convenience, we introduce the standard PN expansion parameter

$$x = (\omega M)^{2/3}. \tag{53}$$

Then the separation r of the black holes, at leading post-Newtonian order and to all orders in spin, denoted by LO-S$^\infty$, is split into even- and odd-in-spin parts

$$r_{\text{LO-S}^\infty}(x) = r_{\text{even}}(x) + r_{\text{odd}}(x), \tag{54}$$

where

$$\begin{aligned} r_{\text{even}}(x) &= \sqrt{\frac{M^2}{x^2} + a_0^2}, \\ r_{\text{odd}}(x) &= -\, r_{\text{even}}(x) \frac{x^{3/2} M}{3} \frac{\sigma^* + 2a_0}{M^2 + x^2 a_0^2}. \end{aligned} \tag{55}$$

Expanding this result yields a leading PN spin expansion of the form described in TABLE I.

Finally, the conserved energy of the BBH, assuming invariance under time translations of (41), reduces to

$$E(\omega) = \mu \left(\frac{1}{2} r^2 \omega^2 + r^2 \omega^3 \sigma^* - \frac{M}{\sqrt{r^2 - a_0^2}} \right). \tag{56}$$

Specializing to circular orbits (i.e., $r = r_{\text{LO-S}^\infty}$), the conserved energy of the BBH to all orders in spin at leading post-Newtonian order, is

$$\begin{aligned} E_{\text{LO-S}^\infty}(x) = -\frac{\mu x}{2} \Big\{ & 1 + \frac{x^{3/2}}{3M}(7a_0 + \delta a_-) - \frac{x^2 a_0^2}{M^2} \\ & - \frac{x^{7/2} a_0^2}{M^3}(a_0 - \delta a_-) \Big\}, \end{aligned} \tag{57}$$

where $\nu = \mu M^{-1}$, $\delta = M^{-1}(m_1 - m_2)$ and $\mathbf{a}_- = \mathbf{a}_1 - \mathbf{a}_2$, with $\boldsymbol{\ell} \cdot \mathbf{a}_- = a_-$.

In this special case, the coefficients A_p, from TABLE I, vanish for $p > 3$. Thus, remarkably, the spin-expansion of the binding energy of the binary terminates after cubic-in-spin contributions. Therefore, our result coincides

		S^0	S^1	S^2	S^3	S^4	...	S^{p} [a]	S^{p+1} [a]	...
LO	0PN	$A_0 \cdot 1$		$A_2 \cdot x^2\chi^2$		$A_4 \cdot x^4\chi^4$	...	$A_p \cdot x^p\chi^p$		...
	0.5PN		$A_1 \cdot x^{3/2}\chi$		$A_3 \cdot x^{7/2}\chi^3$		...		$A_{p+1} \cdot x^{p+3/2}\chi^{p+1}$	...
NLO	1PN	$B_0 \cdot x$		$B_2 \cdot x^3\chi^2$		$B_4 \cdot x^5\chi^4$	...	$B_p \cdot x^{p+1}\chi^p$		...
	1.5PN		$B_1 \cdot x^{5/2}\chi$		$B_3 \cdot x^{9/2}\chi^3$		...		$B_{p+1} \cdot x^{p+5/2}\chi^{p+1}$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋱

[a] Here p is even.

TABLE I. The results presented in this paper are expansions *separately* in the traditional post-Newtonian parameter $\epsilon_{\rm PN} \sim v^2 \sim m/r \sim x$ and the spin expansion parameter $\epsilon_{\rm spin} \sim a/r \sim x\chi$ (In this table, χ serves as book keeping parameter for the spin expansion). Different rows correspond to different powers of $\epsilon_{\rm PN}$ (LO stands for leading-order, and NLO for next-to-leading order), and different columns to different powers of $\epsilon_{\rm spin}$. Let p be even. Then there is an absolute leading order term of the expansion in $x \sim \epsilon_{\rm PN}$, given by $A_p \cdot \epsilon^p_{\rm spin} = A_p \cdot x^p \chi^p$, with coefficient $A_p = A_p(m_1, m_2, \mathbf{a}_1, \mathbf{a}_2)$. This is the leading post-Newtonian term at that order in spin. Similary, at (p+1)th order in spin, the absolute leading order term, in the expansion of $x \sim \epsilon_{\rm PN}$, is $A_{p+1} \cdot \sqrt{\epsilon_{\rm PN}}\epsilon^{p+1}_{\rm spin} = A_{p+1} \cdot x^{p+3/2}\chi^{p+1}$. In this work we only focus on the leading PN order, i.e., $B_p \equiv 0$ for all p, which we denote by LO-S$^\infty$. Every expression (e.g., conerved energy, total energy flux etc.), can be written as a sum over p of all LO terms with different coefficients A_p (up to a multiplicative function depending on masses and x).

with that of [32] in the special case of aligned spins. However, we extended the validity of the expression presented there to all orders in spin. Note that the truncation at cubic order in spin hinges on the use of x as the variable in the binding energy. Compare also to the result for neutron stars to quartic order in spin [20], where the constants $C_{\mathcal{E},\ell}$ and $C_{\mathcal{B},\ell}$ are not unity as for black holes and A_4 does not vanish. In general, for neutron stars all coefficients A_p in $E_{\text{LO-S}^\infty}(x)$ are nonzero.

The justification for the choice of coefficients in (17) can now be presented. We define the dimensionless Kerr spin-parameter $\boldsymbol{\chi}_A = m_A \mathbf{a}_A$ of the individual black holes in the considered BBH. Then, in the limit of $m_2/m_1 \to 0$, the conserved energy $E_{\text{LO-S}^\infty}$ reduces to the binding energy associated with geodesic motion in Kerr spacetime [47] (characterized by spin-parameter $m\chi$), under the identification $\boldsymbol{\chi}_1 \cdot \boldsymbol{\ell} \to \chi$. A small deviation of the coefficients $C_{\mathcal{E},\ell}$ and $C_{\mathcal{B},\ell}$ from unity would have lead to additional terms in the test-body limit of $E_{\text{LO-S}^\infty}$. The resulting binding energy would have not matched the one for geodesic motion in the Kerr solution. Thus, $C_{\mathcal{E},\ell} = C_{\mathcal{B},\ell} = 1$ is the unique choice, to approximate the Kerr solution at 1PM order.

Additionally, one can equivalently describe the conservative dynamics with the conserved angular momentum. In the special case of spin-aligned circular motion the total and the orbital angular momentum are conserved. The orbital angular momentum $L(\omega) = \boldsymbol{L} \cdot \boldsymbol{\ell}$ is always orthogonal to the orbital plane:

$$L(\omega) = \mu \left(\omega r^2 + \frac{3}{2}\omega^2 r^2 \sigma^* - \frac{2Ma_0}{\sqrt{r^2 - a_0^2}} \right). \tag{58}$$

Using $r_{\text{LO-S}^\infty}(x)$, the spin expansion of the orbital angular momentum in terms of the PN parameter x terminates, similar to the binding energy, at cubic-in-spin contributions:

$$L_{\text{LO-S}^\infty}(x) = \mu x^{-1/2} \Big\{ M - \frac{5}{12} x^{3/2}(7a_0 + \delta a_-) + \frac{x^2 a_0^2}{M} + \frac{3x^{7/2}}{4M^2} a_0^2 (a_0 - \delta a_-) \Big\}. \tag{59}$$

IV. FAR ZONE MODES AND ENERGY FLUX

In this section, the far zone dynamics of the binary black hole is analyzed. Again, this is done in the leading post-Newtonian approximation scheme employed before, where all spin-induced multipole moments are considered. The gravitational wave modes and the total energy flux emitted by the binary are determined at future null infinity. To do so, a set of source multipole moments, of the complete BBH, is constructed. Our results for these source moments agree with those presented in [32] (before inserting the solution for the orbital separation r). Utilizing these source moments, we obtain the total gravitational wave energy flux and the gravitational wave modes at the leading PN order and to all orders in the black holes' spin.

A. Source multipole moments

We consider the mass and current type source multipole moments, $\mathcal{I}_L(\tilde{t})$ and $\mathcal{J}_L(\tilde{t})$ respectively, at leading post-Newtonian order. Here we introduced the Euclidean distance R from the above defined center of mass of the system to a far zone spacetime point, which enables us to include the retardation by $\tilde{t} = t - R$. Notice, differentiation with respect to $\tilde{t}$ and t are equivalent and will be denoted by an overdot as before.

The complete energy-momentum distribution of the BBH, the sum of the individual contributions (33), is

$$T^{00} = \Big\{ \mathcal{D}_C[\mathbf{a}_1] - v_i^{(1)} \mathcal{D}_S^i[\mathbf{a}_1] \Big\} m_1\delta_1 + (1 \leftrightarrow 2) + \mathcal{O}(v^2),$$
$$T^{0j} = \frac{1}{2}\Big\{ 2v_1^j \mathcal{D}_C[\mathbf{a}_1] - \mathcal{D}_S^j[\mathbf{a}_1] \Big\} m_1\delta_1 + (1 \leftrightarrow 2) + \mathcal{O}(v^2),$$
$$T^{ij} = - v_1^{(i}\mathcal{D}_S^{j)}[\mathbf{a}_1] m_1\delta_1 + (1 \leftrightarrow 2) + \mathcal{O}(v^2). \tag{60}$$

The slow-motion approximation is achieved by discarding nonlinear velocity contributions.

In the following, angular brackets $\langle \dots \rangle$ denote a symmetric-trace-free (STF) projection of the respective indices. Furthermore, from the above energy-momentum tensor (60) can be seen that $\ddot{T}^{ij} \sim \mathcal{O}(v^3)$ and $\dot{T}^{ij} \sim \mathcal{O}(v^2)$ contribute at next-to-leading order. Considering this, the multipole moments of the post-Newtonian source reduce to [8]

$$\mathcal{I}_L = \mathrm{FP}\int d^3x \int_{-1}^{1} dz \Big\{ \delta_\ell(z) x_{\langle L\rangle}(T^{00} + T^{ij}\delta_{ij}) - \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)}\delta_{\ell+1}(z) x_{\langle aL\rangle}\dot{T}^{0a}\Big\}, \tag{61}$$
$$\mathcal{J}_L = \mathrm{FP}\int d^3x \int_{-1}^{1} dz \; \epsilon^a{}_{b\langle i_\ell}\Big\{ \delta_\ell(z) x_{L-1\rangle a} T^{0b}\Big\}.$$

In these expressions, the energy-momentum tensor $T^{\mu\nu} = T^{\mu\nu}(\mathbf{x}, \tilde{t} + zr)$ is a function of the extended time, which, together with the associated weighting function $\delta_l(z)$, takes the finite size of the source, and the resulting time retardation, into account. As argued above, at leading post-Newtonian order the finite-size-retardation vanishes, since

$$\int_{-1}^{1} dz\, \delta_\ell(z) T^{\mu\nu}(\mathbf{x}, \tilde{t} + zr) = T^{\mu\nu}(\mathbf{x}, \tilde{t}) + \mathcal{O}(v^2). \tag{62}$$

In principle, the total energy flux, as well as the gravitational wave modes, depend on all source multipole moments. However, as discussed below, only the mass- and current-quadrupole and -octopole contain leading PN information.

In that context, the mass-type moments of the compact binary are

$$\mathcal{I}^{ij} = m_1 z_1^{\langle ij\rangle} + \frac{4}{3}\left(2v_1^a S_1^b \epsilon_{ab}{}^{\langle i} z_1^{j\rangle} - z_1^a S_1^b \epsilon_{ab}{}^{\langle i} v_1^{j\rangle}\right) - \frac{1}{m_1} S_1^{\langle i} S_1^{j\rangle} + (1 \leftrightarrow 2) + \mathcal{O}(v^2)$$
$$\mathcal{I}^{ijk} = m_1 z_1^{\langle ijk\rangle} + \frac{3}{2}\left(3v_1^a S_1^b \epsilon_{ab}{}^{\langle i} z_1^{jk\rangle} - 2z_1^a S_1^b \epsilon_{ab}{}^{\langle i} v_1^j z_1^{k\rangle}\right) - \frac{3}{m_1} S_1^{\langle i} S_1^j z_1^{k\rangle} - \frac{3}{2m_1^2} v_1^a S_1^b \epsilon_{ab}{}^{\langle i} S_1^j S_1^{k\rangle} + (1 \leftrightarrow 2) + \mathcal{O}(v^2) \tag{63a}$$

and similarly the current-type moments are

$$\mathcal{J}^{ij} = m_1 z_1^a v_1^b \epsilon_{ab}{}^{\langle i} z_1^{j\rangle} + \frac{3}{2} S_1^{\langle i} z_1^{j\rangle} + \frac{1}{2m_1} 2v_1^a S_1^b \epsilon_{ab}{}^{\langle i} S_1^{j\rangle} + (1 \leftrightarrow 2) + \mathcal{O}(v^2),$$
$$\mathcal{J}^{ijk} = m_1 z_1^a v_1^b \epsilon_{ab}{}^{\langle i} z_1^{jk\rangle} + 2S_1^{\langle i} z_1^{jk\rangle} + \frac{1}{m_1}\left(2v_1^a S_1^b \epsilon_{ab}{}^{\langle i} S_1^j z_1^{k\rangle} - z_1^a v_1^b \epsilon_{ab}{}^{\langle i} S_1^j S_1^{k\rangle}\right) - \frac{2}{3m_1^2} S_1^{\langle i} S_1^j S_1^{k\rangle} + (1 \leftrightarrow 2) + \mathcal{O}(v^2). \tag{63b}$$

B. The total gravitational wave energy flux

The total GW energy flux can be directly obtained from the source multipole moments computed in the last section with the well-known formula [52]

$$\mathcal{F} = \sum_{\ell=2}^{\infty}\Big\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!}\dot{\mathcal{U}}_L\dot{\mathcal{U}}^L + \frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}\dot{\mathcal{V}}_L\dot{\mathcal{V}}^L\Big\}. \tag{64}$$

$\mathcal{U}_L$ and $\mathcal{V}_L$ depend on the ℓ-th time derivative of the source moments $\mathcal{I}_L$ and $\mathcal{J}_L$, as well as auxiliary source moments W_L, X_L, Y_L and Z_L as described in [8]. However, at leading post-Newtonian order: $\mathcal{U}_L = \mathcal{I}_L^{(\ell)}$ and $\mathcal{V}_L = \mathcal{J}_L^{(\ell)}$, with $f^{(\ell)}(\tilde{t}) = d^\ell/dt^\ell f(t-R)$. In the spin-aligned configuration, the only time-dependent quantities in (63) are $\boldsymbol{v}(\tilde{t}) = v\boldsymbol{\lambda}(\tilde{t})$ and $\boldsymbol{r}(\tilde{t}) = r\boldsymbol{n}(\tilde{t})$, where $\boldsymbol{\lambda}^{(\ell)} \sim \mathcal{O}(x^{3\ell/2})$ and $\boldsymbol{n}^{(\ell)} \sim \mathcal{O}(x^{3\ell/2})$. Taking also the spin contributions from $r = r_{\text{LO-S}^\infty}(x)$ into account,[4] only the source quadrupole moments contribute at leading PN order. Therefore, expression (64) reduces to

$$\mathcal{F} = \frac{1}{5}\dddot{\mathcal{I}}_{ij}\dddot{\mathcal{I}}^{ij} + \frac{16}{45}\dddot{\mathcal{J}}_{ij}\dddot{\mathcal{J}}^{ij}. \tag{65}$$

Carefully combining the individual leading PN order contributions at each spin order, the total gravitational wave energy flux of the binary black hole simplifies to

$$\begin{aligned}\mathcal{F}_{\text{LO-S}^\infty} = \frac{\mu^2 x^5}{M^2}\Bigg[\frac{32}{5} - \frac{8x^{3/2}}{5M}\Big\{8a_0 + 3\delta a_-\Big\} + \frac{2x^2}{5M^2}\Big\{32a_0^2 + a_-^2\Big\} - \frac{4x^{7/2}}{15M^3}\Big\{16a_0^3 + 2a_0a_-^2 + 52\delta a_0^2 a_- + \delta a_-^3\Big\} \\ + \frac{2x^4a_0^2}{5M^4}\Big\{16a_0^2 + a_-^2\Big\} + \frac{2a_0^2x^{11/2}}{15M^5}\Big\{64a_0^3 + a_0a_-^2 - 68\delta a_0^2 a_- - 3\delta a_-^3\Big\}\Bigg].\end{aligned} \tag{66}$$

Similar to the conserved energy, the total energy flux assumes the pattern described in TABLE I, where in this case $A_{p>5} = 0$. $\mathcal{F}_{\text{LO-S}^\infty}$ reproduces the results presented in [32] up to cubic-in-spin effects. Again, the infinite sets of spin-induced multipolar interactions of the two black holes remarkably cancel out at higher than quintic-in-spin contributions. Hence, the total energy flux conveys full information about the spin effects at leading PN order in the first five terms of the spin expansion.

C. Far zone gravitational wave modes $h^{\ell m}$

The angular distribution of the energy flux, as well as the frequencies, are encoded in the gravitational wave modes $h^{\ell m}$. In the proceedings, we follow the conventions of [53]. As outlined above, we choose to describe the radiative dynamics in Cartesian coordinates on the background, with the center of mass of the BBH at the origin. Then, the defined spatial triad $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ can be written as

$$\begin{aligned}\boldsymbol{n} &= (\cos\omega\tilde{t}, \sin\omega\tilde{t}, 0),\\ \boldsymbol{\lambda} &= (-\sin\omega\tilde{t}, \cos\omega\tilde{t}, 0),\\ \boldsymbol{\ell} &= (0,0,1).\end{aligned} \tag{67}$$

Additionally, we define the radially outwards pointing 3-vector $\boldsymbol{N}$ by

$$\boldsymbol{N} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta), \tag{68}$$

such that $\boldsymbol{R} = R\boldsymbol{N}$ (see also FIG. 1, with $\omega\tilde{t} = \phi$). Furthermore, the STF spherical harmonics $\mathcal{Y}_L^{\ell m}$ are defined by $Y^{\ell m}(\Theta,\Phi) = \mathcal{Y}_L^{\ell m} N^L$, with the usual spherical harmonics: $Y^{\ell m}(\Theta,\Phi)$.

[4] Together with $v = \omega r = x^{3/2}M^{-1}r(x)$.

The gravitational wave modes $h^{\ell m}$ are the projections of the polarization waveforms $h_+ - ih_\times$ onto the spin weighted spherical harmonics ${}_{-2}Y^{\ell m}(\Theta,\Phi)$ with weight $s = -2$:

$$h_+ - ih_\times = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} h^{\ell m}\ {}_{-2}Y^{\ell m}(\Theta,\Phi). \tag{69}$$

Making use of the Wigner d-function, the spin weighted spherical harmonics are

$${}_{-s}Y^{\ell m}(\Theta,\Phi) = (-1)^s\sqrt{\frac{2\ell+1}{4\pi}}\,d^\ell_{m,s}(\Theta)e^{im\Phi}, \tag{70}$$

where

$$\begin{aligned}d^\ell_{m,s}(\Theta) &= N_{\ell,m}\\ &\times\sum_{k=k_{\max}}^{k_{\min}}\frac{(-1)^k(\sin\Theta/2)^{2k+s-m}(\cos\Theta/2)^{2\ell+m-s-2k}}{k!(\ell+m-k)!(\ell-s-k)!(s-m+k)!},\end{aligned} \tag{71}$$

with

$$\begin{aligned}N_{\ell,m} &= \sqrt{(\ell+m)!(\ell-m)!(\ell+s)!(\ell-s)!},\\ k_{\max} &= \max(0, m-s),\\ k_{\min} &= \min(\ell+m, \ell-s).\end{aligned} \tag{72}$$

Here s is referred to as the spin weight and ℓ and m are the usual labels know from spherical harmonics. The far zone gravitational wave modes are generally given by

$$h^{\ell m} = \frac{1}{\sqrt{2}R}\left[\,\mathcal{U}^{\ell m}(\tilde{t}) - i\mathcal{V}^{\ell m}(\tilde{t})\right], \tag{73}$$

with

$$\mathcal{U}^{\ell m} = \frac{16\pi}{(2\ell+1)!!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}}\,\mathcal{U}^L\mathcal{Y}_L^{\ell m *}, \tag{74}$$

$$\mathcal{V}^{\ell m} = -\frac{32\pi\ell}{(2\ell+1)!!}\sqrt{\frac{(\ell+2)}{2\ell(\ell+1)(\ell-1)}}\,\mathcal{V}^L\mathcal{Y}_L^{\ell m *}, \tag{75}$$

where * denotes complex conjugation.

We present the gravitational wave modes $h^{\ell m}$, such that the polarization waveform (69) only contains leading order post-Newtonian information at each order in spin. However, expanding $h_+ - ih_\times$ in $\epsilon_{\rm PN}$ (as described in TABLE I) we find that $h^{\ell\leq 3,m}$ contribute exclusively – only the mass- and current quadrupole, as well as their octopolar companions, are needed. In order to organize the presentation, we factorize the time- and distance-dependence, $\tilde{t}$ and R respectively, out as follows:

$$h^{\ell m}_{\text{LO-S}^\infty} = \frac{\sqrt{\pi}}{M^3 R}\hat{h}^{\ell m} e^{-im\omega\tilde{t}}. \tag{76}$$

Finally, at leading PN order at all orders in spin, the associated $\hat{h}^{\ell m}$ modes are $\hat{h}^{20} = 0 = \hat{h}^{30}$ and

$$\hat{h}^{22} = -\frac{8\mu x}{3\sqrt{5}}\Big\{3M^3 - M^2 x^{3/2}(3a_0 + a_-\delta) + 3x^2 a_0^2 M + 2x^{7/2}a_0^2(a_0 - a_-\delta)\Big\}, \tag{77a}$$

$$\hat{h}^{21} = -\frac{2i\mu x^{3/2}}{3\sqrt{5}}\frac{1}{\sqrt{M^2 + x^2 a_0^2}}\Big\{4\delta M^4 - 6M^3 x^{1/2} a_- + 2M^2 x^2\Big[2a_0 a_- + 4\delta a_0^2 + \delta a_-^2\Big] - 6Mx^{5/2}a_0^2 a_- + x^4 a_0^2\Big[3\delta a_-^2 + 4\delta a_0^2 - a_0 a_-\Big]\Big\}, \tag{77b}$$

$$\hat{h}^{33} = 3i\sqrt{\frac{6}{7}}\mu\delta x^{3/2}(M^2 + x^2 a_0^2)^{3/2}, \tag{77c}$$

$$\hat{h}^{32} = -\frac{8\mu x^{5/2}}{3\sqrt{7}}(a_0 - \delta a_-)(M^2 + x^2 a_0^2), \tag{77d}$$

$$\hat{h}^{31} = -\frac{i}{3}\sqrt{\frac{2}{35}}\mu x^{3/2}(M^2 + x^2 a_0^2)^{1/2}\Big\{M^2\delta + a_0 x^2(a_0\delta - 4a_-)\Big\}. \tag{77e}$$

The remaining modes are obtained exploiting the relation $h^{\ell,-m} = (-1)^\ell h^{\ell m *}$. The spin expansion of even-in-m modes terminates at a finite order. The polarization waveform (69), together with (77), contains all possible spin effects at leading post-Newtonian order in the case of the circular motion and the black holes' spins aligned with the orbital angular momentum.

V. CONCLUSION

We determined the binding energy, the gravitational wave modes and total energy flux emitted by a spinning non-precessing binary black hole in quasi-circular motion at leading post-Newtonian orders at all orders in spin. Our results include contributions of arbitrarily large PN order, counting in $1/c^2$. In particular, we obtained for the first time the quartic-in-spin contributions to the 4PN waveform and total energy flux, along with all higher-order-in-spin contributions at the corresponding leading PN orders. Remarkably, the binding energy, the total energy flux, as well as some of the gravitational wave modes only contain a finite number of non-zero spin contributions at leading post-Newtonian order. For instance, we showed that previously found results for the binding energy are now valid, without additional corrections, to all orders in spin.

Conversely, the modes where all powers in spin appear are nevertheless rather compact, which can be used to improve the resummation of modes, e.g., in the synergetic EOB waveform model [54]. Though our results are only valid for aligned spins, they can still be used to approximate waveforms from precessing binaries [36]. We leave the investigation of precessing systems, or analysis of possibly similar resummations of all spin orders at next-to-leading post-Newtonian order, to future work.

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Appendix: Action of $\mathcal{D}_C[\mathbf{a}]$ and $\mathcal{D}^i_S[\mathbf{a}]$ on r^{-1}

In the following, the action of the differential operators $\mathcal{D}_C[\mathbf{a}]$ and $\mathcal{D}^i_S[\mathbf{a}]$, defined in (37) and (38) respectively, on $r^{-1} = |\mathbf{z}_1 - \mathbf{z}_2|^{-1}$ are presented in detail. Recall from Sec. III B, the vectors $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ pose a positively oriented triad (i.e., $\mathbf{n}\times\boldsymbol{\lambda} = \boldsymbol{\ell}$) on the spatial slices of spacetime, where $\boldsymbol{r} = r\boldsymbol{n}$. Let us assemble the tools first [55]. In the this appendix, we use $\partial_i = (\partial/\partial \mathbf{z}_1)_i$; therefore, the simple identity

$$\partial_M r^{-1} = (-1)^m (2m-1)!! \frac{n_{\langle M \rangle}}{r^{m+1}}, \tag{A.1}$$

where $\langle \dots \rangle$ indicates a symmetric trace-free (STF) projection, holds. Let $P_\ell(x)$ be the ℓth Legendre polynomial, then the STF contraction of two arbitrary unit vectors $\mathbf{h}$ and $\mathbf{h}'$, with relative angle $\alpha = \mathbf{h} \cdot \mathbf{h}'$, can be shown to be

$$h'_{\langle M \rangle} h^{\langle M \rangle} = \frac{m! P_m(\alpha)}{(2m-1)!!}. \tag{A.2}$$

Lastly, we recall that

$$P_m(0) = \begin{cases} \frac{(-1)^n}{4^n}\binom{2n}{n}, & m = 2n \\ 0, & \text{otherwise}, \end{cases} \tag{A.3}$$

with binomial coefficient $\binom{k}{l}$. Due to spin alignment, we only need to consider arbitrary vectors $\mathbf{k} = k\boldsymbol{\ell}$, where $k = \mathbf{k} \cdot \boldsymbol{\ell}$. Thus, using (37), we find

$$\mathcal{D}_C[\mathbf{k}] r^{-1} = \sum_{m=0}^{\infty} (-1)^m k^{2m} \frac{P_{2m}(\boldsymbol{n} \cdot \boldsymbol{\ell})}{r^{2m+1}}. \tag{A.4}$$

Since $P_{2m}(\boldsymbol{n} \cdot \boldsymbol{\ell})$ is independent of $\boldsymbol{r}$, the circular motion restriction can be used at this stage (even when additional derivatives need to be taken). We obtain, after making use of (A.3) and resumming the resulting series,

$$\begin{aligned} \mathcal{D}_C[\mathbf{k}] r^{-1} &= \sum_{m=0}^{\infty} \binom{2n}{n} \left(\frac{k}{2}\right)^{2m} r^{-2m-1} \\ &= \frac{1}{\sqrt{r^2 - k^2}}. \end{aligned} \tag{A.5}$$

Any additional spatial differentiation of this result with ∂ can be done simply by using the first line of (A.5). For instance, the expression $\partial^j \mathcal{D}_C[\mathbf{k}] r^{-1}$ can be computed by noting $\partial_i r = n_i$. Using (A.1), and resumming the resulting expression, we acquire

$$\partial^j \mathcal{D}_C[\mathbf{k}] r^{-1} = \frac{-r^j}{(r^2 - k^2)^{3/2}}. \tag{A.6}$$

The action of $\mathcal{D}_S^i[\mathbf{k}]$ on r^{-1} is obtained similarly. Using (38), the differential operator is given by

$$\mathcal{D}_S^i[\mathbf{k}] r^{-1} = k^j \epsilon_j{}^{id} \sum_{m=0}^{\infty} (-1)^m k^{2m} \frac{n_d P_{2m}(\boldsymbol{n} \cdot \boldsymbol{\ell})}{r^{2m+4}}. \tag{A.7}$$

For example, applied to the case $v^j \partial_j \mathcal{D}_S^i[\mathbf{k}] r^{-1}$, using (A.1) and resumming, we find

$$v^j \partial_j \mathcal{D}_S^i[\mathbf{k}] r^{-1} = \frac{v k r^i}{r^3 \sqrt{r^2 - k^2}}, \tag{A.8}$$

where $v = |\boldsymbol{v}| = v\boldsymbol{\lambda}$. The necessary expressions for (48) can gathered in this way. We find:

$$\partial^j v_i \mathcal{D}_S^i[\mathbf{k}] r^{-1} = -\, r^j \frac{vk}{r^3} \frac{k^2 - 2r^2}{(r^2 - k^2)^{3/2}}, \tag{A.9}$$

$$v^i \sigma^{*k} \epsilon_{kl}{}^j \partial_i \partial^l \mathcal{D}_C[\mathbf{k}] r^{-1} = r^j \frac{v\sigma^*}{r} \frac{1}{(r^2 - k^2)^{3/2}}. \tag{A.10}$$

Here, as defined in the text, $\sigma^* = \boldsymbol{\ell} \cdot \boldsymbol{\sigma}^*$.

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